粘性的双曲型保存則の解の漸近挙動

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0. まえがき

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の研究成果報告書である。本研究は、

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によってなされた。

さて、双曲型保存則の方程式系は、非線型波として、衝撃波と希薄波をもつ。実在の系では何らかの粘性効果が働き、衝撃波が平滑化された粘性的衝撃波、希薄波、及び散逸波が現れる。この粘性的双曲型保存系の Cauchy 問題の解は、適当な初期条件によって、これらの非線型波、またはそれらの重ね合せに漸近することが期待される。

この研究で扱った問題は、(1) 双曲型保存系に通常の Newton 粘性による項が付いた方程式系の Cauchy 問題、半空間における境界値問題、(2) 双曲型保存系に摩擦による効果を考慮した方程式系の Cauchy 問題、半空間における境界値問題に大別される。

第 1 章では、Newton 粘性を考えた双曲型保存系が扱われた。1.1 節では、境界で流れの出入りがある一次元圧縮性粘性流が考察され、予想される漸近挙動が分類され (松村)、その一部は理論的証明が与えられた (松村－西原)。1.2 節では、単独保存則の方程式系で、粘性的衝撃波への漸近が取扱われ、粘性的衝撃波に対する境界の効果 (西原)、粘性的衝撃波の大域的安定性とその漸近の速さ (西原－Zhao) が得られた。

第 2 章では、摩擦による効果がある場合、即ち、Porous Media 中の一次元圧縮性流の方程式系が考察された。この場合、Darcy の法則から導かれる散逸波に漸近することがわかっているが、その漸近の速さ、Asymptotic profile が得られた (西原－Yang, 西原－Wang－Yang, 西原－西川)。この方法は、熱弾性体の方程式系にも応用された (西原－西畑)。

いずれの論文の内容も、学会、シンポジウム等で発表され、論文誌に投稿され、一部はすでに活字化されている。

この報告書で、三年間の研究期間中に得られた結果を全て収めた。未解決の数多くの課題の整理にもなると思われるし、代表者 (西原) が、研究代表者として科学研究費の補助を初めて得られた memory ともなると考えたからでもある。この報告書が今後の研究の発展の糧にもなれば幸いと思っている。また、本研究の遂行にあたって、各方面の方々からご協力を頂いたことに深く感謝いたします。

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1. Viscous conservation laws

1.1 Inflow problem

Inflow and outflow problems in the half space for a one-dimensional isentropic model system of compressible viscous gas

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Abstract

We consider asymptotic behaviors in time of solutions to the initial boundary value problems in the half space for a one-dimensional isentropic model system of compressible viscous gas. In particular, we focus our attention on inflow(or outflow) problems where the velocity on the boundary is given as a constant inward(or outward) flow, and try to classify all asymptotic behaviors of the solutions. It turns out that depending on the data both on the boundary and at far field (especially depending on whether the state is subsonic, transonic, or supersonic), the asymptotic state variously consists of rarefaction waves, viscous shock waves, and also stationary boundary layer. Moreover, we give a survey of our recent results on some particular cases which justify our classification.

1 Introduction

The one-dimensional and isentropic motion of compressible viscous gas which fills the half space is described by the following system in the Eulerian coordinates:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = \mu u_{xx}, \quad x > 0,\ t > 0, \\ p(\rho) = a\rho^\gamma, \end{cases} \tag{1.1}$$

where $\rho(>0)$ is the density, u is the velocity, p is the pressure, all coefficients $\mu(>0)$, $a(>0)$,and $\gamma(\geq 1)$ are assumed to be constant. We study the initial boundary value problems to the system (1.1) with the initial data

$$(\rho, u)(0, x) = (\rho_0, u_0)(x), \quad x > 0, \quad \inf_{x>0} \rho_0(x) > 0, \tag{1.2}$$

the boundary condition at far field $x = +\infty$

$$\lim_{x\to\infty} (\rho, u)(t, x) = (\rho_+, u_+), \quad t > 0, \tag{1.3}$$

and also one of the following three types of conditions on the boundary $x = 0$.

Case 1 (zero velocity on the boundary):

$$u(t,0) = 0, \quad t > 0. \tag{1.4$_1$}$$

Case 2 (negative velocity on the boundary):

$$u(t,0) = u_- < 0, \quad t > 0. \tag{1.4$_2$}$$

Case 3 (positive velocity on the boundary):

$$\begin{cases} u(t,0) = u_- > 0, \\ \rho(t,0) = \rho_- > 0, \end{cases} \quad t > 0. \tag{1.4$_3$}$$

Here $\rho_\pm$ and $u_\pm$ are prescribed constants, and we of course assume the initial data satisfy the boundary conditions (1.3) and one of (1.4) as compatibility condition. It is noted that the Case 1 means the wall is impermeable, and the Case 2 (resp. Case 3) means the outflow (resp. inflow) constantly exists on the wall. It is also noted that in the Cases 1 and 2 the condition on the density can't be imposed, but in the Case 3 it has to be imposed, so that the corresponding problem to the first equation of (1.1) (conservation of mass density) is well posed as a hyperbolic equation for the mass density ρ. In what follows, we call the initial boundary value problem (1.1) $\sim$ (1.4)$_1$, (1.1) $\sim$ (1.4)$_2$, and (1.1) $\sim$ (1.4)$_3$, the impermeable wall problem, outflow problem, and inflow problem, respectively. In this article, we are interested in the time-global existence of the solutions of these three types of initial boundary value problems, especially the large time asymptotic behaviors of the solutions in connection with the various combinations of the data on the boundary and at far field.

There have been many works on the asymptotic behaviors of solutions to the Cauchy problems for the system (1.1) where the boundary conditions at far fields $x = \pm\infty$ are given by

$$\lim_{x \to \pm\infty} (\rho, u)(t,x) = (\rho_\pm, u_\pm), \quad t > 0. \tag{1.5}$$

Refer to [9]$\sim$[11], [14], [7], etc., and the references therein. All these results show that the large time asymptotic behaviors of solutions of the Cauchy problem with (1.5) are basically same as that of Riemann problem to the hyperbolic part of (1.1)(Euler equation), if only we replace the shock wave with discontinuity by the corresponding smooth viscous shock wave. Hence the asymptotic behaviors are naturally classified into eight different patterns of combination of the rarefaction and viscous shock waves. On the other hand, in the cases of IBVP, the influence of viscosity is expected to emerge not only in smoothing effect on discontinuous shock wave, but also in forming a boundary layer. As for the question when the boundary layer forms, we propose a criterion as follows by considering the Riemann problem for the Euler equation, where the initial right state (ρ_+, u_+) is given by the far field state (1.3), and the left state (ρ_-, u_-) is given by all possible state which is consistent with the boundary condition (1.4) on $x = 0$ (that is, in the cases of the impermeable wall and outflow problems, ρ_- is freely given). When the left state is uniquely determined so that the value on the boundary $x = 0$ of the solution of the Riemann problem is consistent with the boundary condition, that is when the solution consists of only elementary waves with positive speed, we expect no boundary layers emerge. On the other hand, when the value of the Riemann problem's solution on the boundary is not consistent with the boundary condition for any admissible left state, that is the case when the solution includes an elementary wave with negative speed or a stationary shock wave, we expect a boundary layer which smoothly but steeply compensate the gap comes up. Roughly speaking, inconsistency of the incoming or standing hyperbolic waves with the boundary data as the hyperbolic-parabolic system does form a boundary layer. Thus, we can easily imagine the situation crucially depends on whether the state either on the boundary and at far field is subsonic, transonic, or supersonic, because the characteristic wave speeds of the Euler equation are given by $u \pm c(\rho)$, where $c(\rho)$ is so called the sound speed. Recently the impermeable wall problems are investigated by Matsumura & Mei [8] and Matsumura & Nishihara [12], and it turns out that the asymptotic value of ρ_- is uniquely determined by the given far field data (ρ_+, u_+) so that no boundary layers appear and the solution eventually tends toward an outgoing rarefaction wave in the case $u_- < u_+$ ([12]), and a properly shifted viscous shock wave in the case $u_- > u_+$ ([8]). This means the asymptotic behaviors of solutions are basically classified only into two patterns, making a remarkable contrast to the Cauchy problem. On the other hand, the inflow and outflow problems are expected to be far more complicated, and in fact, there have been few results on them. In this paper, we construct the boundary layer by the stationary solution, and try to classify the all possible large-time asymptotic behaviors of solutions of the inflow and outflow problems through the criterion mentioned above. Moreover, we give a survey of our just recent results on some typical cases, which justify our classification.

The paper is organized as follows. In the Section 2, we recall the Riemann problem to the Euler equation and introduce some notations. We recall the arguments on the Cauchy problem in the Section 3, and summarize the recent results on the impermeable wall problem in the Section 4. We study the outflow problem in the Section 5, and finally the inflow problem in the Section 6.

2 Riemann Problem

We recall the Riemann problem to the hyperbolic part of (1.1), that is, Euler equation. In what follows, we use the equivalent system to (1.1) which is described in terms of the specific volume $v\,(:= 1/\rho)$ and the velocity u:

$$\begin{cases} (\frac{1}{v})_t + (\frac{u}{v})_x = 0, \\ (\frac{u}{v})_t + (\frac{u^2}{v} + p(v))_x = \mu u_{xx}, \quad x > 0,\ t > 0, \\ p(v) = av^{-\gamma}. \end{cases} \tag{2.1}$$

Now we consider the Riemann problem on $\mathbf{R}$ to the hyperbolic part of (2.1) for given constant states $(v_\pm, u_\pm)$, $v_\pm > 0$:

$$\begin{cases} (\frac{1}{v})_t + (\frac{u}{v})_x = 0, \\ (\frac{u}{v})_t + (\frac{u^2}{v} + p(v))_x = 0, \quad x \in \mathbf{R},\ t > 0, \\ (v, u)(0, x) = \begin{cases} (v_-, u_-) & x < 0 \\ (v_+, u_+) & x > 0. \end{cases} \end{cases} \tag{2.2}$$

For smooth v and u, the system (2.2) is rewritten in the form

$$\begin{pmatrix} v \\ u \end{pmatrix}_t + A(v, u) \begin{pmatrix} v \\ u \end{pmatrix}_x = 0, \tag{2.3}$$

where

$$A(v, u) = \begin{pmatrix} u & -v \\ vp'(v) & u \end{pmatrix}.$$

The eigenvalues $\{\lambda_i\}_{i=1}^2$ of $A(v, u)$ and corresponding right eigenvectors $\{r_i\}_{i=1}^2$ are given by

$$\lambda_i(v, u) = u + v\lambda_i^L(v), \quad r_i(v) = \begin{pmatrix} 1 \\ -\lambda_i^L(v) \end{pmatrix}, \tag{2.4}$$

where $\lambda_1^L(v) = -|p'(v)|^{1/2}$, and $\lambda_2^L(v) = |p'(v)|^{1/2}$. We also define the sound speed $c(v)$ by

$$c(v) := v|p'(v)|^{\frac{1}{2}} = \sqrt{a\gamma} v^{-\frac{\gamma+1}{2}}.$$

Here note that $\lambda_i^L(v)$ are corresponding to the eigenvalues of the system in Lagrange coordinates setting (cf. (6.6)). Thanks to the properties $p'(v) < 0$ and $p''(v) > 0$, the system (2.2) is strictly hyperbolic and its both characteristic fields are genuinely nonlinear for $v > 0$. Let $\Omega = \{(v, u) \in \mathbf{R}_+ \times \mathbf{R}\}$ be the phase space of (v, u). In what follows, let us abbreviate (v, u) to w, $(v_\pm, u_\pm)$ to $w_\pm$, and so on, for simplicity. Since the integral curves of the right eigenvectors are described by the ordinary differential equations $du/dv = -\lambda_i^L(v)$ in Ω, the rarefaction curves $R_1(w_-)$ and $R_2(w_-)$ for any fixed left state $w_- \in \Omega$ are defined by

$$R_i(w_-) = \{w = (v, u) \in \Omega |\, u = u_- - \int_{v_-}^{v} \lambda_i^L(s)\, ds,\ u \geq u_-\} \quad i = 1, 2.$$

Similarly, for any fixed right state $w_+ \in \Omega$, the rarefaction curves $\tilde{R}_i(w_+)$ are defined. When $w_+ \in R_i(w_-)\,(i = 1, 2)$, or equivalently $w_- \in \tilde{R}_i(w_+)$, the solution of the Riemann problem (2.2) consists of two constant states $w_\pm$ and one centered rarefaction wave which continuously connects the left and right states. We denote this solution by

$$W_i^R(x/t; w_-, w_+) = (V_i^R, U_i^R)(x/t; w_-, w_+), \quad i = 1, 2,$$

and simply call it $i-$rarefaction wave. On the other hand, if $w_\pm$ and $s \in \mathbf{R}$ satisfy the Rankine-Hugoniot condition

$$\begin{cases} -s(\frac{1}{v_+} - \frac{1}{v_-}) + (\frac{u_+}{v_+} - \frac{u_-}{v_-}) = 0, \\ -s(\frac{u_+}{v_+} - \frac{u_-}{v_-}) + (\frac{u_+^2}{v_+} - \frac{u_-^2}{v_-} + p(v_+) - p(v_-)) = 0, \end{cases} \tag{2.5}$$

then it is known the Riemann problem has a shock wave solution with the shock speed s. By elementary calculations to the R-H condition (2.5), the value u_+ and the shock speed s should be given in terms of u_- and $v_\pm$ by

$$\begin{cases} u_+ = u_- - (v_+ - v_-)s_i^L(v_-, v_+), \\ s = s_i(w_-, v_+) := u_- + v_- s_i^L(v_-, v_+), \quad i = 1, 2, \end{cases} \tag{2.6}$$

where

$$s_1^L(v_-, v_+) = -\left|\frac{p(v_+) - p(v_-)}{v_+ - v_-}\right|^{\frac{1}{2}}, s_2^L(v_-, v_+) = -s_1^L(v_-, v_+). \tag{2.7}$$

Combining the above together with the entropy condition

$$\lambda_i(w_+) < s_i < \lambda_i(w_-), \tag{2.8}$$

which is simply equivalent to $u_+ < u_-$ in our cases, we can define the shock curves $S_1(w_-)$ and $S_2(w_-)$ for any fixed left state $w_- \in \Omega$ by

$$S_i(w_-) = \{w = (v, u) \in \Omega | \, u = u_- - (v - v_-)s_i^L(v_-, v), \;\; u \leq u_-\}.$$

Similarly, for any fixed right state $w_+ \in \Omega$, the shock curves $\tilde{S}_i(w_+)$ are defined. When $w_+ \in S_i(w_-)\,(i = 1, 2)$, or equivalently $w_- \in \tilde{S}_i(w_+)$, the solution of the Riemann problem (2.2) consists of two constant states $w_\pm$ and one shock discontinuity which connects the left and right states and propagates with the shock speed s_i. We denote this solution by

$$W_i^S(x - s_i t; w_-, w_+) = (V_i^S, U_i^S)(x - s_i t; w_-, w_+), \quad i = 1, 2,$$

and simply call it $i-$shock wave.

For any fixed right state $w_+ \in \Omega$, the state space Ω is divided into four domains by the rarefaction curves $\{R_i(w_-)\}_{i=1}^2$ and shock curves $\{S_i(w_-)\}_{i=1}^2$, that is, $RR(w_-)$ with the boundary $R_1(w_-)$ and $R_2(w_-)$, $RS(w_-)$ with the boundary $R_1(w_-)$ and $S_2(w_-)$, $SR(w_-)$ with the boundary $S_1(w_-)$ and $R_2(w_-)$, and $SS(w_-)$ with the boundary $S_1(w_-)$ and $S_2(w_-)$. When w_+ is located in one of the above four domains, the Riemann problem's solution is given by the combination of the corresponding $1-$ and $2-$waves. For example, if $w_+ \in RS(w_-)$, then there exists a unique intermediate state $\bar{w} \in R_1(w_-)$ satisfying $w_+ \in S_2(\bar{w})$, and the solution is exactly given by

$$W_1^R(x/t; w_-, \bar{w}) + W_2^S(x - s_2 t; \bar{w}, w_+) - \bar{w}.$$

Thus, all patterns of the solution of the Riemann problem are classified into eight cases depending on where w_+ is located, that is, $\{R_i(w_-)\}_{i=1}^2$, $\{S_i(w_-)\}_{i=1}^2$, $RR(w_-)$, $RS(w_-)$, $SR(w_-)$, and $SS(w_-)$.

3 Viscous Shock Waves

We recall the viscous shock waves of (2.1) and the known results on the Cauchy problem and also the impermeable wall problem. The viscous shock wave W

with the shock speed $s \in \mathbf{R}$ is defined by special solution of (2.1) which has the form

$$w(x,t) = (V, U)(\xi), \quad \xi = x - st, \tag{3.1}$$

and smoothly connects the states $w_\pm \in \Omega$ so that $W(\pm\infty) = w_\pm$. Substituting (3.1) to (2.1), we have the system of ordinary differential equations

$$\begin{cases} -s(\frac{1}{V})' + (\frac{U}{V})' = 0, \\ -s(\frac{U}{V})' + (\frac{U^2}{V} + p(V))' = \mu U'', \quad \xi \in \mathbf{R}. \end{cases} \tag{3.2}$$

It is known that under the R-H condition (2.3) and the entropy condition (2.4), which implies $w_+ \in S_i(w_-)$ $(i = 1 \,\text{or}\, 2)$, the equation is reduced to

$$\begin{cases} U(\xi) = u_\pm - (V(\xi) - v_\pm)s_i^L(v_-, v_+), \quad \xi = x - s_i t, \\ \mu s_i^L V'(\xi) = -p(V(\xi)) + p(v_\pm) - (V(\xi) - v_\pm)(s_i^L)^2, \quad \xi \in \mathbf{R}, \\ V(\pm\infty) = v_\pm, \end{cases} \tag{3.3}$$

where the shock speed $s = s_i(w_-, v_+)$ is given by $u_- + v_- s_i^L(v_-, v_+)$. Due to the convexity of p, it is easy to show the system (3.3) has a unique solution up to shift. We denote this solution by

$$W_i^{VS}(x - s_i t; w_-, w_+) = (V_i^{VS}, U_i^{VS})(x - s_i t; w_-, w_+), \quad i = 1, 2,$$

and simply call it $i-$viscous shock wave.

Now we recall the results on the asymptotic behaviors of solutions of the initial value problem to the system (2.1) with the initial and far fields conditions

$$(v, u)(0, x) = (v_0, u_0)(x), \quad x \in \mathbf{R}, \tag{3.3}$$

$$\lim_{x \to \pm\infty} (v, u)(t, x) = (v_\pm, u_\pm), \quad t > 0. \tag{3.4}$$

As we mentioned in the Section 1, the asymptotic sates are basically same as that of Riemann problem to the hyperbolic part, if only we replace the shock waves by the corresponding smooth viscous shock waves. Therefore, for any fixed $w_- \in \Omega$, depending on where w_+ is located, the solution is expected to tend toward the following asymptotic states:

$$\begin{aligned}
& W_i^R(x/t; w_-, w_+), && \text{for}\, w_+ \in R_i(w_-), \\
& W_i^{VS}(x - s_i t + \alpha_i; w_-, w_+), && \text{for}\, w_+ \in S_i(w_-), \\
& W_1^R(x/t; w_-, \bar{w}) + W_2^R(x/t; \bar{w}, w_+) - \bar{w}, && \text{for}\, w_+ \in RR(w_-), \\
& W_1^{VS}(x - s_1 t + \alpha_1; w_-, \bar{w}) + \\
& W_2^{VS}(x - s_2 t + \alpha_2; \bar{w}, w_+) - \bar{w}, && \text{for}\, w_+ \in SS(w_-), \\
& W_1^{VS}(x - s_1 t + \alpha_1; w_-, \bar{w}) + W_2^R(x/t; \bar{w}, w_+) - \bar{w}, && \text{for}\, w_+ \in SR(w_-), \\
& W_1^R(x/t; w_-, \bar{w}) + W_2^{VS}(x - s_2 t + \alpha_2; \bar{w}, w_+) - \bar{w}, && \text{for}\, w_+ \in RS(w_-).
\end{aligned}$$

As for these asymptotic behaviors, there have been many works where all results are stated for the equivalent problem to (1.1) described in the Lagrangian mass coordinates, in which the system can be handled in easier way than that in the Eulerian coordinates. Here let us make a survey in terms of the Lagrangian coordinates. Matsumura & Nishihara [9] first treated the case $w_+ \in S_i(w_-)$ and succeeded in showing that if $(\gamma - 1)|w_+ - w_-|$ is suitably small, the viscous shock wave $W_i^{VS}(x - s_i t; w_-, w_+)$ is asymptotically stable for small initial perturbations with zero integrals, so that the phase shift α does not occur. For the initial data with non-zero integrals, Liu [2][3] studied the criterion how to determine the phase shift based on his deep insight, and Szepessy & Xin [14] succeeded in showing, for the system with artificial viscosity terms which make the system uniformly parabolic, that if $|w_+ - w_-|$ and the initial perturbations are suitably small, then the solution tends toward the $W_i^{VS}(x - s_i t + \alpha; w_-, w_+)$ where the phase shift α is uniquely determined by the Liu's criterion. Although the introduction in [14] reads the physical viscosity case can be similarly treated, it seems not to be so trivial. The arguments by Szepessy [15] on the asymptotic stability of the viscous shock wave to the Broadwell model are believed enough to compensate the gap. However the complete proof for our original system is not available in any publications up to now. Therefore we really hope to give a complete and simpler proof to our physical viscosity case, even if we use a special feature of our system. In the case $w_+ \in SS(w_-)$, combining the arguments in [9] together with that in [2][3], we can show as a kind of exercise that if $|w_+ - w_-|$ is suitably small and the initial data is suitably close to $W_1^{VS}(x; w_-, \bar{w}) + W_2^{VS}(x; \bar{w}, w_+) - \bar{w}$, then the solution tends toward

$$W_1^{VS}(x - s_1 t + \alpha_1; w_-, \bar{w}) + W_2^{VS}(x - s_2 t + \alpha_2; \bar{w}, w_+) - \bar{w}$$

where the shifts α_1 and α_2 are uniquely determined by the initial data. In the cases $w_+ \in (R_1 \cup R_2 \cup RR)(w_-)$, Matsumura and Nishihara [10][11][12] succeeded in obtaining the complete results that the asymptotic state is given by the corresponding either simple rarefaction wave or combination of two rarefaction waves of the Euler equation without any smallness conditions on $|w_+ - w_-|$, γ, and the initial data. Finally we should emphasize the cases $w_+ \in (SR \cup RS)(w_-)$ are still entirely open. These cases would be very interesting and challenging, because we could expect the interaction between the tails of the rarefaction and viscous shock waves are persistent and subtle enough to make controlling the phase shift of the viscous shock very entangled.

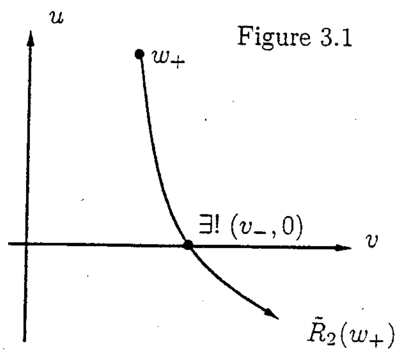

Figure 3.1

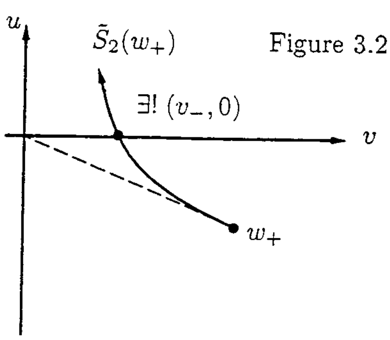

Figure 3.2

Let us turn to the case of the impermeable wall problem (1.1) $\sim$ $(1.4)_1$. Following our criterion how to expect the asymptotic state as mentioned in the introduction, let us consider the corresponding Riemann problem (2.2) where the left state v_- is freely given, that is, all candidates of the left states w_- form the v-axis in Ω. When $u_+ > 0$, as the Figure 3.1 shows, we can see that the Riemann solution includes an incoming wave which is not consistent with the boundary condition $(1.4)_1$ for all w_- on the v axis except the unique point such that $(v_-, 0) \in \tilde{R}_2(w_+)$, equivalently $w_+ \in R_2(v_-, 0)$. For this v_-, the Riemann solution consists of only an outgoing rarefaction wave $W_2^R(x/t; w_-, w_+)$, and we do expect it as the asymptotic state of the solution. In fact, this conjecture was completely proved by Matsumura & Nishihara [12] without any smallness conditions on $|w_+ - w_-|$, γ, and the initial data. Similarly when $u_+ < 0$, as the Figure 3.2 shows, we can see that $(v_-, 0) \in \tilde{S}_2(w_+)$ is the only the option for which the Riemann solution is consistent with the boundary condition, and thus we expect the asymptotic state to be the corresponding viscous shock wave $W_2^{VS}(x - s_2 t + \alpha; w_-, w_+)$ with a proper phase shift α. Here we should note the shock speed s_2 is positive no matter how negatively large u_+ is. (This fact is showed as follows: For a shock wave with the zero shock speed s, the left state w_- turns out to be located on the straight line connecting the origin and the right state w_+, due to the first equation of the R-H condition (2.5). This implies the shock wave has a positive shock speed s_2 if $w_- \in \tilde{S}_2(w_+)$ is located above the straight line, and has the negative speed s_2 if below the line (cf. Figure 3.2).) Recently Matsumura & Mei [8] succeeded in obtaining a positive result on this conjecture that if the viscous shock wave is suitably far away from the boundary at the initial time and if the initial perturbations are small enough, then the solution tends toward a properly shifted viscous shock wave whose phase shift is uniquely and explicitly determined by the initial perturbations. Thus, the asymptotic behaviors of the solutions for the impermeable wall problem can be basically classified into only two cases, whether u_+ is positive or negative, in contrast with the Cauchy problem, eight cases.

Finally in this section, we should make some remarks on the phase shift just mentioned above to compare our case with that of the scalar viscous conservation laws, whose IBVP with Dirichlet zero boundary condition were studied in Liu & Yu [6], and Liu & Nishihara [5]. Even in these scalar cases, in order to locate the phase shift, they needed the laborious analysis (pointwise estimates via Green function in [6], technical weighted estimates in [5]), since the shift can't be determined explicitly because of the viscosity term. So we had thought our system case is much more difficult in many aspects. However it turns out that our system with physical viscosity on the half space has several better features than those both for the scalar cases with boundary and also for the systems without boundary. In particular, our system is not uniformly parabolic, i.e., there is no viscosity term for the density, and we can't impose the boundary value of the density, which usually gives various difficulties. This is really the reason why we can specify the phase shift α of $W_2^{VS}(x - s_2 t + \alpha)$ only by the hyperbolic equation for the density, and we can expect that value of $v(0, t)$ on the boundary is automatically controlled to tend to the value v_- by the structure of the system itself so that the whole solution $w(x, t)$ tends to $W_2^{VS}(x - s_2 t + \alpha)$ with the same α. Let us show how to specify the shift α more precisely. Let denote $W_2^{VS}(x - s_2 t + \alpha; w_-, w_+)$ simply by $W(x - st + \alpha)$. Then, by the equation of the conservation of density, we deduce

$$(\frac{1}{v} - \frac{1}{V})_t + (\frac{u}{v} - \frac{U}{V})_x = 0. \tag{3.5}$$

Integrating (3.5) with respect to both x and t, we have

$$\int_0^\infty \frac{1}{v} - \frac{1}{V}\,dx = \int_0^\infty \frac{1}{v_0(x)} - \frac{1}{V(x+\alpha)}\,dx - \int_0^t \frac{U(-s\tau+\alpha)}{V(-s\tau+\alpha)}\,d\tau. \tag{3.6}$$

If we assume that $v - V$ tends to zero well enough, the right hand side of (3.6) should satisfy

$$\int_0^\infty \frac{1}{v_0(x)} - \frac{1}{V(x+\alpha)}\,dx - \int_0^\infty \frac{U(-s\tau+\alpha)}{V(-s\tau+\alpha)}\,d\tau = 0. \tag{3.7}$$

Set the left hand side of (3.7) by $I(\alpha)$ and differentiate it with respect to α as

$$\frac{dI(\alpha)}{d\alpha} = -\frac{1}{v_+} - \frac{1}{s}\left(-s\frac{1}{V(\alpha)} + \frac{U(\alpha)}{V(\alpha)}\right). \tag{3.8}$$

Since the first equation of (3.2) gives

$$-s\frac{1}{V(\alpha)} + \frac{U(\alpha)}{V(\alpha)} = -s\frac{1}{v_-}, \quad \alpha \in \mathbf{R}^1, \tag{3.9}$$

we deduce from (3.8) and (3.9) that

$$I(\alpha) = -\left(\frac{1}{v_+} - \frac{1}{v_+}\right)\alpha + I(0).$$

Thus it turns out that the phase shift α should be given by the formula

$$\alpha = \frac{1}{\rho_+ - \rho_-}\left\{\int_0^\infty \frac{1}{v_0(x)} - \frac{1}{V(x)}\,dx - \int_0^\infty \frac{U(-s\tau)}{V(-s\tau)}\,d\tau\right\}.$$

4 Boundary Layers

Now we are ready to start to consider the Inflow and Outflow Problems. As mentioned in the introduction, the classification of the large time asymptotic states are expected to depend crucially on whether the state either on the boundary and at far field is subsonic, transonic, or supersonic. Hence we divide the domain Ω into 5 regions (see Figure 4.1)

$$\begin{aligned}
\Omega^+_{super} &= \{w \in \Omega \,|\, u > c(v)\},\\
\Gamma^+_{trans} &= \{w \in \Omega \,|\, u = c(v)\},\\
\Omega_{sub} &= \{w \in \Omega \,|\, |u| < c(v)\},\\
\Gamma^-_{trans} &= \{w \in \Omega \,|\, u = -c(v)\},\\
\Omega^-_{super} &= \{w \in \Omega \,|\, u < -c(v)\},
\end{aligned}$$

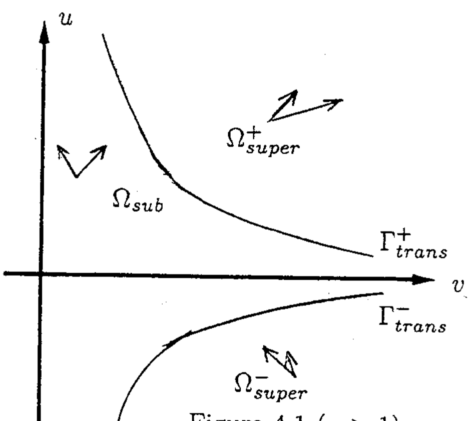

Figure 4.1 ($\gamma > 1$)

where we should note that

$$
\begin{aligned}
w \in \Omega^+_{super} &\quad \rightarrow \quad 0 < \lambda_1(w) < \lambda_2(w),\\
w \in \Gamma^+_{trans} &\quad \rightarrow \quad 0 = \lambda_1(w) < \lambda_2(w),\\
w \in \Omega_{sub} &\quad \rightarrow \quad \lambda_1(w) < 0 < \lambda_2(w),\\
w \in \Gamma^-_{trans} &\quad \rightarrow \quad \lambda_1(w) < \lambda_2(w) = 0,\\
w \in \Omega^-_{super} &\quad \rightarrow \quad \lambda_1(w) < \lambda_2(w) < 0.
\end{aligned}
$$

Now let us pick up an example from the Outflow Problem(1.1) $\sim$ (1.4)$_2$ which show that any corresponding Riemann solution is not consistent with the boundary condition (1.4)$_2$. For fixed $u_- < 0, v_+ > 0$, and $u_+ > 0$, consider the Riemann problem with free v_- as in the previous sections.

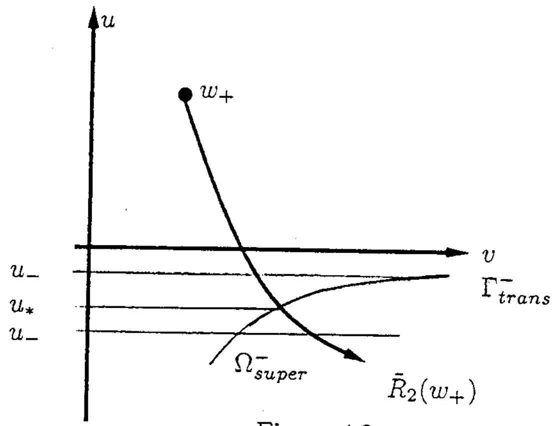

Figure 4.2

As the Figure 4.2 shows, 2-rarefacton curve $\tilde{R}_2(w_+)$ intersects Γ^-_{trans} at a point denoted by $w_* = (v_*, u_*)$. If $u_* \leq u_- < 0$, as in the impermeable wall problem, there exist a unique $v_- \in \tilde{R}_2(w_+)$ so that we can expect the solution tend toward $W_2^R(x/t; w_-, w_+)$ and no boundary layer appear. However, if $u_- < u_*$ then, even for the most plausible option $v_- \in \tilde{R}_2(w_+)$, the Riemann solution $W_2^R(x/t; w_-, w_+)$ includes a part $W_2^R(x/t; w_-, w_*)$ which goes into the boundary, so is not consistent with the boundary condition (1.4)$_2$. In this situation, we do expect the interaction between the incoming wave and the boundary condition select a new v_- and form a boundary layer solution which connects w_- and w_*. In this paper, we refer to stationary solution $W(x)$ of (1.1) with the conditions $W(0) = w_-$ and $W(+\infty) = w_+$ as boundary layer solution connecting w_- and w_+, and denote it by

$$W^{BL}(x; w_-, w_+) = (V^{BL}(x; w_-, w_+), U^{BL}(x; w_-, w_+)).$$

More precisely, the boundary layer $W^{BL}(x; w_-, w_+)$ is defined by the solution of

$$
\begin{cases}
(\frac{U}{V})' = 0, \\
(\frac{U^2}{V} + p(V))' = \mu U'', \quad x > 0, \\
W(0) = w_-, \quad W(+\infty) = w_+.
\end{cases}
\tag{4.1}
$$

Let us investigate the existence of $W(x)$ and its properties. We first consider the case corresponding to the Outflow Problem, that is, $u_- < 0$ (the following arguments on stationary solution are basically due to Kawashima & Nishibata[1]). The first equation of (4.1) immediately implies

$$\frac{U(x)}{V(x)} = \frac{u_+}{v_+} = \frac{u_-}{v_-}, \quad x > 0, \tag{4.2}$$

in particular, that for a fixed w_+, the boundary state w_- should be located on the line $BL(w_+)$ which passes through the origin and w_+,

$$BL(w_+) = \{w \in \Omega \,|\, u = -s_0 v, \ s_0 = -\frac{u_+}{v_+}\}.$$

Integrating the second equation of (4.1) with the aid of (4.2), we have

$$\begin{cases} s_0 \mu V' = -s_0^2(V - v_+) + p(v_+) - p(V), \quad x > 0, \\ V(0) = v_-, \quad V(+\infty) = v_+. \end{cases} \tag{4.3}$$

Here note that $s_0 = -u_\pm / v_\pm > 0$. We can check whether V' has right sign or not on the interval between v_- and v_+ in (4.3) as follows (refer to Figure 4.3). First fix w_+, then draw the line $BL(w_+)$ which always intersects the transonic line Γ^-_{trans}, $u/v = -|p'(v)|^{1/2}$. Denote the intersection point by $w_* = (v_*, u_*)$. Consider the graph of $p(v)$ and draw the tangential line at $(v_*, p(v_*))$. It easily turns out that the slope of the tangential line exactly equals to $-s_0^2$. Then draw the line with the same slope $-s_0^2$ which passes through the point $(v_+, p(v_+))$, denote this line by $l(w_+) = \{(v, q(v))\}$. Note $q(v)$ exactly equals to $-s_0^2(v - v_+) + p(v_+)$. Thus for any $w_- \in BL(w_+)$, we can easily check the sign of V' on the interval between v_- and v_+ in (4.3) by seeing how the two graphs of $p(v)$ and $q(v)$ intersect. If $w_+ \in \Omega^-_{super}$, we can see that the 2−shock curve $\tilde{S}_2(w_+)$ intersects the line $BL(w_+)$ at not only w_+ but another point, be denoted by $\tilde{w} = (\tilde{v}, \tilde{u})$, correspondingly the line $l(w_+)$ intersects the graph of $p(v)$ at not only $v = v_+$ but $v = \tilde{v}$. Then it easily turns out

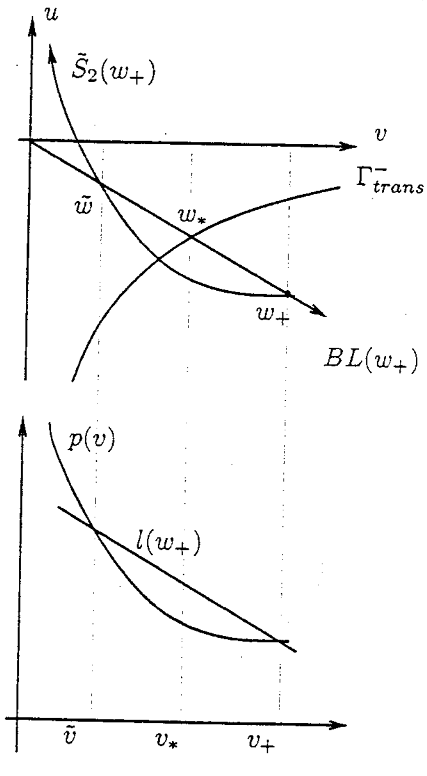

Figure 4.3

that V' has a right sign for $u_- < \tilde{u}$, that is, there exists the boundary layer solution $W^{BL}(x; w_-, w_+)$, and have a wrong sign for $0 > u_- > \tilde{u}$. As for $u_- = \tilde{u}$, although the sign is right, there exists no boundary layer solution but the stationary viscous shock wave $W_2^S(x + \alpha; w_-, w_+)$. If $w_+ \in \Gamma^-_{trans}$, we can also see that the 2−shock curve $\tilde{S}_2(w_+)$ tangentially intersects the line $BL(w_+)$ at w_+ and has no other intersection points. Then V' has a right sign only for $u_- < u_+$, and the boundary layer solution tends toward w_+ at the algeblaic rate $O(1/x)$ as x goes to $+\infty$, because the line $l(w_+)$ coincides with the tangential line of the graph of $p(v)$ at $v = v_* = v_+$ (then we say the boundary layer solution is "degenerate"). Thus by these arguments above, we have the following

Proposition 4.1. *Suppose $u_+ < 0$ and $w_- \in BL(w_+)$.*
i) If $w_+ \in \Omega^-_{super}$ and $0 > u_- \geq \tilde{u}$, then there exists no boundary layer solution.
ii) If $w_+ \in \Omega^-_{super}$ and $\tilde{u} > u_- > u_+$, then there exists a unique boundary layer solution of (4.1) and positive constants C and δ such that

$$V'(x) > 0, \quad x > 0, \textit{and} \quad |W(x) - w_+| \leq C\exp(-\delta x), \ x \to +\infty.$$

iii) If $w_+ \in \Omega^-_{super}$ and $u_+ > u_-$, then there exists a unique boundary layer solution of (4.1) and positive constants C and δ such that

$$V'(x) < 0, \quad x > 0, \textit{and} \quad |W(x) - w_+| \leq C\exp(-\delta x), \ x \to +\infty.$$

iv) If $w_+ \in \Gamma^-_{trans}$ and $0 > u_- \geq u_+$, then there exists no boundary layer solution.
v) If $w_+ \in \Gamma^-_{trans}$ and $u_+ > u_-$, then there exists a unique degenerate boundary layer solution of (4.1) and a positive constants C such that

$$V'(x) < 0, \quad x > 0, \textit{and} \quad |W(x) - w_+| \leq \frac{C}{x}, \ x \to +\infty.$$

vi) If $w_+ \in \Omega^-_{sub}$, then there exists no boundary layer solution.

Next we consider the boundary layer solution which satisfy (4.2)(4.3) corresponding to the Inflow Problem, that is, $u_- > 0$. In this case, note $s_0 = -u_\pm/v_\pm < 0$, and fix $w_-(u_- > 0)$ at first for later arguments. As in the prvious case, draw the line

$$BL(w_-) = \{w \in \Omega \,|\, u = -s_0 v, \ s_0 = -\frac{u_-}{v_-}\}.$$

which always intersects the transonic line Γ^+_{trans}, $u/v = |p'(v)|^{1/2}$, denote the intersection point by $w_* = (v_*, u_*)$, and then check the sign of V' in (4.3) for any $w_+ \in BL(w_-)$. In particular, if $w \in \Omega^+_{super}$, the 1−shock curve $S_1(w_-)$ intersects the line $BL(w_-)$ whose intersection point is denoted by $\tilde{w}$, and V' has a right sign only for $0 < u_+ < \tilde{u}$.

Proposition 4.2 *Suppose $u_- > 0$ and $w_+ \in BL(w_-)$.*
i) If $w_- \in \Omega_{sub} \cup \Gamma^+_{trans}$ and $0 < u_+ < u_-$, then there exists a unique boundary layer solution of (4.1) and positive constants C and δ such that

$$V'(x) < 0, \quad x > 0, \textit{and} \quad |W(x) - w_+| \leq C\exp(-\delta x), \ x \to +\infty.$$

ii) If $w_- \in \Omega_{sub}$ and $u_- < u_+ < u_*$, then there exists a unique boundary layer solution of (4.1) and positive constants C and δ such that

$$V'(x) > 0, \quad x > 0, \textit{and} \quad |W(x) - w_+| \leq C\exp(-\delta x), \ x \to +\infty.$$

iii) If $w_- \in \Omega_{sub}$ and $u_+ = u_*$, then there exists a unique degenerate boundary layer solution of (4.1) *and positive constants C and δ such that*

$$V'(x) > 0, \quad x > 0, \, and \quad |W(x) - w_+| \leq \frac{C}{x}, \; x \to +\infty.$$

iv) If $w_- \in \Omega_{sub} \cup \Gamma^+_{trans}$ and $u_* < u_+$, then there exists no boundary layer solution.

v) If $w_- \in \Omega^+_{super}$ and $0 < u_+ < \tilde{u}$, then there exists a unique boundary layer solution of (4.1) *and positive constants C and δ such that*

$$V'(x) < 0, \quad x > 0, \, and \quad |W(x) - w_+| \leq C \exp(-\delta x), \; x \to +\infty.$$

vi) If $w_- \in \Omega^+_{super}$ and $\tilde{u} \leq u_+$, then there exists no boundary layer solution.

5 Outflow Problem

In this section, using the arguments in the previous sections, we try to classify all asymptotic behaviors of solutions of the Outflow Problem (1.1)$\sim$(1.4)$_2$. We primarily divide into three cases depending on where w_+ is located.

• $u_+ > 0$ (see Figure 5.1):

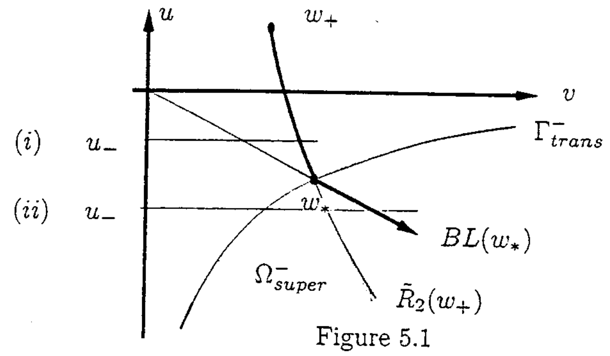

Figure 5.1

(i) $0 > u_- \geq u_*$: There exists a unique v_- such that $w_- \in \tilde{R}_2(w_+)$ and the asymptotic state is expected to be

$$W_2^R(x/t; w_-, w_+). \tag{5.1}$$

(ii) $u_* > u_-$: There exists a unique v_- such that $w_- \in \tilde{BL}_2(w_*)$ and the asymptotic state is expected to be

$$W^{BL}(x; w_-, w_*) + W_2^R(x/t; w_*, w_+) - w_*. \tag{5.2}$$

where note that the boundary layer solution $W^{BL}(x; w_-, w_*)$ is degenerate.

• $u_+ < 0,\ w_+ \in \Omega_{sub} \cup \Gamma^-_{trans}$ (see Figure 5.2):

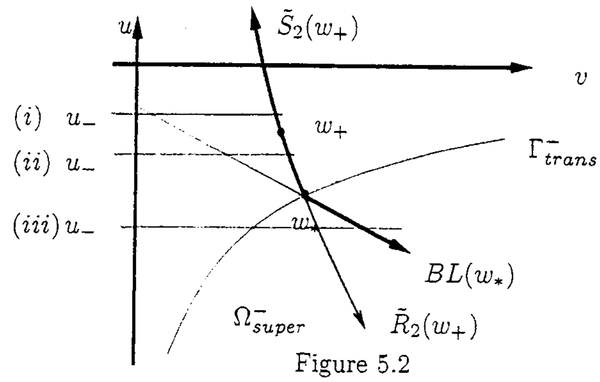

Figure 5.2

(i) $0 > u_- > u_+$: There exists a unique v_- such that $w_- \in \tilde{S}_2(w_+)$ and the asymptotic state is expected to be

$$W_2^{VS}(x - s_2 t + \alpha; w_-, w_+) \quad \text{for some } \alpha. \tag{5.3}$$

(ii) $u_+ > u_- \geq u_*$: There exists a unique v_- such that $w_- \in \tilde{R}_2(w_+)$ and the asymptotic state is expected to be

$$W_2^R(x/t; w_-, w_+). \tag{5.4}$$

(iii) $u_* > u_-$: There exists a unique v_- such that $w_- \in \tilde{BL}_2(w_*)$ and the asymptotic state is expected to be

$$W^{BL}(x; w_-, w_*) + W_2^R(x/t; w_*, w_+) - w_*. \tag{5.5}$$

where note that the boundary layer solution $W^{BL}(x; w_-, w_*)$ is degenerate.

• $u_+ < 0,\ w_+ \in \Omega^-_{super}$ (see Figure 5.3):

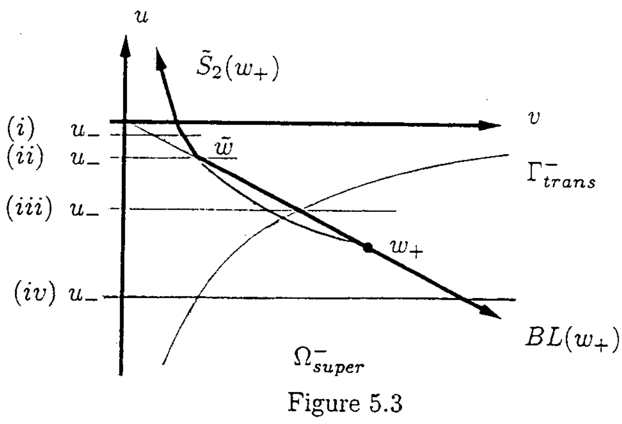

Figure 5.3

(i) $0 > u_- > \tilde{u}$: There exists a unique v_- such that $w_- \in \tilde{S}_2(w_+)$ and the asymptotic state is expected to be

$$W_2^{VS}(x - s_2 t + \alpha; w_-, w_+) \quad \text{for some } \alpha. \tag{5.6}$$

(ii) $u_- = \tilde{u}$: There exists a unique v_- such that $w_- \in \tilde{S}_2(w_+)$ and the asymptotic state is expected to be

$$W_2^{VS}(x+\alpha(t); w_-, w_+) \quad \text{for some } \alpha(t) \nearrow +\infty. \tag{5.7}$$

(iii) $\tilde{u} > u_- > u_+$: There exists a unique v_- such that $w_- \in \tilde{BL}_2(w_+)$ and the asymptotic state is expected to be

$$W^{BL}(x; w_-, w_+) \quad \text{with } (V^{BL})' < 0. \tag{5.8}$$

(iv) $u_+ > u_-$: There exists a unique v_- such that $w_- \in BL(w_+)$ and the asymptotic state is expected to be

$$W^{BL}(x; w_-, w_+) \quad \text{with } (V^{BL})' > 0. \tag{5.9}$$

Among above cases, let us make some comments on the behavior (5.7) which is subtle and interesting. In this case, the corresponding viscous shock wave $W_2^{VS}(x; w_-, w_+)$ is stationary, so there is always a gap between the value w_- and $W_2^{VS}(0+\alpha; w_-, w_+)$ on the boundary for any constant shift α, which makes us expect the shift α rather depend on t and increase up to $+\infty$ as $t \to +\infty$ in order for the gap to decrease to zero. This case is just corresponding to that for the Burgers equation in Liu & Yu [6].

Only the results rigorously proved concerning above asymptotic behaviors are given by Kawashima & Nishibata [1]. They proved that if $w_+ \in \Omega^-_{super}$, $w_- \in BL(w_+)$, and $|w_+ - w_-|$ is suitably small, then the boundary layer solution $W^{BL}(x; w_-, w_+)$ is asymptotically stable. It should be noted that they employ a method where the monotonicity of the boundary layer solution is not used, so they can treat the both cases $w_- < w_+$ and $w_+ < w_-$ (see [1] for details). All other cases are open problems! Finally in this section, we should point out a difficulty the Outflow Problem faces. If we employ, so called, the Lagrangian mass coordinates system, which usually makes the form of equations simpler and the treatment of the equations easier, the problem becomes a free boundary value problem which makes the treatment of boundary more difficult. On the other hand, as we will see in the next section, the Inflow Problem become a corresponding IBVP with a prescribed moving boundary ($x = s_0 t, s_0 = -u_-/v_-$) in the Lagrangian mass coordinates, since the both values of velocity and density on the boundary are given.

6 Inflow Problem

In this section, we try to classify all asymptotic behaviors of solutions of the Inflow Problem (1.1)$\sim$(1.4)$_3$, and show some cases are rigorously proved. Thanks to all arguments in the Section 3 and 4, we primarily classify into two cases in terms of the location of w_-, that is, $w_- \in \Omega_{sub} \cup \Gamma^+_{trans}$ or $w_- \in \Omega^+_{super}$, and then for each w_- classify into many sub-cases depending on where w_+ is located. For $w_- \in \Omega_{sub} \cup \Gamma^+_{trans}$, we basically divide the phase space Ω of w_+ into 13 regions, and for $w_- \in \Omega^+_{super}$, 14 regions.

• $w_- \in \Omega_{sub} \cup \Gamma^+_{trans}$ (see Figure 6.1):

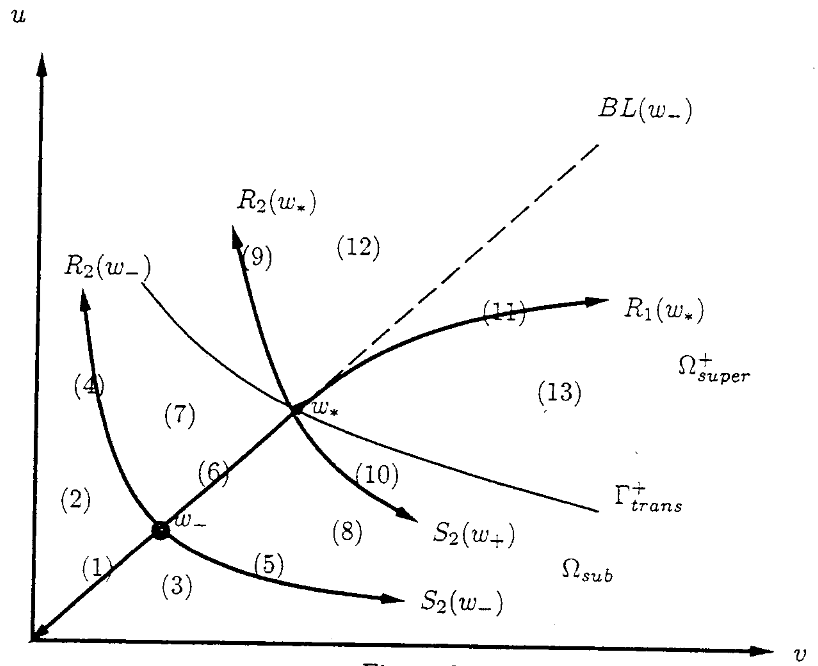

Figure 6.1

(1) $w_+ \in BL(w_-), 0 < u_+ < u_-$: The asymptotic state is expected to be

$$W^{BL}(x; w_-, w_+).$$

(2) $w_+ \in Region(2)$: There exists a unique $\bar{w} \in BL(w_-)\,(\bar{u} < u_-)$ such that $w_+ \in R_2(\bar{w})$ and the asymptotic state is expected to be

$$W^{BL}(x; w_-, \bar{w}) + W_2^R(x/t; \bar{w}, w_+) - \bar{w}.$$

(3) $w_+ \in Region(3)$: There exists a unique $\bar{w} \in BL(w_-)\,(\bar{u} < u_-)$ such that $w_+ \in S_2(\bar{w})$ and the asymptotic state is expected to be

$$W^{BL}(x; w_-, \bar{w}) + W_2^{VS}(x - s_2 t + \alpha; \bar{w}, w_+) - \bar{w}.$$

(4) $w_+ \in R_2(w_-)$: The asymptotic state is expected to be

$$W_2^R(x/t; w_-, w_+).$$

(5) $w_+ \in S_2(w_-)$: The asymptotic state is expected to be

$$W_2^{VS}(x - s_2 t + \alpha; w_-, w_+).$$

(6) $w_+ \in BL(w_-), u_- < u_+ \le u_*$: The asymptotic state is expected to be

$$W^{BL}(x; w_-, w_+),$$

where note that if $w_+ = w_*$, then the boundary layer $W^{BL}(x; w_-, w_+)$ is degenerate.

(7) $w_+ \in Region(7)$: There exists a unique $\bar{w} \in BL(w_-)$ $(\bar{u} > u_-)$ such that $w_+ \in R_2(\bar{w})$ and the asymptotic state is expected to be

$$W^{BL}(x; w_-, \bar{w}) + W_2^R(x/t; \bar{w}, w_+) - \bar{w}.$$

(8) $w_+ \in Region(8)$: There exists a unique $\bar{w} \in BL(w_-)$ $(\bar{u} > u_-)$ such that $w_+ \in S_2(\bar{w})$ and the asymptotic state is expected to be

$$W^{BL}(x; w_-, \bar{w}) + W_2^{VS}(x - s_2 t + \alpha; \bar{w}, w_+) - \bar{w}.$$

(9) $w_+ \in R_2(w_*)$: The asymptotic state is expected to be

$$W^{BL}(x; w_-, w_*) + W_2^R(x/t; w_*, w_+) - w_*.$$

(10) $w_+ \in S_2(w_*)$: The asymptotic state is expected to be

$$W^{BL}(x; w_-, w_*) + W_2^S(x - s_2 t + \alpha; w_*, w_+) - w_*.$$

(11) $w_+ \in R_1(w_*)$: The asymptotic state is expected to be

$$W^{BL}(x; w_-, w_*) + W_1^R(x/t; w_*, w_+) - w_*.$$

(12) $w_+ \in Region(12), i.e., RR(w_*)$: There exists a unique $\bar{w} \in R_1(w_*)$ such that $w_+ \in R_2(\bar{w})$ and the asymptotic state is expected to be

$$W^{BL}(x; w_-, w_*) + (W_1^R(x/t; w_*, \bar{w}) - w_*) + (W_2^R(x/t; \bar{w}, w_+) - \bar{w}).$$

(13) $w_+ \in Region(13), i.e., RS(w_*)$: There exists a unique $\bar{w} \in R_1(w_*)$ such that $w_+ \in S_2(\bar{w})$ and the asymptotic state is expected to be

$$W^{BL}(x; w_-, w_*) + (W_1^R(x/t; w_*, \bar{w}) - w_*) + (W_2^{VS}(x - s_2 t + \alpha; \bar{w}, w_+) - \bar{w}).$$

Here note that in the cases (9) to (13), the boundary layer $W^{BL}(x; w_-, w_*)$ is degenerate one. In Matsumura & Nishihara [13], we succeeded in giving rigorous proofs for some cases. First in the case (5), since the V' has a right sign, we can prove that the boundary layer solution $W^{BL}(x; w_-, w_+)$ is asymptotically stable. Second in the cases (2)(7), combining the arguments in Kawashima & Nishibata [1] with that in Liu, Matsumura & Nishihara [4], we can show that if $|w_+ - w_-|$ is suitably small, the combination of two waves $W^{BL}(x; w_-, \bar{w}) + W_2^R(x/t; \bar{w}, w_+) - \bar{w}$ is asymptotically stable. Finally in the case (12), we can prove that if $|w_+ - w_*|$ is suitably small the combination of three waves is asymptotically stable (note $|w_- - w_*|$ is not necessarily small). We shall state this case more precisely in the last part of this section. All other cases are basically open. Among them, in the cases (3)(8), we should note that the phase shift α in $W^{BL}(x; w_-, \bar{w}) + W_2^{VS}(x - s_2 t + \alpha; \bar{w}, w_+) - \bar{w}$ is explicitly determined by the initial perturbation by the entirely similar argument as that at the end of the Section 3, however the whole proof is not completed yet.

• $w_- \in \Omega^+_{super}$ (see Figure 6.2):

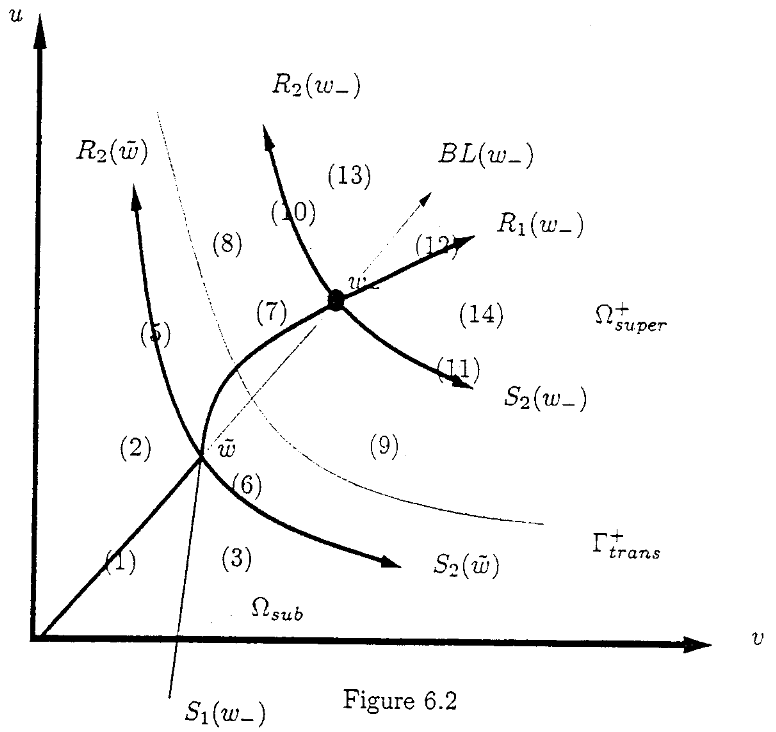

Figure 6.2

(1) $w_+ \in BL(w_-), 0 < u_+ < \bar{u}$: The asymptotic state is expected to be

$$W^{BL}(x; w_-, w_+).$$

(2) $w_+ \in Region(2)$: There exists a unique $\bar{w} \in BL(w_-)$ such that $w_+ \in R_2(\bar{w})$ and the asymptotic state is expected to be

$$W^{BL}(x; w_-, \bar{w}) + W_2^R(x/t; \bar{w}, w_+) - \bar{w}.$$

(3) $w_+ \in Region(3)$: There exists a unique $\bar{w} \in BL(w_-)$ such that $w_+ \in S_2(\bar{w})$ and the asymptotic state is expected to be

$$W^{BL}(x; w_-, \bar{w}) + W_2^{VS}(x - s_2 t + \alpha; \bar{w}, w_+) - \bar{w}.$$

(4) $w_+ = \bar{w}$: The asymptotic state is expected to be

$$W_1^{VS}(x + \alpha(t); w_-, \bar{w}) \quad \text{for some } \alpha(t) \nearrow +\infty.$$

(5) $w_+ \in R_2(\bar{w})$: The asymptotic state is expected to be

$$W_1^{VS}(x + \alpha(t); w_-, \bar{w}) + W_2^R(x/t; \bar{w}, w_+) - \bar{w}$$

for some $\alpha(t) \nearrow +\infty$.

(6) $w_+ \in S_2(\bar{w})$: The asymptotic state is expected to be

$$W_1^{VS}(x + \alpha(t); w_-, \bar{w}) + W_2^{VS}(x - s_2 t + \alpha_2; \bar{w}, w_+) - \bar{w}$$

for some $\alpha(t) \nearrow +\infty$ and $\alpha_2 \in \mathbf{R}$.

(7) $w_+ \in S_1(w_-)$: The asymptotic state is expected to be

$$W_1^{VS}(x - s_1 t + \alpha; w_-, w_+).$$

(8) $w_+ \in Region(8)$: There exists a unique $\bar{w} \in S_1(w_-)\,(\tilde{u} < \bar{u} < u_-)$ such that $w_+ \in R_2(\bar{w})$ and the asymptotic state is expected to be

$$W_1^{VS}(x - s_1 t + \alpha; w_-, \bar{w}) + W_2^R(x/t; \bar{w}, w_+) - \bar{w}.$$

(9) $w_+ \in Region(9)$: There exists a unique $\bar{w} \in S_1(w_-)\,(\tilde{u} < \bar{u} < u_-)$ such that $w_+ \in S_2(\bar{w})$ and the asymptotic state is expected to be

$$W_1^{VS}(x - s_1 t + \alpha_1; w_-, \bar{w}) + W_2^{VS}(x - s_2 t + \alpha_2; \bar{w}, w_+) - \bar{w}.$$

(10) $w_+ \in R_2(w_-)$: The asymptotic state is expected to be

$$W_2^R(x/t; w_-, w_+).$$

(11) $w_+ \in S_2(w_-)$: The asymptotic state is expected to be

$$W_2^{VS}(x - s_2 t + \alpha; w_-, w_+).$$

(12) $w_+ \in R_1(w_-)$: The asymptotic state is expected to be

$$W_1^R(x/t; w_-, w_+).$$

(13) $w_+ \in Region(13), i.e., RR(w_-)$: There exists a unique $\bar{w} \in R_1(w_-)$ such that $w_+ \in R_2(\bar{w})$ and the asymptotic state is expected to be

$$W_1^R(x/t; w_-, \bar{w}) + W_2^R(x/t; \bar{w}, w_+) - \bar{w}.$$

(14) $w_+ \in Region(14), i.e., RS(w_-)$: There exists a unique $\bar{w} \in R_1(w_-)$ such that $w_+ \in S_2(\bar{w})$ and the asymptotic state is expected to be

$$W_1^R(x/t; w_-, \bar{w}) + W_2^{VS}(x - s_2 t + \alpha; \bar{w}, w_+) - \bar{w}.$$

Basically the above 14 cases are open. In the cases (7) to (14), we can easily imagine the behaviors of solutions are similar to that for the Cauchy problem since the situation is totally supersonic. However, because of presence of the boundary, the mathematical proofs are not completed yet even in the cases (10)(12)(13) which are completely solved for the Cauchy problem. The cases (4)(5)(6) should be subtle and might be even more difficult than the case (5.7) of the Outflow problem. In the cases (1)(2)(3), it is interesting to see that although supersonic is the state around the boundary, the state at far field is subsonic enough to create an incoming wave to the boundary which eventually forms a boundary layer.

In the remaining part of this section, we state more precisely about the sub-case (12) in the primary case $w_- \in \Omega_{sub}$ where the asymptotic behavior is expected to be a combination of a boundary layer solution and two rarefaction waves. First let us see that by transformation from the Eulerian coordinates (x, t) to the Lagrangian mass coordinates (ξ, t), we can make the original problem easier to handle and become a corresponding IBVP with the moving boundary $(x = s_0 t,\ s_0 = -u_-/v_- < 0)$. In fact, if we keep in mind that the mass flows in through the boundary at a rate of $\rho_- u_-$, we may define the transformation $x = x(\xi, t)$ for $\xi \geq 0$ by

$$\begin{cases} \frac{\partial x(\xi,t)}{\partial t} = u(t, x(\xi, t)), & t > 0,\ \xi > 0, \\ x(\xi, t) = x_0(\xi), \end{cases} \tag{6.1}$$

where $x_0(\xi)$ is given by the relation

$$\xi = \int_0^{x_0(\xi)} \rho(y, 0)\, dy, \tag{6.2}$$

and for $\xi < 0$ by

$$\begin{cases} \frac{\partial x(\xi,t)}{\partial t} = u(t, x(\xi, t)), & t > t_0(\xi),\ \xi < 0, \\ x(\xi, t_0(\xi)) = 0, \end{cases} \tag{6.3}$$

where $t_0(\xi)$ is given by the relation

$$-\xi = \int_0^{t_0(\xi)} (\rho u)(0, \tau)\, d\tau = (\rho_- u_-) \cdot t_0(\xi). \tag{6.4}$$

By elementary calculations, we deduce from (6.1)$\sim$(6.4) that

$$\xi = \int_{x(\xi,0)}^{x(\xi,t)} \rho(y, t)\, dy, \quad \xi \geq s_0 t, \tag{6.5}$$

which implies

$$\frac{\partial x(\xi, t)}{\partial \xi} = v(t, x(\xi, t)), \quad \frac{\partial x(\xi, t)}{\partial t} = u(t, x(\xi, t)), \quad \xi \geq s_0 t. \tag{6.5}$$

Due to the relations (6.5), we can rewrite the original problem (1.1)$\sim$(1.4)$_3$ in the following form where we use again x instead of ξ:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \mu(u_x/v)_x, \quad x > s_0 t,\ t > 0, \\ p(v) = a v^{-\gamma}, \end{cases} \tag{6.6}$$

with the initial data

$$(v, u)(0, x) = (v_0, u_0)(x), \quad x > 0, \quad \inf_{x>0} v_0 > 0, \tag{6.7}$$

the boundary condition at far field $x = +\infty$

$$\lim_{x\to\infty} (v, u)(t, x) = (v_+, u_+), \quad t > 0, \tag{6.8}$$

and also the condition on the moving boundary $x = s_0 t$

$$(v, u)(t, s_0 t) = (v_-, u_-), \quad t > 0. \tag{6.9}$$

This time the hyperbolic part of (6.6) is written for smooth v and u in the form

$$\begin{pmatrix} v \\ u \end{pmatrix}_t + A^L(v) \begin{pmatrix} v \\ u \end{pmatrix}_x = 0, \tag{6.10}$$

where

$$A^L(v) = \begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix}.$$

It is easy to see that the eigenvalues $\{\lambda_i^L(v)\}_{i=1}^2$ of $A^L(v)$ and corresponding right eigenvectors $\{r_i\}_{i=1}^2$ are just the same as in (2.4),

$$\lambda_1^L(v) = -\lambda_2^L(v) = -|p'(v)|^{1/2}, \quad r_i(v) = \begin{pmatrix} 1 \\ -\lambda_i^L(v) \end{pmatrix}, \tag{6.11}$$

which implies the rarefaction curves $R_i(w_-)$(or $\tilde{R}_i(w_+)$) are naturally the same as that for the Eulerian case. As for the Rankine-Hugoniot condition, we have

$$\begin{cases} -s(v_+ - v_-) - (u_+ - u_-) = 0, \\ -s(u_+ - u_-) + (p(v_+) - p(v_-)) = 0, \end{cases} \tag{6.12}$$

which implies that

$$\begin{cases} u_+ = u_- - (v_+ - v_-) s_i^L(v_-, v_+), \\ s = s_i^L(v_-, v_+), \quad i = 1, 2, \end{cases} \tag{6.13}$$

where $s_i^L(v_-, v_+)$ is as in (2.7). Thus the shock curves $S_i(w_-)$(or $\tilde{S}_i(w_+)$) and the line $BL(w_-)$ are also naturally the same as in the Eulerian case, excepting the shock speeds $s_i(w_-, v_+)$ are replaced by $s_i^L(v_-, v_+)$. Let us use the same notations W_i^R, W_i^S, W_i^{VS}, and W^{BL} to represent rarefaction wave, shock wave, viscous shock wave ,and boundary layer solution respectively. Among them, we especially should note that boundary layer solution is no longer stationary, but defined by a travelling wave solution with the same propagation speed ($s_0 = -u_-/v_-$) as the moving boundary, that is, $W^{BL}(x - s_0 t; w_-, w_+)$ is defined by the solution $W(\xi)$ ($\xi = x - s_0 t$) of

$$\begin{cases} -s_0 V' - U' = 0, \\ -s_0 U' + (p(V))' = \mu (U'/V)', \quad \xi > 0, \\ W(0) = w_-, \quad W(+\infty) = w_+, \end{cases}$$

which is equivalent to, for $w_+ \in BL(w_-)$,

$$\begin{cases} U = s_0 V, \\ s_0 \mu V'/V = -s_0^2 (V - v_+) + p(v_+) - p(V), \quad \xi > 0, \\ V(0) = v_-, \quad V(+\infty) = v_+. \end{cases} \tag{6.15}$$

Since the right hand side of the second equation of (6.15) is exactly the same as that of (4.3), the Proposition 4.1 and 4.2 hold as they are also for (6.15).

Now we fix any w_* on Γ^+_{trans} and suppose $w_- \in BL(w_*)\,(u_- < u_*)$ and $w_+ \in RR(w_*)$. We also assume that the initial data in (6.7) satisfy

$$w_0 - w_+ \in H^1(\mathbf{R}_+), \tag{6.16}$$

and the compatibility condition

$$w_0(0) = w_-. \tag{6.17}$$

Then, as stated above, the solution of the Inflow Problem (6.6) $\sim$ (6.9) is expected to tend toward a combination of three elementary waves

$$W^{asymp}(t,x) := W^{BL}(x - s_0 t, w_-, w_*) + W_1^R(x/t; w_*, \bar{w}) - w_* +$$

$$+ W_2^R(x/t; \bar{w}, w_+) - \bar{w}.$$

Roughly speaking, since the asymptotic state W^{asymp} has a right sign (that is, $V_t^{asymp} = U_x^{asymp} > 0$) to adapt the L^2 energy method, we can prove the following theorem by combining the arguments in Matsumura & Nishihara [12], how to handle the rarefaction waves for the viscous p-system, together with that in Liu, Matsumura & Nishihara [4], how to dispose the interactions between rarefaction waves and boundary layer solutions.

Theorem 6.1 (Matsumura & Nishihara [13]) *Suppose $w_* \in \Gamma^+_{trans}$, $w_- \in BL(w_*)\,(u_- < u_*)$ and $w_+ \in RR(w_*)$. Assume also* (6.16) *and* (6.17). *Then, for any fixed w_* and w_-, there exists a positive constant δ_0 such that if $\| w_0 - (W^{BL} - w_*) - w_+ \|_{H^1} + |w_+ - w_*| < \delta_0$, then the Inflow Problem* (6.6) $\sim$ (6.9) *has a unique time global solution w satisfying $w - w_+ \in C([0,\infty); H^1)$ and the asymptotic behavior*

$$\lim_{t\to\infty} \sup_{x \geq s_0 t} |w(t,x) - W^{asymp}(t,x)| = 0.$$

Finally let us comment on some future's problems. We first should extend the above arguments to the full system including the conservation of energy. We next should extend the arguments to cases, even a 2×2 viscous p-system model, with a free boundary on which inflow or outflow occurs as a result of phase transitions, chemical reactions, etc. Eventually, we hope we could unify the arguments of fluid dynamical aspects and that of Stephan type problems, for example, hopefully could argue on the interactions of free boundary of phase and fluid dynamical waves as shock waves, rarefaction waves and contact discontinuities.

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Large-time behaviors of solutions to an inflow problem in the half space for a one-dimensional system of compressible viscous gas*

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Dedicated to Professors Takaaki Nishida and Masayasu Mimura
on the occasion of their sixtieth birthday

Abstract

The "inflow problem" for a one-dimensional compressible barotropic flow on the half-line $\mathbf{R}_+ = (0, +\infty)$ is investigated. Not only classical waves but also the new wave, which is called the "boundary layer solution", arise. Large time behaviors of the solutions to be expected have been classified in terms of the boundary values by [A. Matsumura, Inflow and outflow problems in the half space for a one-dimensional isentropic model system of compressible viscous gas, to appear in Proceedings of IMS Conference on Differential Equations from Mechnics, Hong Kong, 1999]. In this paper we give the rigorous proofs of the stability theorems on both the boundary layer solution and a superposition of the boundary layer solution and the rarefaction wave.

1 Introduction

In this paper we consider the "inflow problem" recently proposed by the first author [6] for a one-dimensional compressible barotropic flow on the half-line $\mathbf{R}_+ = (0, \infty)$, which is an initial-boundary value problem in the Eulerian coordinate $(\tilde{x}, t)$:

$$\begin{cases} \tilde{\rho}_t + (\tilde{\rho}\tilde{u})_{\tilde{x}} = 0, \quad (\tilde{x}, t) \in \mathbf{R}_+ \times \mathbf{R}_+ \\ (\tilde{\rho}\tilde{u})_t + (\tilde{\rho}\tilde{u}^2 + \tilde{p})_{\tilde{x}} = \mu \tilde{u}_{\tilde{x}\tilde{x}} \\ (\tilde{\rho}, \tilde{u})|_{\tilde{x}=0} = (\rho_-, u_-) \quad \text{with} \quad u_- > 0 \\ (\tilde{\rho}, \tilde{u})|_{t=0} = (\tilde{\rho}_0, \tilde{u}_0)(\tilde{x}) \to (\rho_+, u_+) \quad \text{as} \quad \tilde{x} \to +\infty. \end{cases} \tag{1.1}$$

Here, $\tilde{\rho}(> 0)$ is the density, $\tilde{u}$ is the velocity, $\tilde{p} = \tilde{p}(\tilde{\rho}) = \tilde{\rho}^\gamma$(the adiabatic constant $\gamma \geq 1$) is the pressure, and μ is the viscosity constant. The condition

$$\tilde{\rho}_0(\tilde{x}) > 0, \quad \rho_\pm > 0 \tag{1.2}$$

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is assumed, and so the flow does not include the vacuum state at the initial time. The compatibility condition is

$$(\tilde{\rho}_0, \tilde{u}_0)(0) = (\rho_-, u_-). \tag{1.3}$$

The assumption $u_- > 0$ implies that, through the boundary $\tilde{x} = 0$, the fluid with the density ρ_- flows into the region under consideration with its speed $u_- > 0$, and hence the problem is called the *inflow problem*. In the case of $u_- < 0$ the problem is called the *outflow problem*.

When $u_- = 0$ and hence the condition $\tilde{\rho}|_{\tilde{x}=0} = \rho_-$ is removed, the problem becomes an initial-boundary value problem with fixed boundary:

$$\begin{cases} \tilde{\rho}_t + (\tilde{\rho}\tilde{u})_{\tilde{x}} = 0, \quad (\tilde{x}, t) \in \mathbf{R}_+ \times \mathbf{R}_+ \\ (\tilde{\rho}\tilde{u})_t + (\tilde{\rho}\tilde{u}^2 + \tilde{p})_{\tilde{x}} = \mu \tilde{u}_{\tilde{x}\tilde{x}} \\ \tilde{u}|_{\tilde{x}=0} = 0 \\ (\tilde{\rho}, \tilde{u})|_{t=0} = (\tilde{\rho}_0, \tilde{u}_0)(\tilde{x}) \to (\rho_+, u_+) \quad \text{as} \quad \tilde{x} \to +\infty. \end{cases}$$

This is changed to the problem in the Lagrangian coordinate (x, t) :

$$\begin{cases} v_t - u_x = 0, \quad (x, t) \in \mathbf{R}_+ \times \mathbf{R}_+ \\ u_t + p(v)_x = \mu(\dfrac{u_x}{v})_x \\ u|_{x=0} = 0(:= u_-) \\ (v, u)|_{t=0} = (v_0, u_0)(x) \to (v_+, u_+) := (1/\rho_+, u_+) \quad \text{as} \quad x \to +\infty. \end{cases} \tag{1.4}$$

Here, $v = 1/\rho$, u and $p = p(v) = v^{-\gamma} (\gamma \geq 1)$ are, respectively, the specific volume, velocity and pressure denoted in the Lagrangian coordinate. Matsumura and Nishihara [11] and Matsumura and Mei [7] have shown that the solution (v, u) to (1.4), roughly speaking, tends to the rarefaction wave as t tends to infinity when $u_+ > u_- = 0$, and the viscous shock wave when $u_+ < u_- = 0$.

We now concentrate on the case $u_- > 0$, the inflow problem (1.1). In the case $u_- < 0$, see [6] and also [3]. In [6] (1.1) is treated, but we here transform (1.1) to the problem in the Lagrangian coordinate:

$$\begin{cases} v_t - u_x = 0, \quad x > s_- t, \ t > 0 \\ u_t + p(v)_x = \mu(\dfrac{u_x}{v})_x \\ (v, u)|_{x=s_- t} = (v_-, u_-), \quad v_- = 1/\rho_-, \ u_- > 0 \\ (v, u)|_{t=0} = (v_0, u_0)(x) \to (v_+, u_+) = (1/\rho_+, u_+) \quad \text{as} \quad x \to \infty, \end{cases} \tag{P}$$

where

$$s_- = -\frac{u_-}{v_-} < 0. \tag{1.5}$$

See Figure 1.1.

The change $(\tilde{x}, t) \to (x, t)$ is given by

$$\begin{cases} \dfrac{\partial \tilde{x}(x,t)}{\partial t} = \tilde{u}(\tilde{x}(x,t), t) \quad t > 0, \ x > 0 \\ \\ \tilde{x}(x, 0) = \tilde{x}_0(x), \end{cases}$$

with

$$\int_0^{\tilde{x}_0(x)} \tilde{\rho}(y', 0)dy' = x,$$

where $(\tilde{x}, t) \in \Sigma_1 = \{(\tilde{x}, t);\ \tilde{x} > \tilde{x}(0, t)\}$, and by

$$\begin{cases} \dfrac{\partial \tilde{x}(x,t)}{\partial t} = \tilde{u}(\tilde{x}(x,t), t), \quad t > t_0(x),\ x < 0 \\ \\ \tilde{x}(x, t_0(x)) = 0, \end{cases}$$

with

$$-x = \int_0^{t_0(x)} (\tilde{\rho}\tilde{u})(0, \tau)d\tau = (\rho_- u_-) \cdot t_0(x),$$

where $(\tilde{x}, t) \in \Sigma_2 = \{(\tilde{x}, t);\ 0 < \tilde{x} < \tilde{x}(0, t)\}$. See Figure 1.2.

From the definition it follows that

$$x \geq -\rho_- u_- t = -\frac{u_-}{v_-}t = s_- t$$

and that

$$\int_{\tilde{x}(0,t)}^{\tilde{x}(x,t)} \tilde{\rho}(y', t)dy' = x \quad \text{for} \quad (\tilde{x}, t) \in \Sigma_i (i = 1, 2).$$

Hence, for $f(x, t) = \tilde{f}(\tilde{x}(x, t), t)$

$$\frac{\partial}{\partial t} f(x,t) = (\frac{\partial}{\partial t} + u\frac{\partial}{\partial \tilde{x}})\tilde{f}(\tilde{x}, t), \quad \frac{\partial}{\partial x} f(x,t) = v\frac{\partial}{\partial \tilde{x}}\tilde{f}(\tilde{x}, t),$$

which yields (P) with (1.5).

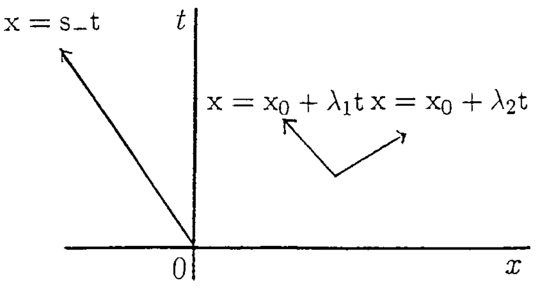

Figure 1.1

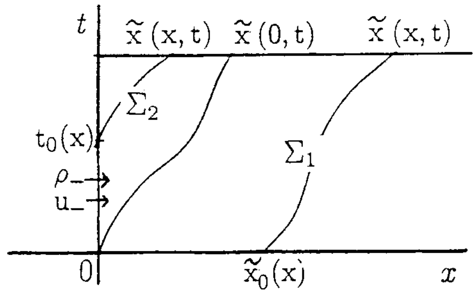

Figure 1.2

We now consider the inflow problem (P) described in the Lagrangian coordinate.

The characteristic speeds of the corresponding hyperbolic system without viscosity are $\lambda_i(v) = (-1)^i\sqrt{-p'(v)}$ $(i = 1, 2)$. Compare them with the speed $s_- = -\frac{u_-}{v_-}$ of the moving boundary. Since the sound speed $c(v)$ is defined by

$$c(v) = v\sqrt{-p'(v)} = \sqrt{\gamma}v^{-(\gamma-1)/2} \tag{1.6}$$

(note that $v\sqrt{-p'(v)} = \sqrt{\tilde{p}'(\tilde{\rho})}$), comparing $|u|$ with $c(v)$ instead of $|u|/v$ with $|\lambda_i(v)|$, we devide the (v, u)-space into three resions:

$$\begin{array}{rcl} \Omega_{sub} & = & \{(v,u);\ |u| < c(v),\ v > 0,\ u > 0\} \\ \Gamma_{trans} & = & \{(v,u);\ |u| = c(v),\ v > 0,\ u > 0\} \\ \Omega_{super} & = & \{(v,u);\ |u| > c(v),\ v > 0,\ u > 0\}. \end{array} \tag{1.7}$$

Call them the subsonic, transonic, and supersonic resions, respectively. See Figure 1.3.

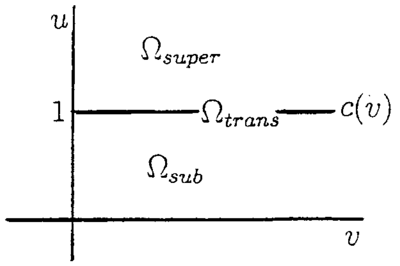

Figure 1.3($\gamma = 1$)

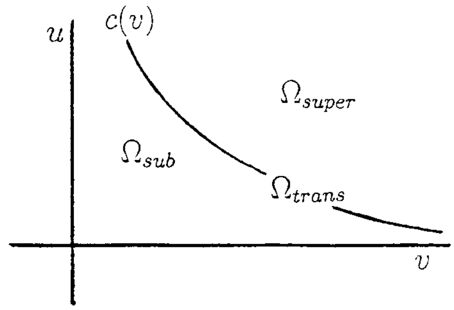

Figure 1.3($\gamma > 1$)

When $(v_-, u_-) \in \Omega_{sub}$, $\lambda_1(v_-) < s_-(< 0)$, and hence the existence of a traveling wave solution $(V, U)(x - s_-t)$ with $(V, U)(0) = (v_-, u_-)$, $(V, U)(+\infty) = (v_+, u_+)$ is expected. Substitute this into $(\mathrm{P})_{1,2}$ (this means the first and second equations in (P)) to have

$$\begin{cases} -s_-V' - U' = 0, \quad ' = d/d\xi, \quad \xi = x - s_-t > 0 \\ -s_-U' + p(V)' = \mu(\dfrac{U'}{V})' \\ (V, U)(0) = (v_-, u_-), \quad (V, U)(+\infty) = (v_+, u_+). \end{cases} \tag{1.8}$$

We call the solution (V, U) the boundary layer solution, or BL-solution simply. Seek for the condition for the existence of the BL-solution. When (V, U) exists, the integration of (1.8) over $(0, \infty)$ and (ξ, ∞) yields

$$\begin{cases} -s_-(v_+ - v_-) - (u_+ - u_-) = 0 \\ -s_-(u_+ - u_-) + p(v_+) - p(v_-) = -\mu\dfrac{U'(0)}{v_-} \end{cases} \tag{1.9}$$

and

$$\begin{cases} -s_-(V - v_+) - (U - u_+) = 0 \\ -s_-(U - u_+) + p(V) - p(v_+) = \mu\dfrac{U'}{V}. \end{cases} \tag{1.10}$$

From $(1.9)_1$ and $(1.10)_1$

$$s_- = -\frac{U(\xi)}{V(\xi)} = -\frac{u_-}{v_-} = -\frac{u_+}{v_+}, \tag{1.11}$$

and hence we define BL-line through $(v_-, u_-) \in \Omega_{sub}$ by

$$BL(v_-, u_-) = \{(v, u) \in \Omega_{sub} \cup \Gamma_{trans};\ \frac{u}{v} = \frac{u_-}{v_-} = -s_-\}.$$

Denote $\Gamma_{trans} \cap BL(v_-, u_-) = \{(v_*, u_*)\}$. By (1.10) we have the ordinary differential

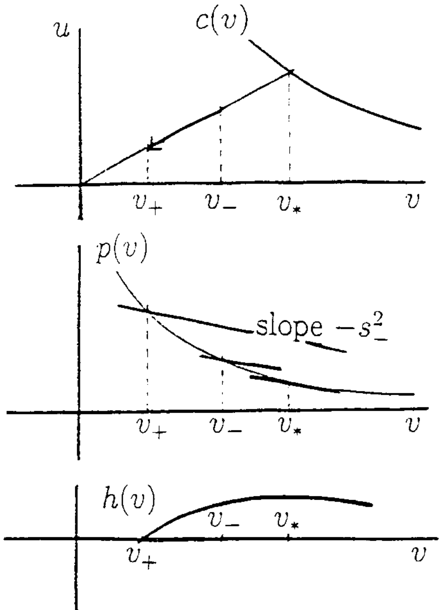

Figure 1.4($v_+ < v_-$)

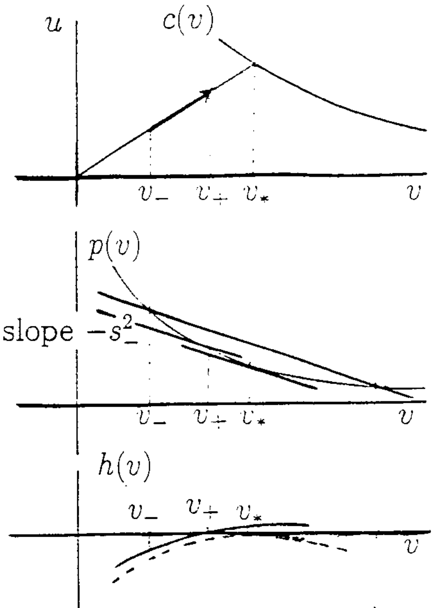

Figure 1.4($v_+ > v_-$)

equation of V:

$$\begin{cases} \mu \dfrac{dV}{d\xi} = \dfrac{V}{s_-}\{-s_-^2(V - v_+) - (p(V) - p(v_+))\} := \dfrac{V}{s_-} h(V) \\ V(0) = v_-, \quad V(+\infty) = v_+. \end{cases} \tag{1.12}$$

To the contrary, for $(v_+, u_+) \in BL(v_-, u_-)$ there exists a solution (V, U) to (1.12). Because we find that $h(v_+) = 0$, $h(v) < 0$ for $v_+ < v < v_-$ if $v_+ < v_-$ and $h(v) > 0$ for $v_- < v_+ (\leq v_*)$ if $v_- < v_+$. See Figure 1.4.

Noting that $h'(v_*) = 0$ and $h''(v_*) \neq 0$, we have the following lemma.

Lemma 1. 1 (Boundary Layer Solution) *Let* $(v_-, u_-) \in \Omega_{sub}$ *and* $(v_+, u_+) \in BL(v_-, u_-)$. *Then, there exists a unique solution* $(V, U)(\xi)$ *to (1.8), which satisfies*

$$\begin{aligned} &|(V(\xi) - v_+, U(\xi) - u_+)| \leq C \exp(-c|\xi|) && \text{if} \quad v_+ < v_* \\ &|(V(\xi) - v_+, U(\xi) - u_+)| \leq C|\xi|^{-1} && \text{if} \quad v_+ = v_*. \end{aligned}$$

On the other hand, since $0 > \lambda_1(v) > s_-$ in Ω_{super}, the 1-characteristic field is away from the moving boundary. The 2-characteristic field is, of course, away from the boundary. Hence, the behaviors of solutions are expected to be same as those for the Cauchy problem. By noting that $c'(v_*) > -\lambda_2(v_*)$ for $1 < \gamma < 3$, the large time behaviors to be expected devide the (v, u)-space as the following figure, Figure 1.5.

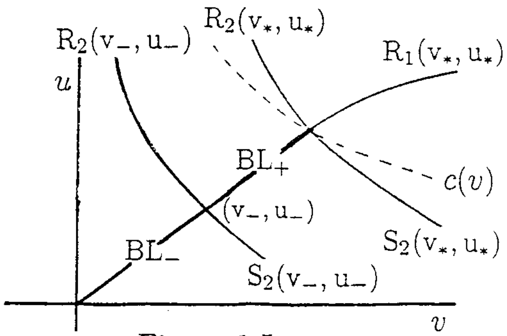

Figure 1.5

Here,

$$\begin{aligned}
BL_+(v_-,u_-) &= \{(v,u) \in BL(v_-,u_-); v_- < v \le v_*\} \\
BL_-(v_-,u_-) &= \{(v,u) \in BL(v_-,u_-); 0 < v < v_-\} \\
R_1(v_*,u_*) &= \{(v,u); u = u_* - \int_{v_*}^{v} \lambda_1(s)ds,\ v > v_*\} \\
R_2(v_-,u_-) &= \{(v,u); u = u_- - \int_{v_-}^{v} \lambda_2(s)ds,\ v < v_*\} \\
R_2(v_*,u_*) &= \{(v,u); u = u_* - \int_{v_*}^{v} \lambda_2(s)ds,\ v < v_*\} \\
S_2(v_-,u_-) &= \{(v,u); u = u_- - s_2(v - v_-),\ v > v_-\} \\
S_2(v_*,u_*) &= \{(v,u); u = u_* - s_*(v - v_*),\ v > v_*\},
\end{aligned}$$

together with $s_2 = \sqrt{-(p(v) - p(v_-))/(v - v_-)}$ and $s_* = \sqrt{-(p(v) - p(v_*))/(v - v_*)}$.

Our aim is to investigate the stability of the BL-solution or a superposition of the BL-solution and nonlinear waves. Our results are, roughly speaking, as follows.

(I) if $(v_+, u_+) \in BL_+(v_-, u_-)$, then the BL-solution is stable.

(II) If $(v_+, u_+) \in BL_-(v_-, u_-)$, then the BL-solution is stable provided that $|(v_+ - v_-, u_+ - u_-)|$ is small. That is, the BL-solution is necessary to be weak.

(III) if $(v_+, u_+) \in BL_+R_2(v_-, u_-)$, then there exists $(\bar{v}, \bar{u}) \in BL_+(v_-, u_-)$ such that $(v_+, u_+) \in R_2(\bar{v}, \bar{u})$, and the superposition of the BL-solution connecting (v_-, u_-) with $(\bar{v}, \bar{u})$ and the 2-rarefaction wave connecting $(\bar{v}, \bar{u})$ with (v_+, u_+) is stable provided that $|(v_+ - \bar{v}, u_+ - \bar{u})|$ is small, where

$$BL_+R_2(v_-,u_-) = \{(v,u);\ u > -s_-v,\ \ u > u_- - \int_{v_-}^{v} \lambda_2(s)ds,\ \ u \le u_* - \int_{v_*}^{v} \lambda_2(s)ds\}.$$

That is, the BL-solution is not necessarily weak and the rarefaction wave is weak.

(IV) if $(v_+, u_+) \in BL_-R_2(v_-, u_-)$, then the superposition of the BL-solution and the

2-rarefaction wave is stable provided that $|(v_+ - v_-, u_+ - u_-)|$ is small, where

$$BL\_R_2(v_-, u_-) = \{(v,u);\ u > -s_-v,\ \ u < u_- - \int_{v_-}^{v} \lambda_2(s)ds\}.$$

In this case, both the BL-solution and the rarefaction wave are weak.
(V) if $(v_*, u_*) \in BL_+(v_-, u_-)$, $(v_+, u_+) \in R_1R_2(v_*, u_*)$ and $|(v_+ - v_*, u_+ - u_*)|$ is small, then the superposition of the BL-solution, 1-rarefaction wave and 2-rarefaction wave is stable. Here,

$$R_1R_2(v_*, u_*) = \{(v,u);\ u > u_* - \int_{v_*}^{v} \lambda_i(s)ds,\ i = 1,2\}.$$

Similar to (III), the BL-solution is not necessarily weak.

In later sections we will give the proofs of (I)-(V), with which it is interesting to compare those of results [8,9,10] on the Cauchy problem for the viscous p-system:

$$\begin{cases} v_t - u_x = 0 \quad (x,t) \in \mathbf{R} \times \mathbf{R}_+ \\ u_t + p(v)_x = \mu(\dfrac{u_x}{v})_x \\ (v,u)|_{t=0} = (v_0, u_0)(x) \to (v_\pm, u_\pm) \ \text{ as } \ x \to \pm\infty \end{cases} \tag{1.13}$$

(For more general systems see [1,4,12,13] etc. and references therein). In these papers the signs of first order derivative of rarefaction waves and viscous shock waves are crucial. The cases (I) and (III) are, respectively, similar to the cases $(v_+, u_+) \in R_1(v_-, u_-)$ and $\in R_1R_2(v_-, u_-)$. Hence, global results on the present problem are expected, but we could not control the values from the boundary for large data. In the case of (II) it seems to be available to take the perturbation of integrated form $(\phi, \psi)(x,t) = -\int_\xi^\infty (v - V, u - U)dy$. However, in general, $\int_0^\infty (v_0 - V, u_0 - U)(y)dy \neq (0,0)$. Even if we assume that $\int_0^\infty (v_0 - V, u_0 - U)(y)dy = (0,0)$, we could not control the values from the boundary even for small data. So, we put the perturbation $(\phi, \psi) = (v - V, u - U)$. This is no integrated form, and the sign of (V_ξ, U_ξ) is not good. However, we can overcome this for the weak BL-solution, applying the discussion by Kawashima and Nikkuni [2].

Related to this case, when

$$(v_+, u_+) \in BL\_S_2(v_-, u_-) = \{(v,u);\ u < -s_-v,\ \ u < u_- - s_2(v - v_-)\},$$

the asmptotic state is conjectured to be $(V,U)(x - s_-t) + (V_2^S, U_2^S)(x - s_2t + \alpha) - (\bar{v}, \bar{u})$ together with a suitable shift α, where $(\bar{v}, \bar{u}) \in BL_-(v_-, u_-)$ such that $(v_+, u_+) \in S_2(\bar{v}, \bar{u})$, and (V,U) is the BL-solution connecting (v_-, u_-) with $(\bar{v}, \bar{u})$ and (V_2^S, U_2^S) is 2-viscous shock wave connecting $(\bar{v}, \bar{u})$ with (v_+, u_+). In the final section how to determine the shift α will be discussed.

Our plan of this paper is as follows. After stating the notations, in Section 2 we show the cases $(I), (II)$. In Section 3 the cases (III)-(V) will be treated. In the final section we will present the concluding remarks.

Notations. Throughout this paper several positive generic constants are denoted by $c_i(a, b, \cdots)$, $C_i(a, b, \cdots)(i = 0, 1, 2, \cdots)$ depending on $a, b, \cdots$, or simply by c_i, C_i, c, C

without confusions. Denote $f(x) \sim g(x)$ as $x \to a$ when $C^{-1}g < f < Cg$ in a neighborhood of a. For function spaces, $L^p(\Omega)$, $1 \le p \le \infty$ be an usual Lebesgue space on $\Omega \subset \mathbf{R} = (-\infty, \infty)$ with its norm

$$\|f\|_{L^p(\Omega)} = (\int_\Omega |f(x)|^p dx)^{1/p}, \ 1 \le p < \infty, \quad \|f\|_{L^\infty(\Omega)} = \sup_\Omega |f(x)|.$$

$H^l(\Omega)$ denotes the l-th order Sobolev space with its norm

$$\|f\|_l = (\sum_{j=0}^{l} \|\partial_x^j f\|^2)^{1/2}, \quad \text{where} \quad \|\cdot\| := \|\cdot\|_{L^2(\Omega)}.$$

$H_0^1(\Omega)$ is a closure of $C_0^\infty(\Omega)$ with respect to H^1-norm, so that $f \in H_0^1(\Omega)$ satisfies $f(\partial\Omega) = 0$. The domain Ω will be often abbrieviated without confusions.

2 Stability of the boundary layer solution

2.1 The case $(v_+, u_+) \in BL_+(v_-, u_-)$

Assume that

$$(v_-, u_-) \in \Omega_{sub} \quad \text{and} \quad (v_+, u_+) \in BL_+(v_-, u_-), \tag{2.1}$$

then Lemma 1.1 gives a unique boundary layer solution $(V, U)(\xi)$, $\xi = x - s_- t \ge 0$, $s_- = -u_-/v_-$ satisfying (1.8) or (1.12). Note that

$$V_\xi = \frac{V}{\mu s_-} h(V) > 0, \quad h(V) < 0, \quad h''(V) > 0 \ \text{ for } \ v_- < V < v_+. \tag{2.2}$$

We put the perturbation $(\phi, \psi)(\xi, t)$ by

$$(v, u)(x, t) = (V, U)(\xi) + (\phi, \psi)(\xi, t), \tag{2.3}$$

so that the reformulated problem is

$$\begin{cases} \phi_t - s_-\phi_\xi - \psi_\xi = 0, \quad \xi > 0, \quad t > 0 \\ \psi_t - s_-\psi_\xi + (p(V+\phi) - p(V))_\xi = \mu(\dfrac{U_\xi + \psi_\xi}{V + \phi} - \dfrac{U_\xi}{V})_\xi \\ (\phi, \psi)|_{\xi=0} = (0, 0) \\ (\phi, \psi)|_{t=0} = (\phi_0, \psi_0)(\xi) := (v_0 - V, u_0 - U)(\xi), \end{cases} \tag{2.4}$$

from (P) and (1.8). The solution space is

$$X_{m,M}(0, T) = \{(\phi, \psi) \in C([0, T]; H_0^1) \mid \phi_\xi \in L^2(0, T; L^2), \ \ \psi_\xi \in L^2(0, T; H^1)$$
$$\text{with} \quad \sup_{[0,T]} \|(\phi, \psi)(t)\|_1 \le M, \quad \inf_{\mathbf{R}_+ \times [0,T]} (V + \phi)(\xi, t) \ge m\},$$

for positive constants m, M.

To obtain the stability theorem, we combine the time-local existence of the solution (ϕ, ψ) to (2.4) with the a priori estimates. Those are given as follows.

Proposition 2. 1 (Local existence) *Let* (ϕ_0, ψ_0) *be in* $H_0^1(\mathbf{R}_+)$. *If* $\|\phi_0, \psi_0\|_1 \le M$, *and* $\inf_{\mathbf{R}_+ \times [0,T]}(V+\phi)(\xi,t) \ge m$, *then there exists* $t_0 = t_0(m, M) > 0$ *such that (2.4) has a unique solution* $(\phi, \psi) \in X_{\frac{1}{2}m, 2M}(0, t_0)$.

Proposition 2. 2 (A priori estimates) *Let* (ϕ, ψ) *be in* $X_{\frac{1}{2}m,\varepsilon}(0,T)$. *Then, for a suitably small* $\varepsilon > 0$, *there exists a constant* $C_0 > 0$ *such that*

$$\|(\phi,\psi)(t)\|_1^2 + \int_0^t (\psi_\xi(0,\tau)^2 + \|\sqrt{V_\xi}\phi(\tau)\|^2 + \|\phi_\xi(\tau)\|^2 + \|\psi_{\xi\xi}(\tau)\|^2)d\tau \le C_0\|\phi_0, \psi_0\|_1^2.$$

Remark 2.1 If ε is suitably small, then $\inf_{\mathbf{R}_+\times[0,T]}(V+\phi)(\xi,t) \ge m/2$ is automatically satisfies by the Sobolev inequality. Hence we denote $X_{m,\varepsilon}(0,T)$ simply by $X_\varepsilon(0,T)$.

The following stability theorem is from these two Propositions, which is on the same line as in [7-11].

Theorem 2. 1 (Stability of BL-solution in case of $(v_+, u_+) \in BL_+(v_-, u_-)$**)**
If $\|v_0 - V, u_0 - U\|_1$ *is suitably small together with the compatibility condition* $(v_0 - V, u_0 - U)(0) = (0,0)$, *then there exists a unique solution* (v, u) *to (P), which satisfy* $(v - V, u - U) \in C([0,\infty); H_0^1)$ *and*

$$\sup_{\xi\ge 0}|(\phi,\psi)(\xi,t)| = \sup_{x \ge s_-t}|(v,u)(x,t) - (V,U)(x - s_-t)| \to 0 \quad as \quad t \to \infty.$$

We first devote ourselves to the proof of Proposition 2.2, which will be done by a series of Lemmas. At the end of this section we will mention the local existence theorem.

Multiply $(2.4)_1$(first equation of (2.4)) and $(2.4)_2$ by $-(p(V+\phi) - p(V))$ and ψ, respectively, and add these two equations to have a divergence form

$$\begin{aligned} &\{\frac{1}{2}\psi^2 + \Phi(v,V)\}_t \\ &+\{-s_-\Phi(v,V) - \frac{s_-}{2}\psi^2 + (p(v) - p(V))\psi - \mu(\frac{u_\xi}{v} - \frac{U_\xi}{V})\psi\}_\xi \\ &+\{\mu\frac{\psi_\xi^2}{v} - \mu s_-\frac{V_\xi\phi\psi_\xi}{vV} - s_-V_\xi(p(V+\phi) - p(V) - p'(V)\phi)\} = 0, \end{aligned} \tag{2.5}$$

where

$$\Phi(v,V) = p(V)\phi - \int_V^{V+\phi} p(\eta)d\eta. \tag{2.6}$$

Here and after we will often use the notation $(v,u) = (V+\phi, U+\phi)$, though the unknown functions are ϕ and ψ. Since $p''(V) > 0$, put

$$p(V+\phi) - p(V) - p'(V)\phi = f(v,V)\phi^2, \tag{2.7}$$

then $f(v,V) \ge 0$. Noting that $-s_-V_\xi > 0$, we regard the last three terms in (2.5) as the quadratic equation:

$$\begin{aligned} Q &:= \mu\frac{\psi_\xi^2}{v} - \mu s_-\frac{V_\xi\phi\psi_\xi}{vV} - s_-V_\xi f(v,V)\phi^2 \\ &= (\sqrt{\mu}\frac{\psi_\xi}{\sqrt{v}})^2 - \frac{\sqrt{-\mu s_- V_\xi}}{V\sqrt{vf(v,V)}}\cdot\sqrt{\mu}\frac{\psi_\xi}{\sqrt{v}}\cdot\sqrt{-s_-V_\xi f(v,V)}\phi + (\sqrt{-s_-V_\xi f(v,V)}\phi)^2. \end{aligned}$$

The discriminant of Q is

$$D = \frac{-\mu s_- V_\xi}{V^2 v f(v,V)} - 4 = \frac{-h(V)}{V v f(v,V)} - 4. \tag{2.8}$$

Since $v_+ > v_-$,

$$-h(V) = s_-^2 (V - v_+) + p(V) - p(v_+) < p(V) = V^{-\gamma}. \tag{2.9}$$

Moreover, by putting $X = V/v$,

$$\begin{aligned} V v f(v,V) &= \frac{Vv(v^{-\gamma} - V^{-\gamma} + \gamma V^{-\gamma-1}(v - V))}{(v-V)^2} \\ &= V^{-\gamma} \cdot \frac{X^{\gamma+1} - (\gamma+1)X + \gamma}{(X-1)^2} \geq \gamma V^{-\gamma}, \end{aligned} \tag{2.10}$$

because $X^{\gamma+1} - (\gamma+1)X + \gamma \geq \gamma(X-1)^2$ for $X \geq 0$. By (2.8) - (2.10),

$$D \leq \frac{V^{-\gamma}}{\gamma V^{-\gamma}} - 4 = \frac{1}{\gamma} - 4 \leq -3. \tag{2.11}$$

Thus, integrating (2.5) over $(0,\infty) \times (0,t)$, we have the following basic lemma.

Lemma 2. 1 (Basic estimate) *For the solution $(\phi, \psi) \in X_{2\varepsilon}(0,T)$, it holds that*

$$\begin{aligned} &\frac{1}{2}\|\psi(t)\|^2 + \int_0^\infty \Phi(v,V)(\xi,t)d\xi \\ +C^{-1}&\int_0^t \int_0^\infty \{\frac{\psi_\xi^2}{v} + |\frac{V_\xi \phi \psi_\xi}{vV}| + (p(V+\phi) - p(V) - p'(V)\phi)V_\xi\} d\xi d\tau \\ &\leq \frac{1}{2}\|\psi_0\|^2 + \int_0^\infty \Phi(v_0,V)(\xi)d\xi \leq C\|\phi_0, \psi_0\|^2. \end{aligned}$$

Remark 2.2 The method used here is similar to that in [11]. But, (2.11) is sharper than the corresponding one in [11]. Note that the basic estimate is obtained without smallness condition on the data.

Next, following [11], change ϕ to $\tilde{v} := v/V$. Since

$$p(V+\phi) - p(V) - p'(V)\phi = V^{-\gamma}(\tilde{v}^{-\gamma} - 1 + \gamma(\tilde{v} - 1))$$

and

$$\Phi(v,V) = V^{-\gamma+1}\tilde{\Phi}(\tilde{v}),$$

where

$$\tilde{\Phi}(\tilde{v}) = \begin{cases} \tilde{v} - 1 - \ln \tilde{v} \quad (\gamma = 1) \\ \tilde{v} - 1 + \dfrac{1}{\gamma - 1}(\tilde{v}^{-\gamma+1} - 1) \quad (\gamma > 1), \end{cases} \tag{2.12}$$

Lemma 2.1 is rewritten as follows.

Lemma 2. 2 *It follows that*

$$\frac{1}{2}\|\psi(t)\|^2+\int_0^\infty V^{-\gamma+1}\tilde{\Phi}(\tilde{v}(\xi,t))d\xi$$

$$+C^{-1}\int_0^t\int_0^\infty\{\frac{\psi_\xi^2}{v}+|\frac{V_\xi\phi\psi_\xi}{vV}|+\frac{V_\xi}{V^\gamma}(\tilde{v}^{-\gamma}-1+\gamma(\tilde{v}-1))\}d\xi d\tau$$
$$\leq C\|\phi_0,\psi_0\|^2.$$

Eq. $(2.4)_2$ is also written as

$$(\mu\frac{\tilde{v}_\xi}{\tilde{v}}-\psi)_t-s_-(\mu\frac{\tilde{v}_\xi}{\tilde{v}}-\psi)_\xi+\frac{\gamma\tilde{v}_\xi}{V^\gamma\tilde{v}^{\gamma+1}}+\frac{\gamma V_\xi}{V^{\gamma+1}}(\tilde{v}^{-\gamma}-1)=0. \tag{2.13}$$

Multiplying (2.13) by $\tilde{v}_\xi/\tilde{v}$, we have a divergence form

$$\begin{aligned}&\{\frac{\mu}{2}(\frac{\tilde{v}_\xi}{\tilde{v}})^2-\psi(\frac{\tilde{v}_\xi}{\tilde{v}})\}_t\\&+\{\psi\frac{v_t}{v}-\frac{\gamma h(V)}{s_-\mu V^\gamma}(\frac{\tilde{v}^{-\gamma}-1}{\gamma}+\ln\tilde{v})-\frac{\mu s_-}{2}(\frac{\tilde{v}_\xi}{\tilde{v}})^2\}_\xi+\frac{\gamma\tilde{v}_\xi^2}{V^\gamma\tilde{v}^{\gamma+2}}\\&=\frac{\psi_\xi^2}{v}+\frac{s_-\phi\psi_\xi V_\xi}{vV}-\frac{\gamma V_\xi}{s_-\mu}\frac{h'(V)V^\gamma-h(V)\gamma V^{\gamma-1}}{V^{2\gamma}}\{\frac{\tilde{v}^{-\gamma}-1}{\gamma}+\ln\tilde{v}\}.\end{aligned}\tag{2.14}$$

By (2.2)

$$|\text{the final term of (2.14)}|\leq C\frac{V_\xi}{V^\gamma}(\tilde{v}^{-\gamma}-1+\gamma(\tilde{v}-1)).$$

Hence, the right hand side of (2.14) is controllable by Lemma 2.2. Thus, integrating (2.14) over $(0,\infty)\times(0,t)$ yields the following lemma.

Lemma 2. 3 *It holds that*

$$\begin{aligned}&\|\frac{\tilde{v}_\xi}{\tilde{v}}(t)\|^2+\int_0^t\int_0^\infty\frac{\tilde{v}_\xi^2}{\tilde{v}^{\gamma+2}}d\xi d\tau\\&\leq C(\|\phi_0\|_1^2+\|\psi_0\|^2)+C\int_0^t(\frac{\tilde{v}_\xi}{\tilde{v}})^2(0,\tau)d\tau.\end{aligned}\tag{2.15}$$

We have tried to control the final term of (2.15), $C\int_0^t(\frac{\tilde{v}_\xi}{\tilde{v}})^2(0,\tau)d\tau$, without smallness condition, in a similar fashion to [11]. But, we could not break through it. However, we can control it provided that the initial data is small. Since

$$(\frac{\tilde{v}_\xi}{\tilde{v}})^2(0,\tau)=\frac{1}{v_-^2}\phi_\xi^2(0,\tau)=\frac{1}{u_-^2}\psi_\xi^2(0,\tau) \tag{2.16}$$

(the validity of this equation will be stated later), it is necessary to estimate $\int_0^t\|\psi_{\xi\xi}(\tau)\|^2d\tau$, which is controllable for small the initial data.

We now assume that

$$N(T):=\sup_{0\leq t\leq T}\|(\phi,\psi)(t)\|_1\leq 2\varepsilon\ll 1.$$

Multiplying $(2.4)_2$ by $-\psi_{\xi\xi}$, we have

$$(\frac{1}{2}\psi_\xi^2)_t + (-\psi_t\psi_\xi + \frac{s_-}{2}\psi_\xi^2)_\xi + \mu\frac{\psi_{\xi\xi}^2}{v}$$

$$= \{-\mu\frac{\psi_\xi(V_\xi+\phi_\xi)}{(V+\phi)^2} + \mu(\frac{U_\xi}{V+\phi} - \frac{U_\xi}{V})_\xi - (p(V+\phi)-p(V))_\xi\}(-\psi_{\xi\xi})$$

and, after integrating the resultant equation over $(0,\infty)\times(0,t)$,

$$\begin{aligned}&\|\psi_\xi(t)\|^2 + \int_0^t (\psi_\xi(0,\tau)^2 + \|\psi_{\xi\xi}(\tau)\|^2)d\tau \\ &\le C\|\psi_{0\xi}\|^2 + C\int_0^t\int_0^\infty (\phi_\xi^2 + V_\xi\phi^2 + \psi_\xi^2)d\xi d\tau.\end{aligned} \tag{2.17}$$

Here, we have estimated the amount $(\phi_\xi\psi_\xi)^2$ as

$$\int_0^t\int_0^\infty (\phi_\xi\psi_\xi)^2 d\xi d\tau \le \int_0^t \|\psi_\xi\|\|\psi_{\xi\xi}\|\|\phi_\xi\|^2 d\tau$$

$$\le \nu\int_0^t \|\psi_{\xi\xi}\|^2 d\tau + C_\nu N(T)^2\int_0^t \|\phi_\xi(\tau)\|^2 d\tau$$

for a small constant $\nu > 0$. By Lemma 2.1, (2.17) is reduced to

$$\begin{aligned}&\|\psi_\xi(t)\|^2 + \int_0^t (\psi_\xi(0,\tau)^2 + \|\psi_{\xi\xi}(\tau)\|^2)d\tau \\ &\le C(\|\phi_0\|^2 + \|\psi_0\|_1^2) + C\int_0^t \|\phi_\xi(\tau)\|^2 d\tau.\end{aligned} \tag{2.18}$$

For a small constant $\lambda > 0$, $(2.15) + (2.18)\cdot\lambda$ together with (2.16) yields

$$\begin{aligned}&\|\frac{\tilde{v}_\xi}{\tilde{v}}(t)\|^2 + \lambda\|\psi_\xi(t)\|^2 + \int_0^t (\|\tilde{v}_\xi(\tau)\|^2 + \lambda\psi_\xi(0,\tau)^2 + \lambda\|\psi_{\xi\xi}(\tau)\|^2)d\tau \\ &\le C\|\phi_0,\psi_0\|_1^2 + C\int_0^t ((\frac{\tilde{v}_\xi}{\tilde{v}})^2(0,\tau) + \lambda\|\phi_\xi(\tau)\|^2)d\tau \\ &\le C\|\phi_0,\psi_0\|_1^2 + \int_0^t (\nu\|\psi_{\xi\xi}(\tau)\|^2 + C_\nu\|\psi_\xi(\tau)\|^2 + C\lambda\|\phi_\xi(\tau)\|^2)d\tau.\end{aligned} \tag{2.19}$$

Since

$$\|\tilde{v}_\xi(t)\|^2 = \int_0^\infty (\frac{\phi_\xi}{V} - \frac{V_\xi\phi}{V^2})^2 d\xi$$

$$\ge \int_0^\infty (\frac{\phi_\xi^2}{2V^2} - \frac{(V_\xi\phi)^2}{V^4})d\xi \ge c_0\|\phi_\xi(t)\|^2 - C\|\phi(t)\|^2$$

and

$$\int_0^t \|\tilde{v}_\xi(\tau)\|^2 d\tau \ge c_0\int_0^t \|\phi_\xi(\tau)\|^2 d\tau - C\int_0^t\int_0^\infty V_\xi\phi^2 d\xi d\tau,$$

we fix λ such that $C\lambda \le c_0/2$ and ν such that $\nu \le \lambda/2$. Then, the following lemma holds.

Lemma 2. 4 *If $N(T) = \sup_{0\le t\le T} \|(\phi,\psi)(t)\|_1$ is suitably small, then*

$$\|(\phi_\xi,\psi_\xi)(t)\|^2 + \int_0^t (\psi_\xi(0,\tau)^2 + \|(\phi_\xi,\psi_{\xi\xi})(\tau)\|^2)d\tau \le C\|\phi_0,\psi_0\|_1^2.$$

Combining Lemmas 2.1-2.4 completes the proof of Proposition 2.2.

We now mention the unique existence of the local solution to (2.4), the proof of Proposition 2.1, briefly. By $(2.4)_1$, ϕ has the explicit form

$$\phi(\xi,t) = \begin{cases} \displaystyle\int_{t+\frac{\xi}{s_-}}^t \psi_\xi(\xi + s_-(t-\tau),\tau)d\tau, & 0\le\xi\le -s_-t \\ \displaystyle\phi_0(\xi+s_-t) + \int_0^t \psi_\xi(\xi+s_-(t-\tau),\tau)d\tau, & \xi \ge -s_-t. \end{cases} \qquad (2.20)_{\phi_0}$$

Eq.$(2.4)_2$ is regarded as the initial-boundary value problem for the linear parabolic equatioin of ψ:

$$\begin{cases} \psi_t - \mu(\dfrac{\psi_\xi}{V+\phi})_\xi = g := g(\psi_\xi,\phi,\phi_\xi) \\ \psi(0,t) = 0 \\ \psi(\xi,0) = \psi_0(\xi), \end{cases} \qquad (2.21)_{\phi_0}$$

where

$$g(\psi_\xi,\phi,\phi_\xi) = s_-\psi_\xi - (p(V+\phi)-p(V))_\xi + \mu(\frac{U_\xi}{V+\phi} - \frac{U_\xi}{V})_\xi. \qquad (2.22)$$

To use the iteration method, we approximate $(\phi_0,\psi_0)\in H_0^1$ by $(\phi_{0k},\psi_{0k})\in H^3\cap H_0^1$ such that

$$(\phi_{0k},\psi_{0k}) \to (\phi_0,\psi_0) \ \text{ strongly in } \ H^1$$

as $k\to\infty$. We may assume

$$\|\phi_{0k},\psi_{0k}\|_1 \le \frac{3}{2}M, \quad \inf_{\mathbf{R}_+}(V+\phi_{0k}) \ge \frac{2}{3}m$$

for any $k\ge 1$. We first define the sequence $\{(\phi^{(n)},\psi^{(n)})\} := \{(\phi_k^{(n)},\psi_k^{(n)})\}$ for each k so that

$$(\phi^{(0)},\psi^{(0)})(\xi,t) = (\phi_{0k},\psi_{0k})(\xi), \qquad (2.23)$$

and, for a given $(\phi^{(n-1)},\psi^{(n-1)})(\xi,t)$, $\psi^{(n)}$ is a solution to

$$\begin{cases} \psi_t^{(n)} - \mu(\dfrac{\psi_\xi^{(n)}}{V+\phi^{(n-1)}})_\xi = g^{(n-1)} := g(\psi_\xi^{(n-1)},\phi^{(n-1)},\phi_\xi^{(n-1)}) \\ \psi^{(n)}(0,t) = 0 \\ \psi^{(n)}(\xi,0) = \psi_{0k}(\xi), \end{cases} \qquad (2.21)'$$

and

$$\phi^{(n)}(\xi,t) = \begin{cases} \displaystyle\int_{t+\frac{\xi}{s_-}}^t \psi_\xi^{(n)}(\xi+s_-(t-\tau),\tau)d\tau, & 0\le\xi\le -s_-t \\ \displaystyle\phi_{0k}(\xi+s_-t) + \int_0^t \psi_\xi^{(n)}(\xi+s_-(t-\tau),\tau)d\tau, & \xi\ge -s_-t. \end{cases} \qquad (2.20)'$$

From the linear theory, if $g \in C^0([0,T]; H^2)$, $\psi_0 \in H^3 \cap H_0^1$, then there exists a unique solution ψ to $(2.21)_{\phi_0}$ satisfying

$$\psi \in C([0,T]; H^3 \cap H_0^1) \cap C^1([0,T]; H^1) \cap L^2(0,T; H^4).$$

Using this, if $(\phi^{(n-1)}, \psi^{(n-1)}) \in X_{\frac{1}{2}m,2M}$, then we have

$$\begin{aligned} \|(\phi^{(n)}, \psi^{(n)})(t)\|^2 &\leq ((\frac{3}{2}M)^2 + C(m,M)t_0) \exp(C(m,M)t_0) \\ &\leq (2M)^2 \quad \text{if} \quad 0 < t_0 := t_0(m,M) \ll 1 \end{aligned} \tag{2.24}$$

and also

$$\int_0^{t_0} \|\psi_\xi^{(n)}(\tau)\|_1^2 d\tau \leq C(m,M)M^2.$$

Hence, direct estimates on $(2.20)'$ give

$$\| \int_{t+\frac{\xi}{s_-}}^t \psi_\xi^{(n)}(\xi + s_-(t-\tau), \tau) d\tau \|_1 \leq C\sqrt{t_0}M$$

and

$$\| \int_0^t \psi_\xi^{(n)}(\xi + s_-(t-\tau), \tau) d\tau \|_1 \leq C\sqrt{t_0}M.$$

Hence, for a suitable small t_0 we have

$$\sup_{0 \leq t \leq t_0} \|\phi^{(n)}(t)\|_1 \leq 2M \quad \text{and} \quad \inf_{\mathbf{R}_+ \times [0,t_0]} (V + \phi)(\xi, t) \geq \frac{1}{2}m. \tag{2.25}$$

By (2.24) - (2.25), $(\phi^{(n)}, \psi^{(n)}) \in X_{\frac{1}{2}m,2M}(0,t_0)$. Since $\|\phi_{0k}, \psi_{0k}\|_3 \leq C_k$, $(\phi^{(n)}, \psi^{(n)})$ can be shown to be the Cauchy sequence in $C([0,t_0]; H^2)$, by a standard way. Thus we have a solution $(\phi_k, \psi_k) \in X_{\frac{1}{2}m,2M}(0,t_0) \cap C([0,t_0]; H^2)$ to $(2.20)_{\phi_{0k}}$ and $(2.21)_{\phi_{0k}}$ by $\lim_{n\to\infty}(\phi^{(n)}, \psi^{(n)}) = \lim_{n\to\infty}(\phi_k^{(n)}, \psi_k^{(n)})$. Here, we note that

$$\psi_k \in C^1([0,t_0]; L^2) \cap L^2(0,t_0; H^3), \tag{2.26}$$

since $g((\psi_k)_\xi, \phi_k, (\phi_k)_\xi) \in C([0,t_0]; H^1)$ and $(\phi_{0k}, \psi_{0k}) \in H^2 \cap H_0^1$. Again, showing that (ϕ_k, ψ_k) is a Cauchy sequence in $C([0,t_0]; H^1)$ (taking t_0 smaller than the previous one if necessary), we obtain the desired unique-local solution $(\phi, \psi) \in X_{\frac{1}{2}m,2M}(0,t_0)$. We omit the details.

Here we state the validity of (2.16). As we see above, $(\phi_k, \psi_k) \to (\phi, \psi)$ as $k \to \infty$, and (ϕ_k, ψ_k) is a solution to (2.4) with its initial data (ϕ_{0k}, ψ_{0k}). Hence, by $(2.20)_{\phi_{0k}}$

$$\phi_k(\xi, t) = \int_{t+\frac{\xi}{s_-}}^t (\psi_k)_\xi(\xi + s_-(t-\tau), \tau) d\tau, \quad 0 \leq \xi \leq s_- t.$$

Since

$$(\phi_k)_t(\xi, t) = (\psi_k)_\xi(\xi, t) - (\psi_k)_\xi(0, t + \frac{\xi}{s_-}) + s_- \int_{t+\frac{\xi}{s_-}}^t (\psi_k)_{\xi\xi}(\xi + s_-(t-\tau), \tau) d\tau,$$

(2.26) shows that $(\phi_k)_t(\xi,t) \to 0$ as $\xi \to 0$. Hence, by $(2.4)_1$

$$-s_-(\phi_k)_\xi(0,t) - (\psi_k)_\xi(0,t) = 0.$$

The solution ψ is in $L^2(0,t_0;H^2)$ and so $\psi_\xi(0,t)$ has a meaning for almost all $t \geq 0$ and

$$\psi_\xi(0,t) = \lim_{k\to\infty}(\psi_k)_\xi(0,t).$$

Thus, there exists $\lim_{k\to\infty}(\phi_k)_\xi(0,t)$. Therefore, Lemma 2.3 is first obtained for (ϕ_k,ψ_k) or $\tilde{v}_k = V + \phi_k$, and then for (ϕ,ψ) after letting k tend to infinity.

2.2 The case $(v_+,u_+) \in BL_-(v_-,u_-)$

In this section we assume that

$$(v_-,u_-) \in \Omega_{sub} \quad \text{and} \quad (v_+,u_+) \in BL_-(v_-,u_-), \tag{2.27}$$

then Lemma 1.1 gives a unique BL-solution $(V,U)(\xi)$ satisfying (1.8) or (1.12) together with

$$V_\xi = \frac{V}{\mu s_-}h(V) < 0, \quad h(V) > 0, \quad h'(V) > 0 \ \text{ for } \ v_+ < V < v_-. \tag{2.28}$$

We put the perturbation $(\phi,\psi)(\xi,t)$ by

$$(v,u)(x,t) = (V,U)(\xi) + (\phi,\psi)(\xi,t), \quad \xi = x - s_-t, \tag{2.29}$$

so that the reformulated problem is

$$\begin{cases} \phi_t - s_-\phi_\xi - \psi_\xi = 0, \quad \xi > 0, \quad t > 0 \\ \psi_t - s_-\psi_\xi + (p(V+\phi) - p(V))_\xi = \mu(\dfrac{U_\xi + \psi_\xi}{V+\phi} - \dfrac{U_\xi}{V})_\xi \\ (\phi,\psi)|_{\xi=0} = (0,0) \\ (\phi,\psi)|_{t=0} = (\phi_0,\psi_0)(\xi) := (v_0 - V, u_0 - U)(\xi), \end{cases} \tag{2.30}$$

which is formally same as (2.4). However, the sign of V_ξ is negative, and so Lemma 2.1 does not hold. Nevertheless, we seek for the solution (ϕ,ψ) to (2.30) in the same solution space

$$X_\varepsilon(0,T) = \{(\phi,\psi) \in C([0,T];H^1) \mid \phi_x \in L^2(0,T;L^2), \quad \psi_x \in L^2(0,T;H^1)$$
$$\text{with} \quad \sup_{[0,T]} \|(\phi,\psi)(t)\|_1 \leq \varepsilon\},$$

for a suitably small $\varepsilon > 0$(Cf. Remark 2.1). Then, the a priori estimates are obtained as follows.

Proposition 2. 3 (A priori estimates) *Let $\delta := |v_+ - v_-, u_+ - u_-|$ be suitably small and (ϕ,ψ) be a solution to (2.30) in $X_\varepsilon(0,T)$ for a suitably small $\varepsilon > 0$. Then, there exists a constant C_1 such that*

$$\|(\phi,\psi)(t)\|_1^2 + \int_0^t (\psi_\xi(0,\tau)^2 + \|\phi_\xi(\tau)\|^2 + \|\psi_\xi(\tau)\|_1^2)d\tau \leq C_1\|\phi_0,\psi_0\|_1^2.$$

Combining the local existence theorem with Proposition 2.3 we have the stability theorem.

Theorem 2. 2 (Stability of BL-solution in case of $(v_+, u_+) \in BL_-(v_-, u_-)$) *If $|v_+ - v_-, u_+ - u_-| + \|v_0 - V, u_0 - U\|_1$ is suitably small with the compatibility condition $(v_0 - V, u_0 - U)(0) = (0,0)$, then there exists a unique solution (v, u) to (P), which satisfies $(v - V, u - U) \in C([0,\infty); H_0^1)$ and*

$$\sup_{x \geq s_- t} |(v,u)(x,t) - (V,U)(x - s_- t)| \to 0 \quad as \quad t \to \infty.$$

We only show the a priori estimates. Assume that

$$N(T) = \sup_{0 \leq t \leq T} \|(\phi, \psi)(t)\|_1 \leq \varepsilon \leq \varepsilon_0 (\ll 1),$$

where ε_0 is chosen as

$$\sup_{\mathbf{R}} |f(x)| \leq C\|f\|^{1/2}\|f_x\|^{1/2} \leq C\varepsilon_0 < \frac{v_+}{2}$$

so that $V + \phi \geq v_+/2$. Multiplying $(2.30)_2$ by ψ and $(2.30)_1$ by $-(p(V+\phi) - p(V))$ and adding those equations, we have

$$\begin{aligned} &\frac{d}{dt}\int_0^\infty (\frac{1}{2}\psi^2 + \Phi(v,V))d\xi + \int_0^\infty \mu\frac{\psi_\xi^2}{v}d\xi \\ &\leq C\int_0^\infty |V_\xi|\phi^2 d\xi + \nu\int_0^\infty \psi_\xi^2 d\xi + C_\nu \int_0^\infty |V_\xi|^2\phi^2 d\xi \end{aligned}$$

for a small constant $\nu > 0$, and hence

$$\begin{aligned} &\|\phi(t)\|^2 + \|\psi(t)\|^2 + \int_0^t \|\psi_\xi(\tau)\|^2 d\tau \\ &\leq C\|\phi_0, \psi_0\|^2 + C\int_0^t\int_0^\infty |V_\xi|\phi(\xi,\tau)^2 d\xi d\tau. \end{aligned} \tag{2.31}$$

Here, we estimate the last term using the idea by Kawashima and Nikkuni [2]. Since

$$\phi(\xi, t) = \phi(0,t) + \int_0^\xi \phi_\xi(\eta, t)d\eta \leq \xi^{1/2}\|\phi_\xi(t)\|,$$

the last term of (2.31) is estimated as

$$|\text{the last term}| \leq C\int_0^t \|\phi_\xi(\tau)\|^2 \int_0^\infty \xi(-V_\xi(\xi))d\xi d\tau \leq C\delta\int_0^t \|\phi_\xi(\tau)\|^2 d\tau.$$

Hence, (2.31) yields

$$\|(\phi,\psi)(t)\|^2 + \int_0^t \|\psi_\xi(\tau)\|^2 d\tau \leq C(\|\phi_0, \psi_0\|^2 + \delta\int_0^t \|\phi_\xi(\tau)\|^2 d\tau). \tag{2.32}$$

Similar fashion to (2.14) yields

$$\|\phi_\xi(t)\|^2 + \int_0^t \|\phi_\xi(\tau)\|^2 d\tau \leq C(\|\phi_0, \psi_0\|_1^2 + \int_0^t \phi_\xi(0,t)^2 d\tau + \int_0^t\int_0^\infty |V_\xi(\xi)|\phi^2 d\xi d\tau).$$

Noting that

$$C\int_0^t \phi_\xi(0,\tau)^2 d\tau = \frac{C}{s_-^2}\int_0^t \psi_\xi(0,\tau)^2 \le \nu\int_0^t \|\psi_{\xi\xi}(\tau)\|^2 d\tau + C_\nu\int_0^t \|\psi_\xi(\tau)\|^2 d\tau,$$

we have

$$\begin{aligned} &\|\phi_\xi(t)\|^2 + \int_0^t \|\phi_\xi(\tau)\|^2 d\tau \\ &\le C\|\phi_0,\psi_0\|_1^2 + \int_0^t \{\nu\|\psi_{\xi\xi}(\tau)\|^2 + C\delta\|\phi_\xi(\tau)\|^2 + C_\nu\|\psi_\xi(\tau)\|^2\} d\tau. \end{aligned} \tag{2.33}$$

The same estimate as (2.17) gives

$$\begin{aligned} &\|\psi_\xi(t)\|^2 + \int_0^t (\psi_\xi(0,\tau)^2 + \|\psi_{\xi\xi}(\tau)\|^2) d\tau \\ &\le C\|\phi_0,\psi_0\|_1^2 + C\int_0^t (\|\phi_\xi(\tau)\|^2 + \|\psi_\xi(\tau)\|^2) d\tau. \end{aligned} \tag{2.34}$$

By (2.33) and (2.34) for a fixed number $\lambda > 0$ such as $1 - C\lambda \ge 1/2$ and $\nu = \lambda/2$,

$$\begin{aligned} &\|(\phi_\xi,\psi_\xi)(t)\|^2 + \int_0^t (\psi_\xi(0,\tau)^2 + \|\phi_\xi(\tau)\|^2 + \|\psi_{\xi\xi}(\tau)\|^2) d\tau \\ &\le C(\|\phi_0,\psi_0\|_1^2 + \int_0^t \|\psi_\xi(\tau)\|^2 d\tau). \end{aligned} \tag{2.35}$$

Again, adding (2.35)$\cdot\lambda(\lambda > 0)$ to (2.32), we have

$$\begin{aligned} &\|(\phi,\psi)(t)\|^2 + \lambda\|(\phi_\xi,\psi_\xi)(t)\|^2 \\ &+ \int_0^t \{(1 - C\lambda)\|\psi_\xi(\tau)\|^2 + \lambda(\psi_\xi(0,\tau)^2 + \|\phi_\xi(\tau)\|^2 + \|\psi_{\xi\xi}(\tau)\|^2\} d\tau \\ &\le C\|\phi_0,\psi_0\|_1^2 + C\delta\int_0^t \|\phi_\xi(\tau)\|^2 d\tau. \end{aligned}$$

Taking λ as $1 - C\lambda \ge 1/2$ and restrict δ as $\lambda - C\delta \ge \lambda/2$, then we obtain the desired a priori estimate, which completes the proof of Proposition 2.3.

3 Superposition of BL-solution and rarefaction wave

In this section we investigate the case

$$\begin{aligned} &(v_-,u_-) \in \Omega_{sub}, \quad (v_*,u_*) \in BL_+(v_-,u_-) \cap \Gamma_{trans} \\ &\text{and} \quad (v_+,u_+) \in R_1R_2(v_*,u_*). \end{aligned} \tag{3.1}$$

That is, we show (V) in Section 1. The cases (III) and (IV) are similar to (V).

In the case of (3.1), there is $(\bar{v},\bar{u}) \in R_1(v_*,u_*)$ such that $(v_+,u_+) \in R_2(\bar{v},\bar{u})$, and there are the 1-rarefaction wave $(v_1^R,u_1^R)(x/t)$ connecting (v_*,u_*) with $(\bar{v},\bar{u})$ and the 2-rarefaction wave $(v_2^R,u_2^R)(x/t)$ connecting $(\bar{v},\bar{u})$ with (v_+,u_+), which are weak solutions to

$$\begin{cases} v_t - u_x = 0, & (x,t) \in \mathbf{R} \times (0,\infty) \\ u_t + p(v)_x = 0 \end{cases} \tag{3.2}$$

with the Riemann initial data

$$(v,u)|_{t=0} = (v_{01}^R, u_{01}^R)(x) = \begin{cases} (v_*, u_*) & x < 0 \\ (\bar{v}, \bar{u}) & x > 0 \end{cases} \tag{3.3$_1$}$$

and

$$(v,u)|_{t=0} = (v_{02}^R, u_{02}^R)(x) = \begin{cases} (\bar{v}, \bar{u}) & x < 0 \\ (v_+, u_+) & x > 0. \end{cases} \tag{3.3$_2$}$$

To construct the smooth approximate rarefaction wave $(\tilde{V}_i, \tilde{U}_i)(x,t)$, $x \in \mathbf{R}$, and its restriction $(V_i, U_i)(\xi, t) := (\tilde{V}_i, \tilde{U}_i)(x,t)|_{x \geq s_- t}$, we prepare the following lemma.

Lemma 3. 1 ([9]) *Let $w_+ > w_-$ and $\tilde{w} = w_+ - w_-$. Then the Cauchy problem*

$$\begin{cases} w_t + ww_x = 0, \quad x \in \mathbf{R}, \quad t > 0 \\ w|_{t=0} = w_0(x) := \dfrac{w_+ + w_-}{2} + \dfrac{w_+ - w_-}{2} \tanh x \end{cases} \tag{3.4}$$

has a unique, smooth and global solution $w(x,t) = w(x,t; w_-, w_+)$, which satisfies the followings:

$$(i) \quad w_- < w(x,t) < w_+, \quad w_x > 0$$

$$(ii) \quad \begin{aligned} &\|w_x(t)\|_{L^p(\mathbf{R})} \leq C_p \min(\tilde{w}, \tilde{w}^{1/p} t^{-1+1/p}) \\ &\|w_{xx}(t)\|_{L^p(\mathbf{R})} \leq C_p \min(\tilde{w}, t^{-1}) \end{aligned}$$

$$(iii) \quad \begin{aligned} &\text{if } 0 < w_-(< w_+), \text{ then, for any } x \leq 0, \\ &|w(x,t) - w_-| \leq \tilde{w} \exp\{-2(|x| + |w_-|t)\} \\ &|w_x(x,t)| \leq 2\tilde{w} \exp\{-2(|x| + |w_-|t)\} \end{aligned}$$

$$(iv) \quad \begin{aligned} &\text{if } 0 > w_+(> w_-), \text{ then, for any } x \geq 0, \\ &|w(x,t) - w_+| \leq \tilde{w} \exp\{-2(|x| + |w_+|t)\} \\ &|w_x(x,t)| \leq 2\tilde{w} \exp\{-2(|x| + |w_+|t)\} \end{aligned}$$

$$(v) \quad \lim_{t \to \infty} \sup_{\mathbf{R}} |w(x,t) - w^R(x/t)| = 0,$$

where

$$w^R(x/t) = \begin{cases} w_- & x \leq w_- t \\ x/t & w_- t < x < w_+ t \\ w_+ & x \geq w_+ t. \end{cases}$$

Remark 3.1 If

$$w_0(x) = \frac{w_+ + w_-}{2} + \frac{w_+ - w_-}{2} \kappa_q \int_0^x (1+y^2)^{-q} dy, \quad \kappa_q \int_0^\infty (1+y^2)^{-q} dy = 1,$$

instead of $(3.4)_2$, then

$$\|w_{xx}(t)\|_{L^p(\mathbf{R})} \le C_{pq}\tilde{w}^{-\frac{p-1}{2pq}}t^{-1-\frac{p-1}{2pq}}$$

([10,11]). This was available for strong rarefaction wave. However, the result in this paper is concerning the weak rarefaction wave, and $(3.4)_2$ is chosen.

The characteristic speed $\lambda_i(v)$, $i = 1, 2$ of the hyperbolic system (3.2) are $\lambda_i(v) = (-1)^i\sqrt{-p'(v)}$, and hence the smooth approximations $(\tilde{V}_i, \tilde{U}_i)(x,t)$ to $(v_i^R, u_i^R)(x/t)$ are given by

$$\begin{cases} \tilde{V}_1(x,t) = \lambda_1^{-1}(w(x,t;2\lambda_1(v_*) - \lambda_1(\bar{v}), \lambda_1(\bar{v}))) \\ \tilde{U}_1(x,t) = u_* - \displaystyle\int_{v_*}^{\tilde{V}_1(x,t)} \lambda_1(s)ds \end{cases} \tag{3.5$_{i=1}$}$$

and

$$\begin{cases} \tilde{V}_2(x,t) = \lambda_2^{-1}(w(x,t;\lambda_2(\bar{v}), \lambda_2(v_+))) \\ \tilde{U}_2(x,t) = \tilde{u} - \displaystyle\int_{\bar{v}}^{\tilde{V}_2(x,t)} \lambda_2(s)ds, \end{cases} \tag{3.5$_{i=2}$}$$

which satisfy, for $i = 1,\ 2$,

$$\begin{cases} \tilde{V}_{it} - \tilde{U}_{ix} = 0, \quad x \in \mathbf{R}, \quad t > 0 \\ \tilde{U}_{it} + p(\tilde{V}_i)_x = 0. \end{cases} \tag{3.6}$$

Then, define, for $i = 1,\ 2$,

$$(V_i, U_i)(\xi, t) = (\tilde{V}_i, \tilde{U}_i)(x,t)|_{x \ge s_-t}, \quad \xi = x - s_-t \ge 0, \tag{3.7}$$

and the following lemma holds.

Lemma 3. 2 *Let $\delta_1 = |v_+ - v_*, u_+ - u_*|$. Then $(V_i, U_i)(\xi, t)$ defined by (3.7) satisfy*

$(i)\quad U_{i\xi} > 0, \quad |V_{i\xi}| \le CU_{i\xi} \le C\delta_1$

$(ii)\quad \|(V_{i\xi}, U_{i\xi})(t)\|_{L^p(\mathbf{R}_+)} \le C_p\delta_1^{1/p}(1+t)^{-1+1/p}$
$\quad\quad \|(V_{i\xi\xi}, U_{i\xi\xi})(t)\|_{L^p(\mathbf{R}_+)} \le C_p \min(\delta_1, (1+t)^{-1})$

$(iii)\quad |(V_1 - \bar{v}, U_1 - \bar{u})(\xi,t)| + |(V_{1\xi}, U_{1\xi})(\xi,t)| \le C\delta_1 \exp\{-c(|\xi + s_-t| + t)\},$
for $\xi \ge -s_-t$
$\quad\quad |(V_2 - \bar{v}, U_2 - \bar{u})(\xi,t)| + |(V_{2\xi}, U_{2\xi})(\xi,t)| \le C\delta_1 \exp\{-c(|\xi + s_-t| + t)\},$
for $0 \le \xi \le -s_-t$

$(iv)\quad \displaystyle\lim_{t\to\infty} \sup_{\mathbf{R}_+} |(V_i, U_i)(\xi,t) - (v_i^R, u_i^R)(\frac{\xi + s_-t}{t})| = 0, \quad i = 1,\ 2$

and also

$$\begin{cases} V_{it} - s_-V_{i\xi} - U_{i\xi} = 0, \quad \xi \in \mathbf{R}_+, \quad t > 0 \\ U_{it} - s_-U_{i\xi} + p(V_i)_\xi = 0 \end{cases} \tag{3.8}$$

with

$$(V_1, U_1)(0,t) = (v_*, u_*), \quad (V_1, U_1)(\infty, t) = (\bar{v}, \bar{u}) \tag{3.9}$$

and

$$|(V_2 - \bar{v}, U_2 - \bar{u})(0,t)| \le C\delta_1 \exp(-ct), \quad (V_2, U_2)(\infty, t) = (v_+, u_+). \tag{3.10}$$

All results except for (3.9) are direct consequence from (3.5)-(3.7) and Lemma 3.1. Eq. (3.9) follows from the choice of $w_- = 2\lambda_1(v_*) - \lambda_1(\bar{v})$ in $(3.5)_{i=1}$. It is natural to take $w_- = \lambda_1(v_*)$. However, from our choice

$$\frac{w_+ + w_-}{2} = \frac{\lambda_1(\bar{v}) + 2\lambda_1(v_*) - \lambda_1(\bar{v})}{2} = s_-$$

(note that $\lambda_1(v_*) = s_-$). Hence, $w_0(0) = s_-$ in (3.4), which means $w(x,t)|_{x=s_-t} = s_-$. By the definition $(3.5)_{i=1}$, $(\tilde{V}_1, \tilde{U}_1)(s_-t, t) = (v_*, u_*)$ and hence (3.9) holds.

On the other hand, the BL-solution $(V_0, U_0)(\xi)$ connecting (v_-, u_-) and (v_*, u_*) is given by Lemma 1.1:

$$\begin{cases} -s_-V_{0\xi} - U_{0\xi} = 0, \quad \xi \in \mathbf{R}_+ \\ -s_-U_{0\xi} + p(V_0)_\xi = \mu(\dfrac{U_{0\xi}}{V_0})_\xi \\ (V_0, U_0)(0) = (v_-, u_-), \quad |(V_0 - v_*, U_0 - u_*)(\xi)| \le C\delta_0(1+\xi)^{-1}, \end{cases} \tag{3.11}$$

where $\delta_0 = |(v_* - v_-, u_* - u_-)|$.

Putting

$$\begin{pmatrix} V \\ U \end{pmatrix}(\xi, t) = \begin{pmatrix} V_0(\xi) + V_1(\xi,t) + V_2(\xi,t) - v_* - \bar{v} \\ U_0(\xi) + U_1(\xi,t) + U_2(\xi,t) - u_* - \bar{u} \end{pmatrix}, \tag{3.12}$$

we set the perturbation (ϕ, ψ) by $(v,u)(x,t) = (V+\phi, U+\psi)(\xi,t)$. Then the reformulated problem is, from (1.5), (3.8)-(3.12),

$$\begin{cases} \phi_t - s_-\phi_\xi - \psi_\xi = 0, \quad \xi \in \mathbf{R}_+, \quad t > 0 \\ \psi_t - s_-\psi_\xi + (p(V+\phi) - p(V))_\xi = \mu(\dfrac{U_\xi + \psi_\xi}{V+\phi} - \dfrac{U_\xi}{V})_\xi + G_\xi \\ (\phi,\psi)(0,t) = (\bar{v} - V_2(0,t), \bar{u} - U_2(0,t)) =: (b_V, b_U)(t) \\ (\phi,\psi)(\xi,0) = (v_0(\xi) - V(\xi,0), u_0(\xi) - U(\xi,0)) =: (\phi_0,\psi_0)(\xi), \end{cases} \tag{3.13}$$

where

$$|(b_V, b_U)(t)| \le C\delta_1 \exp(-ct), \tag{3.14}$$

$$(\phi_0, \psi_0)(0) = (b_V, b_U)(0) \tag{3.15}$$

and

$$\begin{aligned} G &= -(p(V) - p(V_0) - p(V_1) - p(V_2) + p(v_*) + p(\bar{v})) + \mu(\frac{U_\xi}{V} - \frac{U_{0\xi}}{V_0}) \\ &=: -G_1 + G_2. \end{aligned} \tag{3.16}$$

The equation (3.13) is almost same as (2.4) together with $U_\xi > 0$(note that $U_\xi = -V_\xi/s_- > 0$ in (2.4)) except for the term G_ξ. Therefore, for the solution $(\phi,\psi)(\xi,t)$, $0 \le t \le T$, to (3.13) with

$$\sup_{0 \le t \le T} \|(\phi,\psi)(t)\|_1 \le \varepsilon \le \varepsilon_0 \ll 1,$$

we have

$$\begin{aligned} &\|(\phi,\psi)(t)\|_1^2 + \int_0^t (\|\sqrt{U_\xi}\phi(\tau)\|^2 + \|\psi_\xi(\tau)\|^2) d\tau \\ &\le C(\|\phi_0,\psi_0\|^2 + \delta_1) + C\delta_1 \int_0^t \psi_\xi(0,\tau)^2 d\tau + C\int_0^t \int_0^\infty G_\xi \psi \, d\xi d\tau. \end{aligned} \tag{3.17}$$

by a similar way to that in Subsection 2.1. The second to the last term comes from the boundary value and is controlable by combining the estimates of higher order derivatives provided δ_1 is small. We must control the last term in (3.17).

Since

$$G_{1\xi} = p'(V)(V_{0\xi} + V_{1\xi} + V_{2\xi}) - p'(V_0)V_{0\xi} - p'(V_1)V_{1\xi} - p'(V_2)V_{2\xi}$$
$$= (p'(V) - p'(V_0))V_{0\xi} + (p'(V) - p'(V_1))V_{1\xi} + (p'(V) - p'(V_2))V_{2\xi},$$

by noting the signs of $V_1 - v_*$ etc.,

$$\begin{aligned} |G_{1\xi}| &\leq C\{(V_1 - v_*)V_{0\xi} + (v_* - V_0)V_{1\xi}\} \\ &\quad + C\{|V_2 - \bar{v}|(|V_{0\xi}| + |V_{1\xi}|) + |V_{2\xi}|(|V_0 - v_*| + |V_1 - \bar{v}|)\} \\ &=: G_{11\xi} + |G_{12\xi}|. \end{aligned} \tag{3.18}$$

The second term $|G_{12\xi}|$ is easily controlled. Because the wave V_2 is away from the waves V_0 and V_1. In fact,

$$\begin{aligned} \|G_{12\xi}(t)\|^2 &= \left(\int_0^{-s_-t} + \int_{-s_-t}^{\infty}\right)|G_{12\xi}|^2(\xi,t)d\xi \\ &\leq C \sup_{0\leq\xi\leq -s_-t}\{(|V_{0\xi}(\xi)|^2 + |V_{1\xi}(\xi,t)|^2)|V_2(\xi,t) - \bar{v}|\} \int_0^{-s_-t} |V_2(\xi,t) - \bar{v}|d\xi \\ &\quad + C \sup_{0\leq\xi\leq -s_-t}\{(|V_0 - v_*|^2 + |V_1 - \bar{v}|^2)|V_{2\xi}(\xi,t)\} \int_0^{-s_-t} |V_{2\xi}(\xi,t)|d\xi \\ &\quad + C \sup_{\xi\geq -s_-t}\{|V_2 - \bar{v}|^2(|V_{0\xi}(\xi)| + |V_1(\xi,t)|\} \int_{-s_-t}^{\infty} (|V_{0\xi}(\xi)| + |V_1(\xi,t)|)d\xi \\ &\quad + C \sup_{\xi\geq -s_-t}\{(|V_0 - v_*|^2 + |V_1 - \bar{v}|^2)|V_{2\xi}(\xi,t)|\} \int_{-s_-t}^{\infty} |V_{2\xi}(\xi,t)|d\xi \\ &\leq C(\delta_1 + \delta_0)\delta_1(1+t)^3. \end{aligned}$$

Here we have used $V_{0\xi} = \frac{V_0}{\mu s_-}h(V_0) \sim -\frac{V_0}{\mu s_-}(V_0 - v_*)^2$ and $|V_0(\xi) - v_*| \leq C\delta_0|\xi|^{-1}$ as $\xi \to \infty$. Hence

$$C\left|\int_0^t \int_0^{\infty} G_{12\xi}\psi d\xi d\tau\right| \leq C(\delta^{1/2} + \delta^{1/2})\delta^{1/2} \int_0^t (1+\tau)^{-3/2}\|\psi(\tau)\|d\tau,$$

which is controllable by the Gronwall inequality. The term $G_{11\xi}$ is rather difficult to be treated, because the waves V_0 and V_1 are contact for all time with each other. See Figure 3.1. However, this situation is similar to that in [5, Section 4] and the method used there is applicable to our case.

Putting $a := -s_- + \lambda_1(\bar{v}) > 0$, we have

$$
\begin{aligned}
& C|\int_0^t\int_0^\infty G_{11\xi}\psi d\xi d\tau| \\
& \leq C\int_0^t \sup_{\mathbf{R}_+}|\psi|(\int_0^{at}+\int_{at}^\infty)\{(V_1-v_*)V_{0\xi}+(v_*-V_0)V_{1\xi}\}d\xi d\tau \\
& \leq C\int_0^t \|\psi\|^{1/2}\|\psi_\xi\|^{1/2}\{[(V_1-v_*)(V_0-v_*)]_{\xi=0}^{at}-2\int_0^{at}V_{1\xi}(V_0-v_*)d\xi \\
& \quad +[(v_*-V_0)(V_1-v_*)]_{\xi=at}^\infty+2\int_{at}^\infty V_{0\xi}(V_1-v_*)d\xi\}d\tau \\
& \leq C\int_0^t \|\psi\|^{1/2}\|\psi_\xi\|^{1/2}\{\delta_1^{1/8}(1+\tau)^{-7/8}\delta_0\int_0^{a\tau}+\delta_0\delta_1(1+\tau)^{-1})d\tau \\
& \leq \nu\int_0^t \|\psi_\xi(\tau)\|^2 d\tau + C_\nu\int_0^t \delta_1^{1/6}\delta_0^{3/4}\|\psi(\tau)\|^{2/3}(1+\tau)^{-7/6}(\ln(2+\tau))^{4/3}d\tau \\
& \leq \nu\int_0^t \|\psi_\xi(\tau)\|^2 d\tau + C\delta_0^{4/3}\delta_1^{1/6}.
\end{aligned}
$$

Here we have used, by Lemma 3.2 (ii),

$$\|V_{1\xi}(\cdot,t)\|_{L^\infty} \leq \|V_{1\xi}(\cdot,t)\|_{L^\infty}^{\frac{1}{8}+\frac{7}{8}} \leq \delta_1^{1/8}(1+t)^{-7/8}.$$

Estimating $C\int_0^t\int_0^\infty G_{2\xi}\psi d\xi d\tau$ by a similar way to the above, we have, for any fixed δ_0,

$$
\begin{aligned}
& \|(\phi,\psi)(t)\|_1^2+\int_0^t(\|\sqrt{U_\xi}\phi(\tau)\|^2+\|\psi_\xi(\tau)\|^2)d\tau \\
& \leq C(\|\phi_0,\psi_0\|^2+\delta_1^{1/6})+C\delta_1\int_0^t \psi_\xi(0,\tau)^2 d\tau.
\end{aligned}
$$

The estimates of higher order derivatives are also obtained, though the calculations are rather tedious. Thus, we have the following theorem.

Theorem 3. 1 (the case of $(v_+,u_+) \in BL_+R_1R_2(v_-,u_-)$**)** *Define* $(V,U)(\xi,t)$ *by (3.12). Then, if both* $\|\phi_0,\psi_0\|_1 = \|v_0 - V(\cdot,0), u_0 - U(\cdot,0)\|_1$ *and* $\delta_1 = |v_+ - v_*, u_+ - u_*|$ *are suitably small, then there exists a unique solution* $(\phi,\psi) \in C([0,\infty);H^1)$ *to (3.13) and hence a solution* (v,u) *to (P) which satisfies*

$$\sup_{\xi\geq 0}|(\phi,\psi)(\xi,t)| = \sup_{x\geq s_-t}|(v,u)(x,t)-(V,U)(x-s_-t,t)| \to 0 \ (t\to\infty).$$

Remark 3.2 This result implies that the BL-solution is not necessary to be weak though the rarefaction waves are necessarily weak. We do not know whether the weakness is removed.

Remark 3.3 The case of $(v_+,u_+) \in BL_+R_2(v_-,u_-)$ is treated in a similar fashion to the above case, and the assertion (III) holds. However, in the case of $(v_+,u_+) \in BL_-R_2(v_-,u_-)$ the situation is similar to that in Subsection 2.2, and hence the BL-solution is necessary to be weak, and (IV) holds.

4 Concluding remarks

Except for the cases treated in Sections 2 and 3, all other cases are open. In this section we discuss the cases concerning the viscous shock wave, which are not yet solved, either.

Compared to the case of the corresponding Cauchy problem, we consider the case

$$(v_-, u_-) \in \Omega_{sub}, \quad (v_+, u_+) \in BL\_S_2(v_-, u_-), \tag{4.1}$$

where a superposition of the BL-solution and the viscous shock wave is expected to be an asymptoics of the solution to (P). In this case, there is $(\bar{v}, \bar{u}) \in BL_-(v_-, u_-)$ such that $(v_+, u_+) \in S_2(\bar{v}, \bar{u})$, and there are the BL-solution $(V_0, U_0)(\xi)$ satisfying

$$\begin{cases} -s_- V_{0\xi} - U_{0\xi} = 0, \quad \xi \in \mathbf{R}_+ \\ -s_- U_{0\xi} + p(V_0)_\xi = \mu(\dfrac{U_{0\xi}}{V_0})_\xi \\ (V_0, U_0)(0) = (v_-, u_-), \quad (V_0, U_0)(\infty) = (\bar{v}, \bar{u}) \end{cases} \tag{4.2}$$

with $s_- = -u_-/v_-$ and the 2-viscous shock wave $(\tilde{V}_2, \tilde{V}_2)(x - s_2 t + \alpha)$ connecting $(\bar{v}, \bar{u})$ with (v_+, u_+), and its restriction $(V_2, U_2)(\xi, t; \alpha) := (\tilde{V}_2, \tilde{V}_2)(x - s_2 t + \alpha)|_{x \geq s_- t} = (\tilde{V}_2, \tilde{V}_2)(\xi - (s_2 - s_-)t + \alpha)|_{\xi \geq 0}$, which satisfies

$$\begin{cases} V_{2t} - s_- V_{2\xi} - U_{2\xi} = 0, \quad \xi \in \mathbf{R}_+, \quad t > 0 \\ U_{2t} - s_- U_{2\xi} + p(V_2)_\xi = \mu(\dfrac{U_{2\xi}}{V_2})_\xi \\ (V_2 - \bar{v}, U_2 - \bar{u})|_{\xi=0} =: (b_V, -b_U)(t), \quad |(b_V, -b_U)(t)| \leq C\delta_1 \exp(-ct) \\ (V_2, U_2)|_{\xi=\infty} = (v_+, u_+), \end{cases} \tag{4.3}$$

where α is a shift and $s_2 = \sqrt{-\frac{p(v_+) - p(\bar{v})}{v_+ - \bar{v}}} > 0$ with $\delta_1 = |v_+ - \bar{v}, u_+ - \bar{u}|$. Hence the solution (v, u) to (P) is expected to tend to

$$(V, U)(\xi, t; \alpha) = (V_0(\xi) + V_2(\xi, t; \alpha) - \bar{v}, U_0(\xi) + U_2(\xi, t; \alpha) - \bar{u}). \tag{4.4}$$

The key point is how to determine α, which are suggested by the method in [4]. The perturbation $(v - V, u - U)$ satisfies

$$\begin{cases} (v - V)_t - s_-(v - V)_\xi - (u - U)_\xi = 0 \\ (u - U)_t - s_-(u - U)_\xi + (p(v) - p(V_0) - p(V_2) + p(\bar{v}))_\xi \\ \quad = \mu(\dfrac{u_\xi}{v} - \dfrac{U_{0\xi}}{V_0} - \dfrac{U_{2\xi}}{V_2})_\xi \\ (v - V, u - U)|_{\xi=0} = (-b_V, b_U)(t) \\ (v - V, u - U)|_{t=0} \\ \quad = (v_0 - V_0 - V_2(\cdot, 0; \alpha) + \bar{v}, u_0 - U_0 - U_2(\cdot, 0; \alpha) + \bar{u})(\xi). \end{cases} \tag{4.5}$$

Integrating $(4.5)_1$ in ξ over $\mathbf{R}_+$, we have

$$\frac{d}{dt} \int_0^\infty (v - V)(\xi, t) d\xi = s_- b_V(t) - b_U(t). \tag{4.6}$$

Expecting $\int_0^\infty (v - V)(\xi, t) d\xi|_{t=\infty} = 0$ yields

$$-\int_0^\infty (v_0(\xi) - V(\xi, 0; \alpha)) d\xi = \int_0^\infty (s_- b_V(t) - b_U(t)) dt$$

$$= -(s_2 - s_-)\int_0^\infty (V_2(\alpha - (s_2 - s_-)t) - \bar{v})dt$$

and hence

$$\begin{aligned} I(\alpha) := &\int_0^\infty (v_0(\xi) - V_0(\xi) - V_2(\xi+\alpha) + \bar{v})d\xi \\ &-(s_2 - s_-)\int_0^\infty (V_2(\alpha - (s_2 - s_-)t) - \bar{u})dt = 0. \end{aligned} \tag{4.7}$$

Since

$$\begin{aligned} I'(\alpha) &= \int_0^\infty -V_2'(\xi+\alpha)d\xi - (s_2 - s_-)\int_0^\infty V_2'(\alpha - (s_2 - s_-)t)dt \\ &= -(v_+ - V_2(\alpha)) + \bar{v} - V_2(\alpha) = \bar{v} - v_+, \end{aligned}$$

the equality $0 = I(\alpha) = I(0) + (\bar{v} - v_+)\alpha$ determines α by

$$\begin{aligned} \alpha = -\frac{1}{\bar{v} - v_+}\{&\int_0^\infty (v_0(\xi) - V_0(\xi) - V_2(\xi) + \bar{v})d\xi \\ &-(s_2 - s_-)\int_0^\infty (V_2(-(s_2 - s_-)t) - \bar{v})dt\}. \end{aligned} \tag{4.8}$$

Again, integrating (4.6) over $(0,t)$, we have

$$\begin{aligned} \int_0^\infty (v(\xi,t) - V(\xi,t;\alpha))d\xi &= \int_0^\infty (v_0(\xi) - V(\xi,0;\alpha)d\xi + \int_0^t (s_- b_V(\tau) - b_U(\tau))d\tau \\ &= (s_2 - s_-)\int_t^\infty (V_2(\alpha - (s_2 - s_-)\tau) - \bar{v})d\tau \ (= -\hat{b}_V(t)) \\ &\to 0 \quad \text{exponentially as} \quad t \to \infty. \end{aligned} \tag{4.9}$$

Thus, putting the perturbation in the integrated form by

$$(\phi,\psi)(\xi,t) = -\int_\xi^\infty (v(\eta,t) - V(\eta,t;\alpha), u(\eta,t) - U(\eta,t;\alpha))d\eta,$$

we reach the reformulated problem

$$\begin{cases} \phi_t - s_-\phi_\xi - \psi_\xi = 0, \quad \xi \in \mathbf{R}_+, \quad t > 0 \\ \psi_t - s_-\psi_\xi + p(V + \phi_\xi) - p(V_0) - p(V_2) + p(\bar{v}) \\ \quad = \mu(\dfrac{U_\xi + \psi_{\xi\xi}}{V + \phi_\xi} - \dfrac{U_{0\xi}}{V_0} - \dfrac{U_{2\xi}}{V_2}) \\ \phi|_{\xi=0} = \hat{b}_V(t), \quad \psi_\xi|_{\xi=0} = -b_U(t) \\ (\phi,\psi)|_{t=0} = (\phi_0,\psi_0)(\xi) := -\displaystyle\int_\xi^\infty (v_0 - V(\cdot,0;\alpha), u_0 - U(\cdot,0;\alpha))(\xi)d\xi. \end{cases} \tag{4.10}$$

This setting seems to be reasonable. However, we could not prove the global existence theorem on (4.10) even if both $|v_+ - v_-, u_+ - u_-|$ and $\|\phi_0,\psi_0\|_2$ were small. The difficulty was to control the value $\psi(0,t)$ from the boundary.

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1.2 Stability of viscous shock wave

Boundary Effect on Stationary Viscous Shock Wave for Scalar Viscous Conservation Laws

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Abstract

The initial-boundary value problem on the negative half-line $\mathbf{R}_-$

$$\begin{cases} u_t + f(u)_x = u_{xx}, & (x,t) \in \mathbf{R}_- \times (0,\infty) \\ u(0,t) = u_+, & \\ u(x,0) = u_0(x) = \begin{cases} \to u_- & x \to -\infty \\ = u_+ & x = 0 \end{cases} \end{cases} \tag{$*$}$$

is considered, subsequently to [T.-P. Liu and K. Nishihara, *J. Differential Equations* **133** (1997), 57 - 82]. Here, the flux f is a smooth function satisfying $f(u_\pm) = 0$ and the Oleinik shock condition $f(u) < 0$ for $u_+ < u < u_-$ if $u_+ < u_-$ or $f(u) > 0$ for $u_+ > u > u_-$ if $u_+ < u_-$. In this situation the corresponding Cauchy problem on the whole line $\mathbf{R} = (-\infty, \infty)$ to $(*)$ has a stationary viscous shock wave $\phi(x + x_0)$ for any fixed x_0. Our aim in this paper is to show that the solution $u(x,t)$ to $(*)$ behaves as $\phi(x + d(t))$ with $d(t) = O(\ln t)$ as $t \to \infty$ under the suitable smallness conditions. When $f = u^2/2$, the fact was shown by [T.-P. Liu and S.-H. Yu, *Arch. Rational Mech. Anal.* **139** (1997), 57 - 82], based on the Hopf-Cole transformation. Our proof is based on the weighted energy method.

1 Introduction

Consider the initial-boundary value problem to scalar viscous conservation laws on the negative half-line $\mathbf{R}_- = (-\infty, 0)$, subsequently to [3] :

$$\begin{cases} u_t + f(u)_x = u_{xx}, & (x,t) \in \mathbf{R}_- \times (0,\infty) \\ u(0,t) = u_+, & t \in (0,\infty) \\ u(x,0) = u_0(x) = \begin{cases} \to u_- & x \to -\infty \\ = u_+ & x = 0. \end{cases} \end{cases} \tag{1.1}$$

Here, the flux f is a smooth function of u satisfying

$$f(u_-) = f(u_+) = 0 \tag{1.2}$$

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and the Oleinik shock condition

$$f(u)\begin{cases} < 0 & u_+ < u < u_- \quad \text{if} \quad u_+ < u_- \\ > 0 & u_+ > u > u_- \quad \text{if} \quad u_+ > u_-. \end{cases} \tag{1.3}$$

Our concern is the large-time behavior of the solution u to (1.1). Under the conditions (1.2) and (1.3) there is a unique stationary viscous shock wave $\phi(\cdot)$ on $\mathbf{R} = (-\infty, \infty)$ up to a shift d_0 defined by the solution of the ordinary differential equation

$$\phi_x = f(\phi), \quad \phi(\pm\infty) = u_\pm. \tag{1.4}$$

Moreover, ϕ satisfies as $x \to \pm\infty$ for some positive constant $c_\pm$

$$|\phi(x) - u_\pm| \sim \begin{cases} \exp(-c_\pm|x|) & \text{if} \quad f'(u_\pm) \neq 0 \\ |x|^{-1/k_\pm} & \text{if} \quad |f(\phi) - f(u_\pm)| \sim |\phi - u_\pm|^{1+k_\pm} \ (k_\pm > 0), \end{cases} \tag{1.5}$$

where $g(x) \sim h(x)$ as $x \to a$ means that $g(x)/h(x) \to const.(\neq 0)$ as $x \to a$. Refer to [1, 5]. In the initial-boundary value problem, for any constant shift d_0, $\phi(x + d_0)|_{x\le 0}$ has a "boundary gap" $\phi(d_0) - u_+$ at $x = 0$. Hence, the solution u to (1.1) is expected to be pushed backward to compensate the "boundary gap", that is, to tend to $\phi(x + d(t))$ with $d(t) \to \infty$ as $t \to \infty$. In fact, when $f = u^2/2$, Liu and Yu [4] have shown that

$$u(x,t) \sim \phi(x + d(t)), \quad d(t) \sim \log t \quad \text{as} \quad t \to \infty, \tag{1.6}$$

by using the Hopf-Cole transformation. In the case $f(u_-) > f(u_+) = 0$ or $s > 0$, the viscous shock wave $\phi(x - st + d_0)$ goes away from the boundary. Hence the boundary gap automatically vanishes as t tends to infinity, and the effect of boundary is rather small. In this sense the case $s = 0$ is more interesting. On the other hand, when $s > 0$, that is, the wave goes into the boundary, there is a unique stationary solution $\phi(x)$ satisfying $\phi(-\infty) = u_-, \phi(0) = u_+$, which is shown to be stable. See [3].

Our aim in this paper is that the large-time behavior (1.6) also holds for general flux function f satisfying (1.2) and (1.3) provided that the shock is non-degenerate, *i.e.*

$$f'(u_+) < s < f'(u_-)$$

with $s = 0$. When $f'(u_+) = s$ or $f'(u_-) = s$, the shock is called to be degenerate. The degenrate shock case is also important, but it is only conjectured that $d(t) \sim t^{1/3}$ as $t \to \infty$.

We now restrict our case to

$$u_+ < u_- \quad \text{and} \quad f(\phi) < 0 \quad \text{for} \quad u_+ < \phi < u_-. \tag{1.7}$$

Let us put

$$\phi = \phi(\xi), \quad \xi = x + d(t), \quad x \le 0. \tag{1.8}$$

Then ϕ satisfies

$$\phi_t - d'(t)\phi_x + f(\phi)_x = \phi_{xx}, \quad \phi|_{x=0} = \phi(d(t)), \tag{1.9}$$

and hence the perturbation $u(x,t) - \phi(x + d(t))$ does

$$\begin{cases} (u-\phi)_t + d'(t)\phi_x + (f(u) - f(\phi))_x = (u-\phi)_{xx} \\ (u-\phi)|_{x=0} = u_+ - \phi(d(t)). \end{cases} \tag{1.10}$$

To determine the shift $d(t)$, integrate (1.10) over $(-\infty, 0)$ to have

$$\frac{d}{dt}\int_{-\infty}^{0}(u-\phi)dx + d'(t)(\phi(d(t))-u_-) - f(\phi(d(t))) = (u-\phi)_x|_{x=0}. \tag{1.11}$$

We define $d(t)$ by

$$\begin{cases} d'(t)(\phi(d(t))-u_-) - f(\phi(d(t))) = (u-\phi)_x|_{x=0} \\ d(0) = d_0 \end{cases} \tag{1.12}$$

with

$$\int_{-\infty}^{0}(u_0(x)-\phi(x+d_0))dx = 0, \tag{1.13}$$

so that

$$\int_{-\infty}^{0}(u(x,t)-\phi(x+d(t)))dx = 0 \quad \text{for} \quad \text{any} \quad t \geq 0.$$

Thus, setting the perturbation by

$$v(x,t) = \int_{-\infty}^{x}(u(y,t)-\phi(y+d(t)))dy, \tag{1.14}$$

we obtain the reformulated problem as a system of (v, d) by integrating (1.10) over $(-\infty, x)$:

$$\begin{cases} v_t + d'(t)(\phi(\xi)-u_-) + f(\phi(\xi)+v_x) - f(\phi(\xi)) = v_{xx}, \quad x \in \mathbf{R}_-, \quad t > 0 \\ v|_{t=0} = v_0(x) := \displaystyle\int_{-\infty}^{x}(u_0(y)-\phi(y+d_0))dy, \quad x \in \mathbf{R}_- \\ v|_{x=-\infty} = v|_{x=0} = 0, \quad t > 0, \end{cases} \tag{1.15}$$

and

$$\begin{cases} d'(t)(\phi(d(t))-u_-) - f(\phi(d(t))) = v_{xx}(0,t), \\ d(0) = d_0, \end{cases} \tag{1.16}$$

where $\xi = x + d(t)$.

In later sections we devote ourselves to investigate (1.15) and (1.16).

2 Preliminaries and Theorem

In our setting the behavior $d(t)$ defined by (1.16) is important. So, as an approximatioin of $d(t)$, define $d_0(t)$ by the solution in the case of $v_{xx}(0,t) \equiv 0$ in (1.16):

$$d_0'(t)(\phi(d_0(t))-u_-) = f(\phi(d_0(t))), \quad d_0(0) = d_0. \tag{2.1}$$

By (1.7), $d_0'(t) > 0$ and $d_0(t) \geq d_0$. Hence, $\phi(d_0(t)) - u_- \leq -c < 0$ and

$$0 < C^{-1}(\phi(d_0(t))-u_+) \leq d_0'(t) \leq C(\phi(d_0(t))-u_+). \tag{2.2}$$

Here and hereafter, denote several generic constants by c, C and c_i, $C_i (i = 1, 2, \cdots)$. By (1.5)

$$C^{-1} \exp(-c_+ d_0(t)) \le \frac{d}{dt} d_0(t) \le C \exp(-c_+ d_0(t)) \tag{2.3}$$

and

$$\log(\exp(c_+ d_0) + c_+ C^{-1} t) \le c_+ d_0(t) \le \log(\exp(c_+ d_0) + c_+ C t). \tag{2.4}$$

Since $\log(\exp(c_+ d_0) + c_+ C^{-1} t) \ge C_1 + \log(\exp(c_+ d_0) + t)$ etc., we have the following lemma.

Lemma 2. 1 *Define $D_0 = \exp(c_+ d_0)$. Then the solution $d_0(t)$ to (2.1) satisfies*

$$C_1 + \log(D_0 + t) \le c_+ d_0(t) \le C_2 + \log(D_0 + t), \tag{2.5}$$

$$C^{-1}(D_0 + t)^{-1} \le d_0'(t) \le C(D_0 + t)^{-1}. \tag{2.6}$$

Therefore, if $v_{xx}(0,t)$ is small and decays faster that t^{-1}, then $d(t)$ is expected to behave as (2.5). Hence, for a given small constant $\delta > 0$, we a priori assume

$$(D_0 + t)^{1+\delta} |v_{xx}(0,t)| \le 1, \quad 0 \le t \le T, \tag{2.7}$$

and, in particular,

$$|v_{xx}(0,t)| \le D_0^{-1-\delta} = e^{-(1+\delta)c_+ d_0}.$$

Then

$$\begin{aligned} -f(\phi(d_0)) - v_{xx}(0,t) &\ge C^{-1}|\phi(d_0) - u_+| - |v_{xx}(0,t)| \\ &\ge C^{-1} c e^{-c_+ d_0} - e^{-(1+\delta)c_+ d_0} > 0 \end{aligned} \tag{2.8}$$

if d_0 is sufficiently large. Hence, we take $d_0 \gg 1$ such that (2.8) holds. From (1.16) and (2.8), $d'(0) > 0$ and $d(t) \ge d_0$ for $0 \le t \le t_0$. If $d(\bar{t}) = d_0$ for some time $\bar{t} \le T$, then $d'(\bar{t}) > 0$ and $d(t) \ge d_0$ for $\bar{t} \le t \le \bar{t} + t_0$. Hence

$$d(t) \ge d_0 \quad \text{for} \quad 0 \le t \le T, \tag{2.9}$$

and

$$0 < C^{-1} \le u_- - \phi(d(t)) \le C. \tag{2.10}$$

Thus the difference $d(t) - d_0(t)$ satisfies

$$\begin{aligned} (d - d_0)'(t) &= -\frac{f(\phi(d(t)))}{u_- - \phi(d(t))} + \frac{f(\phi(d_0(t)))}{u_- - \phi(d_0(t))} - \frac{v_{xx}(0,t)}{u_- - \phi(d(t))} \\ &\sim -(d - d_0)'(t)(1 + (D_0 + t)^{-1}) - \frac{v_{xx}(0,t)}{u_- \phi(d(t))}. \end{aligned}$$

Solving this ordinary differential inequality in a similar fashion to (2.2) - (2.4), we have

Lemma 2. 2 *For a given $v_{xx}(0,t)$ satisfing (2.7) with d_0 satisfying (2.8), it holds that*

$$(D_0 + t)^{\delta} |d(t) - d_0(t)| \le C, \tag{2.11}$$

$$|c_+ d(t)| \le C + \log(D_0 + t), \tag{2.12}$$

$$|d'(t)| \le C(D_0 + t)^{-1}.$$

These estimates imply that we can obtain the desired solution (v, d) provided that the initial shift d_0 is large and that the initial disturbance $v_0(x)$ is sufficiently small.

To state our theorem, introduce some notations. $L^2 = L^2(\mathbf{R}_-)$ is a usual Lebesgue space with its norm $\|\cdot\|$. For the weight function $\bar{W}_a(x) := e^{ac_-|x|}$, $a > 0$, define the function space

$$L^2_a = L^2_a(\mathbf{R}_-) = \{g \in L^2; \int_{-\infty}^0 \bar{W}_a(x)|g(x)|^2 dx < +\infty\}.$$

When $u \in C([0,T]; L^2_a)$, it holds that

$$|v(t)|_a := (\int_{-\infty}^0 W_a(\xi)|v(x,t)|^2 dx)^{1/2} < +\infty \tag{2.14}$$

for each $t \in [0,T]$, where

$$W_a(\xi) = (-\phi_x(x+d(t)))^{-a} \sim \begin{cases} e^{ac_-|x+d(t)|} & x+d(t) < 0 \\ e^{ac_+|x+d(t)|} & x+d(t) > 0, \end{cases} \tag{2.15}$$

Then the solution space of (v, d) is defined by

$$X(0,T) = \{(v,d);\ v, v_x \in C([0,T]; L^2_{3\delta}),\ (v_{xx}, v_{xxx}) \in C([0,T]; L^2_{1+6\delta}),\ d - d_0 \in C([0,T])\}.$$

When $(v, d) \in X(0,T)$, we use the notation

$$|v(t)|_{3;\delta} := |(v, v_x)(t)|_{3\delta} + |(v_{xx}, v_{xxx})(t)|_{1+6\delta}, \tag{2.16}$$

which is well-defined.

Thus, our theorem is stated as follows.

Theorem 1 *Let $\delta > 0$ be a small constant. Then, if $D_0 := e^{c_+ d_0}$ is large, and both $(v_0, v_{0x}) \in L^2_{3\delta}$ and $(v_{0xx}, v_{0xxx}) \in L^2_{1+6\delta}$ are sufficiently small, then there exists a unique solution $(v,d) \in X(0,\infty)$, which satisfies*

$$\begin{aligned} &(D_0+t)^{1-4\delta}|v(t)|^2_{3;\delta} + (D_0+t)^{2+2\delta}|v_{xx}(0,t)|^2 \\ &+\int_0^t ((D_0+\tau)^{1-4\delta}(|v(\tau)|^2_{3\delta} + |v_x(\tau)|^2_{3;\delta}) + (D_0+\tau)^{2+2\delta}|v_{xx}(0,\tau)|)d\tau \\ &\le C(D_0^{1-4\delta}|v_0|^2_{3;\delta} + \delta^{-2}D_0^{-\delta/2}). \end{aligned} \tag{2.17}$$

Corollary 1 *Under the same conditions as in Theorem 1, a unique solution $u(x,t) = \phi(x + d(t)) + v_x(x,t)$ to (1.1) satisfies*

$$\sup_{\mathbf{R}_-} |u(x,t) - \phi(x+d(t))| \le C(D_0+t)^{-\frac{1}{2}+2\delta},$$

and

$$C_1 + \log(D_0+t) \le c_+ d(t) \le C_2 + \log(D_0+t).$$

The proof of Thorem 1 is done by the combination of the local existence and the a priori estimates. Since $\phi(\xi) - u_- \sim e^{-c_-|x|}$ as $x \to -\infty$, there exists a unique local solution $(v,d) \in X(0,t_0)$ if $|v_0|_{3;\delta} \le C$ by a standard way. Hence, we devote ourselves to the a priori estimates in the next section.

3 A priori estimates

Let T be

$$T = \sup\{t_0; (v,d)) \in X(0,t_0) \text{ is a solution of (1.15), (1.16) } \textit{with}$$
$$N(t) \leq 1 \text{ and } (D_0+t)^{1+1\delta}|v_{xx}(0,t)| \leq 1, \quad 0 \leq t \leq t_0\}, \tag{3.1}$$

where

$$N(t)^2 = \sup_{0\leq\tau\leq t}\{(D_0+\tau)^{1-4\delta}|v(\tau)|^2_{3;\delta} + (D_0+\tau)^{2+2\delta}|v_{xx}(0,t)|^2$$
$$+\int_0^T((D_0+s)^{1-4\delta}(|v(s)|^2_{3\delta} + |v_x(s)|^2_{3;\delta}) + (D_0+s)^{2+2\delta}|v_{xx}(0,s)|^2)ds\ \}. \tag{3.2}$$

Then we show

Proposition 3. 1 (A priori estimates) *Let* $(v,d) \in X(0,T)$ *with* d_0 *satisfying (2.8). Then (2.17) holds for* $0 \leq t \leq T$ *provided that* $D_0^{1-4\delta}|v_0|^2_{3;\delta} + \delta^{-2}D_0^{-\delta/2}$ *is sufficiently small.*

Proposition 3.1 implies $T = \infty$ by combining the local existence theorem. To prove Proposition 3.1 we show

$$N(T)^2 \leq C(D_0^{1-4\delta}|v_0|^2_{3;\delta} + \delta^{-2}D_0^{-\delta/2} + N(T)^3, \tag{3.3}$$

which is derived in the later subsections.

3.1 Basic estimate

Linearize (1.15) around $\phi(\xi)$ to have

$$L(v) := v_t + f'(\phi(\xi))v_x - v_{xx} = d'(t)(u_- - \phi(\xi)) + F, \tag{3.4}$$

where

$$F = -(f(\phi(\xi) + v_x) - f(\phi(\xi)) - f'(\phi(\xi))v_x) = O(v_x^2). \tag{3.5}$$

Since the flux function f is not necessarily convex nor concave, we introduce the weight function

$$w_a(\phi(\xi)) = \frac{\{(\phi-u_+)(u_- - \phi)\}^{1-a}}{-f(\phi)}, \quad 0 < a < 1. \tag{3.6}$$

The weight function of this type was first introduced in [5]. Note that

$$w_a(\phi(\xi)) \sim (-\phi_x(\xi))^{-a} = W_a(\phi(\xi)) \sim e^{ac_-|\xi|} \text{ as } x \to -\infty \tag{3.7}$$

and that, at $x = 0$,

$$w_a(\phi(d(t))) \sim (-\phi_x(d(t)))^{-a} \sim e^{ac_+d(t)} \sim (D_0+t)^a \tag{3.8}$$

as $t \to \infty$ by Lemma 2.2. Moreover, putting

$$\bar{u} = \frac{u_+ + u_-}{2},$$

we have

$$
\begin{aligned}
(w_a f)'(\phi) &= -2(1-a)\{(\phi-u_+)(u_- -\phi)\}^{-a}(\bar{u}-\phi) \\
&\sim \begin{cases} (1-a)(-\phi_x(\xi))^{-a}(>0) & \text{as} \quad \xi \to -\infty \\ -(1-a)(-\phi_x(\xi))^{-a}(<0) & \text{as} \quad \xi \to +\infty, \end{cases}
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
&(w_a f)''(\phi) \\
&= 2(1-a)\{(\phi-u_+)(u_- -\phi)\}^{-1-a}\{2a(\bar{u}-\phi)^2 + (\phi-u_+)(u_- -\phi)\} \\
&\sim a(1-a)(-\phi_x(\xi))^{-1-a}(>0) \quad \text{as} \quad \xi \to \pm\infty,
\end{aligned} \tag{3.10}
$$

and

$$
\begin{aligned}
|w_a'(\phi)| &= \{(\phi-u_+)(u_- -\phi)\}^{-a}\left|\frac{2(1-a)(\bar{u}-\phi)}{f(\phi)} + f'(\phi)\frac{(\phi-u_+)(u_- -\phi)}{f(\phi)}\right| \\
&\le C(1-a)(-\phi_x(\xi))^{-1-a}.
\end{aligned} \tag{3.11}
$$

Multiplying (3.4) by $w_{3\delta}(\phi(\xi))v$, we have

$$
\begin{aligned}
&(\frac{1}{2}w_{3\delta}(\phi)v^2)_t + \{\frac{1}{2}(w_{3\delta}f)'(\phi)v^2 - w_{3\delta}(\phi)vv_x\}_x \\
&\quad -\frac{1}{2}\{(w_{3\delta}f)''(\phi) + w_{3\delta}'(\phi)d'(t)\}\phi_x v^2 + w_{3\delta}(\phi)v_x^2 \\
&= d'(t)(u_- - \phi(\xi))w_{3\delta}(\phi)v + w_{3\delta}(\phi)vF.
\end{aligned} \tag{3.12}
$$

The estimate of the second to last term is important in this basic estimates. By using $v(0,t)=0$, (3.4) - (3.11) and (2.13), integrating (3.12) over $(-\infty, 0)$ yields

$$
\begin{aligned}
&(\frac{1}{2}\int_{-\infty}^0 w_{3\delta}(\phi)v^2dx)_t + \int_{-\infty}^0 c_0(-\phi_x)^{-3\delta}\{(\delta - CD_0^{-1})v^2 + v_x^2\}dx \\
&\le d'(t)\int_{-\infty}^0 (u_-\phi(\xi))w_{3\delta}(\phi)vdx + C\int_{-\infty}^0 w_{3\delta}(\phi)|v|v_x^2dx.
\end{aligned} \tag{3.13}
$$

The right-hand side of (3.13) is estimated as follows:

$$|d'(t)\int_{-\infty}^{0}(u_- - \phi(\xi))w_{3\delta}v dx| = |\int_{-\infty}^{-d(t)} + \int_{-d(t)}^{0}|$$

$$\leq \int_{-\infty}^{-d(t)}(\nu\delta(-\phi_x)^{-3\delta}v^2 + C_\nu\delta^{-1}d'(t)^2(-\phi_x)^{-3\delta}|u_- - \phi(\xi)|^2)dx$$

$$+|d'(t)|\int_{-d(t)}^{0}|\int_x^0 \frac{\partial}{\partial x}\{(u_- - \phi(\xi))w_{3\delta}(\phi)v\}dy|dx$$

$$\leq \nu\delta\int_{-\infty}^{0}(-\phi_x)^{-3\delta}v^2 dx + C_\nu\delta^{-1}d'(t)^2$$

$$+C|d'(t)|\int_{-d(t)}^{0}(\delta^{-1}\int_x^0(-\phi_x)^{-3\delta}dy)^{1/2}(\int_x^0(-\phi_x)^{-3\delta}(\delta v^2 + v_x^2)dy)^{1/2}dx$$

$$\leq \nu\int_{-\infty}^{0}(-\phi_x)^{-3\delta}(\delta v^2 + v_x^2)dx + C\delta^{-1}|d'(t)|^2 d(t)^3 e^{3\delta c + d(t)}$$

for a small constant $\nu > 0$, and

$$C\int_{-\infty}^{0}w_{3\delta}(\phi)|v|v_x^2 dx \leq CD_0^{-\frac{1}{2}(1-4\delta)}N(t)\int_{-\infty}^{0}w_{3\delta}(\phi)v_x^2 dx.$$

Hence, for a large D_0

$$\begin{aligned}&(\frac{1}{2}\int_{-\infty}^{0}w_{3\delta}(\phi)v^2 dx)_t + c_0\int_{-\infty}^{0}(-\phi_x)^{-3\delta}(\delta v^2 + v_x^2)dx \\ &\leq C\delta^{-1}|d'(t)|^2 d(t)^3 e^{3\delta c + d(t)}.\end{aligned} \tag{3.14}$$

Since

$$C\delta^{-1}\int_0^t |d'(\tau)|^2 d(\tau)^3 e^{3\delta c + d(\tau)}(D_0 + \tau)^{1-4\delta}d\tau$$

$$\leq C\delta^{-1}\int_0^t (D_0 + \tau)^{-2}(C + \log(D_0 + \tau))^3(D_0 + \tau)^{d\delta + 1 - 4\delta}d\tau$$

$$\leq C\delta^{-2}D_0^{-\delta/2},$$

multiplying (3.14) by $(D_0 + t)^{1-4\delta}$ and integrating the resultant inequality over $(0, t)$, we obtain the following lemma.

Lemma 3. 1 *It holds that*

$$(D_0 + t)^{1-4\delta}|v(t)|_{3\delta}^2 + \int_0^t (D_0 + \tau)^{1-4\delta}(\delta|v(\tau)|_{3\delta}^2 + |v_x(\tau)|_{3\delta}^2)d\tau$$

$$\leq C(D_0^{1-4\delta}|v_0|_{3\delta}^2 + \delta^{-2}D_0^{-\delta/2} + CN(T)^3).$$

3.2 Estimate of higher order derivative (I)

Differentiate (3.4) in x to have

$$
\begin{aligned}
L(v_x) &= v_{xt} + f'(\phi(\xi))v_{xx} - v_{xxx} \\
&= f''(\phi(\xi))\phi_x(\xi)v_x - d'(t)\phi_x(\xi) + F_x,
\end{aligned} \tag{3.16}
$$

where

$$
\begin{aligned}
F_x &= -(f'(\phi+v_x) - f'(\phi) - f''(\phi)v_x)\phi_x - (f'(\phi+v_x) - f'(\phi))v_{xx} \\
&= O(|\phi_x|v_x^2 + |v_x v_{xx}|).
\end{aligned} \tag{3.17}
$$

Multiplying (3.16) by $w_{3\delta}(\phi)v_x$ we have

$$
\begin{aligned}
&(\frac{1}{2}w_{3\delta}v_x^2)_t + \{\frac{1}{2}(w_{3\delta}f)'(\phi)v_x^2 - w_{3\delta}(\phi)v_x v_{xx}\}_x \\
&-\frac{1}{2}\{(w_{3\delta}f)''(\phi) + w_{3\delta}'(\phi)d'(t)\}\phi_x v_x^2 + w_{3\delta}(\phi)v_{xx}^2 \\
&\le C|\phi_x|w_{3\delta}(\phi)v_x^2 + C|d'(t)|(-\phi_x)^{1-3\delta}|v_x| + |F_x \cdot w_{3\delta}(\phi)v_x|,
\end{aligned} \tag{3.18}
$$

which is similar to (3.12). The second to last term in (3.18) has not difficulty, different from (3.12). Here we must evaluate the terms from the boundary, which are bounded by

$$
C(-\phi_x(d(t)))^{-3\delta}\{(-\phi_x(d(t)))^2 + (-\phi_x(d(t)))\cdot C(D_0+t)^{-1-\delta}\} \le C(D_0+t)^{-2+3\delta}, \tag{3.19}
$$

because

$$
v_x(0,t) = v_+ - \phi(d(t)) \quad \text{and} \quad |v_{xx}(0,t)| \le C(D_0+t)^{-1-\delta}.
$$

Integrating (3.18) over $(-\infty, 0)$, we have

$$
\begin{aligned}
&(\frac{1}{2}\int_{-\infty}^0 w_{3\delta}(\phi)v_x^2 dx)_t + \int_{-\infty}^0 c_0(-\phi_x(\xi))^{-3\delta}(\delta v_x^2 + v_{xx}^2)dx \\
&\le C\{(D_0+t)^{-2+3\delta} + |v_x(t)|_{3\delta}^2 + (D_0+t)^{-2} + \int_{-\infty}^0 |F_x \cdot w_{3\delta}(\phi)v_x|dx\}
\end{aligned} \tag{3.20}
$$

Since

$$
C\int_0^t (D_0+\tau)^{1-4\delta}\cdot(D_0+\tau)^{-2+3\delta}d\tau \le C\delta^{-1}D_0^{-\delta/2},
$$

multiplying (3.20) by $(D_0+t)^{1-4\delta}$ and using Lemma 3.1, we obtain

Lemma 3. 2 *It holds that*

$$
\begin{aligned}
&(D_0+t)^{1-4\delta}|v_x(t)|_{3\delta}^2 + \int_0^t (D_0+\tau)^{1-4\delta}(\delta|v_x(\tau)|_{3\delta}^2 + |v_{xx}(\tau)|_{3\delta}^2)d\tau \\
&\le C(D_0^{1-4\delta}|v_0, v_{0x}|_{3\delta}^2 + \delta^{-2}D_0^{-\delta/2} + N(T)^3).
\end{aligned} \tag{3.21}
$$

3.3 Estimate of higher order derivative (II)

Differentiate (3.16) in x to have

$$\begin{aligned} L(v_{xx}) &= v_{xxt} + f'(\phi(\xi))v_{xxx} - v_{xxxx} \\ &= O(|\phi_x(\xi)|(|v_x| + |v_{xx}|)) - d'(t)\phi_{xx}(\xi) + F_{xx}, \end{aligned} \tag{3.22}$$

where

$$F_{xx} = O(|\phi_x(\xi)|(v_x^2 + |v_x v_{xx}|) + |v_x v_{xxx}| + v_{xx}^2). \tag{3.23}$$

First, we see the relation of boundary values. Since $u = \phi(\xi) + v_x$ satisfies $u_t + f(u)_x - u_{xx} = 0$, the integration $\int_{-\infty}^0 (u_t + f(u)_x - u_{xx})_x dx = 0$ gives $f'(u_+)u_x|_{x=0} = u_{xx}|_{x=0}$ and hence

$$\begin{aligned} v_{xxx}(0,t) &= f'(u_+)v_{xx}(0,t) + f'(u_+)\phi_x(d(t)) - \phi_{xx}(d(t)) \\ &= f'(u_+)v_{xx}(0,t) + O(1)(-\phi_x(d(t)))^2. \end{aligned} \tag{3.24}$$

Here, we adopt the additional weight function

$$z_{9\delta}(\xi) = e^{9\delta c_\pm \langle \xi - \xi_* \rangle} \sim (-\phi_x(\xi))^{-9\delta} \quad \text{as} \quad \xi \to \pm\infty,$$

where $\phi(\xi_*) = \bar{u}$ and $c_\pm$ is, respectively, chosen when $\xi - \xi_* > 0$ or < 0. Multiply (3.22) by $w_{1-3\delta}(\phi(\xi))z_{9\delta}(\xi)v_{xx} \sim (-\phi_x(\xi))^{-1-6\delta}v_{xx}$ as $\xi \to \pm\infty$, we have, with $a = 1 - 3\delta$,

$$\begin{aligned} &(\frac{1}{2}z_{9\delta}w_a(\phi)v_{xx}^2)_t + [z_{9\delta}(\xi)\{\frac{1}{2}(w_a f)'(\phi)v_{xx}^2 - w_a(\phi)v_{xx}v_{xxx}\}]_x \\ &-z_{9\delta}(\xi)[A(x,t) + d'(t)B(x,t)]v_{xx}^2 + z_{9\delta}(\xi)w_a(\phi)v_{xxx}^2 \\ &\le |z_{9\delta}(\xi)\cdot 9\delta c_\pm \frac{\xi - \xi_*}{\langle \xi - \xi_* \rangle} w_a(\phi)v_{xx}v_{xxx}| \\ &+C(-\phi_x)^{-6\delta}(v_x^2 + v_{xx}^2) + C|d'(t)|(-\phi_x)^{-6\delta}|v_{xx}| \\ &+C|F_{xx}|(-\phi_x)^{-1-6\delta}|v_{xx}|, \end{aligned} \tag{3.25}$$

where

$$A(x,t) = \frac{1}{2}(9\delta c_\pm \frac{\xi - \xi_*}{\langle \xi - \xi_* \rangle}(w_a f)'(\phi) + (w_a f)''(\phi)\phi_x(\xi)) \tag{3.26}$$

and

$$B(x,t) = w_a'(\phi)\phi_x(\xi) + 9\delta c_\pm \frac{\xi - \xi_*}{\langle \xi - \xi_* \rangle} w_a(\phi). \tag{3.27}$$

By (3.9) and the definition of ξ_*

$$A(x,t) \ge c_0\delta(-\phi_x(\xi))^{-1+3\delta}$$

(cf. [1, 5]), and $d'(t)B(x,t)$ is absorbed into $A(x,t)$ if $D_0 \gg 1$. Moreover,

$$
\begin{aligned}
&\int_{-\infty}^{0} [z_{9\delta}(\xi)\{\frac{1}{2}(w_a f)'(\phi)v_{xx}^2 - w_a(\phi)v_{xx}v_{xxx}\}]_x dx \\
&\geq (-\phi_x(d(t))^{-1-6\delta}(C^{-1}\frac{-6\delta}{2} - \frac{1}{2}f'(u_+))v_{xx}(0,t)^2 - C(-\phi_x(d(t)))^{3-6\delta} \\
&\geq c_0(D_0+t)^{1+6\delta}v_{xx}(0,t)^2 - C(D_0+t)^{-3+6\delta}
\end{aligned}
\tag{3.28}
$$

for a small fixed $\delta > 0$ by $f'(u_+) < 0$. Thus, multiplying (3.25) by $(D_0+t)^{1-4\delta}$ and using (3.26) - (3.28) and Lemmas 3.1 - 3.2, we obtain the third lemma.

Lemma 3. 3 *It holds that*

$$
\begin{aligned}
&(D_0+t)^{1-4\delta}|v_{xx}(t)|_{1+6\delta}^2 + \int_0^t (D_0+\tau)^{2+2\delta}v_{xx}(0,\tau)^2 d\tau \\
&+\int_0^t (D_0+\tau)^{1-4\delta}(\delta|v_{xx}(\tau)|_{1+6\delta}^2 + |v_{xxx}(\tau)|_{1+6\delta}^2)d\tau \\
&\leq C(D_0^{1-4\delta}(|v_0, v_{0x}|_{3\delta}^2 + |v_{0xx}|_{1+6\delta}^2) + \delta^{-2}D_0^{-\delta/2} + N(T)^3).
\end{aligned}
\tag{3.29}
$$

3.4 Estimate of higher order derivative (III)

Finally, differentiate (3.22) once more in x to have

$$
\begin{aligned}
L(v_{xxx}) &= v_{xxxt} + f'(\phi(\xi)))v_{xxxx} - v_{xxxxx} \\
&= O(|\phi_x|(|v_x| + |v_{xx}| + |v_{xxx}|)) - d'(t)\phi_{xxx}(\xi) + F_{xxx},
\end{aligned}
\tag{3.30}
$$

where

$$
F_{xxx} = O(1)(|\phi_x|(v_x^2 + |v_x v_{xxx}| + v_{xx}^2) + |v_{xx}v_{xxx}| + |v_x v_{xxxx}| + |v_{xx}|^3). \tag{3.31}
$$

Integrating $(u_{xt} + f''(u)u_x^2 + f'(u)u_{xx} - u_{xxx})_x$ over $(-\infty, 0)$, we have the relation at $x=0$:

$$
\begin{aligned}
&\frac{d}{dt}v_{xx}(0,t) + f'(u_+)v_{xxx}(0,t) - v_{xxxx}(0,t) \\
&= O(1)(v_{xx}(0,t)^2 + |d'(t)||\phi_x(d(t))| + |\phi_x(d(t))|^2) := \Gamma(t).
\end{aligned}
\tag{3.32}
$$

Similarly to the preceding subsection, multiply (3.30) by $z_{9\delta}(\xi)w_a(\phi)v_{xxx}$, $a = 1-3\delta$, then we have

$$
\begin{aligned}
&(\frac{1}{2}z_{9\delta}(\xi)w_a(\phi)v_{xxx}^2)_t + [z_{9\delta}(\xi)\{\frac{1}{2}(w_a f)'(\phi)v_{xxx}^2 - w_a(\phi)v_{xxx}v_{xxxx}\}]_x \\
&-z_{9\delta}(\xi)[A(x,t) + d'(t)B(x,t)] + z_{9\delta}(\xi)w_a(\phi)v_{xxxx}^2 \\
&\leq |z_{9\delta}(\xi)\cdot 9\delta c_\pm \frac{\xi-\xi_*}{\langle\xi-\xi_*\rangle}w_a(\phi)v_{xxx}v_{xxxx}| \\
&+C(-\phi_x)^{-6\delta}(v_x^2 + v_{xx}^2 + v_{xxx}^2) + |d'(t)|(-\phi_x)^{-6\delta}|v_{xxx}| + |z_{9\delta}w_a(\phi)v_{xxx}F_{xxx}|.
\end{aligned}
\tag{3.33}
$$

Here,

$$
\begin{aligned}
&-z_{9\delta}w_a(\phi)v_{xxx}v_{xxxx}|_{x=0}\\
&=-z_{9\delta}w_x(\phi)|_{x=0}\cdot(f'(u_+)v_{xx}(0,t)\\
&\quad+O(1)(-\phi_x(d(t)))^2\cdot(\frac{d}{dt}v_{xx}(0,t)+f'(u_+)v_{xxx}(0,t)-\Gamma(t))\\
&=\frac{d}{dt}[z_{9\delta}w_a(\phi)|_{x=0}(\frac{-f'(u_+)}{2}v_{xx}(0,t)^2-v_{xx}(0,t)\cdot O(1)(-\phi_x(d(t)))^2)]\\
&\quad+z_{9\delta}w_a(\phi)|_{x=0}(-f'(u_+)v_{xxx}(0,t)^2+v_{xxx}(0,t)\Gamma(t)).
\end{aligned}
\tag{3.34}
$$

Noting that $f'(u_+)<0$, $v_{xxx}(0,t)\sim -f'(u_+)v_{xx}$ and $z_{9\delta}w_a(\phi)|_{x=0}\sim(D_0+t)^{1+6\delta}$, and multiplyiing (3.33) by $(D_0+t)^{1-4\delta}$, we obtain the final lemma.

Lemma 3. 4 *It holds that*

$$
\begin{aligned}
&(D_0+t)^{1-4\delta}|v_{xxx}(t)|^2_{1+6\delta}+(D_0+t)^{2+2\delta}|v_{xx}(0,t)|^2\\
&+\int_0^t((D_0+\tau)^{2+2\delta}|v_{xx}(0,\tau)|^2+(D_0+\tau)^{1-4\delta}(\delta|v_{xxx}(\tau)|^2_{1+6\delta}+|v_{xxxx}(\tau)|^2_{1+6\delta})d\tau\\
&\le C(D_0^{1-4\delta}|v_0|_{3;\delta}+\delta^{-2}D_0^{-\delta/2}+N(T)^3).
\end{aligned}
\tag{3.35}
$$

Adding (3.15), (3.21), (3.29) and (3.35) we obtain (3.3), which proves Proposition 3.1.

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Convergence Rates to Viscous Shock Profile for General Scalar Viscous Conservation Laws with Large Initial Disturbance

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Abstract

This paper is concerned with the convergence rates to viscous shock profile for general scalar viscous conservation laws. Compared with former results in this direction, the main novelty in this paper lies in the fact that the initial disturbance can be chosen arbitrarily large. This answers positively an open problem proposed by A. Matsumura in [12] and K. Nishihara in [16]. Our analysis is based on the L^1-stability results obtained by H. Freistühler and D. Serre in [1].

1 Introduction and the statement of our main results

This paper is concerned with the convergence rates to viscous shock profile of solutions to the Cauchy problem for general scalar viscous conservation laws

$$u_t + f(u)_x = u_{xx}, \quad x \in \mathbf{R}, \quad t > 0 \tag{1.1}$$

with initial data

$$u(t,x)|_{t=0} = u_0(x), \quad x \in \mathbf{R}, \tag{1.2}$$

where $f(u) \in C^2(\mathbf{R})$ on the domain under our consideration and the initial data $u_0(x)$ is asymptotically constant as $x \to \pm\infty$:

$$u_0(x) \to u_\pm \quad \text{as} \quad x \to \pm\infty. \tag{1.3}$$

The traveling wave $u(x - st) \equiv \phi(\xi)$ is called a viscous shock profile to (1.1) - (1.3) if it satisfies

$$-s\phi_\xi + f(\phi)_\xi = \phi_{\xi\xi}, \quad \phi(\xi) \to u_\pm \quad \text{as} \quad \xi \to \pm\infty. \tag{1.4}$$

Here the constants $u_\pm$ and s (shock speed) satisfy the Rankine-Hugoniot condition

$$-s(u_+ - u_-) + f(u_+) - f(u_-) = 0 \tag{1.5}$$

and the generalized entropy condition

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$$h(u) \equiv -s(u-u_\pm)+f(u)-f(u_\pm)\begin{cases} <0, & \text{if} \quad u_+<u<u_-, \\ >0, & \text{if} \quad u_-<u<u_+. \end{cases} \tag{1.6}$$

That is, the viscous shock profile ϕ is a solution to

$$\phi_\xi = h(\phi), \quad \phi(\pm\infty)=u_\pm.$$

It is noted that the condition (1.6) implies

$$f'(u_+) \le s \le f'(u_-) \tag{1.7}$$

which includes the following cases: *the nondegenerate shock condition*

$$f'(u_+) < s < f'(u_-) \tag*{$(1.8)_1$}$$

and *the degenerate shock condition*

$$f'(u_+) = s < f'(u_-), \quad f'(u_+) < s = f'(u_-) \quad \text{or} \quad s = f'(u_\pm). \tag*{$(1.8)_2$}$$

We call the shock satisfying $(1.8)_1$ *Lax shock* (regardless of viscous or inviscid case) while those satisfying $(1.8)_2$ are called *marginal shock*. In what follows, for the marginal shock, we only pay our attention to the case $f'(u_+)=s<f'(u_-)$ since the other cases can be treated similarly.

Stability results have a long history starting with the paper of A. M. Il'in and O. A. Oleinik [4], in which they proved that the viscous shock profile in the case of a convex flux function is indeed stable. Since then, a lot of good results have been obtained by employing various methods (All references [1-18] are on this line. Especially, see the survey paper [12]).

To go directly to the main point of this paper, we only review two results which are closely related to ours. The most general result on the nonlinear stability of the viscous shock profile is given by H. Freistühler and D. Serre in [1].

Theorem 1. 1 **(L^1–stability)** *Let $\phi(\xi):\mathbf{R}\to\mathbf{R}$ be a bounded viscous shock profile of (1.1), (1.2). Then for any $u_0(x)$ satisfying $u_0(x)-\phi(x)\in L^1(\mathbf{R})$, the Cauchy problem (1.1), (1.2) admits a unique solution $u(t,x)$ satisfying*

$$\lim_{t\to+\infty}\left\|u(t,x)-\phi(x-st+\delta)\right\|_{L^1}=0, \tag{1.9}$$

where

$$\delta := \frac{\int_{\mathbf{R}}(u_0(x)-\phi(x))dx}{u_+-u_-}. \tag{1.10}$$

Although the results obtained in Theorem 1.1 are quite perfect, no decay rates have been obtained. On the other hand, A. Matsumura and K. Nishihara [13], M. Nishikawa [17] have obtained the following decay properies via the L^2–energy method. Notations are given in Remark 1.1 below.

Theorem 1. 2 (Decay rates) *(I). When $f'(u_+) < s < f'(u_-)$, suppose that $u_0(x) - \phi(x)$ is integrable and that*

$$U_0(x) := \int_{-\infty}^{x} \Big\{u_0(z) - \phi(z+\delta)\Big\}dz \in H^2 \cap L^2_\alpha(\mathbf{R}).$$

Then there exists a sufficiently small positive constant ε_1 such that if $\|U_0(x)\|_2 < \varepsilon_1$, the Cauchy problem (1.1), (1.2) has a unique global solution $u(t,x)$ satisfying

$$\sup_{x\in\mathbf{R}} \Big|u(t,x) - \phi(x - st + \delta)\Big| \le O(1)(1+t)^{-\frac{\alpha}{2}}\Big(\|u_0 - \phi\|_1 + |U_0|_\alpha\Big). \tag{1.11}$$

(II). When $f'(u_+) = s < f'(u_-)$, suppose that $f(u) \in C^{n+1}(\mathbf{R})$ such that

$$f''(u_+) = \cdots = f^{(n)}(u_+) = 0 \quad \text{and} \quad f^{(n+1)}(u_+) \neq 0 \quad \text{for some} \quad n \ge 1 \tag{1.12}$$

and that $u_0(x) - \phi(x)$ is integrable and $U_0(x) \in H^2 \cap L^2_{\alpha,\langle\xi\rangle_+}\left(0 < \alpha < \frac{2}{n}\right)$, then there exists a sufficiently small positive constant $\varepsilon_1 > 0$ such that if $\|U_0\|_2 + |U_0|_{\langle\xi\rangle_+} < \varepsilon_1$, the Cauchy problem (1.1), (1.2) has a unique global solution $u(t,x)$ satisfying

$$\sup_{x\in\mathbf{R}} \Big|u(t,x) - \phi(x - st + \delta)\Big| \le O(1)(1+t)^{-\frac{\alpha}{4}}\left(\|u_0 - \phi\|_1 + |U_0|_{\alpha,\langle\xi\rangle_+}\right). \tag{1.13}$$

Here

$$\langle\xi\rangle_+ := \begin{cases} \sqrt{1+\xi^2}, & \xi \ge 0, \\ 1, & \xi < 0. \end{cases} \tag{1.14}$$

Remark 1. 1 (Notations) *Here in the above and in what follows, by C or $O(1)$, we denote several generic constants and for each $\tau \ge 0$, $C(t-\tau)$ (or $C_i(t-\tau)$ for some $i \in \mathbf{Z}^+$) will be used to denote some generic function which is continuous with respect to t on $[\tau, \infty)$. For two functions $f(x)$ and $g(x)$, $f(x) \sim g(x)$ as $x \to a$ means*

$$C^{-1}f(x) \le g(x) \le Cf(x) \tag{1.15}$$

in the neighborhood of a. $H^l(\mathbf{R})(l \ge 0)$ denotes the usual Sobolev space with norm $\|\cdot\|_l$ and $\|\cdot\|_0 = \|\cdot\|$ will denote the usual L^2-norm. For the weighted function $w(x) > 0$, $L^2_w(\mathbf{R})$ denotes the space of measurable functions $f(x)$ satisfying $\sqrt{w(x)}f(x) \in L^2(\mathbf{R})$ with the norm

$$|f|_w := \left(\int_{\mathbf{R}} w(x)|f(x)|^2 dx\right)^{\frac{1}{2}}. \tag{1.16}$$

When $C^{-1} \le w(x) \le C$, we note that $L^2_w(\mathbf{R}) = L^2(\mathbf{R})$ with $|\cdot|_w = \|\cdot\|$. When $w(x) \sim \langle x\rangle^\alpha = (1+x^2)^{\frac{\alpha}{2}}$, we write $L^2_w(\mathbf{R}) = L^2_\alpha(\mathbf{R})$ and $|\cdot|_w = |\cdot|_\alpha$ without confusion. Moreover, if $w(x)$ is replaced by $\langle x\rangle^\alpha w(x)$, we denote the space by $L^2_{\alpha,w}(\mathbf{R})$ with the norm

$$|f|_{\alpha,w} := \left(\int_{\mathbf{R}} \langle x\rangle^\alpha w(x)|f(x)|^2 dx\right)^{\frac{1}{2}}. \tag{1.17}$$

From the above two results, it is easy to find that in Theorem 1.1, the initial disturbance can be chosen arbitrarily large but no decay rates can be obtained. In Theorem 1.2, some decay rates have been obtained but, due to the limitation of their arguments, its initial disturbance should be small in certain Sobolev space. Thus it is of interest how to get the decay without smallness condition. In fact, such a problem is one of the open problems proposed by A. Matsumura in [12] and K. Nishihara in [16]. Our main purpose of this paper is to give a positive answer to this problem.

Theorem 1. 3 (Main results) *Let the initial data $u_0(x) - \phi(x) \in L^1 \cap L^\infty(\mathbf{R})$ and $U_0(\xi) \in L^2(\mathbf{R})$, then the following attsertions hold.*
(I). When $f'(u_+) < s < f'(u_-)$, the estimates (1.11) holds provided $U_0(\xi) \in L^2_\alpha(\mathbf{R})$;
(II). When $f'(u_+) = s < f'(u_-)$, the estimate (1.13) holds provided that the assumption (1.12) holds and $U_0(\xi) \in L^2_{\alpha,\langle\xi\rangle_+}(\mathbf{R})$ with $0 < \alpha < \frac{2}{n}$.

Remark 1. 2 *When $f'(u_+) < s = f'(u_-)$ or $s = f'(u_\pm)$, then $L^2_{\alpha,\langle\xi\rangle_+}(\mathbf{R})$ in (II) of Theorem 1.3 should be replaced by $L^2_{\alpha,\langle\xi\rangle_-}(\mathbf{R})$ or $L^2_{\alpha,\langle\xi\rangle}(\mathbf{R}) \equiv L^2_{\alpha+1}(\mathbf{R})$ respectively while the same results also hold. Here*

$$\langle\xi\rangle_- := \begin{cases} \sqrt{1+\xi^2}, & \xi \leq 0, \\ 1, & \xi > 0. \end{cases}$$

Remark 1. 3 *Compared with the results obtained in [13, 17], the regularity assumptions on the initial data is also weaker than those in [13, 17].*

Remark 1. 4 *As pointed out by A. Matsumura and K. Nishihara in [13], for the Lax shock, the decay rates obtained in Theorem 1.3 is expected to be optimal in the L^2-setting. In fact, when $f(u) = \frac{1}{2}u^2$, by exploiting an explicit formula, K. Nishihara showed in [15] that if*

$$|U_0(x)| \leq O(1)|x|^{-\frac{\alpha}{2}} \quad \text{as} \quad |x| \to +\infty,$$

then

$$\sup_{x\in\mathbf{R}} |u(t,x) - \phi(x - st + \delta)| \leq O(1)t^{-\frac{\alpha}{2}},$$

which is an optimal decay rate in general.

Before concluding this section, we give main ideas in deducing our main result, Theorem 1.3. Decay rates (1.11), (1.13) in Theorem 1.2 have been obtainde by the weighted energy method developed by Kawashima, Matsumura and Nishihara *etc.* in [6,7,13]. In their method, to obtain the *a priori* estimates is a key point under the *a priori* assumption

$$N(t) := \sup_{0\leq s\leq t} \|U(s,\cdot)\|_2 \leq \varepsilon$$

for sufficiently small positive constant ε, so that the initial disturbance $U_0(\xi)$ should be small. However, we found that the *a priori* estimates are available provided that $\|U(t,\cdot)\|_{L^\infty}$ is small. The L^1-stability theorem, Theorem 1.1, by H. Freistüler and D. Serre in [1] also shows that $\|U(t,\cdot)\|_{L^\infty} \to 0$ as $t \to \infty$. Therefore we can apply the weighted energy method on $[T_1, \infty) \times \mathbf{R}$ for some large T_1.

Our plan is as follows. In Section 2, we give some preliminary results. The proof of our main results will be given in Section 3.

2 Preliminary lemmas

In this section, we give some preliminary lemmas which will be used in proving our main results in the next section.

First, the existence of viscous shock profiles $\phi(\xi)$ follows from Kawashima and Matsumura [5].

Lemma 2. 1 (Existence of the viscous shock profile)
(i). If the Cauchy problem (1.1), (1.2) admits viscous shock profile $\phi(x-st)$ connecting u_- and u_+, then u_-, u_+ and s must satisfy the Rankine-Hugoniot condition (1.5) and the generalized entropy condition (1.6);
(ii). Conversely, suppose that (1.5) and (1.6) hold, then there exists a viscous shock profile $\phi(x-st)$ of (1.1), (1.2) which connects u_- and u_+ and is unique up to a shift in $\xi = x - st$ and is monotone in ξ. Moreover, if

$$h(\phi) \sim |\phi - u_\pm|^{1+k_\pm} \tag{2.1}$$

as $\phi \to u_\pm$ with $k_\pm \geq 0$, then it holds

$$\begin{cases} |\phi(\xi) - u_\pm| \sim \exp\left(-C_\pm|\xi|\right) & \text{as } \xi \to \pm\infty \text{ if } k_\pm = 0, \\ |\phi(\xi) - u_\pm| \sim |\xi|^{-\frac{1}{k_\pm}} & \text{as } \xi \to \pm\infty \text{ if } k_\pm \neq 0, \end{cases} \tag{2.2}$$

for some positive constant $C_\pm$.

Note that $k_\pm = n$ in (2.1) if $h'(u_\pm) = \cdots = h^{(n)}(u_\pm) = 0$ and $h^{(n+1)}(u_\pm) \neq 0$ which are corresponding to (1.12).

We now define the shift δ of the viscous shock profile $\phi(x-st)$ as

$$\int_{\mathbf{R}} \Big(u_0(x) - \phi(x+\delta)\Big)dx = 0 \tag{2.3}$$

and set

$$U_0(x) := \int_{-\infty}^{x} \Big(u_0(z) - \phi(z+\delta)\Big)dz. \tag{2.4}$$

It is easy to see that δ satisfies (1.10) and, without loss of generality, we may take $\delta = 0$. Following A. Matsumura and K. Nishihara [13], we put the perturbation

$$u(t,x) = \phi(\xi) + U_\xi(t,\xi), \quad \xi = x - st, \tag{2.5}$$

then the problem (1.1), (1.2) is reformulated to

$$U_t - U_{\xi\xi} + h'(\phi)U_\xi = F(t,\xi), \tag{2.6}$$

$$U(t,\xi)|_{t=0} = U_0(\xi) \equiv \int_{-\infty}^{\xi} \Big(u_0(z) - \phi(z)\Big)dz, \tag{2.7}$$

where

$$F(t,\xi) := -\Big\{f(\phi + U_\xi) - f(\phi) - f'(\phi)U_\xi\Big\}. \tag{2.8}$$

Note that $\phi(\xi) \in L^\infty(\mathbf{R})$ and $U_{0\xi}(\xi) \in L^\infty(\mathbf{R})$. From the well-known result on the global solvability of the Cauchy problem to scalar parabolic equations [1], we have that

Lemma 2. 2 (Global existence to the Cauchy problem (2.6), (2.7)) *Suppose that $f(u) \in C^1(\mathbf{R})$, $u_0(x) - \phi(x) \in L^1 \cap L^\infty(\mathbf{R})$, then the Cauchy problem (2.6), (2.7) admits a unique global smooth solution $U(t,\xi)$ satisfying*

$$|U_\xi(t,\xi)| \le C_1, \tag{2.9}$$

where

$$C_1 := \|u_0(x) - \phi(x)\|_{L^\infty} + \|\phi(x)\|_{L^\infty}. \tag{2.10}$$

From the Duhamel principle, the solution $U(t,\xi)$ to the Cauchy problem (2.6), (2.7) has the following integral representation

$$U(t,\xi) = K(t,\xi) * U_0(\xi) + \int_0^t K(t-s,\xi) * G(s,\xi)ds. \tag{2.11}$$

Here $*$ denotes the convolution in space and

$$\begin{cases} K(t,\xi) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{|\xi|^2}{4t}\right), \\ G(t,\xi) = -\Big(f(\phi(\xi) + U_\xi(t,\xi)) - f(\phi(\xi)) - sU_\xi\Big). \end{cases} \tag{2.12}$$

Having obtained the above integral representation, we can deduce that

Lemma 2. 3 *In addition to the assumptions stated in Lemma 2.2, we assume further that $U_0(\xi) \in L^2(\mathbf{R}), f(u) \in C^1(\mathbf{R})$, then we have for each $\tau > 0, T > 0$ and $i = 0,1$ that*

$$\left\| \frac{\partial^i}{\partial \xi^i} U(t,\xi) \right\|_{L^2} \le C_2(t) t^{-\frac{1}{2}}, \quad \tau \le t \le T. \tag{2.13}$$

Proof: Notice that

$$\frac{\partial^i}{\partial \xi^i} U(t,\xi) = \frac{\partial^i}{\partial \xi^i} K(t,\xi) * U_0(\xi) + \int_0^t \frac{\partial^i}{\partial \xi^i} K(t-s,\xi) * G(s,\xi) ds, \quad i = 0,1 \tag{2.14}$$

and

$$G(t,\xi) = O(1) \left| U_\xi(t,\xi) \right|. \tag{2.15}$$

By Hausdorff-Young's inequality and (2.9) we have that, for $i = 0,1$,

$$\begin{aligned} \left\| \tfrac{\partial^i}{\partial \xi^i} U(t,\xi) \right\|_{L^2} &\le \left\| \tfrac{\partial^i}{\partial \xi^i} K(t,\xi) \right\|_{L^1} \|U_0(\xi)\|_{L^2} + \int_0^t \left\| \frac{\partial^i}{\partial \xi^i} K(t-s,\xi) \right\|_{L^1} \|G(s,\xi)\|_{L^2} ds \\ &\le O(1) t^{-\frac{i}{2}} \|U_0(\xi)\|_{L^2} + O(1) \int_0^t (t-s)^{-\frac{i}{2}} \left\| U_\xi(s,\xi) \right\|_{L^2} ds, \end{aligned} \tag{2.16}$$

and hence

$$\sum_{i=0}^{1}\left\|\frac{\partial^i}{\partial\xi^i}U(t,\xi)\right\|_{L^2}\le O(1)\left(1+t^{-\frac{1}{2}}\right)\|U_0(\xi)\|_{L^2}+O(1)\int_0^t\left(1+(t-s)^{-\frac{1}{2}}\right)\|U_\xi(s,\xi)\|_{L^2}\,ds. \quad (2.17)$$

Thus the singular Gronwall inequality gives

$$\left\|\frac{\partial^i}{\partial\xi^i}U(t,\xi)\right\|_{L^2}\le C(t)t^{-\frac{1}{2}}\|U_0(\xi)\|_{L^2}, \quad (2.17)$$

which is the desired estimates (2.13). This completes the proof of Lemma 2.3.

Remark 2. 1 *Since the viscous shock profile $\phi(\xi)$ satisfies*

$$\phi_{\xi\xi}=(f(\phi)-s\phi)_\xi\equiv h'(\phi)\phi_\xi, \quad (2.18)$$

we can deduce that, if $f(u)\in C^k(\mathbf{R})$ for some positive integer $k>0$, then

$$\left|\frac{\partial^{k+1}}{\partial\xi^{k+1}}\phi(\xi)\right|\le O(1). \quad (2.19)$$

Combining the above observation with the technique used in the proof of Lemma 2.3, we have the following lemma.

Lemma 2. 4 *In addition to the assumptions listed in Lemma 2.3, we assume further that $f(u)\in C^k(\mathbf{R})$ for some positive integer k, then we have*

$$\left\|\frac{\partial^i}{\partial\xi^i}U(t,\xi)\right\|_{L^2}\le C_3\left(\frac{\tau}{2},t-\frac{\tau}{2}\right)\|U_0(\xi)\|_{L^2},\quad \tau\le t\le T,\quad i=0,1,\cdots,k. \quad (2.20)$$

Proof: We only treat the case $k=2$. The case $k>2$ can be shown by employing the induction method. In the case $k=2$, from Lemma 2.3, we only need to estimate $\|U_{\xi\xi}(t,\xi)\|_{L^2}$. We first have that for each $\tau_1>0$

$$U(t,\xi)=K(t-\tau_1,\xi)*U(\tau_1,\xi)+\int_{\tau_1}^t K(t-s,\xi)*G(s,\xi)ds. \quad (2.21)$$

and hence

$$U_{\xi\xi}(t,\xi)=K_{\xi\xi}(t-\tau_1,\xi)*U(\tau_1,\xi)+\int_{\tau_1}^t K_\xi(t-s,\xi)*G_\xi(s,\xi)ds. \quad (2.22)$$

Since

$$G_\xi(t,\xi)=O(1)\left|\phi(\xi)\right|\left|U_\xi(t,\xi)\right|+O(1)\left|U_{\xi\xi}(t,\xi)\right|, \quad (2.23)$$

we have from (2.9), (2.19), Lemma 2.3 and Hausdorff-Young's inequality that for $t>\tau_1$

$$
\begin{aligned}
\|U_{\xi\xi}(t,\xi)\|_{L^2} &\le O(1)(t-\tau_1)^{-1}\|U(\tau_1,\xi)\|_{L^2} \\
&\quad + O(1)\int_{\tau_1}^{t}(t-s)^{-\frac{1}{2}}\left(\|U_\xi(s,\xi)\|_{L^2}+\|U_{\xi\xi}(s,\xi)\|_{L^2}\right)ds \\
&\le O(1)(t-\tau_1)^{-1}\tau_1^{-\frac{1}{2}}C_2(\tau_1)\|U_0(\xi)\|_{L^2} + O(1)\int_{\tau_1}^{t}C_2(s)(t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}\|U_0(\xi)\|_{L^2}ds \\
&\quad + O(1)\int_{\tau_1}^{t}(t-s)^{-\frac{1}{2}}\|U_{\xi\xi}(s,\xi)\|_{L^2}ds \\
&\le O(1)(t-\tau_1)^{-1}C(t,\tau_1)\|U_0(\xi)\|_{L^2} + O(1)\int_{\tau_1}^{t}(t-s)^{-\frac{1}{2}}\|U_{\xi\xi}(s,\xi)\|_{L^2}ds.
\end{aligned} \tag{2.24}
$$

Thus the singular Gronwall's inequality deduces

$$
\|U_{\xi\xi}(t,\xi)\|_{L^2} \le (t-\tau_1)^{-1}C(t,t-\tau_1,\tau_1)\|U_0(\xi)\|_{L^2}. \tag{2.25}
$$

Here $C(t,t-\tau_1,\tau_1)$ is a continuous, monotonically increasing function of t and $t-\tau_1$.

By (2.25), if we take $\tau_1 = \frac{\tau}{2}$ for each given $\tau > 0$, then we have

$$
\|U_{\xi\xi}(t,\xi)\|_{L^2} \le \left(t-\frac{\tau}{2}\right)^{-1} C\left(t,t-\frac{\tau}{2},\frac{\tau}{2}\right)\|U_0(\xi)\|_{L^2}, \tag{2.26}
$$

which shows (2.20) with $k=2$ and completes the proof of Lemma 2.4.

Our final result in this section is concerned with the weighted energy estimate on the solution $U(t,\xi)$ obtained in Lemma 2.2.

Lemma 2. 5 *In addition to the assumptions in Lemma 2.2, suppose further that* $U_0(\xi)\in L^2_{\overline{w}}(\mathbf{R})$, *then the solution* $U(t,\xi)$ *obtained in Lemma 2.2 satisfies*

$$
\left\|\sqrt{\overline{w}(\xi)}U(t,\xi)\right\|_{L^2} \le C_4(t)\left\|\sqrt{\overline{w}(\xi)}U_0(\xi)\right\|_{L^2} \tag{2.27}
$$

provided that the weighted function $\overline{w}(\xi)$ *satisfies*

$$
\left|\frac{\overline{w}'(\xi)}{\overline{w}(\xi)}\right| \le O(1)\overline{w}(\xi). \tag{2.28}
$$

Multiplying (2.6) by $\overline{w}(\xi)U(t,\xi)$ and integrating the resultant equation with respect to t and ξ over $[0,t]\times\mathbf{R}$. If (2.28) holds, then we can employ the Gronwall inequality and obtain (2.27). Since this is a standard way, we omit the details.

Remark 2. 2 *It is easy to check that all the weighted functions used in our subsequent analysis satisfying (2.28).*

3 The proof of Theorem 1.3

In this section we devote ourselves to the proof of our main result, Theorem 1.3. The non-degenerate shock case can be treated easier than the degenerate shock case. Hence we deal with the case $s = f'(u_+) < f'(u_-)$. Without loss of generality, we assume $u_+ < u_-$ and $h(\phi) < 0$ for $\phi \in (u_+, u_-)$. Consequently, there is a unique number $\xi_* \in \mathbf{R}$ such that

$$\phi(\xi_*) = \overline{u} := \frac{u_+ + u_-}{2}. \tag{3.1}$$

To overcome the nonconvexity of $f(u)$, as in [13], the weight $w(\xi)$ is chosen as

$$w(\phi) := \frac{(\phi - u_+)(\phi - u_-)}{h(\phi)}. \tag{3.2}$$

It is easy to find that

$$w(\phi(\xi)) \sim \begin{cases} C, & \text{if } f'(u_+) < s < f'(u_-) \\ \xi & \text{if } f'(u_\pm) = s \end{cases} \tag{3.3}$$

as $\xi \to \pm\infty$ and

$$\frac{d^2}{d\phi^2}\Big(h(\phi)w(\phi)\Big) = 2. \tag{3.4}$$

For the weight function $w(\xi)$ chosen above, we have the following basic energy estimates.

Lemma 3. 1 *Let $U(t,\xi)$ be the solution of the Cauchy problem (2.6), (2.7) obtained in Lemma 2.5, then it follows that*

$$\frac{1}{2}|U(t)|^2_{w(\phi)} + \int_{T_1}^t \left\|\sqrt{-\phi_\xi}U(s)\right\|^2 ds + \left(1 - C_5 \sup_{[T_1,t]} \|U(t)\|_{L^\infty}\right)\int_{T_1}^t |U_\xi(s)|^2_{w(\phi)} ds \le C_6(T_1). \tag{3.5}$$

Proof: Multiplying (2.6) by $w(\phi(\xi))U(t,\xi)$, we have

$$\begin{aligned} &\left(\tfrac{1}{2}w(\phi)U^2(t)\right)_t + \left(\tfrac{1}{2}(wh)'(\phi)U^2(t) - w(\phi)U(t)U_\xi(t)\right)_\xi \\ &+ w(\phi)U_\xi^2(t) - \tfrac{1}{2}(wh)''(\phi)\phi_\xi U^2(t) = w(\phi)U(t)F(t). \end{aligned} \tag{3.6}$$

Here we have used the fact that $\phi_\xi(\xi) = h(\phi(\xi))$.

Noticing $\phi_\xi(\xi) < 0$ and $F(t,\xi) = O(1)\,|U_\xi(t,\xi)|^2$, we can get (3.5) from (2.27) and (3.4) immediately by integrating (3.6) with respect to t and ξ over $[T_1, t] \times \mathbf{R}$. This completes the proof of Lemma 3.1.

The next lemma is concerned with the improvement of the estimate (3.5)

Lemma 3. 2 *For $0 < \beta \le \alpha < \frac{2}{n}(n \ge 1)$, we have that the solution $U(t,\xi)$ of the Cauchy problem (2.6), (2.7) satisfies*

$$
\begin{aligned}
&\int_{\mathbf{R}} w(\phi)^{1+\beta}U^2(t)d\xi + \int_{T_1}^{t}\int_{\xi>0} w(\phi)^{\beta-1}U^2(s)d\xi ds \\
&\quad + \left(1 - C_7 \sup_{[T_1,t]} \|U(t)\|_{L^\infty}\right) \int_{T_1}^{t}\int_{\mathbf{R}} w(\phi)^{\beta}|\phi_\xi|U^2(s)d\xi ds \\
&\quad + \int_{T_1}^{t}\int_{\mathbf{R}} w(\phi)^{1+\beta}U_\xi^2(s)d\xi ds \le C_8(T_1) + C_9 \sup_{[T_1,t]} \|U(t)\|_{L^\infty} \int_{T_1}^{t} |U_\xi(s)|^2_{w(\phi)} ds.
\end{aligned} \tag{3.7}
$$

Proof: The proof of Lemma 3.2 follows essentially the arguments developed by A. Matsumura and K. Nishihara in [13]. Thus, we only give a sketch of the proof, and the difference between our arguments and those in [13] will be emphasized.

Multiplying (2.6) by $2w(\phi)^{1+\beta}U(t)$, we have similar to the proof of Lemma 6.1 in [13] that

$$
\begin{aligned}
&\left(w(\phi)^{1+\beta}U^2(t)\right)_t + (\cdots)_\xi + 2(1-\varepsilon)w(\phi)^{1+\beta}U_\xi^2(t) \\
&\quad +2\left\{ - 2w(\phi)^{\beta}\phi_\xi + \beta w(\phi)^{\beta-1}h(\phi)\left(2(\overline{u}-\phi) - \tfrac{\beta w'(\phi)h(\phi)}{2\varepsilon}\right)\right\}U^2(t) \\
&\le 2w(\phi)^{1+\beta}|U(t)F(t)|.
\end{aligned} \tag{3.8}
$$

Here $\varepsilon \in (0,1)$ is an arbitrarily chosen constant.

On the other hand, if $\delta = \phi(\xi) - u_+ > 0$ and $\tilde{u} = u_- - u_+ > 0$, then

$$
\begin{aligned}
I(\xi) &:= \beta w(\phi(\xi))^{\beta-1}w'(\phi(\xi))h(\phi(\xi))\left(2(\overline{u}-\phi(\xi)) - \tfrac{\beta w'(\phi(\xi))h(\phi(\xi))}{2\varepsilon}\right) \\
&= \beta w(\phi(\xi))^{\beta-1}(\tilde{u}n + O(\delta))\left(\tilde{u}\left(1-\tfrac{\beta n}{2\epsilon}\right) + O(\delta)\right)
\end{aligned} \tag{3.9}
$$

as $\xi \to +\infty$.

Since $\beta \le \alpha < \frac{2}{n}$, we can always choose $\varepsilon \in (0,1)$ such that $1 - \frac{\beta n}{2\varepsilon} > 0$. Consequently, there are positive constants C_{10} and R_1 such that

$$
I(\xi) \ge C_{10} \quad \text{for} \quad \xi \ge R_1. \tag{3.10}
$$

Noticing also $C^{-1} \le w(\phi(\xi)) \le C, C^{-1} \le w'(\phi(\xi)) \le C$ as $\xi \to -\infty$, we have from (3.5) that

$$
\begin{aligned}
\int_{T_1}^{t}\int_{\xi\le R_1} 2I(\xi)U^2(s,\xi)d\xi ds &\le O(1)\int_{T_1}^{t} |\phi_\xi(\xi)|U^2(s,\xi)d\xi ds \\
&\le C(T_1) + O(1)\sup_{[T_1,t]}\|U(t,\xi)\|_{L^\infty}\int_{T_1}^{t} |U_\xi(s,\xi)|^2_{w(\phi)} ds
\end{aligned} \tag{3.11}
$$

and

$$
\int_{T_1}^{t}\int_{\mathbf{R}} w(\phi(\xi))^{1+\beta}|U(s,\xi)F(s,\xi)|d\xi ds \le O(1)\sup_{[T_1,t]}\|U(t,\xi)\|_{L^\infty}\int_{T_1}^{t}\int_{\mathbf{R}} w(\phi(\xi))^{1+\beta}U_\xi^2(s,\xi)d\xi ds. \tag{3.12}
$$

Integrating (3.8) with respect to t and ξ over $[T_1,t]\times\mathbf{R}$, we can immediately get (3.7) from (3.10)-(3.12). This completes the proof of Lemma 3.2.

Lemma 3. 3 *For each given $\alpha > 0$, the solution $U(t,\xi)$ to the Cauchy problem (2.6), (2.7) satisfies for $\beta \in [0,\alpha]$*

$$\begin{aligned}
&(1+t)^\gamma |U(t)|^2_{\beta,w(\phi)} + \left(1 - C_{11}\sup_{[T_1,t]}\|U(t,\xi)\|_{L^\infty}\right)\int_{T_1}^t (1+s)^\gamma |U_\xi(s)|^2_{\beta,w(\phi)}ds \\
&\quad +\beta\int_{T_1}^t (1+s)^\gamma |U(s)|^2_{\beta-1}ds \\
&\le C_{12}(T_1)\Big\{1+\gamma\int_{T_1}^t (1+s)^{\gamma-1}|U(s)|^2_{\beta,w(\phi)}ds \\
&\quad +\beta\int_{T_1}^t (1+s)^\gamma \int_{\mathbf{R}} \langle \xi-\xi_*\rangle^{\beta-1} w(\phi(\xi))|U(s,\xi)U_\xi(s,\xi)|d\xi ds\Big\}.
\end{aligned} \tag{3.13}_{\gamma,\beta}$$

Proof: Putting $\langle \xi-\xi_*\rangle := \sqrt{1+(\xi-\xi_*)^2}$ and multiplying (2.6) by $2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w(\phi(\xi))U(t,\xi)$, we get

$$\begin{aligned}
&\Big((1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w(\phi)U^2(t)\Big)_t + \Big(2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta U(t)U_\xi(t)\,(w(\phi)+(wh)'(\phi))\Big)_\xi \\
&\quad +2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w(\phi)U_\xi^2 - \gamma(1+t)^{\gamma-1}\langle\xi-\xi_*\rangle^\beta w(\phi)U^2(t) + (1+t)^\gamma\langle\xi-\xi_*\rangle^{\beta-1}A_\beta(\xi)U^2(t) \\
&\quad +2\beta(1+t)^\gamma\langle\xi-\xi_*\rangle^{\beta-2}(\xi-\xi_*)w(\phi)U(t)U_\xi(t) = 2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w(\phi)U(t)F(t).
\end{aligned} \tag{3.14}$$

Here

$$\begin{aligned}
A_\beta(\xi) &:= -\langle\xi-\xi_*\rangle\phi_\xi(\xi)(wh)''(\phi(\xi)) - \beta\tfrac{\xi-\xi_*}{\langle\xi-\xi_*\rangle}(wh)'(\phi(\xi)) \\
&= -2\langle\xi-\xi_*\rangle\phi_\xi(\xi) - 2\beta\tfrac{\xi-\xi_*}{\langle\xi-\xi_*\rangle}(\phi(\xi)-\overline{u}).
\end{aligned} \tag{3.15}$$

Due to (3.1), there exists a positive constant C_0 independent of β such that

$$A_\beta(\xi) \ge C_0\beta \quad \text{for any } \xi\in\mathbf{R}. \tag{3.16}$$

Integrating (3.14) with respect to t and ξ over $[T_1,t]\times\mathbf{R}$, it is deduced by (3.16) that

$$\begin{aligned}
&(1+t)^\gamma|U(t)|^2_{\beta,w(\phi)} + 2\int_{T_1}^t(1+s)^\gamma|U_\xi(s)|^2_{\beta,w(\phi)}ds + C_0\beta\int_{T_1}^t(1+s)^\gamma|U(s)|^2_{\beta-1}ds \\
&\le (1+T_1)^\gamma|U(T_1)|^2_{\beta,w(\phi)} + \gamma\int_{T_1}^t(1+s)^{\gamma-1}|U(s)|^2_{\beta,w(\phi)}ds \\
&\quad +2\beta\int_{T_1}^t\int_{\mathbf{R}}(1+s)^\gamma\langle\xi-\xi_*\rangle^{\beta-1}w(\phi)|U(s)U_\xi(s)|d\xi ds \\
&\quad +2\int_{T_1}^t\int_{\mathbf{R}}(1+s)^\gamma\langle\xi-\xi_*\rangle^\beta w(\phi)|U(s)F(s)|d\xi ds.
\end{aligned} \tag{3.17}$$

Due to

$$\int_{T_1}^t\int_{\mathbf{R}}(1+s)^\gamma\langle\xi-\xi_*\rangle^\beta w(\phi)|U(s)F(s)|d\xi ds \le C_{11}\sup_{[T_1,t]}\|U(t,\xi)\|_{L^\infty}\int_{T_1}^t(1+s)^\gamma|U_\xi(s)|^2_{\beta,w(\phi)}ds, \tag{3.18}$$

we can get $(3.13)_{\gamma,\beta}$ immediately by substituting (3.18) into (3.17), which completes the proof of Lemma 3.3.

Now from the L^1-stability result, Theorem 1.1, by H. Freistühler and D. Serre in [1], we conclude that

$$\lim_{t\to\infty} \|U(t,\xi)\|_{L^\infty} \leq \lim_{t\to\infty} \int_{\mathbf{R}} |u(t,x) - \phi(x-st)|dx = 0. \tag{3.19}$$

Thus if we choose $\overline{T}_1$ sufficiently large such that

$$\sup_{[\overline{T}_1,\infty]} \|U(t,\xi)\|_{L^\infty} < \frac{1}{2}\min\left\{\frac{1}{C_5}, \frac{1}{C_7}, \frac{1}{C_{11}}\right\}, \tag{3.20}$$

then we have from (3.20) and Lemma 3.1-Lemma 3.3 that

Corollary 3. 1 *For $\overline{T}_1$ chosen as above and $0 < \beta \leq \alpha < \frac{2}{n}(n \geq 1)$, the solution $U(t,\xi)$ to the Cauchy problem (2.6), (2.7) satisfies for $t \geq \overline{T}_1$*

$$\frac{1}{2}|U(t)|^2_{w(\phi)} + \int_{\overline{T}_1}^t \left\|\sqrt{-\phi_\xi}U(s)\right\|^2 ds + \int_{\overline{T}_1}^t |U_\xi(s)|^2_{w(\phi)}ds \leq C_6(\overline{T}_1), \tag{3.21}$$

$$\begin{aligned}&\int_{\mathbf{R}} w(\phi)^{1+\beta}U^2(t)d\xi + \int_{\overline{T}_1}^t\int_{\xi>0} w(\phi)^{\beta-1}U^2(s)d\xi ds + \int_{\overline{T}_1}^t\int_{\mathbf{R}} w(\phi)^{\beta}|\phi_\xi|U^2(s)d\xi ds\\ &\quad+\int_{\overline{T}_1}^t\int_{\mathbf{R}} w(\phi)^{1+\beta}U_\xi^2(s)d\xi ds \leq C_8(\overline{T}_1)\end{aligned} \tag{3.22$_\beta$}$$

and

$$\begin{aligned}&(1+t)^\gamma|U(t)|^2_{\beta,w(\phi)} + \int_{\overline{T}_1}^t (1+s)^\gamma|U_\xi(s)|^2_{\beta,w(\phi)}ds + \beta\int_{\overline{T}_1}^t (1+s)^\gamma|U(s)|^2_{\beta-1}ds\\ &\leq C_{12}(\overline{T}_1)\Big\{1 + \gamma\int_{\overline{T}_1}^t (1+s)^{\gamma-1}|U(s)|^2_{\beta,w(\phi)}ds\\ &\quad+\beta\int_{\overline{T}_1}^t (1+s)^\gamma\int_{\mathbf{R}}\langle\xi-\xi_*\rangle^{\beta-1}w(\phi(\xi))|U(s,\xi)U_\xi(s,\xi)|d\xi ds\Big\}\end{aligned} \tag{3.23$_{\gamma,\beta}$}$$

The proof of (II) of Theorem 1.3 follows from (3.21), $(3.22)_\beta$ and $(3.23)_{\gamma,\beta}$, in a similar fashion to that in [13, 17]. For completeness, we give the outline.

First, letting $\gamma = 0$ and $\beta \leq \alpha$ in $(3.23)_{\gamma,\beta}$, we can estimate the corresponding last term as in the following

$$\begin{aligned}&\left|\text{last term in } (3.23)_{0,\beta}\right| \leq \tfrac{\beta}{2}\int_{\overline{T}_1}^t |U(s)|^2_{\beta-1}ds\\ &\quad+O(1)\int_{\overline{T}_1}^t\int_{\mathbf{R}}\langle\xi-\xi_*\rangle^{\beta-1}w(\phi(\xi))^2U_\xi^2(s,\xi)d\xi ds := I_1 + I_2.\end{aligned} \tag{3.24}$$

Noticing

$$w(\phi(\xi))\begin{cases}\sim\xi & \text{as } \xi\to+\infty,\\ \sim \text{Const.} & \text{as } \xi\to-\infty,\end{cases} \tag{3.25}$$

we can find two positive constants $R_2 > 0$ and $R_3 > 0$ such that

$$
\begin{aligned}
I_2 &\le \tfrac{1}{2}\int_{\overline{T}_1}^{t}\int_{\xi<-R_3}\langle\xi-\xi_*\rangle^{\beta} w(\phi)U_\xi^2(s)d\xi ds \\
&\quad +O(1)\int_{\overline{T}_1}^{t}\int_{\xi>R_2} w(\phi)^{\beta+1}U_\xi^2(s)d\xi ds + O(1)\int_{\overline{T}_1}^{t}\int_{-R_3\le\xi\le R_2} U_\xi^2(s)d\xi ds \\
&\le \tfrac{1}{2}\int_{\overline{T}_1}^{t}|U_\xi(s)|^2_{\beta,w(\phi)}ds + O(1)\int_{\overline{T}_1}^{t}\int_{\mathbf{R}} w(\phi)^{1+\beta}U_\xi^2(s)d\xi ds + O(1)\int_{\overline{T}_1}^{t}\int_{\mathbf{R}} U_\xi^2(s)d\xi ds \\
&\le C(\overline{T}_1) + \tfrac{1}{2}\int_{\overline{T}_1}^{t}|U_\xi(s)|^2_{\beta,w(\phi)}ds.
\end{aligned}
\tag{3.26}
$$

Here we have used (3.21) and $(3.22)_\beta$.

Substituting (3.26) and (3.24) into $(3.23)_{0,\beta}$ and letting $\beta=\alpha$, we have for $\alpha < \frac{2}{n}(n\ge 1)$ that

$$
|U(t)|^2_{\alpha,w(\phi)} + \int_{\overline{T}_1}^{t}\left(|U(s)|^2_{\alpha-1} + |U_\xi(s)|^2_{\alpha,w(\phi)}\right)ds \le C_{13}(\overline{T}_1)
\tag{3.27}
$$

provided that $t \ge \overline{T}_1$.

Next, we consider $(3.23)_{\gamma,\beta}$ with $\gamma = \frac{\alpha}{2}+\varepsilon$ and $\beta = 0$

$$
\begin{aligned}
&(1+t)^{\frac{\alpha}{2}+\varepsilon}|U(t)|^2_{w(\phi)} + \int_{\overline{T}_1}^{t}(1+s)^{\frac{\alpha}{2}+\varepsilon}|U_\xi(s)|^2_{w(\phi)}ds \\
&\quad\le C_{12}(\overline{T}_1)\Big(1+\int_{\overline{T}_1}^{t}(1+s)^{\frac{\alpha}{2}+\varepsilon-1}|U(s)|^2_{w(\phi)}ds\Big).
\end{aligned}
\tag*{$(3.23)_{\frac{\alpha}{2}+\varepsilon,0}$}
$$

Here $\varepsilon > 0$ is chosen sufficiently small such that

$$
\varepsilon < \frac{\alpha}{2},\quad \frac{\alpha}{2}+\varepsilon < \frac{1}{n} \le 1.
$$

Since

$$
\begin{aligned}
&\int_{\overline{T}_1}^{t}(1+s)^{\frac{\alpha}{2}+\varepsilon-1}|U(s)|^2_{w(\phi)}ds \le \int_{\overline{T}_1}^{t}(1+s)^{\frac{\alpha}{2}+\varepsilon-1}\left(\int_{\xi>0}+\int_{\xi\le 0}\right)w(\phi)U^2(s,\xi)d\xi ds \\
&\quad\le O(1)\int_{\overline{T}_1}^{t}(1+s)^{\frac{\alpha}{2}+\varepsilon-1}\int_{\xi>0}w(\phi)U^2(s)d\xi ds + O(1)\int_{\overline{T}_1}^{t}(1+s)^{\frac{\alpha}{2}+\varepsilon-1}\int_{\xi\le 0}U^2(s)d\xi ds \\
&\quad := J_1 + J_2,
\end{aligned}
\tag{3.28}
$$

we have from (3.25) and (3.22) that

$$
\begin{aligned}
J_1 &\le O(1)\int_{\overline{T}_1}^{t}(1+s)^{\frac{\alpha}{2}+\varepsilon-1}\left(\int_{\xi>0}w(\phi)^{1+\alpha}U^2(s)d\xi\right)^{\frac{2-\alpha}{2}}\left(\int_{\xi>0}w(\phi)^{\alpha-1}U^2(s)d\xi\right)^{\frac{\alpha}{2}}ds \\
&\le C(\overline{T}_1)\int_{\overline{T}_1}^{t}(1+s)^{\frac{\alpha}{2}+\varepsilon-1}\left(\int_{\xi>0}w(\phi)^{\alpha-1}U^2(s)d\xi\right)^{\frac{\alpha}{2}}ds \\
&\le C(\overline{T}_1)\left(\int_{\overline{T}_1}^{t}(1+s)^{-1+\frac{2\varepsilon}{2-\alpha}}ds\right)^{\frac{2-\alpha}{2}}\left(\int_{\overline{T}_1}^{t}\int_{\xi>0}w(\phi)^{\alpha-1}U^2(s)d\xi ds\right)^{\frac{\alpha}{2}} \\
&\le C(\overline{T}_1)(1+t)^{\varepsilon}.
\end{aligned}
\tag{3.29}
$$

As to J_2, if $\alpha \geq 1$ (consequently $n = 1$), we have from $\frac{\alpha}{2} + \varepsilon < 1$ that

$$J_2 \leq O(1) \int_{\overline{T}_1}^{t} \int_{\xi<0} U^2(s,\xi) d\xi ds \leq O(1) \int_{\overline{T}_1}^{t} |U(s)|_{\alpha-1}^2 ds \leq C(\overline{T}_1). \tag{3.30}$$

When $n \geq 2$ (consequently $\alpha < 1$), we have from (3.25) and (3.27) that

$$\begin{aligned}
J_2 &\leq O(1) \int_{\overline{T}_1}^{t} (1+s)^{\frac{\alpha}{2}+\varepsilon-1} \left(\int_{\xi<0} \langle \xi - \xi_* \rangle^{\alpha} U^2(s,\xi) d\xi \right)^{1-\alpha} \left(\int_{\xi<0} \langle \xi - \xi_* \rangle^{\alpha-1} U^2(s,\xi) d\xi \right)^{\alpha} ds \\
&\leq O(1) \int_{\overline{T}_1}^{t} 1+s)^{\frac{\alpha}{2}+\varepsilon-1} \left(\int_{\xi<0} \langle \xi - \xi_* \rangle^{\alpha} w(\phi) U^2(s,\xi) d\xi \right)^{1-\alpha} \left(\int_{\xi<0} \langle \xi - \xi_* \rangle^{\alpha-1} U^2(s,\xi) d\xi \right)^{\alpha} ds \\
&\leq C(\overline{T}_1) \int_{\overline{T}_1}^{t} (1+s)^{\frac{\alpha}{2}+\varepsilon-1} |U(s)|_{\alpha-1}^{2\alpha} ds \\
&\leq C(\overline{T}_1) \left(\int_{\overline{T}_1}^{t} (1+s)^{-\frac{1-\frac{\alpha}{2}-\varepsilon}{1-\alpha}} ds \right)^{1-\alpha} \left(\int_{\overline{T}_1}^{t} |U(s)|_{\alpha-1}^2 ds \right)^{\alpha} \leq C(\overline{T}_1)
\end{aligned} \tag{3.31}$$

since $\varepsilon < \frac{\alpha}{2}$.

Inserting (3.28)-(3.31) into $(3.23)_{\frac{\alpha}{2}+\varepsilon,0}$ deduces

$$(1+t)^{\frac{\alpha}{2}+\varepsilon} |U(t)|_{w(\phi)}^2 + \int_{\overline{T}_1}^{t} (1+s)^{\frac{\alpha}{2}+\varepsilon} |U_\xi(s)|_{w(\phi)}^2 ds \leq C_{14}(\overline{T}_1)(1+t)^{\varepsilon} \tag{3.32}$$

provided that $t \geq \overline{T}_1$. Thus we have the following lemma.

Lemma 3. 4 *Under the conditions (II) in Theorem 1.3, the solution $U(t,\xi)$ to the Cauchy problem (2.6), (2.7) satisfies (3.32) for any $t \geq \overline{T}_1$ and some sufficiently small $\varepsilon > 0$.*

Now we turn to get the decay rates for derivatives of $U(t,\xi)$. We first have

Lemma 3. 5 *In additional to the assumptions listed in Theorem 1.3, suppose that $f(u) \in C^k(\mathbf{R})$ for some $k \in \mathbf{Z}^+$, then, for each fixed $\tau > 0$, the solution $U(t,\xi)$ to the Cauchy problem (2.6), (2.7) satisfies*

$$\sup_{[\tau,\infty)} \left\| \frac{\partial^j}{\partial \xi^j} U(t,\xi) \right\|_{L^\infty} \leq C_{15}(\tau) < \infty, \quad j = 1, \cdots, k. \tag{3.33}$$

Proof: We only prove (3.33) for the case $k = 2$ since the rest can be treated similarly. For each $0 < \tau_1 < \tau_2 < \tau \leq t \leq T$, we have

$$\begin{cases} U_\xi(t,\xi) = K_\xi(t-\tau_1,\xi) * U(\tau_1,\xi) + \int_{\tau_1}^{t} K_\xi(t-s,\xi) * G(s,\xi) ds, \\ U_{\xi\xi}(t,\xi) = K_{\xi\xi}(t-\tau_2,\xi) * U(\tau_2,\xi) + \int_{\tau_2}^{t} K_\xi(t-s,\xi) * G_\xi(s,\xi) ds. \end{cases} \tag{3.34}$$

On the other hand, we have from the L^1-stability result obtained in [1] that

$$\|U(t,\xi)\|_{L^\infty} \leq O(1). \tag{3.35}$$

Consequently from (2.9), (3.34) and (3.35), we have by the iteration arguments used in Lemma 2.4 that for $\tau \le t \le T$

$$\left\| \frac{\partial^j}{\partial \xi^j} U(t,\xi) \right\|_{L^\infty} \le C_{16}(t-\tau_j;\tau_1,\cdots,\tau_{j-1}), \quad j=1,2. \tag{3.36}$$

Having obtained (3.36), we now turn to prove (3.33).

First we notice that (3.36) holds for each given τ_1, τ_2, T. Hence, for each fixed $\tau > 0$, letting $\overline{\tau}_2 = 2\overline{\tau}_1 = \frac{\tau}{2}, T = 2t_1$ (where $t_1 > \tau$ is an arbitrarily given positive constant), we have from (3.36) that

$$\sup_{[\tau,2t_1]} \left\| \frac{\partial^j}{\partial \xi^j} U(t,\xi) \right\|_{L^\infty} \le C_{17}(2t_1-\overline{\tau}_j;\overline{\tau}_1,\cdots,\overline{\tau}_{j-1}), \quad j=1,2. \tag{3.37}$$

Now suppose that for some $1 < m \in \mathbf{Z}^+$

$$\sup_{[\tau,(m+1)t_1]} \left\| \frac{\partial^j}{\partial \xi^j} U(t,\xi) \right\|_{L^\infty} \le C_{17}(2t_1-\overline{\tau}_j;\overline{\tau}_1,\cdots,\overline{\tau}_{j-1}), \quad j=1,2, \tag{3.38}$$

then it holds that

$$\sup_{[mt_1+\tau,(m+2)t_1]} \left\| \frac{\partial^j}{\partial \xi^j} U(t,\xi) \right\|_{L^\infty} \le C_{17}(2t_1-\overline{\tau}_j;\overline{\tau}_1,\cdots,\overline{\tau}_{j-1}), \quad j=1,2. \tag{3.39}$$

In fact, letting T, τ_1, τ_2 in (3.36) be equal to $(m+2)t_1, mt_1+\overline{\tau}_1, mt_1+\overline{\tau}_2$ respectively, we can get (3.39). By setting $t_1 = 2\tau$ and $C_{15}(\tau) = C_{17}\left(4\tau - \frac{j\tau}{4}; \frac{\tau}{4}, \cdots, \frac{(j-1)\tau}{4}\right)$, (3.33) follows easily. This completes the proof of Lemma 3.5.

Since

$$\begin{aligned} |U_\xi(t,\xi)|^2 &= |u(t,\xi)-\phi(\xi)|^2 \le \|(u(t,\xi)-\phi(\xi))_\xi\|_{L^\infty}\|u(t,\xi)-\phi(\xi)\|_{L^1} \\ &= \|U_{\xi\xi}(t,\xi)\|_{L^\infty}\|U(t,\xi)\|_{L^\infty}, \end{aligned}$$

we have from Lemma 3.5 and Theorem 1.1 that

$$\lim_{t\to\infty} \|U_\xi(t,\xi)\|_{L^\infty} = 0. \tag{3.40}$$

Furthermore, from Lemma 2.4, under the assumption that $U_0(\xi) \in L^2(\mathbf{R})$, we have

$$\|U(T_1,\xi)\|_{H^2} \le C_{18}(T_1) \tag{3.41}$$

for each given $T_1 > 0$. With (3.40) and (3.41), we also have the following lemma.

Lemma 3. 6 *Let $l = 1,2$ and assume that the conditions listed in Lemma 3.4 are satisfied, then it holds for any $t \ge \overline{T}_1$ and some sufficiently small $\varepsilon > 0$ that*

$$(1+t)^{\frac{\alpha}{2}+\varepsilon}\left\| \frac{\partial^l}{\partial \xi^l} U(t) \right\|^2 + \int_{\overline{T}_1}^t (1+s)^{\frac{\alpha}{2}+\varepsilon}\left\| \frac{\partial^l}{\partial \xi^l} U_\xi(s) \right\|^2 ds \le C_{19}(\overline{T}_1)(1+t)^{\varepsilon}. \tag{3.42}$$

Combining Lemma 3.4 with Lemma 3.6, we can deduce that

$$\begin{aligned}\sup_{x\in\mathbf{R}} |u(t,x)-\phi(x-st)| &= \sup_{x\in\mathbf{R}} |U_\xi(t,\xi)| \\ &\le C(\overline{T}_1)\|U_\xi(t)\|^{\frac{1}{2}} \|U_{\xi\xi}(t)\|^{\frac{1}{2}} \le C(\overline{T}_1)(1+t)^{-\frac{\alpha}{4}},\end{aligned}$$

which proves (II) of Theorem 1.3.

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2. Compressible flow with frictional damping

Boundary Effect on Asymptotic Behavior of Solutions to the p-System with Linear Damping

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Abstract

We consider the asymptotic behaviour of solutions to the p-system with linear damping on the half-line $\mathbf{R}_+ = (0, \infty)$

$$v_t - u_x = 0, \qquad u_t + p(v)_x = -\alpha u,$$

with the Dirichlet boundary condition $u|_{x=0} = 0$ or the Neumann boundary condition $u_x|_{x=0} = 0$. The initial date $(v_0, u_0)(x)$ has the constant state (v_+, u_+) at $x = \infty$. [L. Hsiao and T.-P. Liu, *Commun. Math. Phys.* **143**(1992), 599-605] have shown that the solution to the corresponding Cauchy problem behaves like diffusion wave, and [K. Nishihara, *J. Differential Equations* **131**, 171-188 and **137**, 384-395] has proved its optimal convergence rate.

Our main concern in this paper is the boundary effect. In the case of null-Dirichlet boundary condition on u, the solution (v, u) is proved to tend to $(v_+, 0)$ as t tends to infinity. Its optimal convergence rate is also obtained by using the Green function of the diffusion equation with constant coefficients. In the case of null-Neumann boundary condition on u, $v(0, t)$ is conservative and $v(0, t) \equiv v_0(0)$ by virtue of the first equation, so that $v(x, t)$ is expected to tend to the diffusion wave $\bar{v}(x, t)$ connecting $v_0(0)$ and v_+. In fact the solution $(v, u)(x, t)$ is proved to tend to $(\bar{v}(x, t), 0)$. In the special case $v_0(0) = v_+$, the optimal convergence rate is also obtained. However, it is not known in the case of $v_0(0) \neq v_+$.

1 Introduction

In this paper we consider the initial-boundary value problem for the p-system with linear damping:

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = -\alpha u, \quad x \in \mathbf{R}_+ = (0, \infty), \quad t > 0, \end{cases} \tag{1.1}$$

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with the initial data

$$(v,u)(x,0) = (v_0,u_0)(x) \to (v_+,u_+),\ v_+ > 0,\ \text{as } x \to \infty, \tag{1.2}$$

and with the Dirichlet boundary or the Neumann boundary condition. Eq. (1.1) models a one-dimensional compressible flow through porous media. Here, $v > 0$ is the specific volume, u is the velocity, the pressure p is a smooth function of v with $p > 0$, $p' < 0$, and α is a positive constant.

For the Cauchy problem to (1.1), the solutions were shown time-asymptotically behave like those of Darcy's law

$$\begin{cases} \bar{v}_t - \bar{u}_x &= 0 \\ p(\bar{v})_x &= -\alpha\bar{u}, \end{cases} \tag{1.3}$$

or

$$\begin{cases} \bar{v}_t = -\frac{1}{\alpha}p(\bar{v})_{xx} \\ p(\bar{v})_x = -\alpha\bar{u} \end{cases} \tag*{(1.3)$'$}$$

by Hsiao and Liu in [4, 5]. A better convergence rate and the optimal convergence rate when $v(\infty,0) = v(-\infty,0)$ were obtained by Nishihara in [14, 15] by the enegy method and the pointwise estimate. For the related problem, see [3, 6] and references therein. See also the book [2] by Hsiao.

Though the initial-boundary value problems on $\mathbf{R}_+$ to the equations of viscous conservation laws have been recently investigated by several authors [7, 8, 9, 12, 13, 16], there are few works on (1.1) as far as we know. Our results discussed below show that even for the case with boundary condition, the Dirichlet or the Neumann boundary condition at $x = 0$, the solutions of (1.3) capture the time-asymptotic behaviour of the solutions to (1.1). In the case of the Dirichlet boundary condition

$$u(0,t) = 0, \tag{1.4}$$

we show that the solution $(v,u)(x,t)$ converges to $(v_+,0)$ as $t \to \infty$. Futhermore, since the solution converges to a constant state, the analysis of [14, 15] can be applied and the optimal convergence rate are obtained. In the case of the Neumann boundary condition

$$u_x(0,t) = 0, \tag{1.5}$$

$(1.1)_1$(the first equation of (1.1)) heuristically yields $\frac{d}{dt}v(0,t) = 0$ and $v(0,t) = v_0(0)$. Hence, when $v_0(0) \neq v_+$, the solution $(v,u)(x,t)$ will be shown to converge to the profile $(\bar{v},\bar{u})$ of (1.3) in the form of $\bar{v} = \psi(\xi)$, $\xi = x/\sqrt{t+1}$, with $\psi(+\infty) = v_+$ and $\psi(0) = v_0(0)$. Eventually, if $v_0(0) = v_+$, $\bar{v}(x,t) \equiv v_+$, then the analysis in [14, 15] can also be applied and the optimal convergence rate is obtained.

Both problems are reformulated to the perturbed problems from the diffusion wave $(\bar{v},\bar{u})(x,t)$ and the auxiliary function $(\hat{v},\hat{u})(x,t)$, which are defined in a similar fashion to those in Hsiao and Liu [4]. These will be stated in later sections, respectively.

Here, we shortly mention the condition (1.5), which corresponds to the Dirichlet condition $v(0,t) = v_-$(given constant) on v from the discussion above. Recently (1.1) with (1.2) and $v(0,t) = g(t)$, $g(t) \to v_+$ has been considered by Marcati and Mei [10]. However, the case $g(t) \equiv v_-(\neq v_+)$ or $g(t) \to v_-(\neq v_+)$ is not treated there.

The content of our paper is as follows. After stating the notations, in Sec. 2 the problem with the Dirichlet boundary condition is reformulated and the results will be stated. In Subsec. 2.2 the proof will be given, and many parts rely on the papers [14, 15]. In Sec. 3 the Neumann boundary problem will be considered.

Notations. We denote several positive constants depending on $a, b, \cdots$ by $C_{a,b,\cdots}$ or only by C without confusion. For function spaces, $L^p = L^p(\mathbf{R}_+)\,(1 \le p \le \infty)$ is a usual Lebesgue space with the norm

$$||f||_{L^p} = (\int_{\mathbf{R}_+} |f(x)|^p dx)^{1/p}, \quad 1 \le p < \infty \quad \text{and} \quad ||f||_{L^\infty} = \sup_{\mathbf{R}_+} |f(x)|.$$

The L^2-norm on $\mathbf{R}_+$ is simply denoted by $\|\cdot\|$. $H^l\,(l \ge 0)$ denotes the usual l-th order Sobolev space on $\mathbf{R}_+$ with its norm

$$||f||_l = (\sum_{j=0}^{l} ||\partial_x^j f||^2)^{1/2}, \quad ||\cdot|| = ||\cdot||_0 = ||\cdot||_{L^2}.$$

2 The Case of the Dirichlet Boundary Condition

2.1 Reformulation of the Problem and Theorems

We first reformulate the problem (1.1), (1.2) with the Dirichlet boundary condition (1.4). Expecting

$$(v, u)(x, t) \to (v_+, 0), \quad t \to \infty, \tag{2.1}$$

we put $u_t \equiv 0$ to have (1.3) or (1.3)' with $u(0,t) = v_x(0,t) = 0, \;\; v(+\infty, t) = v_+$. Approximating this by the solution $\bar{v}(x,t)$ of

$$\bar{v}_t - \kappa \bar{v}_{xx} = 0, \;\; \bar{v}_x(0,t) = 0, \;\; \bar{v}(+\infty, t) = v_+, \tag{2.2}$$

or explicitely

$$\bar{v}(x,t) = v_+ + \frac{\delta_0}{\sqrt{4\kappa\pi(t+1)}} \exp\,(-\frac{x^2}{4\kappa(t+1)}), \tag{2.3}$$

where $\kappa := -p'(v_+)/\alpha > 0$ and δ_0 is defined by

$$\delta_0 = 2(\int_0^\infty (v_0(x) - v_+)dx - \frac{u_+}{\alpha}). \tag{2.4}$$

We set

$$\bar{u}(x,t) = -\frac{p'(v_+)}{\alpha}\bar{v}_x(x,t) = \kappa \bar{v}_x(x,t)$$

so that $\bar{u}|_{x=0} = 0$ because $\bar{v}_x|_{x=0} = 0$.

Thus, $(\bar{v}, \bar{u})(x,t)$, called the diffusion wave, satisfies

$$\begin{cases} \bar{v}_t - \bar{u}_x &= 0 \\ p'(v_+)\bar{v}_x &= -\alpha \bar{u} \\ \bar{u}|_{x=0} = 0, & (\bar{v}, \bar{u})|_{x=\infty} = (v_+, 0). \end{cases} \tag{2.5}$$

Next, expecting $u(+\infty,t)=u_+e^{-\alpha t}$, we define the auxiliary function $(\hat{v},\hat{u})(x,t)$ by

$$(\hat{v},\hat{u})(x,t)=(\frac{u_+m_0(x)}{-\alpha}e^{-\alpha t},\ u_+\int_0^x m_0(y)dy\cdot e^{-\alpha t}), \tag{2.6}$$

where m_0 is a smooth function with compact support such that

$$\int_0^\infty m_0(y)dy=1,\quad \operatorname{supp} m_0\subset \mathbf{R}_+. \tag{2.7}$$

Therefore, $(\hat{v},\hat{u})(x,t)$ satisfies

$$\begin{cases} \hat{v}_t-\hat{u}_x \qquad =0 \\ \hat{u}_t \qquad\qquad =-\alpha\hat{u} \\ \hat{u}|_{x=0}=0,\quad (\hat{v},\hat{u})|_{x=\infty}=(0,u_+e^{-\alpha t}). \end{cases} \tag{2.8}$$

Combining (1.1) with (2.5) and (2.8) we have

$$\begin{cases} (v-\bar{v}-\hat{v})_t-(u-\bar{u}-\hat{u})_x=0 \\ (u-\bar{u}-\hat{u})_t+(p(v)-p(\bar{v}))_x \\ \quad =-\alpha(u-\bar{u}-\hat{u})-\bar{u}_t+(p'(v_+)-p'(\bar{v}))\bar{v}_x. \end{cases} \tag{2.9}$$

By virtue of $(2.9)_1$ and (2.4)

$$\int_0^\infty (v-\bar{v}-\hat{v})(y,t)dy=\int_0^\infty (v_0(x)-v_+)dx-\frac{\delta_0}{2}-\frac{u_+}{\alpha}=0$$

and hence we reach the setting of perturbation

$$\begin{aligned} V(x,t)&=-\int_x^\infty (v-\bar{v}-\hat{v})(y,t)dy \\ z(x,t)&=u(x,t)-\bar{u}(x,t)-\hat{u}(x,t) \end{aligned} \tag{2.10}$$

and the reformulated problem, after the integration of $(2.9)_1$ once over (x,∞),

$$\begin{cases} V_t-z=0 \\ z_t+(p(V_x+\bar{v}+\hat{v})-p(\bar{v}))_x+\alpha z=-\bar{u}_t+(p'(v_+)-p'(\bar{v}))\bar{v}_x \\ (V,z)|_{t=0}=(V_0,z_0)(x) \\ \quad :=(-\int_x^\infty (v_0(y)-\bar{v}(y,0)-\hat{v}(y,0))dy, u_0(x)-\bar{u}(x,0)-\hat{u}(x,0)) \\ z|_{x=0}=0, \end{cases} \tag{RP}$$

or the linearized problem around $\bar{v}$

$$\begin{cases} V_t - z = 0 \\ z_t + (p'(\bar{v})V_x)_x + \alpha z = -F \\ (V,z)|_{t=0} = (V_0, z_0)(x), \quad z|_{x=0} = 0, \end{cases} \tag{LP}$$

where

$$F = \frac{p'(v_+)}{\alpha}\bar{v}_{xt} - (p'(v_+) - p'(\bar{v}))\bar{v}_x + (p(V_x + \bar{v} + \hat{v}) - p(\bar{v}) - p'(\bar{v})V_x)_x. \tag{2.11}$$

Noting that, by (2.4),

$$|\delta_0| \leq 2(\|v_0 - v_+\|_{L^1} + \frac{|u_+|}{\alpha}), \tag{2.12}$$

we obtain the following first theorem.

Theorem 2. 1 (Dirichlet boundary) *Suppose that $v_0 - v_+$ is in L^1, $(V_0, z_0) \in H^3 \times H^2$ and that both $\|v_0 - v_+\|_{L^1} + \|V_0\|_3 + \|z_0\|_2$ and $|u_+|$ are sufficiently small. Then there exists a unique time-global solution $(V,z)(x,t)$ of (RP), which satisfies*

$$V \in C^i([0,\infty);\, H^{3-i}), \quad i = 0,1,2,3$$

$$z \in C^i([0,\infty);\, H^{2-i}), \quad i = 0,1,2$$

and moreover

$$\begin{aligned} &\sum_{k=0}^{3}(1+t)^k\|\partial_x^k V(\cdot,t)\|^2 + \sum_{k=0}^{2}(1+t)^{k+2}\|\partial_x^k z(\cdot,t)\|^2 + \\ &+\int_0^t [\sum_{j=1}^{3}(1+\tau)^{j-1}\|\partial_x^j V(\cdot,\tau)\|^2 + \sum_{j=0}^{2}(1+\tau)^{j+1}\|\partial_x^j z(\cdot,\tau)\|^2]d\tau \\ &\leq C(\|V_0\|_3^2 + \|z_0\|_2^2 + |\delta_0|), \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} &(1+t)^4\|z_t(\cdot,t)\|^2 + (1+t)^5(\|z_{xt}(\cdot,t)^2 + \|z_{tt}(\cdot,t)\|^2) + \\ &+\int_0^t [(1+\tau)^4\|z_{xt}(\cdot,\tau)\|^2 + (1+\tau)^5\|z_{tt}(\cdot,\tau)\|^2]d\tau \\ &\leq C(\|V_0\|_3^2 + \|z_0\|_2^2 + |\delta_0|). \end{aligned} \tag{2.14}$$

The solution (V,z) obtained in Theorem 2.1 satisfies $V|_{x=0} = 0$ by the first equation of (RP) and the boundary condition $z|_{x=0} = 0$, and hence (RP) or (LP) can be rewritten as the problem to the second order wave equation of V with linear damping

$$\begin{cases} V_{tt} + (p'(\bar{v})V_x)_x + \alpha V_t = -F \\ (V, V_t)|_{t=0} = (V_0, z_0)(x), \quad V|_{x=0} = 0. \end{cases} \tag{2.15}$$

Moreover, we rewrite $(2.15)_1$ to the linearized parabolic problem around v_+

$$V_t - \kappa V_{xx} = -\frac{1}{\alpha}(V_{tt} + \tilde{F}) + \frac{1}{\alpha}(p'(v_+) - p'(\bar{v}))\bar{v}_x, \quad \kappa = -\frac{p'(v_+)}{\alpha}, \tag{2.16}$$

to use the Green function of the parabolic equation with null-Dirichlet boundary

$$E(x,t;y) = \frac{1}{\sqrt{4\pi\kappa t}}(e^{-\frac{(x-y)^2}{4\kappa t}} - e^{-\frac{(x+y)^2}{4\kappa t}}), \tag{2.17}$$

where

$$\tilde{F} = \frac{p'(v_+)}{\alpha}\bar{v}_{xt} + (p(V_x + \bar{v} + \hat{v}) - p(\bar{v}) - p'(\bar{v})V_x)_x - ((p'(v_+) - p'(\bar{v}))V_x)_x. \tag{2.18}$$

Hence we have the explicit formula of V:

$$\begin{aligned} V(x,t) = &\int_0^\infty E(x,t;y)V_0(y)dy - \frac{1}{\alpha}\int_0^t\int_0^\infty E(x,t-\tau;y)(V_{tt} + \tilde{F})(y,\tau)dyd\tau \\ &+\frac{1}{\alpha}\int_0^t\int_0^\infty E(x,t-\tau;y)(p'(v_+) - p'(\bar{v}))\bar{v}_x(y,\tau)dyd\tau. \end{aligned} \tag{2.19}$$

Define $\phi(x,t)$ by

$$\begin{aligned} \phi(x,t) = &\int_0^\infty E(x,t;y)(V_0(y) + \frac{1}{\alpha}z_0(y))dy \\ &+\tfrac{1}{\alpha}\int_0^t\int_0^\infty E(x,t-\tau;y)(p'(v_+) - p'(\bar{v}))\bar{v}_x(y,\tau)dyd\tau \end{aligned} \tag{2.20}$$

or the solution of

$$\begin{cases} \phi_t - \kappa\phi_{xx} = \frac{1}{\alpha}(p'(v_+) - p'(\bar{v}))\bar{v}_x, \quad (x,t) \in \mathbf{R}_+ \times \mathbf{R}_+ \\ \phi(x,0) = V_0(x) + \frac{1}{\alpha}z_0(x), \quad \phi(0,t) = 0. \end{cases} \tag{2.20$'$}$$

Then we have the asymptotic profile ϕ of V as $t \to \infty$ in the sense of the following theorem.

Theorem 2. 2 (Asymptotic Profile) *Define ϕ by (2.20) or (2.20)' and suppose that $(V_0, z_0) \in L^1 \times L^1$. Then the solution (V,z) of (RP) obtained in Theorem 2.1 satisfies*

$$\|(V-\phi,\ (V-\phi)_x,\ (V-\phi)_t)(\cdot,t)\|_{L^\infty} = O(t^{-1}\ln t, t^{-3/2}\ln t, t^{-2}\ln t) \tag{2.21}$$

as $t \to \infty$.

Remark 2.1 Since ϕ satisfies

$$\|(\phi,\phi_x,\phi_t)(\cdot,t)\|_{L^\infty} = O(t^{-1/2}, t^{-1}, t^{-3/2}), \tag{2.22}$$

ϕ is generally an asymptotic profile of V as $t \to \infty$, which is on the same line of assertions in [15]. However, in the present case we have the slightly worse term $-((p'(v_+) - p'(\bar{v}))V_x)_x$ in $\tilde{F}$ and hence $\ln t$ in (2.21) are added.

Remark 2.2 All results are obtained under the condition that any data are small. For large data the singularity will generally develop after a finite time and the weak solution must be considered. In such cases the asymptotic behavior of the solutions of (1.1) is unknown in general even for Cauchy problem.

2.2 Proofs of Theorems

First, applying the L^2-energy method we prove Theorem 2.1, which is established by the combination of the local existence result with a priori estimates. For the local existence of the solution (V, z) to (RP) see e.g. Matsumura [11] and references therein.

We now devote ourselves to the a priori estimates of the solution $(V, z)(x, t),\ 0 < t < T$, to the linearized equation (LP) under the a priori assumption

$$N(T) := \sup_{0<t<T} \{\sum_{k=0}^{3} (1+t)^k \|\partial_x^k V(\cdot, t)\|^2 + \sum_{k=0}^{2} (1+t)^{k+2} \|\partial_x^k z(\cdot, t)\|^2\} \leq \varepsilon. \tag{2.23}$$

Since it suffices to establish the estimates for sufficiently smooth solution, the equations in (RP) and $z|_{x=0} = 0$ gives the following boundary conditions for higher order derivatives:

$$V(0, t) = V_{xx}(0, t) = V_t(0, t) = V_{txx}(0, t) = 0 \ \text{ etc.}$$

Therefore, estimates obtained below are formally quite similar to those in Sec. 3 of [14]. The difference between (LP) in [14] and (LP) in this paper is the second term of F:

$$h(x, t) := -(p'(v_+) - p'(\bar{v}))\bar{v}_x \tag{2.24}$$

Since $h(x, t) = O(1)(\bar{v} - v_+)\bar{v}_x$, following decay properties hold:

$$\begin{aligned} &\int_0^\infty |h(x, t)|^2 dx \leq C\delta_0^4 (1+t)^{-5/2} \\ &\int_0^\infty |h_x(x, t)|^2 dx \leq C\delta_0^4 (1+t)^{-7/2}, \ \text{ etc.}, \end{aligned} \tag{2.25}$$

decay rates of which are the same as those in (LP) of [14]. Note that the first term decays faster than those in (2.25). Hence, we briefly repeat the lemmas.

Multiplying $(\text{LP})_2$ by $z + \lambda V (0 < \lambda \ll 1)$ and using $(\text{LP})_1$, we have the first lemma.

Lemma 2. 1 *If $N(T) \leq \varepsilon$ and $|\delta_0|$ are small, then*

$$||V(t)||_1^2 + ||z(t)||^2 + \int_0^t (||V_x(\tau)||^2 + ||z(\tau)||^2) d\tau \leq C(||V_0||_1^2 + ||z_0||^2 + |\delta_0|).$$

Multiplying $(\text{LP})_2$ by $(1+t)z$ and applying Lemma 2.1, we have the second lemma.

Lemma 2. 2 *If $\varepsilon + |\delta_0| \ll 1$, then*

$$(1+t)(||V_x(t)||^2 + ||z(t)||^2) + \int_0^t (1+\tau)||z(\tau)||^2 d\tau \leq C(||V_0||_1^2 + ||z_0||^2 + |\delta_0|).$$

Next, differentiate $(\text{LP})_2$ with respect to x to obtain

$$z_{xt} + (p'(\bar{v})V_x)_{xx} + \alpha z_x = -F_x. \tag{2.26}$$

Multiplying (2.26) by $(1+t)^k (z_x - \lambda V_{xx})\ (0 < \lambda \ll 1),\ \ k = 0, 1$, we have

$$\begin{aligned} &(1+t)(||V_x(t)||_1^2 + ||z_x(t)||^2) + \int_0^t (1+\tau)(||V_{xx}(\tau)||^2 + ||z_x(\tau)||^2) d\tau \\ &\leq C(||V_0||_2^2 + ||z_0||_1^2 + |\delta_0|). \end{aligned} \tag{2.27}$$

Again, multiplying (2.26) by $(1+t)^2 z_x$ and applying (2.27), we have

$$(1+t)^2(||V_{xx}(t)||^2 + ||z_x(t)||^2) + \int_0^t (1+\tau)^2 ||z_x(\tau)||^2 d\tau \le C(||V_0||_2^2 + ||z_0||_1^2 + |\delta_0|), \tag{2.28}$$

which gives the third lemma together with (2.27).

Lemma 2. 3 *If* $\varepsilon + |\delta_0| \ll 1$, *then*

$$(1+t)^2(||V_{xx}(t)||^2 + ||z_x(t)||^2) + \int_0^t [(1+\tau)||V_{xx}(\tau)||^2 + (1+\tau)^2||z_x(\tau)||^2]d\tau$$
$$\le C(||V_0||_2^2 + ||z_0||_1^2 + |\delta_0|).$$

Similar procedure applied to the equation obtained by defferentiating (2.26) with respect to x once more yields

Lemma 2. 4 *If* $\varepsilon + |\delta_0| \ll 1$, *then*

$$(1+t)^3(||V_{xxx}(t)||^2 + ||z_{xx}(t)||^2) + \int_0^t [(1+\tau)^2||V_{xxx}(\tau)||^2 + (1+\tau)^3||z_{xx}(\tau)||^2]d\tau$$
$$\le C(||V_0||_3^2 + ||z_0||_2^2 + |\delta|_0).$$

Proceeding the same procedure as above to

$$z_{tt} + (p'(\bar{v})V_x)_{xt} + \alpha z_t = -F_t, \tag{2.29}$$

we have the following two lemmas.

Lemma 2. 5 *If* $\varepsilon + |\delta_0| \ll 1$, *then*

$$(1+t)^2||z(t)||^2 + (1+t)^3(||z_x(t)||^2 + ||z_t(t)||^2)$$
$$+\int_0^t [(1+\tau)^2||z_x(\tau)||^2 + (1+\tau)^3||z_t(\tau)||^2]d\tau$$
$$\le C(||V_0||_2^2 + ||z_0||_1^2 + |\delta_0|).$$

Lemma 2. 6 *If* $\varepsilon + |\delta_0| \ll 1$, *then*

$$(1+t)^4(||z_{xx}(t)||^2 + ||z_{xt}(t)||^2) + \int_0^t [(1+\tau)^3||z_{xx}(\tau)||^2 + (1+\tau)^4||z_{xt}(\tau)||^2]d\tau$$
$$\le C(||V_0||_3^2 + ||z_0||_2^2 + |\delta_0|).$$

The estimates obtained in the series of the above six lemmas show (2.13).
To obtain (2.14), we differentiate (2.29) with respect to t once more:

$$z_{ttt} + (p'(\bar{v})V_x)_{xtt} + \alpha z_{tt} = -F_{tt}$$

or

$$
\begin{aligned}
&z_{ttt} + (p'(\bar{v})z_{xt})_x + \alpha z_{tt} \\
&= -F_{tt} - (2p''(\bar{v})\bar{v}_t z_x + (p''(\bar{v})\bar{v}_{tt} + p'''(\bar{v})\bar{v}_t^2)V_x)_x \\
&:= -F_{tt} - \tilde{P}_x.
\end{aligned}
\tag{2.30}
$$

We proceed the same procedure as above to (2.30), that is, multiply (2.30) by $(1+t)^k(z_{tt}+\lambda z_t)$ $(0 < \lambda \ll 1)$, $k = 0, 1, \cdots, 4$ and use Lemmas 2.1-2.6. Then we have

$$
\begin{aligned}
&(1+t)^4(\|z_t(t)\|_1^2 + \|z_{tt}(t)\|^2) + \int_0^t (1+\tau)^4(\|z_{xt}(\tau)\|^2 + \|z_{tt}(\tau)\|^2)d\tau \\
&\leq C(\|V_0\|_3^2 + \|z_0\|_2^2 + |\delta_0|).
\end{aligned}
\tag{2.31}
$$

Since $\int_0^t (1+\tau)^5 \int_0^\infty |F_{tt} + \tilde{P}_x|^2 dx d\tau \leq C(\|V_0\|_3^2 + \|z_0\|_2^2 + |\delta_0|)$ is shown by (2.31) after tedious calculations, multiplying of (2.30) by $(1+t)^5 z_{tt}$ and using of (2.31) yield

$$
\begin{aligned}
&(1+t)^5(\|z_{xt}(t)\|^2 + \|z_{tt}(t)\|^2) + \int_0^t (1+\tau)^5 \|z_{tt}(\tau)\|^2 d\tau \\
&\leq C(\|V_0\|_3^2 + \|z_0\|_2^2 + |\delta_0|),
\end{aligned}
$$

which shows (2.14) together with (2.31).

Thus we have completed the proof of Theorem 2.1.

We now turn to the L^∞-estimate assuming that $(V_0, z_0) \in L^1 \times L^1$. The proof is similar to that in [15].

First we show (2.22). The first term of right-hand side in (2.20) clearly satisfies (2.22) since $(V_0, z_0) \in L^1 \times L^1$. The last term is estimated by (2.3) as follows:

$$
\begin{aligned}
&|\text{the last term in (2.21)}| \\
&\leq |\tfrac{1}{\alpha}\int_0^{t/2}\int_0^\infty E(x, t-\tau; y)(p'(v_+)\bar{v} - \int_{v_+}^{\bar{v}} p'(s)ds)_y(y,\tau)dyd\tau| \\
&\quad + C\int_{t/2}^t \int_0^\infty E(x, t-\tau; y)|\bar{v} - v_+||\bar{v}_x|dyd\tau \\
&\leq C\int_0^{t/2} \|E(t-\tau)\|_{L^\infty}\|(\bar{v}-v_+)(\tau)\|^2 d\tau + C\int_{t/2}^t \|E(t-\tau)\|_{L^\infty}\|\bar{v}_x\|_{L^\infty}\|(\bar{v}-v_+)(\tau)\|_{L^1}d\tau \\
&\leq C(\int_0^{t/2}(t-\tau)^{-1}(1+\tau)^{-1/2}d\tau + \int_{t/2}^t (t-\tau)^{-1/2}(1+\tau)^{-1}d\tau) \leq C(1+t)^{-1/2}.
\end{aligned}
$$

Derivatives of ϕ are also estimated similarly.

Since

$$
\begin{aligned}
&-\tfrac{1}{\alpha}\int_0^{t/2}\int_0^\infty E(x,t-\tau;y)V_{tt}(y,\tau)dyd\tau \\
&=\tfrac{1}{\alpha}\int_0^\infty E(x,t;y)z_0(y)dy-\frac{1}{\alpha}\int_0^\infty E(x,t/2;y)z(y,t/2)dy \\
&-\tfrac{1}{\alpha}\int_0^{t/2}\int_0^\infty E_t(x,t-\tau;y)z(y,\tau)dyd\tau,
\end{aligned}
\tag{2.32}
$$

(2.19) and (2.20) with (2.32) give the expression

$$
\begin{aligned}
(V-\phi)(x,t) &= -\frac{1}{\alpha}\int_0^\infty E(x,t/2;y)z(y,t/2)dy \\
&-\tfrac{1}{\alpha}\int_0^{t/2}\int_0^\infty E_t(x,t-\tau;y)z(y,\tau)dyd\tau \\
&-\tfrac{1}{\alpha}\int_{t/2}^{t}\int_0^\infty E(x,t-\tau;y)z_t(y,\tau)dyd\tau \\
&-\tfrac{1}{\alpha}\Big(\int_0^{t/2}+\int_{t/2}^{t}\Big)\int_0^\infty E(x,t-\tau;y)\tilde{F}(y,\tau)dyd\tau \\
&:= I+II+III+(IV_1+IV_2).
\end{aligned}
\tag{2.33}
$$

Since the Green kernel E is given by (2.17), the following estimates hold:

$$
\begin{aligned}
|I| &\le C\|E(t/2)\|\|z(t/2)\|\le C(1+t)^{-\frac{1}{4}-1}, \\
|II| &\le \int_0^{t/2}\|E(t-\tau)\|\|z(\tau)\|d\tau \\
&\le \int_0^{t/2}(1+t-\tau)^{-5/4}(1+\tau)^{-1}d\tau\le C(1+t)^{-5/4}\ln(2+t) \\
|III| &\le C\int_{t/2}^{t}\|E(t-\tau)\|\|z_t(\tau)\|d\tau \\
&\le C\int_{t/2}^{t}(1+t-\tau)^{-1/4}(1+\tau)^{-2}d\tau\le C(1+t)^{-2+\frac{3}{4}}.
\end{aligned}
\tag{2.34}
$$

For IV_1 and IV_2 we recall that $\tilde{F}$ given by (2.18) has the form $\tilde{F}=f_x$. Hence

$$
\begin{aligned}
|IV_1| &\le C\int_0^{t/2}\int_0^{\infty}|E_y(x,t-\tau;y)||f(y,\tau)|dyd\tau \\
&\le C\int_0^{t/2}(t-\tau)^{-1}(\|\bar{v}_t(\tau)\|_{L^1}+\|\hat{v}(\tau)\|_{L^1}+\|V_x(\tau)\|^2+\|\hat{v}(\tau)\|^2 \\
&\quad +\|V_x(\tau)\|_{L^\infty}\|(\bar{v}-v_+)(\tau)\|_{L^1})d\tau \\
&\le Ct^{-1}\int_0^{t/2}((1+\tau)^{-1}+e^{-\alpha\tau}+(1+\tau)^{-3/4})d\tau\le C(1+t)^{-3/4}
\end{aligned}
\tag{2.35}
$$

and

$$
\begin{aligned}
|IV_2| &\le C\int_{t/2}^{t}\{\|E(t-\tau)\|(\|\bar{v}_{xt}(\tau)\|+\|\hat{v}_x(\tau)\|+\|(V_xV_{xx}+\hat{v}\hat{v}_x)(\tau)\|) \\
&\quad +\|E_y(t-\tau)\|\|V_x(\tau)\|_{L^\infty}\|(\bar{v}-v_+)(\tau)\|\}d\tau \\
&\le C\int_{t/2}^{t}(t-\tau)^{-1/4}((1+\tau)^{-7/4}+e^{-\alpha\tau})+(t-\tau)^{-3/4}(1+\tau)^{-3/4-1/4})d\tau \\
&\le C(1+t)^{-3/4}.
\end{aligned}
\tag{2.36}
$$

Combining (2.33) with (2.34)-(2.36) shows that $\|(V-\phi)(t)\|_{L^\infty}=O(t^{-3/4})$. Estimates of $\|(V-\phi)_x(t)\|_{L^\infty}=O(t^{-5/4})$ and $\|(V-\phi)_t(t)\|_{L^\infty}=O(t^{-7/4})$ are obtained in a similar fashion to the above. In particular, $\|(V-\phi)_x(t)\|_{L^\infty}=O(t^{-5/4})$ and (2.22) show that $\|V_x(t)\|_{L^\infty}=O(t^{-1})$. Applying this to (2.35) and (2.36) again, we have

$$|IV_1|\le C(1+t)^{-1}\ln(2+t) \tag{2.35$'$}$$

and

$$|IV_2|\le C(1+t)^{-1}, \tag{2.36$'$}$$

which gives the estimate $\|(V-\phi)(t)\|_{L^\infty}=O(t^{-1}\ln t)$. Derivatives of $V-\phi$ are also obtained, which yields the desired estimate (2.21).

3 The Case of the Neumann Boundary Condition

We now turn to the problem with the Neumann boundary condition (1.5)

$$
\begin{cases}
v_t-u_x=0, \quad (x,t)\in\mathbf{R}_+\times\mathbf{R}_+ \\
u_t+p(v)_x=-\alpha u \\
(v,u)|_{t=0}=(v_0,u_0)(x), \quad u_x|_{x=0}=0.
\end{cases}
\tag{3.1}
$$

Same as the preceding section we first reformulate (3.1). Heuristically, $(3.1)_1$ yields $\frac{d}{dt}v(0,t)=u_x(0,t)=0$ and $v(0,t)=v_0(0)$ for any $t>0$. Hence, we can expect that

$$(v,u)(x,t)\to(\bar{v},0)(x,t) \quad \text{as} \quad t\to\infty, \tag{3.2}$$

where $\bar{v}(x,t)$ is a diffusion wave connecting $v_0(0)$ and v_+.

In the case of $v_0(0) \neq v_+$, putting $u_t = 0$ in $(3.1)_2$ we have

$$u = -\frac{1}{\alpha}p(v)_x \quad \text{and} \quad v_t + \frac{1}{\alpha}p(v)_x = 0. \tag{3.3}$$

To construct the diffusion wave $(\bar{v}, \bar{u})$, it is known that for any constant $v_- > 0$ we have a self-similar solution $\tau = \psi(x/\sqrt{t+1})$ satifying

$$\begin{cases} \tau_t + \frac{1}{\alpha}p(\tau)_{xx} = 0, \quad x \in \mathbf{R} = (-\infty, \infty), \ t > 0 \\ \tau|_{x=\pm\infty} = v_\pm. \end{cases} \tag{3.4}$$

Therefore, for $v_0(0) > 0$ between v_- and v_+, there exists a unique $\bar{v}(x,t)$ in the form of $\psi(x/\sqrt{t+1})|_{x\geq 0}$ satisfying

$$\begin{cases} \bar{v}_t + \frac{1}{\alpha}p(\bar{v})_{xx} = 0, \quad (x,t) \in \mathbf{R}_+ \times \mathbf{R}_+ \\ \bar{v}|_{x=0} = v_0(0), \quad \bar{v}|_{x=\infty} = v_+. \end{cases} \tag{3.5}$$

For these results see [1]. Moreover, $\bar{u}$ is defined by

$$\bar{u}(x,t) = -\frac{1}{\alpha}p(\bar{v})_x$$

so that

$$\bar{u}_x|_{x=0} = \bar{v}_t|_{x=0} = \psi'(x/\sqrt{t+1})(-\frac{x}{2\sqrt{t+1}(t+1)})|_{x=0} = 0. \tag{3.6}$$

Thus we have had

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0 \\ \quad p(\bar{v})_x = -\alpha\bar{u} \\ (\bar{v}, \bar{u}_x)|_{x=0} = (v_0(0), 0), \quad (\bar{v}, \bar{u})|_{x=\infty} = (v_+, 0). \end{cases} \tag{3.7}$$

Similar to that in the Dirichlet boundary problem, the auxiliary function $(\hat{v}, \hat{u})(x,t)$ is defined by

$$(\hat{v}, \hat{u})(x,t) = (\frac{u_0(0) - u_+}{\alpha}m_0(x)e^{-\alpha t}, [(u_0(0) - u_+)\int_x^\infty m_0(y)dy + u_+]e^{-\alpha t}) \tag{3.8}$$

where m_0 is a smooth function satisfying (2.7). Hence $(\hat{v}, \hat{u})$ satisfies

$$\begin{cases} \hat{v}_t - \hat{u}_x = 0 \\ \hat{u}_t \qquad = -\alpha\hat{u} \\ (\hat{u}, \hat{u}_x)|_{x=0} = (u_0(0)e^{-\alpha t}, 0), \quad \hat{v}|_{x=0} = 0 \\ (\hat{v}, \hat{u})|_{x=\infty} = (0, u_+e^{-\alpha t}). \end{cases} \tag{3.9}$$

Combining (3.1) with (3.7) and (3.9) we have

$$\begin{cases} (v-\bar{v}-\hat{v})_t-(u-\bar{u}-\hat{u})_x=0 \\ (u-\bar{u}-\hat{u})_t+(p(v)-p(\bar{v}))_x=-\alpha(u-\bar{u}-\hat{u})-\bar{u}_t \\ (u-\bar{u}-\hat{u})_x|_{x=0}=0 \\ (v-\bar{v}-\hat{v},u-\bar{u}-\hat{u})|_{t=0}=(v_0,u_0)(x)-(\bar{v}+\hat{v},\bar{u}+\hat{u})(x,0) \end{cases} \tag{3.10}$$

Defining the perturbation by

$$\begin{cases} V(x,t)=-\int_x^\infty (v-\bar{v}-\hat{v})(y,t)dy \\ z(x,t)=(u-\bar{u}-\hat{u})(x,t), \end{cases} \tag{3.11}$$

we have the reformulated problem, after the integration of $(3.10)_1$ once over (x,∞),

$$\begin{cases} V_t-z=0 \\ z_t+(p(V_x+\bar{v}+\hat{v})-p(\bar{v}))_x=-\alpha z-\bar{u}_t \\ z_x|_{x=0}=0 \quad (\text{or} \quad V_x|_{x=0}) \\ (V,z)|_{t=0}=(V_0,z_0)(x) \\ \quad :=(-\int_x^\infty (v_0(y)-\bar{v}(y,0)-\hat{v}(y,0)dy, u_0(x)-\bar{u}(x,0)-\hat{u}(x,0)), \end{cases} \tag{NRP}$$

or the second order wave equation of V with damping

$$\begin{cases} V_{tt}+(p(V_x+\bar{v}+\hat{v})-p(\bar{v}))_x+\alpha V_t=-\bar{u}_t \\ V_x|_{x=0}=0, \quad (V,V_t)|_{t=0}=(V_0,z_0). \end{cases} \tag{3.12}$$

Note that, if (V,z) is sufficiently smooth in x,t, (3.10) or (3.12) yields the boundary conditions at $x=0$

$$V_x=V_{tx}=V_{ttx}=p(V_x+\bar{v}+\hat{v})-p(\bar{v})=(p(V_x+\bar{v}+\hat{v})-p(\bar{v}))_{xx}=0, \quad \text{etc.}$$

Therefore, once we have the smooth solutions, we can treat them formally the same as those in the Cauchy problem in [14]. The diffusion wave $\bar{v}$ defined in (3.6) has the same behavior as the self-similar solution τ defined in (3.5). For the diffusion wave see [1] and [4, 14]. Hence, the same L^2-estimates for the local smooth solution (V,z) to (NRP) are obtained. Thus we have the following theorem.

Theorem 3. 1 (The case of $v_0(0)\neq v_+$) *Suppose that v_0-v_+ is in $L^1(\mathbf{R}_+)$ and both $\|V_0\|_3+\|z_0\|_2$ and $\delta_1:=|(v_0(0)-v_+,u_+-u_0(0))|$ are small. Then, there exists a unique time-global solution $(V,z)(x,t)$ of (NRP), which satisfies*

$$V\in C^i([0,\infty);\, H^{3-i}), \quad i=0,1,2,3$$

$$z \in C^i([0,\infty);\, H^{2-i}), \quad i = 0,1,2$$

and moreover

$$\begin{aligned}
&\sum_{k=0}^{3}(1+t)^k\|\partial_x^k V(\cdot,t)\|^2 + \sum_{k=0}^{2}(1+t)^{k+2}\|\partial_x^k z(\cdot,t)\|^2 + \\
&+\int_0^t [\sum_{j=1}^{3}(1+\tau)^{j-1}\|\partial_x^j V(\cdot,\tau)\|^2 + \sum_{j=0}^{2}(1+\tau)^{j+1}\|\partial_x^j z(\cdot,\tau)\|^2]d\tau \\
&\le C(\|V_0\|_3^2 + \|z_0\|_2^2 + \delta_1),
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
&(1+t)^4\|z_t(\cdot,t)\|^2 + (1+t)^5(\|z_{xt}(\cdot,t)^2 + \|z_{tt}(\cdot,t)\|^2) + \\
&+\int_0^t [(1+\tau)^4\|z_{xt}(\cdot,\tau)\|^2 + (1+\tau)^5\|z_{tt}(\cdot,\tau)\|^2]d\tau \\
&\le C(\|V_0\|_3^2 + \|z_0\|_2^2 + \delta_1).
\end{aligned} \tag{3.14}$$

The derivation of (3.14) is similar to that of (2.14) and so the proof of Theorem 3.1 is omitted.

Remark. The L^1-property of $v_0 - v_+$ is a sufficient condition for the definition of $V_0(x)$. The decay rates in (3.13)-(3.14) will be optimal in L^2-setting. In L^1-setting the optimal decay rates are not known, different from Theorem 2.2. However, when $v_0(0) = v_+$, optimal decay rates are obtained as shown below.

We now treat the case of $v_0(0) = v_+$. Taking

$$(\bar{v},\bar{u})(x,t) \equiv (v_+, 0) \tag{3.15}$$

and

$$(\hat{v},\hat{u})(x,t) = (\frac{u_0(0)-u_+}{\alpha}m_0(x)e^{-\alpha t}, [(u_0(0)-u_+)\int_x^\infty m_0(y)dy + u_+]e^{-\alpha t}), \tag{3.16}$$

we have

$$\begin{cases}
(v - v_+ - \hat{v})_t - (u-\hat{u})_x = 0 \\
(u-\hat{u})_t + (p(v) - p(v_+))_x = -\alpha(u-\hat{u}) \\
(u-\hat{u})_x|_{x=0} = 0 \quad (\text{or} \quad (v - v_+ - \hat{v})|_{x=0} = 0) \\
(v - v_+ - \hat{v}, u - \hat{u})|_{t=0} = (v_0 - v_+, u_0)(x) - (\hat{v},\hat{u})(x,0).
\end{cases} \tag{3.17}$$

The definition

$$(V,z)(x,t) = (-\int_x^\infty (v - v_+ - \hat{v})(y,t)dy, u(x,t) - \hat{u}(x,t)) \tag{3.18}$$

gives the reformulated problem

$$\begin{cases} V_t - z = 0 \\ z_t + (p(V_x + v_+ + \hat{v}) - p(v_+))_x = \alpha z \\ z_x|_{x=0} = 0 \ \ (\text{or} \ \ V_x|_{x=0} = 0) \\ (V, z)|_{t=0} = (V_0, z_0)(x) := (-\int_x^\infty (v_0(y) - v_+ - \hat{v}(y,0))dy, u_0(x) - \hat{u}(x,0)) \end{cases} \tag{3.19}$$

or the linearized wave equation of V around v_+

$$\begin{cases} v_{tt} + p'(v_+)V_{xx} + \alpha V_t = -\tilde{F} \\ \quad := -(p(V_x + v_+ + \hat{v}) - p(v_+) - p'(v_+)V_x)_x \\ V_x|_{x=0} = 0, \quad (V, z)|_{t=0} = (V_0, z_0)(x). \end{cases} \tag{3.20}$$

Therefore, if $(V_0, z_0) \in H^3 \times H^2$, then we can obtain the following theorem on the same line as Theorem 3.1.

Theorem 3. 2 (The case of $v_0(0) = v_+$) *Suppose that $v_0 - v_+$ is in $L^1(\mathbf{R}_+)$ and both $\|V_0\|_3 + \|z_0\|_2$ and $\delta_2 := |u_+ - u_0(0)|$ are small. Then, there exists a unique time-global solution $(V, z)(x, t)$ of (3.19), which satisfies*

$$V \in C^i([0,\infty);\ H^{3-i}), \quad i = 0, 1, 2, 3$$
$$z \in C^i([0,\infty);\ H^{2-i}), \quad i = 0, 1, 2$$

and moreover

$$\begin{aligned} &\sum_{k=0}^{3}(1+t)^k\|\partial_x^k V(\cdot,t)\|^2 + \sum_{k=0}^{2}(1+t)^{k+2}\|\partial_x^k z(\cdot,t)\|^2 + \\ &+\int_0^t [\sum_{j=1}^{3}(1+\tau)^{j-1}\|\partial_x^j V(\cdot,\tau)\|^2 + \sum_{j=0}^{2}(1+\tau)^{j+1}\|\partial_x^j z(\cdot,\tau)\|^2]d\tau \\ &\le C(\|V_0\|_3^2 + \|z_0\|_2^2 + \delta_2). \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} &(1+t)^4\|z_t(\cdot,t)\|^2 + (1+t)^5(\|z_{xt}(\cdot,t)^2 + \|z_{tt}(\cdot,t)\|^2) + \\ &+\int_0^t [(1+\tau)^4\|z_{xt}(\cdot,\tau)\|^2 + (1+\tau)^5\|z_{tt}(\cdot,\tau)\|^2]d\tau \\ &\le C(\|V_0\|_3^2 + \|z_0\|_2^2 + \delta_2). \end{aligned} \tag{3.22}$$

Moreover, the same line to Theorem 2.2 can be applied. In the Dirichlet problem, since $\frac{1}{\alpha}(p'(v_+) - p'(\bar{v}))\bar{v}_x$ in (2.16) has not enough decay order, ϕ was defined by (2.20)' with a sourcing term. However, in the present case, since $\bar{v} \equiv v_+$, we define an asymptotic profile $\phi_1(x,t)$ by

$$\phi_1(x,t) = \int_0^\infty E_1(x,t;y)(V_0(y) + \frac{1}{\alpha}z_0(y))dy \tag{3.23}$$

or the solution of the corresponding parabolic equation

$$\begin{cases} \phi_{1t} - \kappa\phi_{1xx} = 0, & \kappa = -\frac{p'(v_+)}{\alpha} \\ \phi_{1x}|_{x=0} = 0, & \phi_1|_{t=0} = V_0(x) + \frac{1}{\alpha}z_0(x), \end{cases} \tag{3.24}$$

where

$$E_1(x,t;y) = \frac{1}{\sqrt{4\kappa\pi t}}(e^{-\frac{(x+y)^2}{4\kappa t}} + e^{-\frac{(x-y)^2}{4\kappa t}}). \tag{3.25}$$

Then we have the expression

$$\begin{aligned} (V-\phi_1)(x,t) &= -\tfrac{1}{\alpha}\int_0^\infty E_1(x,t/2;y)z(y,t/2)dy \\ &-\tfrac{1}{\alpha}\int_0^{t/2}\int_0^\infty E_{1t}(x,t-\tau;y)z(y,\tau)dyd\tau \\ &-\tfrac{1}{\alpha}\int_{t/2}^{t}\int_0^\infty E_1(x,t-\tau;y)z_t(y,\tau)dyd\tau \\ &-\tfrac{1}{\alpha}(\int_0^{t/2}+\int_{t/2}^{t})\int_0^\infty E_1(x,t-\tau;y)\tilde{F}_1(y,\tau)dyd\tau. \end{aligned} \tag{3.26}$$

We note that $\tilde{F}_1$ in (3.26) does not include the bad term likely $-((p'(v_+)-p'(\bar{v}))V_x)_x$ in (2.18). Thus, applying the decay properties obtained in Theorem 3.2 to (3.26), we reach the final Theorem.

Theorem 3. 3 (Asymptotic Profile) *Define ϕ_1 by (3.23) or (3.24) and suppose that $(V_0, z_0) \in L^1 \times L^1$. Then, the solution (V, z) of (3.19) obtained in Theorem 3.2 satisfies*

$$\|(V-\phi_1, (V-\phi_1)_x, (V-\phi_1)_t)(\cdot,t)\|_{L^\infty} = O(t^{-1}, t^{-3/2}, t^{-2}).$$

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L_p-Convergence Rate to Nonlinear Diffusion waves for p-System with Damping

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Abstract. In this paper, we study the p-system with frictional damping and show that the solutions time-asympotically tend to the nonlinear diffusion waves governed by the classical Darcy's law. By introducing an approximate Green function, we obtain the optimal L_p, $2 \leq p \leq +\infty$, convergence rate of the weak solution, which is a perturbation of the nonlinear diffusion wave, to the hyperbolic system.

Key words and phrases: p-system with damping, nonlinear diffusion wave, approximate Green function, L_p estimate.
1991 MR Subject Classification: 35K, 76N, 35L

1 Introduction

In this paper, we are interested in the time-asymptotic behavior of solutions to the p-system with frictional damping

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u, \quad \alpha > 0, \quad p' < 0, \end{cases} \tag{1.1}$$

with the initial data

$$(v,u)(x,0) = (v_0(x), u_0(x)).$$

Here $v(x,t) > 0$ and $u(x,t)$ represent the specific volume and velocity respectively; the pressure $p(v)$ is assumed to be a smooth function of v with $p(v) > 0, p'(v) < C_0 < 0$; and α is a positive constant. The system can be viewed as the isentropic Euler equations in

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Lagrangian coordinates with frictional term $-\alpha u$ in the momentum equation. (1.1) can be used to model the compressible fluid flow through a porous media. In [5], Hsiao and Liu proved that the solutions to (1.1) time asymptotically behave like those governed by the Darcy's law in L_2 and L_∞ norms. That is, as t tends to ∞, the smooth solution $(v(x,t), u(x,t))$ which is away from vacuum will approach to the solutions $(\bar{v}(x,t), \bar{u}(x,t))$ governed by the following system with the same initial data:

$$\begin{cases} \bar{v}_t = -\frac{1}{\alpha} p(\bar{v})_{xx}, \\ p(\bar{v})_x = -\alpha \bar{u}, \end{cases} \tag{1.2}$$

or

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\alpha \bar{u}. \end{cases} \tag{1.2$'$}$$

Recently in [13], Nishihara investigated the same problem and improved the convergence rates in L_2 and L_∞ norms by using energy method. By combining the energy method and the pointwise estimation, for the special case, i.e, $v_0(-\infty) = v_0(\infty)$, L_∞ convergence rates obtained in [13] is optimal. This case is easier because the Green function is the heat kernal and the exact expression of the function $V(x,t)$ can be obtained. Here the function $V(x,t)$ will be introduced in Section 2.

The main purpose of this paper is to obtain a sharper result on the convergence rate in $L_p, 2 \leq p \leq \infty$. These estimates are obtained by using the method introduced by Liu as in [9] which depends on the careful study of some approximate Green function. By introducing an appropriate approximate Green function for the reduced equation and using some estimates obtained from energy estimation, we give the optimal $L_p, 2 \leq p \leq \infty$, convergence rates for the genaral cases.

For the system (1.1), the existence of BV solutions when the end states at $x = \pm\infty$ are the same was proved in [1]. And for the 3×3 system, i.e. (1.1) with another equation for conservation of energy, the similar problem was also studied, see [L7] and reference therein. For the case when vacuum appears, there is no general theory. However, Liu in [8] gave a family of special solutions to (1.1) connecting to vaccum which tend to the Barenblatt solutions time asymptotically. For other results related to the system (1.1), please see [3,4,6,11,14,15] and reference therein.

The rest of the paper is arranged as follows: The main theorem is stated in Section 2; In Section 3, we restate some properties of the diffusion wave of (1.2) and some energy estimates obtained in [13]; In Section 4, the approximate Green function is introduced. The proof of the main theorem is given in the last section.

Throughout this paper we denote the generic constants by c or C. $H^m(\mathbf{R})$ denotes the usual Sobolev space with its norm

$$||f||_m := \sum_{k=0}^{m} ||\partial_x^k f||, \quad ||\cdot|| = ||\cdot||_{L^2(\mathbf{R})}.$$

Moreover, the domain $\mathbf{R}$ will be abbreviated without any ambiguity.

2 The Main Result

We are interested in the behavior of the solution of (1.1) with initial data satisfying

$$(v,u)(x,0) \to (v_\pm, u_\pm) \quad as \quad x \to \pm\infty, \tag{2.1}$$

with v_+ not necessarily equal to v_-. Denote the self-similar solution, diffusion wave, of (1.2) in the form of $\varphi(\frac{x}{\sqrt{t+1}})$ by $\bar{v}(x,t)$ with the same end states as $v(x,0)$:

$$\bar{v}(\pm\infty, t) = v_\pm, \tag{2.2}$$

and set

$$\bar{u}(x,t) \equiv -\frac{1}{\alpha} p(\bar{v})_x. \tag{2.3}$$

Since the u component of the solution is expected to decay exponentially at $x = \pm\infty$. As in [5], the following functions are needed to eliminate the u values at $x = \pm\infty$. The functions $\tilde{u}$ and $\tilde{v}$ are the solution of

$$\begin{cases} \tilde{v}_t - \tilde{u}_x = 0 \\ \tilde{u}_t = -\alpha\tilde{u}. \end{cases} \tag{2.4}$$

with $\tilde{u}(x,t) \to e^{-\alpha t} u_\pm$ as $x \to \infty$. For definiteness, we choose them as those in [5]:

$$\tilde{u}(x,t) = e^{-\alpha t}\Big(u_- + (u_+ - u_-)\int_{-\infty}^{x} m_0(y)dy\Big) \tag{2.5}$$

and

$$\tilde{v}(x,t) = \frac{u_+ - u_-}{-\alpha} e^{-\alpha t} m_0(x), \tag{2.6}$$

where $m_0(x)$ is a smooth function with compact support satisfying

$$\int_{-\infty}^{\infty} m_0(x)dx = 1.$$

Let the initial data $v_0(x)$ be a small perturbation of a diffusion wave, we are going to study how the solution behaves as t tends to ∞. As shown in [5] and [13], there exists a shift x_0 of $\bar{v}(x,t)$ such that at time zero,

$$\int_{-\infty}^{\infty} \Big(v_0(y) - \bar{v}(y + x_0, 0) - \tilde{v}(y,0)\Big)dy = 0.$$

By using the first equation of (1.1), it is easy to show that the following function $V(x,t)$,

$$V(x,t) = \int_{-\infty}^{x} \Big(v(y,t) - \bar{v}(y + x_0, t) - \tilde{v}(y,t)\Big)dy \tag{2.7}$$

satisfies $V(\pm\infty, t) = 0$. Here x_0 is a constant uniquely determined by

$$\int_{-\infty}^{\infty} \Big(v(x,0) - \bar{v}(x + x_0, 0)\Big)dx = \frac{u_+ - u_-}{-\alpha}. \tag{2.8}$$

For later use, we denote

$$U(x,t) = u(x,t) - \bar{u}(x+x_0,t) - \tilde{u}(x,t), \tag{2.9}$$

and $V_0(x) = V(x,0), U_0(x) = U(x,0) = V_t(x,0)$. From (1.1), (1.2), (2.4), (2.7) and (2.8), we can reformulate (1.1) as

$$\begin{cases} V_t - U = 0, \\ U_t + (p(V_x+\bar{v}+\tilde{v}) - p(\bar{v}))_x + \alpha U = \frac{1}{\alpha}p(\bar{v})_{xt}, \\ (V,U)|_{t=0} \equiv (V_0,U_0)(x) \to 0 \quad as \quad x \to \pm\infty. \end{cases} \tag{2.10}$$

By linearizing the second equation of (2.10) about $\bar{v}$, we have the following linearized system

$$\begin{cases} V_t - U = 0, \\ U_t + (p'(\bar{v})V_x)_x + \alpha U = F_1 + F_2, \\ (V,U)|_{t=0} \equiv (V_0,U_0)(x) \to 0 \quad as \quad x \to \pm\infty, \end{cases} \tag{2.11}$$

where $F_j(x,t) = (\tilde{F}_j(x,t))_x (j=1,2)$, and

$$\begin{aligned} \tilde{F}_1(x,t) &= \tfrac{1}{\alpha}p(\bar{v})_t, \\ \tilde{F}_2(x,t) &= -(p(V_x+\bar{v}+\tilde{v}) - p(\bar{v}) - p'(\bar{v})V_x). \end{aligned} \tag{2.12}$$

¿From now on, we will study the system (2.11). For the completeness of the paper and the comparison of the convergence rates, we list the main theorems in [5] and [13] as follows.

Theorem 2. 1 . *(Hsiao and Liu, [5])If* $V_0(x) = V(x,0) \in H^3(\mathbf{R}), \mathbf{V_t(x,0)} \in \mathbf{H^2(R)}$, *and*

$$|u_+ - u_-| + |v_+ - v_-| + \|V_0\|_{H^3} + \|V_t(\cdot,0)\|_{H^2} \le \varepsilon_0$$

for some sufficiently small ε_0, *then there exists a global in time solution* $V(x,t) \in H^3, U(x,t) \in H^2$ *of (2.10), which satisfies*

$$\|(V_x,U)(t)\|_{L_2} + \|(V_x,U)(t)\|_{L_\infty} = O(1)\varepsilon_0(1+t)^{-1/2}. \tag{2.13}$$

Theorem 2. 2 . *(Nishihara, [13])Under the conditions of Theorem 2.1, there exists a global in time solution of (2.12) which satisfies*

$$V(x,t) \in W^{\bar{k},\infty}([0,\infty); H^{3-\bar{k}}), U(x,t) \in W^{k,\infty}([0,\infty); H^{3-k}) \tag{2.14}$$

for $\bar{k} = 0,1,2,3; k = 0,1,2,$ *and*

$$\|\partial_x^k V_x(t)\|_{L_2} = O(1)\varepsilon_0(1+t)^{-(k+1)/2}, \|\partial_x^k U(t)\|_{L_2} = O(1)\varepsilon_0(1+t)^{-(k+2)/2}, \tag{2.15}$$

$$\|V_x(t)\|_{L_\infty} = O(1)\varepsilon_0(1+t)^{-3/4}, \|U(t)\|_{L_\infty} = O(1)\varepsilon_0(1+t)^{-5/4}. \tag{2.16}$$

Moreover, if $v_+ = v_-, u_+ = u_- = 0$ *and* $V_0, U_0 \in L_1(\mathbf{R})$ *with*

$$\int_{-\infty}^{\infty} (v_0(x) - v_-)dx = 0,$$

then

$$\|V_x(t)\|_{L_\infty} = O(1)(1+t)^{-1}, \ \|U(t)\|_{L_\infty} = O(1)(1+t)^{-3/2}. \tag{2.17}$$

Our main result of this paper is the following.

Theorem 2. 3 . *If* $V_0(x) \in H^3(\mathbf{R}) \cap L_1(\mathbf{R}), U_0(x) \in H^2(\mathbf{R}) \cap L_1(\mathbf{R})$, *and*

$$|u_+ - u_-| + |v_+ - v_-| + \|V_0\|_{H^3} + \|U_0\|_{H^2} + \|V_0\|_{L_1} + \|U_0\|_{L_1} \leq \varepsilon_0$$

for some sufficiently small ε_0, *then there exists a global in time solution* $V(x,t)$, $U(x,t)$ *of (2.10), which satisfies (2.14) and*

$$\|\partial_x^k V_x(t)\|_{L_p} = O(1)\varepsilon_0(1+t)^{-(1-1/p)/2-(k+1)/2}, \tag{2.18}$$

$$\|\partial_x^k U(t)\|_{L_p} = O(1)\varepsilon_0(1+t)^{-(1-1/p)/2-k/2-1} \tag{2.19}$$

for any $k \leq 2$ *and* $p \in [2, +\infty)$.

3 Some Known Estimates

In this section we will restate some known properties of the nonlinear diffusion wave $\bar{v}$ to (1.2) and (2.1) and the L_2 estimates of the derivatives of the function $V(x,t)$ (see (2.7)). All these results were obtained in or follow from the papers [5] and [13], we list them here for the convenience of the readers.

First, we list the propeties of $\bar{v}(x,t)$ as follows: The function $\bar{v}(x,t)$ possesses the form

$$\begin{aligned} \bar{v}(x,t) = \varphi(x/\sqrt{1+t}) \equiv \varphi(\xi), \quad -\infty < \xi < +\infty, \\ \varphi(\pm\infty) = v_{\pm}. \end{aligned} \tag{3.1}$$

And the function $\varphi(\xi)$ satisfies

$$\sum_{k=1}^{6} |\varphi^{(k)}(\xi)| + |\varphi(\xi) - v_+|_{\xi>0} + |\varphi(\xi) - v_-|_{\xi<0} \le C|v_+ - v_-|e^{-C\alpha\xi^2}, \tag{3.2}$$

where $\varphi^{(k)}(\xi)$ denotes the derivative of $\varphi(\xi)$ with respect to ξ k times, and C is a positive constant. According to the form of the function $\bar{v}(x,t)$, we have

$$\begin{aligned}
&\bar{v}_x = \tfrac{\varphi'(\xi)}{\sqrt{t+1}}, \quad \bar{v}_t - \tfrac{\xi\varphi'(\xi)}{2(t+1)}, \quad \bar{v}_{xx}\tfrac{\varphi''(\xi)}{t+1}, \quad \bar{v}_{xt} = \tfrac{\varphi'(\xi)+\xi\varphi''(\xi)}{2(t+1)\sqrt{t+1}}, \\
&\bar{v}_{tt} = \tfrac{3\xi\varphi'(\xi)+\xi^2\varphi''(\xi)}{4(t+1)^2}, \quad \bar{v}_{xxx} = \tfrac{\varphi'''(\xi)}{(t+1)\sqrt{t+1}}, \quad \bar{v}_{xtt} = \tfrac{3\varphi'(\xi)+5\xi\varphi''(\xi)+\xi^2\varphi'''(\xi)}{4(t+1)^2\sqrt{t+1}} \\
&\bar{v}_{xtt} = -\tfrac{15\varphi'(\xi)+33\xi\varphi''(\xi)+12\xi^2\varphi'''(\xi)+\xi^3\varphi''''(\xi)}{8(t+1)^3\sqrt{t+1}}
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\textstyle\int |\bar{v}_x(x,t)|^2 dx = O(1)|v_+ - v_-|^2(t+1)^{-1/2}, \\
&\textstyle\int (|\bar{v}_t|^2 + |\bar{v}_{xx}|^2) dx = O(1)|v_+ - v_-|^2(t+1)^{-3/2}, \\
&\textstyle\int (|\bar{v}_{xt}|^2 + |\bar{v}_{xxx}|^2) dx = O(1)|v_+ - v_-|^2(t+1)^{-5/2}, \\
&\textstyle\int |\bar{v}_{tt}|^2 dx = O(1)|v_+ - v_-|^2(t+1)^{-7/2}, \\
&\textstyle\int |\bar{v}_{xtt}|^2 dx = O(1)|v_+ - v_-|^2(t+1)^{-9/2} \\
&\textstyle\int |\bar{v}_{xttt}|^2 dx = O(1)|v_+ - v_-|^2(t+1)^{-13/2}.
\end{aligned} \tag{3.4}$$

Next we will give an L_2 estimate on U_{tt} which will be used later. Before doing this, we restate the following lemma from [13].

Lemma 3. 1 *Suppose that both* $\delta := |v_+ - v_-| + |u_+ - u_-|$ *and* $||V_0||_3 + ||U_0||_2$ *are sufficiently small. Then, there exists a unique global in time solution* $(V,U)(x,t)$ *of (2.10), which satisfies*

$$V \in W^{i,\infty}([0,\infty); H^{3-i}), \quad i = 0,1,2,3,$$

and moreover

$$\sum_{k=0}^{3}(1+t)^k||\partial_x^k V(\cdot,t)||^2+\sum_{k=0}^{2}(1+t)^{k+2}||\partial_x^k U(\cdot,t)||^2+(1+t)^3||\partial_t U(\cdot,t)||^2$$

$$+\int_0^t[\sum_{j=1}^{3}(1+\tau)^{j-1}||\partial_x^j V(\cdot,\tau)||^2+\sum_{j=0}^{2}(1+\tau)^{j+1}||\partial_x^j U(\cdot,\tau)||^2]d\tau \qquad (3.5)$$

$$\leq C(||V_0||_3^2+||U_0||_2^2+\delta).$$

Similar to the proof of the above Lemma in [13], we have the following estimate on U_{tt} after tedious calculations. This estimate for the case when $v_- = v_+$ was also obtained in [13].

Lemma 3. 2 *Under the hypothesis of Lemma 3.1, we have the following estimate for the decay rate of* $||U_{tt}||^2$.

$$(t+1)^5(||U_{tt}||^2+||U_{xt}||^2)+\int_0^t(\tau+1)^5||U_{tt}||^2d\tau$$

$$\leq C(||V_0||_3^2+||U_0||_2^2+\delta). \qquad (3.6)$$

Outline of the Proof. Differentiate the second equation of (2.11) with respect to t twice and multiply it by U_{tt}. By using the properties of the diffusion wave and Lemma 3.1 , after some calculation we have

$$\begin{aligned}
&\tfrac{d}{dt}\int_{-\infty}^{\infty}(U_{tt}^2-p'(\bar{v})U_{xt}^2)dx+\alpha\int_{-\infty}^{\infty}U_{tt}^2dx\\
\leq\ & C\epsilon_0[(t+1)^{-1}\int_{-\infty}^{\infty}U_{xt}^2dx+(t+1)^{-2}\int_{-\infty}^{\infty}U_{xx}^2dx+(t+1)^{-3}\int_{-\infty}^{\infty}U_x^2dx]\\
+\ & C\delta[(t+1)^{-4}\int_{-\infty}^{\infty}V_{xx}^2dx+(t+1)^{-5}\int_{-\infty}^{\infty}V_x^2dx+(t+1)^{-\frac{13}{2}}].
\end{aligned}$$

Multiplying the above inequality by $(t+1)^5$ and integrate from 0 to t, and using the Lemma 3.1 again yields the estimate (3.6).

Q.E.D.

4 Green Function

We rewrite (2.11) as

$$\alpha V_t - (a(x,t)V_x)_x = F - V_{tt}, \tag{4.1}$$

where $a(x,t) = -p'(\bar{v}(x,t)) > 0, F = F_1 + F_2$. Now we will construct an approximate Green function $G(x,t;y,s)$ for (4.1) which is continous and piecewise smooth. It satisfies the basic requirement

$$G(x,t;y,t) = \delta(y-x), \tag{4.2}$$

where δ is the Dirac delta function. Multiplying (4.1) by G and integrating over the region $(y,s) \in \mathbf{R} \times (\mathbf{0}, \mathbf{t})$, (4.2) gives

$$\begin{aligned} V(x,t) = & \int_{-\infty}^{\infty} G(x,t;y,0)V_0(y)dy \\ & + \alpha^{-1} \int_0^t \int_{-\infty}^{\infty} G(x,t;y,s)(F(y,s) - V_{ss}(y,s))dyds \\ & + \int_0^t \int_{-\infty}^{\infty} \Big(G_s(x,t;y,s) + \alpha^{-1}(a(y,s)G_y(x,t;y,s))_y\Big)V(y,s)dyds. \end{aligned} \tag{4.3}$$

If $a(y,s)$ is a constant and G is a Green function of (4.1), we know that the last integral of (4.3) is equal to zero. But it is difficult to give an explicit expression of Green function, such that the last integral of (4.3) is equal to zero. However, we only need to minimize the expression $\alpha G_s + (aG_y)_y$. For this purpose, we choose the following approximate Green function of (4.3):

$$G(x,t;y,s) = \Big(\frac{\alpha}{4\pi a(x,t)(t-s)}\Big)^{1/2} \exp\Big(\frac{-\alpha(x-y)^2}{4A(y,s,t)(t-s)}\Big), \tag{4.4}$$

where $A(y,s,t) = -p'(\varphi(\eta))$, φ is defined in (3.1), and

$$\eta = \begin{cases} y/\sqrt{1+s}, & s > t/2, \\ y/\sqrt{1+t/2}, & s \le t/2. \end{cases}$$

It is clear that the Green function in (4.4) satisfies the condition (4.2). Setting

$$G_D(y,s) = \Big(\frac{\alpha}{4\pi C_0 s}\Big)^{1/2} \exp\Big(\frac{-\alpha y^2}{Ds}\Big), \tag{4.5}$$

for any positive constant $D > 4 \max A(y,s,t) + O(1)\varepsilon$. Denote $\theta(t,s) = \theta_1(t,s) + \theta_2(t,s)$ with

$$\theta_1(t,s) = \begin{cases} (1+s)^{-1/2}, & s > t/2, \\ 0, & s \le t/2, \end{cases}$$

$$\theta_2(t,s) = \begin{cases} 0, & s > t/2, \\ (1+t)^{-1/2}, & s \le t/2. \end{cases} \tag{4.6}$$

Using above notations, if $l \le 1, h \le 1$, we have

$$\begin{aligned} & |\partial_t^l \partial_s^h \partial_x^k \partial_y^m G(x,t;y,s)| \\ = \ & O(1)(\textstyle\sum_{m_1+m_2=m}(t-s)^{(-m_1-k)/2}\theta^{m_2}). \\ & (\theta_2^2 + (t-s)^{-1})^l(\theta_1^2 + (t-s)^{-1})^h G_D(x-y, t-s). \end{aligned} \tag{4.7}$$

For heat kernel $G_D(x,t)$, we have

$$\|G_D(\cdot,t)\|_{L_p} \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}. \tag{4.8}$$

Since the function $G(x,t;y,s)$ is not symmetric with respect to variables x and y, t and s, we need the following formulas:

$$\partial_x G = -\partial_y G - G\Big(\frac{\alpha(x-y)^2}{4A^2(t-s)}A'_y + \frac{a'_x(x,t)}{a(x,t)}\Big),$$

and

$$\partial_t G = -\partial_s G - G\frac{\alpha(x-y)^2}{4A^2(t-s)}(A'_s + A'_t).$$

It follows from (3.2) that

$$|A'_y| + |a'_x| = O(1)\varepsilon_0\theta, \;\; |A'_s| + |A'_t| = O(1)\varepsilon_0\theta^2,$$

where A'_y, A'_t and A'_s represent the derivative of A with respect to y, t, s respectively. Given a function $g(y,s)$ and two constants $0 \leq a < b \leq t$, we have

$$\begin{aligned} &\textstyle\int_a^b \int_{-\infty}^{\infty} \partial_x G(x,t;y,s) g(y,s)dyds \\ = &\textstyle\int_a^b \int_{-\infty}^{\infty} G(x,t;y,s)\partial_y g(y,s)dyds \\ &\textstyle+O(1)\varepsilon_0 \int_a^b \int_{-\infty}^{\infty} \theta(t,s) G_D(x-y,t-s) g(y,s)dyds, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &\textstyle\int_a^b \int_{-\infty}^{\infty} \partial_t G(x,t;y,s) g(y,s)dyds \\ = &\textstyle\int_a^b \int_{-\infty}^{\infty} G(x,t;y,s)\partial_s g(y,s)dyds \\ &\textstyle-\int_{-\infty}^{\infty} G(x,t;y,s) g(y,s)dy|_{s=a}^{s=b} \\ &\textstyle+O(1)\varepsilon_0 \int_a^b \int_{-\infty}^{\infty} \theta^2(t,s) G_D(x-y,t-s) g(y,s)dyds. \end{aligned} \tag{4.10}$$

In general, for $k \geq 1$, we have

$$\begin{aligned} &\textstyle\int_a^b \int_{-\infty}^{\infty} \partial_x^{h+k} G(x,t;y,s) g(y,s)dyds \\ = &\textstyle\int_a^b \int_{-\infty}^{\infty} \partial_x^h G(x,t;y,s)\partial_y^k g(y,s)dyds \\ &\textstyle+O(1)\varepsilon_0 \sum_{\beta<k} \int_a^b \int_{-\infty}^{\infty} (t-s)^{-h/2}\theta^{(k-\beta)}(t,s) G_D \partial_y^\beta g(y,s)dyds, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} &\textstyle\int_a^b \int_{-\infty}^{\infty} \partial_t\partial_x^{h+k} G(x,t;y,s) g(y,s)dyds \\ = &\textstyle\int_a^b \int_{-\infty}^{\infty} \partial_x^h G(x,t;y,s)\partial_s\partial_y^k g(y,s)dyds \\ &\textstyle-\int_{-\infty}^{\infty} \partial_x^{h+k} G(x,t;y,s) g(y,s)dy|_{s=a}^{s=b} \\ &\textstyle+O(1)\varepsilon_0 \sum_{\beta<k} \int_a^b \int_{-\infty}^{\infty} (t-s)^{-h/2}\theta^{2+(k-\beta)}(t,s) G_D \partial_y^\beta g(y,s)dyds. \end{aligned} \tag{4.12}$$

Similarly, we have

$$\begin{aligned} &\textstyle\int_{t/2}^t \int_{-\infty}^{\infty} \partial_x^3 G(x,t;y,s) g(y,s)dyds \\ = &\textstyle\int_{t/2}^t \int_{-\infty}^{\infty} \partial_y^2 G(x,t;y,s)\partial_y g(y,s)dyds \\ &\textstyle+O(1)\varepsilon_0 \sum_{\beta\leq 1} \int_{t/2}^t \int_{-\infty}^{\infty} ((t-s)^{-1/2}+\theta)\theta^{(2-\beta)}(t,s) G_D \partial_y^\beta g(y,s)dyds. \end{aligned} \tag{4.13}$$

Denoting

$$R_G \equiv G_s(x,t;y,s) + \alpha^{-1}(a(y,s)G_y(x,t;y,s))_y,$$

the first term of the right hand of (4.13) can be rewriten as

$$\begin{aligned}
&\int_{t/2}^{t}\int_{-\infty}^{\infty} \partial_y^2 G(x,t;y,s)\partial_y g(y,s)dyds \\
= &\int_{t/2}^{t}\int_{-\infty}^{\infty}(\alpha a^{-1}(y,s)(R_G - \partial_s G) - \tfrac{\partial_y a}{a}(y,s)\partial_y G)\partial_y g(y,s)dyds.
\end{aligned}$$

Using integration by part of variables y and s, we have

$$\begin{aligned}
&\int_{t/2}^{t}\int_{-\infty}^{\infty} a^{-1}(y,s)\partial_s G(x,t;y,s)\partial_y g(y,s)dyds \\
= &\int_{t/2}^{t}\int_{-\infty}^{\infty} a^{-1}(y,s)\partial_y G\partial_s g(y,s)dyds + \int_{-\infty}^{\infty} a^{-1}(y,s)G\partial_y g(y,s)dy|_{s=t/2}^{s=t} \\
&+O(1)\varepsilon_0\int_{t/2}^{t}\int_{-\infty}^{\infty} G_D(t-s,x-y)(\theta^2\partial_y g(y,s) + \theta\partial_s g(y,s))dyds.
\end{aligned}$$

With these estimates, we can get from (4.13) that

$$\begin{aligned}
&\int_{t/2}^{t}\int_{-\infty}^{\infty} \partial_x^3 G(x,t;y,s)g(y,s)dyds \\
= &\int_{t/2}^{t}\int_{-\infty}^{\infty}\Big(\alpha a^{-1}(y,s)R_G - \tfrac{\partial_y a}{a}(y,s)\partial_y G\Big)\partial_y g(y,s)dyds \\
&-\alpha\int_{t/2}^{t}\int_{-\infty}^{\infty} a^{-1}(y,s)\partial_y G\partial_s g(y,s)dyds \\
&-\alpha\int_{-\infty}^{\infty} a^{-1}(y,s)G\partial_y g(y,s)dy|_{s=t/2}^{s=t} \\
&+O(1)\varepsilon_0\sum_{\beta\le 1}\int_{t/2}^{t}\int_{-\infty}^{\infty}((t-s)^{-1/2}+\theta)\theta^{(2-\beta)}(t,s)G_D\partial_y^\beta g(y,s)dyds \\
&+O(1)\varepsilon_0\int_{t/2}^{t}\int_{-\infty}^{\infty}\theta(t,s)G_D\partial_s g(y,s)dyds.
\end{aligned} \tag{4.14}$$

Similarly, we can conclude that

$$\begin{aligned}
&\int_{t/2}^{t}\int_{-\infty}^{\infty} \partial_t\partial_x^2 G(x,t;y,s)g(y,s)dyds \\
= &\int_{t/2}^{t}\int_{-\infty}^{\infty} \partial_y^2 G(x,t;y,s)\partial_s g(y,s)dyds - \int_{-\infty}^{\infty}\partial_x^2 G g(y,s)dy|_{s=t/2}^{s=t} \\
&+O(1)\varepsilon_0\sum_{\beta\le 1}\int_{t/2}^{t}\int_{-\infty}^{\infty}((t-s)^{-1/2}+\theta)\theta^{(3-\beta)}(t,s)G_D\partial_y^\beta g(y,s)dyds.
\end{aligned}$$

Letting $g(y,s) = \partial_y\tilde{g}(y,s)$ and using the same method, we have

$$\begin{aligned}
&\int_{t/2}^{t}\int_{-\infty}^{\infty} \partial_t\partial_x^2 G(x,t;y,s)\partial_y\tilde{g}(y,s)dyds \\
= &\int_{t/2}^{t}\int_{-\infty}^{\infty}(\alpha a^{-1}(y,s)R_G - \tfrac{\partial_y a}{a}(y,s)\partial_y G)\partial_s\partial_y\tilde{g}(y,s)dyds \\
&-\alpha\int_{t/2}^{t}\int_{-\infty}^{\infty} a^{-1}(y,s)\partial_y G\partial_s^2\tilde{g}(y,s)dyds \\
&-\int_{-\infty}^{\infty}(\partial_y^2 G\partial_y\tilde{g}(y,s) + \alpha a^{-1}(y,s)G\partial_s\partial_y\tilde{g}(y,s))dy|_{s=t/2}^{s=t} \\
&+O(1)\varepsilon_0\sum_{\beta\le 1}\int_{t/2}^{t}\int_{-\infty}^{\infty}((t-s)^{-1/2}+\theta)\theta^{(3-\beta)}(t,s)G_D\partial_y^{\beta+1}\tilde{g}(y,s)dyds \\
&+O(1)\varepsilon_0\int_{t/2}^{t}\int_{-\infty}^{\infty}(\theta(t,s)G_D\partial_s^2 + \theta^2(t,s)G_D\partial_s\partial_y)\tilde{g}(y,s)dyds.
\end{aligned} \tag{4.15}$$

We next estimate $R_G \equiv \alpha G_s + (aG_y)_y$. After some calculations, we have

$$R_G = \; O(1)\varepsilon_0\Theta(t,s)\tilde{E}(y,t,s)G_D(x-y,t-s), \tag{4.16}$$

where

$$\Theta(t,s) = \begin{cases} ((1+s)^{-1} + (t-s)^{-1/2}(1+s)^{-1/2}), & s > t/2 \\ ((1+t)^{-1} + (t-s)^{-1/2}(1+s)^{-1/2}), & s \le t/2, \end{cases}$$

$$\tilde{E}(y,t,s)=\begin{cases} E(y,s), & s>t/2,\\ E(y,t), & s\le t/2,\end{cases}$$

with $E(y,\tau)=\exp\left(\frac{-C\alpha y^2}{1+\tau}\right)$. If $s<t/2$, by direct calculation, we know

$$|\partial_t^l\partial_x^k R_G(x,t;y,s)|\le C\varepsilon_0(1+s)^{-\frac{1}{2}}(1+t)^{-(l+\frac{k+1}{2})}E(y,t)G_D(x-y,t-s). \tag{4.17}$$

Notice that $R_G(x,t;y,s)$ is discontinuous at $s=t/2$. When $s=t/2$, we have

$$\lim_{s\to t/2\pm}|\partial_x^k R_G(x,t;y,s)|\le C\varepsilon_0(1+t)^{-(1+\frac{k}{2})}E(y,t/2)G_D(x-y,t/2). \tag{4.18}$$

Moreover, for a given function $g(y,s)$, we have

$$\begin{array}{cl} & \int_{t/2}^t\int_{-\infty}^{\infty}\partial_x^k R_G(x,t;y,s)g(y,s)dyds\\ = & \int_{t/2}^t\int_{-\infty}^{\infty} R_G(x,t;y,s)\partial_y^k g(y,s)dyds\\ & +O(1)\varepsilon_0\sum_{\beta<k}\int_{t/2}^t\int_{-\infty}^{\infty}\Theta\theta^{(k-\beta)}E(y,s)G_D(x-y,t-s)\partial_y^\beta g(y,s)dyds,\end{array} \tag{4.19}$$

and

$$\begin{array}{cl} & \int_{t/2}^t\int_{-\infty}^{\infty}\partial_t\partial_x^k R_G(x,t;y,s)g(y,s)dyds\\ = & \int_{t/2}^t\int_{-\infty}^{\infty} R_G(x,t;y,s)\partial_s\partial_y^k g(y,s)dyds\\ & +\int_{-\infty}^{\infty}\partial_x^k R_G(x,t;y,s)g(y,s)dy|_{s=t/2}^{s=t}\\ & +O(1)\varepsilon_0\sum_{\beta<k}\int_{t/2}^t\int_{-\infty}^{\infty}\Theta\theta^{2+(k-\beta)}E(y,s)G_D(x-y,t-s)\partial_y^\beta g(y,s)dyds.\end{array} \tag{4.20}$$

5 The Proof of Main Theorem

In this section, we will give some estimates on $\partial_t^l\partial_x^k V(x,t)$, $(l\le 1, k+l\le 3)$ by using the approximate Green function. Denoting

$$\begin{array}{rcl} I_1^{l,k} & = & \int_{-\infty}^{\infty}\partial_t^l\partial_x^k G(x,t;y,0)V(y,0)dy,\\ I_2^{l,k} & = & \partial_t^l\int_0^t\int_{-\infty}^{\infty}\partial_x^k G(x,t;y,s)F_1(y,s)dyds,\\ I_3^{l,k} & = & \partial_t^l\int_0^t\int_{-\infty}^{\infty}\partial_x^k G(x,t;y,s)F_2(y,s)dyds,\\ I_4^{l,k} & = & \partial_t^l\int_0^t\int_{-\infty}^{\infty}\partial_x^k G(x,t;y,s)(V_{ss}(y,s))dyds,\\ I_5^{l,k} & = & \partial_t^l\int_0^t\int_{-\infty}^{\infty}\partial_x^k R_G(x,t;y,s)V(y,s)dyds,\end{array} \tag{5.1}$$

from (4.3) we can write

$$\partial_t^l\partial_x^k V(x,t)=I_1^{l,k}+\alpha^{-1}I_2^{l,k}+\alpha^{-1}I_3^{l,k}-\alpha^{-1}I_4^{l,k}+I_5^{l,k}. \tag{5.2}$$

Set

$$B_p^{l,k}(t)=(1+t)^{\frac{1}{2}(1-\frac{1}{p})+l+\frac{k}{2}}, \tag{5.3}$$

and

$$M(t)=\sup_{p\ge 2,0\le s\le t,l+k\le 3,l\le 1}B_p^{l,k}(s)\|\partial_t^l\partial_x^k V(\cdot,s)\|_{L_p}. \tag{5.4}$$

We are going to estimate $I_j^{l,k}$ as follows. Since $\|V_0\|_{L_1} \leq C\varepsilon_0$, $\|U_0\|_{L^1} \equiv \|V_s(\cdot, 0)\|_{L_1} \leq C\varepsilon_0$, (4.7), (4.8) and Hausdoff-Young inequality gives

$$\|I_1^{l,k}\|_{L_p} \leq C\varepsilon_0 \|(1+t)^{-l-k/2} G_D(\cdot, t)\|_{L_p} \leq C\varepsilon_0 (1+t)^{-\frac{1}{2}(1-\frac{1}{p})-l-\frac{k}{2}}. \tag{5.5}$$

For $I_2^{l,k}$, it is easy to see that from (3.1) and (3.2)

$$\|\partial_t^l \partial_x^k \bar{v}\|_{L_p} \leq C\varepsilon_0 (1+t)^{-l-\frac{k}{2}+\frac{1}{2p}}, \tag{5.6}$$

provided that $1 \leq l+k \leq 6$. Thus for $l+k \leq 5$, we have

$$\|\partial_t^l \partial_x^k \tilde{F}_1\|_{L_p} \leq C\varepsilon_0 (1+t)^{-l-\frac{k}{2}-\frac{1}{2}-\frac{1}{2}(1-\frac{1}{p})}. \tag{5.7}$$

By (4.11) and integration by part with respect to variable y, we have

$$\begin{aligned} \|I_2^{0,k}\|_{L_p} \leq \; & C\Big(\int_{t/2}^t \sum_{h\leq k} \| \int_{-\infty}^{\infty} (1+s)^{-(k-h)/2} G_D(\cdot - y, t-s) \partial_y^h F_1(y,s) dy \|_{L_p} ds \\ & + \int_0^{t/2} \| \int_{-\infty}^{\infty} \partial_y \partial_x^k G(\cdot, t; y, s) \tilde{F}_1(y,s) dy \|_{L_p} ds \Big). \end{aligned}$$

Thus, (4.7) and (5.7) give

$$\begin{aligned} \|I_2^{0,k}\|_{L_p} \leq \; & C \int_{t/2}^t ((t-s)^{-\frac{1}{2}(1-\frac{1}{p})} \varepsilon_0 (1+s)^{-1-\frac{k}{2}}) ds \\ & + \int_0^{t/2} (1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k+1}{2}} \varepsilon_0 (1+s)^{-\frac{1}{2}}) ds \\ \leq \; & C\varepsilon_0 (1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}}. \end{aligned}$$

If $l = 1$, then (4.12) yields

$$\begin{aligned} & \|I_2^{1,k}\|_{L_p} \\ \leq \; & C\Big(\int_{t/2}^t \sum_{h\leq k, m\leq 1} \| \int_{-\infty}^{\infty} (1+s)^{-(k-h)/2-(1-m)} G_D \partial_s^m \partial_y^h F_1(y,s) dy \|_{L_p} ds \\ & + \int_0^{t/2} \| \int_{-\infty}^{\infty} \partial_t \partial_y \partial_x^k G(\cdot, t; y, s) \tilde{F}_1(y,s) dy \|_{L_p} ds \\ & + \| \int_{-\infty}^{\infty} \partial_x^k G(\cdot, t; y, t/2) F_1(y, t/2) dy \|_{L_p} \Big). \end{aligned}$$

The estimation of the first and the second terms of the above equality is similar to those of $I_2^{0,k}$. The last term can be estimated by using (4.7), and gives and (5.7)

$$\| \int_{-\infty}^{\infty} \partial_x^k G(\cdot, t; y, t/2) F_1(y, t/2) dy \|_{L_p} \leq C\varepsilon_0 (1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}-1}.$$

Thus, we obtain

$$\|I_2^{l,k}\|_{L_p} \leq C\varepsilon_0 (1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}-l}. \tag{5.8}$$

Next, we first consider $I_5^{l,k}$. When $l = 0, k \leq 3$, by (4.19) we have

$$\begin{aligned} I_5^{0,k} = \; & \int_0^{t/2} \int_{-\infty}^{\infty} \partial_x^k R_G \cdot V dy ds + \int_{t/2}^t \int_{-\infty}^{\infty} R_G \cdot \partial_y^k V dy ds \\ & + O(1)\varepsilon_0 \sum_{\beta<k} \int_{t/2}^t \int_{-\infty}^{\infty} (1+s)^{-(k-\beta)} \Theta E(y,s) G_D \cdot \partial_y^\beta V dy ds. \end{aligned}$$

Using (4.16), (4.17), Hausdoff-Young inequality and Hölder inequality, we have

$$
\begin{aligned}
&\|I_5^{0,k}\|_{L_p}\\
\leq\ & C\varepsilon_0\Big(\int_0^{t/2}(1+s)^{-1/2}(1+t)^{-(k+1)/2}\|G_D\|_{L_p}\|E(\cdot,t)\|_{L_2}\|V\|_{L_2}ds\\
&+\int_{t/2}^t\Theta\|G_D\|_{L_2}\|E(\cdot,s)\|_{L_2}\|\partial_y^k V\|_{L_p}ds\\
&+\sum_{\beta<k}\int_{t/2}^t(1+s)^{-(k-\beta)/2}\Theta\|G_D\|_{L_2}\|E(\cdot,s)\|_{L_2}\|\partial_y^\beta V\|_{L_p}ds\Big)\\
\leq\ & C\varepsilon_0\Big(\int_0^{t/2}(1+t)^{-(k+1)/2}(t-s)^{-\frac{1}{2}(1-\frac{1}{p})}(1+t)^{1/4}(1+s)^{-\frac{1}{2}(1-\frac{1}{2})-\frac{1}{2}}ds\\
&+\sum_{\beta\leq k}\int_{t/2}^t(1+s)^{-(k-\beta)/2}\Theta(t-s)^{-\frac{1}{2}(1-\frac{1}{2})}(1+s)^{1/4}(1+s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{\beta}{2}}ds\Big).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\sum_{\beta\leq k}\int_{t/2}^t(1+s)^{-(k-\beta)/2}\Theta(t-s)^{-\frac{1}{2}(1-\frac{1}{2})}(1+s)^{1/4}(1+s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{\beta}{2}}ds\\
\leq=\ & C\int_{t/2}^t(1+s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}+\frac{1}{4}}(t-s)^{-\frac{1}{4}}((1+s)^{-1}+(t-s)^{-1/2}(1+s)^{-1/2})ds\\
\leq\ & C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}}.
\end{aligned}
$$

We have

$$\|I_5^{0,k}\|\leq C\varepsilon_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}}.$$

If $l=1,k\leq 2$, we have

$$
\begin{aligned}
I_5^{1,k} =\ & \partial_t\Big(\Big(\int_{t/2}^t+\int_0^{t/2}\Big)\int_{-\infty}^\infty\partial_x^k R_G(x,t;y,s)V(y,s)dyds\Big)\\
=\ & \int_{t/2}^t\int_{-\infty}^\infty(\partial_t\partial_x^k R_G)V\,dyds+\int_0^{t/2}\int_{-\infty}^\infty(\partial_t\partial_x^k R_G)V\,dyds\\
&+\int_{-\infty}^\infty(\partial_x^k R_G)V\,dy|_{s=t}-\lim_{s\to t/2+}\int_{-\infty}^\infty(\partial_x^k R_G)V\,dy\\
&+\lim_{s\to t/2-}\int_{-\infty}^\infty(\partial_x^k R_G)V\,dy.
\end{aligned}
$$

(4.20), (4.17) and (4.18) give

$$
\begin{aligned}
&\|I_5^{1,k}\|_{L_p}\\
\leq\ & C\varepsilon_0\Big(\int_0^{t/2}(1+s)^{-1/2}(1+t)^{-(k+3)/2}\|G_D\|_{L_p}\|E(\cdot,t)\|_{L_2}\|V\|_{L_2}ds\\
&+\sum_{\beta\leq k,\alpha\leq 1}\int_{t/2}^t(1+s)^{-(k-\beta)/2-(1-\alpha)}\Theta\|G_D\|_{L_p}\|E(\cdot,s)\|_{L_2}\|\partial_s^\alpha\partial_y^\beta V\|_{L_2}ds\\
&+(1+t)^{-1-k/2}\|G_D(\cdot,t/2)\|_{L_p}\|E(\cdot,t/2)\|_{L_2}\|\partial_y^\beta V(\cdot,t/2)\|_{L_2}\Big).
\end{aligned}
$$

By the similar method as the one for the case of $l=0$, we can get

$$\|I_5^{1,k}\|_{L_p}\leq C\varepsilon_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}-1}.$$

Thus for $l+k\leq 3$ and $l\leq 1$, we have

$$\|I_5^{l,k}\|_{L_p}\leq C\varepsilon_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}-l}. \tag{5.9}$$

We now turn to $I_3^{l,k}$. First we can write

$$\tilde{F}_2=-(p'(\bar{v})\tilde{v}+p''(\bar{v}+\mu(V_x+\tilde{v}))((V_x+\tilde{v})^2/2),$$

with $0 < \mu < 1$. Since $m_0(x)$ is a smooth function with compact support and

$$|\partial_t^l \partial_x^k \tilde{v}(x,t)| \leq C|u_+ - u_-|e^{-\alpha t}\partial_x^k m_0(x),$$

we have

$$|\partial_t^l \partial_x^k \tilde{v}(x,t)| \leq C\varepsilon_0 e^{-\alpha t} e^{-x^2}. \tag{5.10}$$

Using Hölder inequality, (5.3) and (5.4), we have

$$\|\partial_t^l \partial_x^k ((p''(\bar{v} + \mu(\tilde{v} + V_x))V_x^2)\|_{L_p} \leq CM^2(t)(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{3+k}{2}-l}. \tag{5.11}$$

Since we have estimates only on $V(y,s) \in W^{m,\infty}([0,\infty); H^{3-m})$ for $m \leq 3$, (5.11) holds only when $l+k \leq 2$ and $l \leq 1$. When $l+k \leq 2$ and $l \leq 1$, (5.6), (5.10) and (5.11) yield

$$\|\partial_t^l \partial_x^k \tilde{F}_2\|_{L_p} \leq C(\varepsilon_0 + M^2(t))(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-l-\frac{3+k}{2}}; \tag{5.12}$$

When $l = 2, k = 0$, (5.4) and Lemma 3.2 give

$$\|\partial_t^2((p''(\bar{v} + \theta(\tilde{v} + V_x))V_x^2)\|_{L_p} \leq C(\varepsilon_0 + M^2(t))(1+t)^{-\frac{1}{2}(1-\frac{1}{q})-\frac{5}{2}-\frac{1}{2}}, \tag{5.13}$$

where q is a constant satisfying $\frac{1}{q} + \frac{1}{2} = \frac{1}{p} + 1$. Thus, from (5.6), (5.10) and (5.13), we have

$$\|\partial_t^2 \tilde{F}_2\|_{L_p} \leq C(\varepsilon_0 + M^2(t))(1+t)^{-\frac{1}{2}(1-\frac{1}{p})+\frac{1}{4}-3}. \tag{5.14}$$

We now consider $I_3^{l,k}$. When $k \leq 2$ and $l = 0$, it follows from (4.11) that

$$\begin{aligned}
\|I_3^{0,k}\|_{L_p} \leq\ & C\Big(\int_{t/2}^t \|\int_{-\infty}^\infty \partial_x G(\cdot,t;y,s)\partial_y^{k-1}F_2(y,s)dy\|_{L_p}ds \\
& + \int_0^{t/2} \|\int_{-\infty}^\infty \partial_x^k G(\cdot,t;y,s)F_2(y,s)dy\|_{L_p}ds\Big) \\
& + C\varepsilon_0 \int_{t/2}^t \|\int_{-\infty}^\infty \theta(t,s)G_D F_2(y,s)dy\|_{L_p}ds,
\end{aligned}$$

by Hausdoff-Young inequality, (4.7) and (5.12), we have

$$\begin{aligned}
\|I_3^{0,k}\|_{L_p} \leq\ & C(\varepsilon_0 + M^2(t))\Big(\int_0^{t/2}(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}}(1+s)^{-2}ds \\
& + \int_{t/2}^t((t-s)^{-\frac{1}{2}} + (1+s)^{-\frac{1}{2}})(1+s)^{-\frac{1}{2}(1-\frac{1}{p})-2-\frac{k-1}{2}}ds\Big) \\
\leq\ & C(\varepsilon_0 + M^2(t))(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}}.
\end{aligned}$$

When $k \leq 1$ and $l = 1$, it follows from (4.12) that

$$\begin{aligned}
\|I_3^{1,k}\|_{L_p} \leq\ & C\Big(\int_{t/2}^t \|\int_{-\infty}^\infty \partial_x^k G(\cdot,t;y,s)\partial_s F_2(y,s)dy\|_{L_p}ds \\
& + \int_0^{t/2} \|\int_{-\infty}^\infty \partial_s\partial_x^k G(\cdot,t;y,s)F_2(y,s)dy\|_{L_p}ds \\
& + O(1)\varepsilon_0 \int_{t/2}^t \int_{-\infty}^\infty \theta^2(t,s)G_D F_2(y,s)dyds \\
& + \|\int_{-\infty}^\infty \partial_x^k G(\cdot,t;y,t/2)F_2(y,t/2)dy\|\|_{L_p}.
\end{aligned}$$

(4.7) and (5.12) give

$$\|I_3^{1,k}\|_{L_p} \leq\ C(\varepsilon_0 + M^2(t))(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1-\frac{k}{2}}.$$

For $l+k=3$, since $(t-s)^{-h}$ is non-integrable in $[t,t/2]$ if $h\geq 1$ and $V(y,s)\in W^{m,\infty}([0,\infty);H^{3-m})$ for $m\leq 3$, we must replace $\partial_x^2 G(x,t;y,s)$ by $R_G(x,t;y,s)$ and $\partial_s G(x,t;y,s)$ by using (4.14) and (4.15). In fact, it follows from (4.14) that

$$\begin{aligned}
&\|I_3^{0,3}\|_{L_p}\\
\leq\ & \int_0^{t/2}\|\int_{-\infty}^{\infty}\partial_x^3 G(\cdot,t;y,s)F_2(y,s)dy\|_{L_p}ds\\
&+C\int_{t/2}^{t}(\|\int_{-\infty}^{\infty}R_G\partial_y F_2(y,s)dy\|_{L_p}+\|\int_{-\infty}^{\infty}\partial_y G\partial_s F_2(y,s)dy\|_{L_p}\\
&+\|\int_{-\infty}^{\infty}(1+s)^{-1/2}\partial_y G\partial_y F_2(y,s)dy\|_{L_p})ds\\
&+C\|\int_{-\infty}^{\infty}G\partial_y F_2(y,s)dy|_{s=t/2}\|_{L_p}\\
&+C\varepsilon_0\sum_{\beta\leq 1}\int_{t/2}^{t}\|\int_{-\infty}^{\infty}((t-s)^{-1/2}+\theta)\theta^{(2-\beta)}(t,s)G_D\partial_y^\beta F_2(y,s)dy\|_{L_p}ds\\
&+C\varepsilon_0\int_{t/2}^{t}\|\int_{-\infty}^{\infty}\theta(t,s)G_D\partial_s F_2(y,s)dy\|_{L_p}ds.
\end{aligned}$$

Then, (4.7), (5.9) and (5.12) yield get

$$\|I_3^{0,3}\|_{L_p}\leq\ C(\varepsilon_0+M^2(t))(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{3}{2}}.$$

We now estimate $I_3^{1,2}$ by using (4.15)

$$\begin{aligned}
&\|I_3^{1,2}\|_{L_p}\\
\leq\ & \int_0^{t/2}\|\int_{-\infty}^{\infty}\partial_t\partial_x^2 G(\cdot,t;y,s)F_2(y,s)dy\|_{L_p}ds\\
&+C\int_{t/2}^{t}(\|\int_{-\infty}^{\infty}R_G\partial_s\partial_y\tilde{F}_2(y,s)dy\|_{L_p}+\|\int_{-\infty}^{\infty}\partial_y G\partial_s^2\tilde{F}_2(y,s)dy\|_{L_p}\\
&+\|\int_{-\infty}^{\infty}(1+s)^{-1/2}\partial_y G\partial_s\partial_y\tilde{F}_2(y,s)dy\|_{L_p})ds\\
&+C\|\int_{-\infty}^{\infty}(G\partial_s\partial_y+\partial_x^2 G\partial_y)\tilde{F}_2(y,s)dy|_{s=t/2}\|_{L_p}\\
&+C\|\int_{-\infty}^{\infty}G\partial_s\partial_y\tilde{F}_2(y,s)dy|_{s=t}\|_{L_p}\\
&+C\varepsilon_0\sum_{\beta\leq 1}\int_{t/2}^{t}\|\int_{-\infty}^{\infty}((t-s)^{-1/2}+\theta)\theta^{(3-\beta)}(t,s)G_D\partial_y^{\beta+1}\tilde{F}_2(y,s)dy\|_{L_p}ds\\
&+C\varepsilon_0\int_{t/2}^{t}\|\int_{-\infty}^{\infty}(\theta^2 G_D\partial_s\partial_y+\theta G_D\partial_s^2)\tilde{F}_2(y,s)dy\|_{L_p}ds.
\end{aligned}$$

The proof is very similar to the proof of $I_3^{0,3}$, by noticing that we can use (5.14) not (5.12) for $\partial_s^2\tilde{F}_2(y,s)$. It follows that

$$\|I_3^{1,2}\|_{L_p}\leq\ C(\varepsilon_0+M^2(t))(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-2}.$$

In summary, for $l+k\leq 3$ and $l\leq 1$, we have

$$\|I_3^{l,k}\|_{L_p}\leq\ C(\varepsilon_0+M^2(t))(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}-l}. \tag{5.15}$$

Finally, we consider $I_4^{l,k}$. When $l=0,k\leq 2$, by (4.11) and intergration by part to variable s, we have

$$\begin{aligned}
\|I_4^{0,k}\|_{L_P}\leq\ & C\Big(\int_{t/2}^{t}\|\int_{-\infty}^{\infty}\partial_x^{k-1}G(\cdot,t;y,s)\partial_y\partial_s^2 V(y,s)dy\|_{L_p}ds\\
&+\int_{t/2}^{t}\|\int_{-\infty}^{\infty}(1+s)^{-1/2}G_D(\cdot-y,t-s)\partial_s^2 V(y,s)dy\|_{L_p}ds\\
&+\int_0^{t/2}\|\int_{-\infty}^{\infty}\partial_x^k\partial_s G(\cdot,t;y,s)\partial_s V(y,s)dy\|_{L_p}ds\\
&+\|\int_{-\infty}^{\infty}\partial_x^k G(\cdot,t;y,s)\partial_s V(y,s)dy|_{s=0}^{s=t/2}\|_{L_p}\Big).
\end{aligned}$$

Let q be a constant satisfying $\frac{1}{q}+\frac{1}{2}=\frac{1}{p}+1$. By (4.7), Lemmas 3.1 and 3.2, we have

$$\begin{aligned}\|I_4^{0,k}\|_{L_P} \leq\; & C\varepsilon_0\Big(\textstyle\int_{t/2}^t(t-s)^{-\frac{1}{2}(1-\frac{1}{q})-\frac{k-1}{2}}((1+s)^{-\frac{5}{2}}+(1+s)^{-2})ds\\ &+\textstyle\int_0^{t/2}(t-s)^{-\frac{1}{2}(1-\frac{1}{q})-\frac{k}{2}-1}(1+s)^{-1}ds\\ &+(1+t)^{-\frac{1}{2}(1-\frac{1}{q})-\frac{k}{2}-1}\Big)+\|U_0\|_{L_1}(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}}\\ \leq\; & C\varepsilon_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}}.\end{aligned}$$

If $l=1,k\leq 1$, by (4.12) and intergration by part to variable s, we have

$$\begin{aligned}\|I_4^{1,k}\|_{L_P} \leq\; & C\Big(\textstyle\int_{t/2}^t\|\int_{-\infty}^{\infty}\partial_x^kG(\cdot,t;y,s)\partial_s^3V(y,s)dy\|_{L_p}ds\\ &+\textstyle\int_{t/2}^t\|\int_{-\infty}^{\infty}(1+s)^{-1}G_D(\cdot-y,t-s)\partial_s^2V(y,s)dy\|_{L_p}ds\\ &+\textstyle\int_0^{t/2}\|\int_{-\infty}^{\infty}\partial_x^k\partial_s^2G(\cdot,t;y,s)\partial_sV(y,s)dy\|_{L_p}ds\\ &+\|\textstyle\int_{-\infty}^{\infty}\partial_s\partial_x^kG(\cdot,t;y,s)\partial_sV(y,s)dy|_{s=0}^{s=t/2}\|_{L_p}\\ &+\|\textstyle\int_{-\infty}^{\infty}\partial_x^kG(\cdot,t;y,s)\partial_s^2V(y,s)dy|_{s=t/2}\|_{L_p}\Big).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|I_4^{1,k}\|_{L_P} \leq\; & C\varepsilon_0\Big(\textstyle\int_{t/2}^t(t-s)^{-\frac{1}{2}(1-\frac{1}{q})-\frac{k}{2}}(1+s)^{-5/2}ds\\ &+\textstyle\int_0^{t/2}(t-s)^{-\frac{1}{2}(1-\frac{1}{q})-\frac{k}{2}-2}(1+s)^{-1}ds\\ &+(1+t)^{-\frac{1}{2}(1-\frac{1}{q})-\frac{k}{2}-\frac{3}{2}}\Big)+\|U_0\|_{L_1}(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}-1}\\ \leq\; & C\varepsilon_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}-1}.\end{aligned}$$

For $I_4^{0,3}$, we also need to replace $\partial_x^2G(x,t;y,s)$ by $R_G(x,t;y,s)$ and $\partial_sG(x,t;y,s)$. It follows from (4.14) that

$$\begin{aligned}&\|I_4^{0,3}\|_{L_p}\\ \leq\; & \textstyle\int_0^{t/2}\|\int_{-\infty}^{\infty}\partial_x^3G(\cdot,t;y,s)V_{ss}(y,s)dy\|_{L_p}ds\\ &+C\textstyle\int_{t/2}^t(\|\int_{-\infty}^{\infty}R_G\partial_yV_{ss}(y,s)dy\|_{L_p}+\|\int_{-\infty}^{\infty}\partial_yG\partial_sV_{ss}(y,s)dy\|_{L_p}\\ &+\|\textstyle\int_{-\infty}^{\infty}(1+s)^{-1/2}\partial_yG\partial_yV_{ss}(y,s)dy\|_{L_p})ds\\ &+C\|\textstyle\int_{-\infty}^{\infty}G\partial_yV_{ss}(y,s)dy|_{s=t/2}\|_{L_p}\\ &+C\varepsilon_0\textstyle\sum_{\beta\leq 1}\int_{t/2}^t\|\int_{-\infty}^{\infty}((t-s)^{-1/2}+\theta)\theta^{(2-\beta)}(t,s)G_D\partial_y^\beta V_{ss}(y,s)dy\|_{L_p}ds\\ &+C\varepsilon_0\textstyle\int_{t/2}^t\|\int_{-\infty}^{\infty}\theta(t,s)G_D\partial_sV_{ss}(y,s)dy\|_{L_p}ds.\end{aligned}$$

Using Hausdoff-Young inequality, (4.7), (4.16), Lemmas 3.1 and 3.2, we have

$$\|I_4^{0,3}\|_{L_p}\leq\; C\varepsilon_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{3}{2}}.$$

For $l=1,k=2$, we denote $I_4^{1,2}=I_{4,1}^{1,2}+I_{4,2}^{1,2}$, where

$$\begin{aligned}I_{4,1}^{1,2}=\;&\textstyle\int_0^{t-1}\int_{-\infty}^{\infty}\partial_t\partial_x^2G(x,t;y,s)(V_{ss}(y,s))dyds+\int_{-\infty}^{\infty}\partial_x^2G(x,t;y,s)V_{ss}(y,s)dy|_{s=t-1},\\ I_{4,2}^{1,2}=\;&\textstyle\int_{t-1}^t\int_{-\infty}^{\infty}\partial_t\partial_x^2G(x,t;y,s)(V_{ss}(y,s))dyds-\int_{-\infty}^{\infty}\partial_x^2G(x,t;y,s)V_{ss}(y,s)dy|_{s=t-1}\\ &+\partial_x^2\partial_t^2V(x,t).\end{aligned}$$

We first consider $I^{1,2}_{4,1}$. Using (4.12) and integrating by part of the variable s,

$$\begin{aligned}\|I^{1,2}_{4,1}\|_{L^P} \leq\ & C\Big(\int_0^{t/2}\|\int_{-\infty}^{\infty}\partial_x^2\partial_t\partial_s G(\cdot,t;y,s)\partial_s V(y,s)dy\|_{L_p}ds\\ &+\|\int_{-\infty}^{\infty}\partial_x^2\partial_t G(\cdot,t;y,s)\partial_s V(y,s)dy|_{s=0}^{s=t/2}\|_{L_p}\\ &+\int_{t/2}^{t-1}\|\int_{-\infty}^{\infty}\partial_x^2 G(\cdot,t;y,s)\partial_s^3 V(y,s)dy\|_{L_p}ds\\ &+\int_{t/2}^{t-1}\|\int_{-\infty}^{\infty}(1+s)^{-1}(t-s)^{-1}G_D(\cdot-y,t-s)\partial_s^2 V(y,s)dy\|_{L_p}ds\\ &+\|\int_{-\infty}^{\infty}\partial_x^2 G(\cdot,t;y,t/2)\partial_s^2 V(y,t/2)dy\|_{L_p}\Big).\end{aligned}$$

Using Hausdoff-Young inequality, (4.7), Lemmas 3.1 and 3.2 again, we obtain

$$\|I^{1,2}_{4,1}\|_{L^P} \leq C\varepsilon_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-2}.$$

Notice that we can replace $\partial_x^2 G(x,t;y,s)$ by $R_G(x,t;y,s)$ and $\partial_s G(x,t;y,s)$. Similar to the proof of (4.15), we have

$$\begin{aligned}&\|I^{1,2}_{4,2}\|_{L_p}\\ \leq\ & C\int_{t-1}^{t}(\|\int_{-\infty}^{\infty}R_G\partial_s V_{ss}(y,s)dy\|_{L_p}+\|\int_{-\infty}^{\infty}G\partial_s^2 V_{ss}(y,s)dy\|_{L_p}\\ &+\|\int_{-\infty}^{\infty}(1+s)^{-1/2}\partial_y G\partial_s V_{ss}(y,s)dy\|_{L_p})ds\\ &+C\|\int_{-\infty}^{\infty}(G\partial_s V_{ss}(y,s))dy|_{s=t-1}^{s=t}\|_{L_p}+C\|\int_{-\infty}^{\infty}(\partial_y^2 G-\partial_x^2)V_{ss}(y,s)|_{s=t-1}\|_{L_p}\\ &+C\varepsilon_0\sum_{\beta\leq 1}\int_{t-1}^{t}\|\int_{-\infty}^{\infty}((t-s)^{-1/2}+\theta)\theta^{(3-\beta)}(t,s)G_D\partial_y^\beta V_{ss}(y,s)dy\|_{L_p}ds\\ &+C\varepsilon_0\int_{t-1}^{t}\|\int_{-\infty}^{\infty}\theta(t,s)G_D\partial_s V_{ss}(y,s)dy\|_{L_p}ds.\end{aligned}$$

As for (4.11), we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\partial_x^2 G(x,t;y,s)g(y,s)dy\\ =\ &\int_{-\infty}^{\infty}\partial_y^2 G(x,t;y,s)g(y,s)dy\\ &+O(1)\varepsilon_0\sum_{\beta<2}\int_{-\infty}^{\infty}\theta^{(2-\beta)}(t,s)G_D\partial_y^\beta g(y,s)dy.\end{aligned}$$

Thus by Lemmas 3.1 and 3.2, we have

$$\begin{aligned}&\|\int_{-\infty}^{\infty}(\partial_y^2 G-\partial_x^2)V_{ss}(y,s)|_{s=t-1}\|_{L_p}\\ \leq\ & C\varepsilon_0(\|\int_{-\infty}^{\infty}\theta^1(t,s)G_D\partial_y V_{ss}(y,s)dy|_{s=t-1}\|_{L_p}+\|\int_{-\infty}^{\infty}\theta^2(t,s)G_D V_{ss}(y,s)dy|_{s=t-1}\|_{L_p}\\ \leq\ & C\varepsilon_0(1+t)^{-5/2}.\end{aligned}$$

Since

$$V_{ss}(y,s)=F(y,s)-\alpha V_s(y,s)+a_y(y,s)V_y(y,s)+a(y,s)V_{yy}(y,s),$$

The second term in the above inequality can be estimated by using the estimates for $F(y,s)$. Thus, we only need to consider the following three terms.

$$\begin{aligned}R_1 =\ & \int_{t-1}^{t}\int_{-\infty}^{\infty}G(x,t;y,s)V_{sss}(y,s)dyds,\\ R_2 =\ & \int_{t-1}^{t}\int_{-\infty}^{\infty}G(x,t;y,s)V_{yyss}(y,s)dyds,\\ R_3 =\ & \int_{t-1}^{t}\int_{-\infty}^{\infty}G(x,t;y,s)\partial_s^2(a_y(y,s)V_y(y,s))dyds,\end{aligned}$$

It follows from (4.11) that

$$\|R_1\|_{L_p} \leq\ C\int_{t-1}^{t}\|\int_{-\infty}^{\infty}G_D(t-s,\cdot-y)V_{sss}(y,s)dy\|_{L_p}ds$$

By Hausdoff-Young inequality, (4.8), Lemma 3.2, we have

$$\begin{aligned}\|R_1\|_{L_p} \leq & \ C\varepsilon_0\Big(\int_{t-1}^{t}((t-s)^{-\frac{1}{2}(1-\frac{1}{q})})(1+s)^{-\frac{5}{2}}ds\Big)\\ \leq & \ C\varepsilon_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-2}.\end{aligned}$$

Similarly, we can prove that

$$\|R_j\|_{L_p} \leq \ C\varepsilon_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-2},$$

for $j=2,3$. Therefore, (4.7), (5.9), Lemmas 3.1 and 3.2 imply that

$$\|I_{4,2}^{1,2}\|_{L_p} \leq \ C\varepsilon_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-2}.$$

Thus we have

$$\|I_4^{l,k}\|_{L_P} \leq C\varepsilon_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}-l}. \tag{5.16}$$

Combining (5.1), (5.5), (5.8), (5.9), (5.15) and (5.16), we have the estimate

$$\|\partial_t^l\partial_x^k V(\cdot,t)\|_{L_p} \leq C(\varepsilon_0+M^2(t))(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}-l}. \tag{5.17}$$

Thus for $l+k\leq 3$ and $l\leq 1$ we have

$$M(t)\leq C(\varepsilon_0+M^2(t)).$$

Taking ε_0 small enough and using continuity of $M(t)$ and induction, we conclude that $M(t)\leq C\varepsilon_0$, i.e.

$$\|\partial_t^l\partial_x^k V(\cdot,t)\|_{L_p} \leq C\varepsilon_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}-l}. \tag{5.18}$$

Since $U(x,t)=V_t(x,t)$ we have proved Theorem 2.3 from (5.18).

Q.E.D.

According to the above discussion, it is easy to see that we can get the optimal estimates for higher derivatives of the solution if we know that higher derivative of initial data are small enough. In fact, we have the following theorem.

Theorem 5. 1 *If $V_0(x)\in H^{m+1}(\mathbf{R})\cap L_1(\mathbf{R}), U_0(x)\in H^m(\mathbf{R})\cap L_1(\mathbf{R})$, and*

$$|u_+-u_-|+|v_+-v_-|+\|V_0\|_{H^{m+1}}+\|U_0\|_{H^m}+\|V_0\|_{L_1}+\|U_0\|_{L_1}\leq\varepsilon_0$$

for some sufficiently small ε_0, then there exists a global in time solution $V(x,t)$, $U(x,t)$ of (2.10). Moreover, if we have the following estimate

$$\sum_{l=2}^{m}(1+t)^{2l-1}\|\partial_x^{l-2}\partial_t^2V(\cdot,t)\|_{L_2}^2+\sum_{l=3}^{m}(1+t)^{2l-1}\|\partial_x^{l-3}\partial_t^3V(\cdot,t)\|_{L_2}^2\leq C\varepsilon_0, \tag{5.19}$$

for $l\leq m$, then

$$\|\partial_x^k V_x(t)\|_{L_p}=O(1)\varepsilon_0(1+t)^{-(1-1/p)/2-(k+1)/2}, \tag{5.20}$$

$$\|\partial_x^k U(t)\|_{L_p}=O(1)\varepsilon_0(1+t)^{-(1-1/p)/2-k/2-1} \tag{5.21}$$

for any $k\leq m$ and $p\in[2,+\infty)$.

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE SYSTEM OF COMPRESSIBLE ADIABATIC FLOW THROUGH POROUS MEDIA

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Abstract In Hsiao and Serre (Chin. Ann. Math. Ser.B (1995), pp. 1-14), they showed the solution to the following system

$$\begin{cases} v_t - u_x = 0, & (t,x) \in R_+ \times R, \\ u_t + p(v,s)_x = -\alpha u, & \alpha > 0, \\ s_t = 0, \end{cases}$$

with initial data

$$(v,u,s)(0,x) = (v_0,u_0,s_0)(x) \to (\underline{v}, u_\pm, \underline{s}), \qquad \text{as } x \to \pm\infty$$

tend to the following nonlinear parabolic equation time-asymptotically:

$$\begin{cases} \tilde{v}_t = -\frac{1}{\alpha} p(\tilde{v}, s_0)_{xx}, & (t,x) \in R_+ \times R, \\ \tilde{u} = -\frac{1}{\alpha} p(\tilde{v}, s_0)_x, \end{cases}$$

In this papar we got its convergence rate, which will be optimal.

Keywords. asymptotic behavior, the system of compressible adiabatic flow, convergence rate

AMS Subject Classification. Primary 35L65, 35L67, 76L05

1. Introduction. We consider the Cauchy problem for the equation of the form

$$\begin{cases} v_t - u_x = 0, & (t,x) \in R_+ \times R, \\ u_t + p(v,s)_x = -\alpha u, & \alpha > 0, \\ s_t = 0, \end{cases} \tag{1.1}$$

which can be used to model the adiabatic gas flow through porous media. Here v is the specific volume; u denotes the velocity; s stands for the entropy; p denotes the pressure with $p > 0$, $p_v < 0$ for $v > 0$. Typical example of p is $p(v,s) = (\gamma - 1)v^{-\gamma}e^s$ $(\gamma > 1)$.

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In Hsiao and Serre [2, 3], it has been proved that the solution of the Cauchy problem (1.1) with

$$(v,u,s)(0,x)=(v_0,u_0,s_0)(x)\to(v_\pm,u_\pm,s_\pm),\quad v_+=v_-,s_+=s_-\qquad \text{as } x\to\pm\infty \tag{1.2}$$

can be described time-asymptotically by the solution of the following equations

$$\begin{cases} \tilde v_t=-\frac{1}{\alpha}p(\tilde v,s_0)_{xx}, & (t,x)\in R_+\times R,\\ \tilde u=-\frac{1}{\alpha}p(\tilde v,s_0)_x, \end{cases} \tag{1.3}$$

or

$$\begin{cases} \tilde v_t-\tilde u_x=0, & (t,x)\in R_+\times R,\\ p(\tilde v,s_0)_x=-\alpha\tilde u. \end{cases} \tag{1.3'}$$

The system (1.3') is obtained from (1.1) by approximating the momentum equation $(1.1)_2$ (the second equation of (1.1)) with Darcy's law.

In the case of isentropic flow, namely, $s(t,x)\equiv constant$, Hsiao and Liu [4] has proved that the solution to the Cauchy problem (1.1) converges to that to (1.3) with a rate $t^{-\frac{1}{2}}$ in the sense of $L^2\cap L^\infty$-norm. More precisely, for any smooth function $m_0(x)$ with compact support satisfying

$$\int_R m_0(x)dx=1, \tag{1.4}$$

we put

$$\begin{cases} \hat v\equiv-\frac{u_+-u_-}{\alpha}m_0(x)e^{-\alpha t},\\ \hat u\equiv e^{-\alpha t}\left[u_-+(u_+-u_-)\int_{-\infty}^x m_0(y)dy\right], \end{cases} \tag{1.5}$$

and uniquely determine $(\tilde v,\tilde u)(t,x)$ by

$$\int_R\{v_0(x)-\tilde v(0,x)\}dx=\frac{u_+-u_-}{-\alpha}. \tag{1.6}$$

Then it holds that

$$\|(v-\tilde v-\hat v,u-\tilde u-\hat u)(t,\cdot)\|_{L^2\cap L^\infty}=O(t^{-\frac{1}{2}}). \tag{1.7}$$

Moreover, the first author [9] has obtained sharper rates than that. Precisely, if we put $(v-\tilde v-\hat v,u-\tilde u-\hat u)=(V_x,z)$ due to (1.6), it holds that

$$\|(V_x,z)(t,\cdot)\|_{L^2(R)}=O(t^{-\frac{1}{2}},t^{-1}) \tag{1.8}$$

and

$$\|(V_x,z)(t,\cdot)\|_{L^\infty(R)}=O(t^{-\frac{3}{4}},t^{-\frac{5}{4}}), \tag{1.9}$$

which are based on the L^2-energy estimates for the reformulated problem

$$\begin{cases} V_{tt}+\{p_v(\tilde v)V_x\}_x+\alpha V_t=\frac{1}{\alpha}p(\tilde v)_{xt}-\{p(V_x+\tilde v+\hat v)-p(\tilde v)-p_v(\tilde v)V_x\}_x\\ V(0,x)=\int_{-\infty}^x(v-\tilde v-\hat v)(0,y)dy,\ V_t(0,x)=(u-\tilde u-\hat u)(0,x). \end{cases} \tag{1.10}$$

Moreover, the fact that V_{tt} decays fast suggests that V has parabolic structure as $t \to \infty$. In fact, it is also shown that, if $v_+ = v_-$ and $(V, z)(0, x) \in L^1(R) \times L^1(R)$, then

$$\|(V_x, z)(t, \cdot)\|_{L^\infty(R)} = O(t^{-1}, t^{-\frac{3}{2}}). \tag{1.11}$$

In the nonisentropic case, the asymptotic stability in the case of $v_+ = v_-$ and $s_+ = s_-$ has been obtained in [2, 3]. Our purpose in this paper is to obtain its convergence rate, especially its second order term of asymptotic, which is on the same line as in [9-11]. See also Gallay and Raugel [1].

In the case of $v_+ \neq v_-$ and $s_+ = s_-$, Hsiao and Luo [5] has obtained the stability theorem. Furthermore in Marcati and Pan [8], the stability results with convergence rates have been obtained in the following cases: 1. $v_+ = v_-$ and $s_+ = s_-$ 2. $p(v_-, s_-) = p(v_+, s_+)$. Hence the case $s_+ \neq s_-$ has been partly solved. We note that in these papers all data are so small that the solutions are smooth. Since large data generally yields the singularity after a finite time, we need to consider the weak solution to treat large data. See also Hsiao and Luo [6] and the reference therein.

Throughout this paper we denote several generic constants by c or C. By $H^m(R)$ denote the usual Sobolev space with its norm

$$\|f\|_m := \sum_{k=0}^{m} \|\partial_x^k f\|, \quad \|\cdot\| = \|\cdot\|_0 = \|\cdot\|_{L^2(R)}.$$

2. Preliminaries and Theorems. For simplicity, we restrict our case to $v_+ = v_- := \underline{v}$, $u_+ = u_- := 0$, $s_+ = s_- := \underline{s}$, so that $(\hat{v}, \hat{u})(t, x) \equiv (0, 0)$. The constant α is normalized to 1 without loss of generality.

First, let us consider the problem (1.3) in order to reformulate our problem (1.1), (1.2). Since $\tilde{u}$ is defined by $(1.3)_2$, we investigate the Cauchy problem of $\tilde{v}$ to the parabolic equation:

$$\begin{cases} \tilde{v}_t + p(\tilde{v}, s_0)_{xx} = 0, & (t, x) \in R_+ \times R, \\ \tilde{v}(0, x) = \tilde{v}_0(x) \to \underline{v} & (x \to \pm\infty). \end{cases} \tag{2.1}$$

Eq.$(2.1)_1$ has a stationary solution $\bar{v}(x)$ defined by

$$p(\bar{v}(x), s_0(x)) = p(\underline{v}, \underline{s}). \tag{2.2}$$

For a typical form of $p(v, s)$ in gas dynamics, $\bar{v}$ is given by

$$\bar{v}(x) = e^{\frac{1}{\gamma}(s_0(x) - \underline{s})} \underline{v}. \tag{2.3}$$

Our first Proposition is on the property of $\tilde{v}$, which is necessary to investigate the behavior of solutions to (1.1),(1.2).

Proposition 1 (Asymptotic property of $\tilde{v}$).

Suppose that $p(v,s)$ is a smooth function with $p>0, p_v<0$ for $v>0$. If $(\tilde{v}_0-\bar{v}, s_0-\underline{s}) \in H^6(R)\times H^6(R)$ is sufficiently small, then there exists a unique global solution $\tilde{v}(t,x)$ to (2.1), which satisfies that

$$\partial_t^i(\tilde{v}-\bar{v}) \in C([0,\infty); H^{6-2i}(R)), \partial_t^j q_x \in C([0,\infty); H^{5-2j}(R))$$
$$q_{tttx} \in L^2(0,\infty; L^2(R)),$$

where $q = p(\bar{v},s) - p(\tilde{v},s)$ and $i = 0,1,2,3,\ j=0,1,2$, and that

$$\begin{aligned}
&\sum_{k=0}^{3}(1+t)^{2k}\|\partial_t^k(\tilde{v}-\bar{v})(t)\|^2 + \sum_{k=0}^{2}(1+t)^{2k+1}\|\partial_t^k q_x(t)\|^2 \\
&+\int_0^t \left(\sum_{k=1}^{3}(1+\tau)^{2k-1}\|\partial_t^k(\tilde{v}-\bar{v})(\tau)\|^2 + \sum_{k=0}^{3}(1+\tau)^{2k}\|\partial_t^k q_x(\tau)\|^2\right)d\tau \\
&\le C\|(\tilde{v}_0(\cdot)-\bar{v}(\cdot), s_0(\cdot)-\underline{s})\|_6^2.
\end{aligned} \tag{2.4}$$

Moreover, if $(\tilde{v}_0-\bar{v}, s_0-\underline{s}) \in L^1(R)\times L^1(R)$ is assumed, then $\tilde{v}$ satisfies

$$\begin{cases}
\|\tilde{v}(t,\cdot)-(\bar{v}(\cdot)+\bar{\theta}_0(t,\cdot))\|_{L^\infty} \le C(1+t)^{-1}, \\
\|(\tilde{v}-\bar{\theta}_0)_t(t,\cdot)\|_{L^\infty} \le C(1+t)^{-2}, \\
\|(\tilde{v}-\bar{\theta}_0)_{tt}(t,\cdot)\|_{L^\infty} \le C(1+t)^{-\frac{5}{2}}, \\
\|\tilde{v}(t,\cdot)-(\bar{v}(\cdot)+\bar{\theta}_0(t,\cdot))\|_{L^2} \le C(1+t)^{-\frac{3}{4}}, \\
\|(\tilde{v}-\bar{\theta}_0)_t(t,\cdot)\|_{L^2} \le C(1+t)^{-\frac{7}{4}}, \\
\|(\tilde{v}-\bar{\theta}_0)_{tt}(t,\cdot)\|_{L^2} \le C(1+t)^{-\frac{5}{2}},
\end{cases} \tag{2.5}$$

where $\bar{\theta}_0$ is given by the explicit formula

$$\bar{\theta}_0(t,x) = \frac{-p_v(\underline{v},\underline{s})}{-p_v(\bar{v},s)}\int_R G(t,x-y)\{\tilde{v}_0(y)-\bar{v}(y)\}dy, \tag{2.6}$$

by using the Green function of $v_t + p_v(\underline{v},\underline{s})v_{xx} = 0$.

Remark 1. In Proposition 1, we assume that the initial disturbance is in $H^6(R)$ in order to obtain the decay estimates for $\tilde{v}_{ttt}$, q_{tttx} etc., which will be used in the proof of Theorems 1 and 2 below. The function $\bar{\theta}_0(t,x)$ satisfies

$$(\bar{\theta}_0)_t = \frac{\underline{a}}{a(x)}(a(x)\bar{\theta}_0)_{xx}, \tag{2.7}$$

where $a(x) = -p_v(\bar{v}(x), s_0(x))$ and $\underline{a} = -p_v(\underline{v},\underline{s})$. Since $\|(a(\cdot)\bar{\theta}_0, a(\cdot)\bar{\theta}_{0t})(t)\|_{L^\infty} = O(t^{-\frac{1}{2}}, t^{-\frac{3}{2}})$ etc., we can say from (2.5) that $\bar{v}(x)+\bar{\theta}_0(t,x)$ is an asymptotic profile of $\tilde{v}$ as $t\to\infty$. It seems to be curious, because $\bar{\theta}_0$ satisfies (2.6') instead of $(\bar{\theta}_0)_t = (a(x)\bar{\theta}_0)_{xx}$, linearized equation of (2.1) around $\bar{v}$. However, we have adopted $\bar{\theta}_0$ in (2.6) which has an explicit formula.

We also obtain the asymptotic property of $\tilde{u}$.

Proposition 2.
The function $\tilde{u}$ defined by (1.3')$_2$ for $\tilde{v}$ obtained in Proposition 1 satisfies

$$\begin{cases} \|(\tilde{u}-\bar{q}_{0x})(t,\cdot)\|_{L^\infty} \le C(1+t)^{-\frac{3}{2}}\log(2+t), \\ \|(\tilde{u}-\bar{q}_{0x})(t,\cdot)\|_{L^2} \le C(1+t)^{-\frac{5}{4}}, \end{cases} \tag{2.8}$$

where $\bar{q}_0$ is given by

$$\bar{q}_0(t,x) = -p_v(\underline{v},\underline{s})\int_R G(t,x-y)\{\tilde{v}_0(y)-\bar{v}(y)\}dy. \tag{2.9}$$

The proofs of Proposition 1 and 2 will be given in Section 3.

We now turn to the original problem (1.1),(1.2) with $v_\pm = \underline{v}$ and $u_\pm = 0$. If $\tilde{v}_0(x) = \underline{v} + \frac{\delta_0}{\sqrt{4\pi}}\exp(-\frac{(x-x_0)^2}{4})$, $\delta_0 = \int_R(v_0(y)-\underline{v})dy$, then $\int_R(v_0-\tilde{v}_0)(x)dx = 0$. Hence $\int_R(v-\tilde{v})(t,x)dx \equiv 0$ follows from (1.1)$_1$ and (1.3')$_1$. Thus, putting

$$(V,z)(t,x) = \left(\int_{-\infty}^{x}(v-\tilde{v})(t,y)dy, (u-\tilde{u})(t,x)\right), \tag{2.10}$$

we have the reformulated problem

$$\begin{cases} V_t - z = 0, \\ z_t + \{p(V_x+\tilde{v},s)-p(\tilde{v},s)\}_x + z = p(\tilde{v},s)_{xt}, \\ (V,z)(0,x) = (V_0,z_0)(x) \\ \qquad := \left(\int_{-\infty}^{x}\{v_0(y)-\tilde{v}(0,y)\}dy, u_0(x)-\tilde{u}(0,x)\right) \\ \qquad \to 0 \qquad \text{as} \qquad x\to\pm\infty \end{cases} \tag{RP}$$

where $s(t,x) \equiv s_0(x) =: s(x)$.

Hsiao and Serre [2, 3] have shown that $(\tilde{v},\tilde{u})(t,x) \to (\bar{v}(x),0)$ as $t\to\infty$, and that $(V_x,z) = (v-\tilde{v},u-\tilde{u})(t,x) \to 0$ as $t\to\infty$ under suitably smallness conditions. Namely, the solution (v,u) to (1.1), (1.2) tends to $(\bar{v}(x),0)$ as t tends to infinity. In this paper, we obtain those convergence rates by applying not only L^2-energy method but also the Green function of parabolic equation.

Using the property of $\tilde{v}$ in Proposition 1, we obtain the following first main theorem based on the L^2-energy method.

Theorem 1.

In addition to the assumptions in Proposition 1, suppose that $v_0 - \tilde{v}_0 \in L^1(R)$. *If* $(V_0, z_0) \in H^3(R) \times H^2(R)$ *is sufficiently small, then there exists a unique global solution* $(V,z)(t,x)$ *to* (RP), *which satisfies*

$$\begin{aligned}
&\sum_{k=0}^{1}(1+t)^k\|\partial_x^k V(t)\|^2 + \sum_{k=2}^{3}(1+t)^k\|\partial_x^{k-1}P(t)\|^2 + \sum_{k=0}^{1}(1+t)^{k+2}\|\partial_x^k z(t)\|^2 \\
&+ (1+t)^3\|z_{xx}(t)\|^2 + (1+t)^4\|P_{xt}(t)\|^2 + \sum_{k=0}^{1}(1+t)^{2k+3}\|\partial_x^k z_t(t)\|^2 + (1+t)^5\|z_{tt}(t)\|^2 \\
&+ \int_0^t \left\{\|V_x(\tau)\|^2 + (1+\tau)\|P_x(\tau)\|^2 + (1+\tau)^2\|P_{xx}(\tau)\|^2 + (1+\tau)^3\|P_{xt}(\tau)\|^2\right\} d\tau \\
&+ \int_0^t \left\{\sum_{k=0}^{2}(1+\tau)^{k+1}\|\partial_x^k z(\tau)\|^2 + \sum_{k=0}^{1}(1+\tau)^{k+3}\|\partial_t^k z_t(\tau)\|^2 + (1+\tau)^5\|z_{tt}(\tau)\|^2\right\} d\tau \\
&\leq C\left(\|V_0\|_3^2 + \|z_0\|_2^2 + \|(\tilde{v}_0(\cdot) - \bar{v}(\cdot), s_0(\cdot) - \underline{s})\|_6^2\right).
\end{aligned} \tag{2.11}$$

Here $P = P(V_x) = P(V_x; \tilde{v}, s)(t,x) := p(V_x(t,x) + \tilde{v}(t,x), s(x)) - p(\tilde{v}(t,x), s(x))$.

Proof. The proof is given by the same method as in [9].

Remark 2. The decay estimate of (V_x, z) same as (1.9) is derived as follows. By the Sobolev inequality and $C^{-1}|V_x| \leq |P(V_x)| \leq C|V_x|$,

$$\begin{aligned}
\|V_x(t,\cdot)\|_{L^\infty} &\leq C\|P(t; V_x, \tilde{v}, s)\| \\
&\leq C\|P(t; V_x, \tilde{v}, s)\|^{\frac{1}{2}}\|P(t; V_x, \tilde{v}, s)_x\|^{\frac{1}{2}} \\
&\leq C\|V_x(t)\|^{\frac{1}{2}}\|P(t; V_x, \tilde{v}, s)_x\|^{\frac{1}{2}} \\
&\leq C(1+t)^{-\frac{1}{4}}(1+t)^{-\frac{1}{2}} = C(1+t)^{-\frac{3}{4}},
\end{aligned}$$

and

$$\begin{aligned}
\|z(t,\cdot)\|_{L^\infty} &\leq C\|z(t)\|^{\frac{1}{2}}\|z_x(t)\|^{\frac{1}{2}} \\
&\leq C(1+t)^{-\frac{1}{2}}(1+t)^{-\frac{3}{4}} = C(1+t)^{-\frac{5}{4}}.
\end{aligned}$$

Next, we obtain the optimal convergence rate, same as (1.11), assuming $(V_0, z_0) \in L^1(R) \times L^1(R)$. Linearized problem of (RP) is

$$\left\{\begin{array}{l}
V_{tt} + \{p_v(\tilde{v}, s)V_x\}_x + V_t = p(\tilde{v}, s)_{xt} - F_x \\
V(0,x) = V_0(x), V_t(0,x) = z_0(x),
\end{array}\right. \tag{2.12}$$

where $F = p(V_x + \tilde{v}, s) - p(\tilde{v}, s) - p_v(\tilde{v}, s)V_x$. Regarding (2.12) as the parabolic equation of V with "forcing terms", we have the expression

$$
\begin{aligned}
V(t,x) = &\int_R G(t, x-y)V_0(y)dy - \int_0^t \int_R G(t-\tau, x-y)(V_{tt} + F_x)dyd\tau \\
&+ \int_0^t \int_R G(t-\tau, x-y)p(\tilde{v}, s)_{xt}dyd\tau \\
&- \int_0^t \int_R G(t-\tau, x-y)\{(p_v(\tilde{v}, s) - p_v(\underline{v}, \underline{s}))V_x\}_x dyd\tau, \qquad (2.13)
\end{aligned}
$$

which are estimated by using the result in Theorem 1. Thus we have the following theorem.

Theorem 2.

In addition to the assumptions in Theorem 1, suppose that $(V_0, z_0) \in L^1(R) \times L^1(R)$. *Then the global solution* (V, z) *to* (RP) *satisfies*

$$
\left\{
\begin{array}{l}
\|(V_x, z)(t)\|_{L^\infty} = O(t^{-1}, t^{-\frac{3}{2}}) \\
\|(V_x, z)(t)\|_{L^2} = O(t^{-\frac{3}{4}}, t^{-\frac{5}{4}}).
\end{array}
\right. \qquad (2.14)
$$

Combining Propositions 1-2 and Theorem 2, we have the last theorem.

Theorem 3.

Suppose the same assumptions as those in Theorem 2. Then the global solution to (1.1),(1.2) satisfies

$$
\left\{
\begin{array}{l}
\|v(t,\cdot) - (\bar{v}(\cdot) + \bar{\theta}_0(t,\cdot))\|_{L^\infty} \le C(1+t)^{-1}, \\
\|u(t,\cdot) - \bar{q}_{0x}(t,\cdot)\|_{L^\infty} \le C(1+t)^{-\frac{3}{2}} \log(2+t), \\
\|v(t,\cdot) - (\bar{v}(\cdot) + \bar{\theta}_0(t,\cdot))\|_{L^2} \le C(1+t)^{-\frac{3}{4}}, \\
\|u(t,\cdot) - \bar{q}_{0x}(t,\cdot)\|_{L^2} \le C(1+t)^{-\frac{5}{4}}.
\end{array}
\right. \qquad (2.15)
$$

Remark 2. If, eventually, $\int_R (v_0 - \bar{v})(x)dx = 0$, then we can put $(V, z)(t,x) = \Big(\int_{-\infty}^x (v(t,y) - \bar{v}(y))dy, (u - \bar{u})(t,x)\Big)$, which yields simpler problem

$$
\left\{
\begin{array}{l}
V_t - z = 0, \\
z_t + \{p(V_x + \bar{v}, s) - p(\bar{v}, s)\}_x + z = 0.
\end{array}
\right.
$$

In the result, we have $\|v(t,\cdot) - \bar{v}(\cdot)\|_{L^\infty} = O(t^{-1})$ instead of (2.15). Hence, the estimate (2.15) almost implies that the diffusion wave $\bar{\theta}_0(t,x)$ carries on the amount $\int_R (v_0(x) - \bar{v}(x))dx$. In (2.15), we could not remove $\log(2+t)$.

The proofs of Theorems 1-3 will be done in Sections 4-5.

3. Asymptotic behavior of the parabolic equation. In this section, we prove Propositions 1 and 2. By setting θ as

$$\theta := \tilde{v} - \bar{v}, \tag{3.1}$$

the Cauchy problem (2.1) is rewritten as

$$\begin{cases} \theta_t = q(\theta, x)_{xx} \\ \theta|_{t=0} = \theta_0(x) \equiv \tilde{v}_0(x) - \bar{v}(x), \end{cases} \tag{3.2}$$

where

$$q(\theta, x) \equiv p(\bar{v}(x), s(x)) - p(\theta + \bar{v}(x), s(x)). \tag{3.3}$$

Applying the L^2-energy method, we first prove the following proposition.

Proposition 3.1. *Suppose that $p(v, s)$ is smooth function with $p > 0, p_v < 0$ for $v > 0$. If $(\theta_0, s_0 - \underline{s}) \in H^6(R) \times H^6(R)$ is sufficiently small, then there exists a unique global solution $\theta(t, x)$ to* (3.2), *which satisfies that*

$$\partial_t^i \theta \in C([0,\infty); H^{6-2i}(R)), \partial_t^j q_x \in C([0,\infty); H^{5-2j}(R))$$
$$q_{tttx} \in L^2(0,\infty; L^2(R)),$$

where $i = 0, 1, 2, 3$ and $j = 0, 1, 2$, and that

$$\begin{aligned} &\sum_{k=0}^{3} (1+t)^{2k} \|\partial_t^k \theta(t)\|^2 + \sum_{k=0}^{2} (1+t)^{2k+1} \|\partial_t^k q_x(t)\|^2 \\ &+ \int_0^t \left(\sum_{k=1}^{3} (1+\tau)^{2k-1} \|\partial_t^k \theta(\tau)\|^2 + \sum_{k=0}^{3} (1+\tau)^{2k} \|\partial_t^k q_x(\tau)\|^2 \right) d\tau \\ &\leq C \|(\theta_0(\cdot), s_0(\cdot) - \underline{s})\|_6^2. \end{aligned} \tag{3.4}$$

Next, using the Green function, we obtain an asymptotic profile under the assumption of $\theta_0 \in L^1(R)$, which gives the optimal decay rates of θ.

Proposition 3.2.

In addition to the assumptions in Proposition 3.1, suppose that $(\theta_0, s_0 - \underline{s}) \in L^1(R) \times L^1(R)$. Then the global solution $\theta(t, x)$ to (3.2) satisfies

$$\begin{cases} \|(\theta - \bar{\theta}_0)(t, \cdot)\|_{L^\infty} \leq C(1+t)^{-1}, \\ \|(\theta - \bar{\theta}_0)_t(t, \cdot)\|_{L^\infty} \leq C(1+t)^{-2}, \\ \|(\theta - \bar{\theta}_0)_{tt}(t, \cdot)\|_{L^\infty} \leq C(1+t)^{-\frac{11}{4}}, \\ \|(\theta - \bar{\theta}_0)(t, \cdot)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}}, \\ \|(\theta - \bar{\theta}_0)_t(t, \cdot)\|_{L^2} \leq C(1+t)^{-\frac{7}{4}}, \\ \|(\theta - \bar{\theta}_0)_{tt}(t, \cdot)\|_{L^2} \leq C(1+t)^{-\frac{5}{2}}. \end{cases} \tag{3.5}$$

Since

$$\tilde{u} = -p(\theta + \bar{v}, s_0)_x = -\{p(\theta + \bar{v}, s_0) - p(\underline{v}, \underline{s})\}_x$$
$$= -\{p(\theta + \bar{v}, s_0) - p(\bar{v}, s_0)\}_x = q_x,$$

we also estimate $\tilde{u}$ by using the Green function. Differentiating q in t, we obtain $q_t = -p_v(\theta + \bar{v}, s_0)\theta_t$. Substituting this into (3.2), we have

$$q_t = -p_v(\theta + \bar{v}, s_0)q_{xx}. \tag{3.6}$$

Then it hold that

Proposition 3.3.

Suppose the same assumptions as those in Proposition 2. Then the global solution $q(t,x)$ of (3.6) with $\tilde{u} = q_x$ satisfies

$$\begin{cases} \|(q - \bar{q}_0)(t,\cdot)\|_{L^\infty} \leq C(1+t)^{-1}, \\ \|(q - \tilde{q}_0)_x(t,\cdot)\|_{L^\infty} \leq C(1+t)^{-\frac{3}{2}}\log(2+t), \\ \|(q - \bar{q}_0)(t,\cdot)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}}, \\ \|(q - \tilde{q}_0)_x(t,\cdot)\|_{L^2} \leq C(1+t)^{-\frac{5}{4}}. \end{cases} \tag{3.7}$$

The assertions of Proposition 1 follow from Proposition 3.1 and 3.2, and those of Proposition 2 follow from Proposition 3.3. The proofs of Propositions 3.1 and 3.2 will be devided into several steps. Proposition 3.3 will be proved in the end of this section.

Proof of Proposition 3.1.

The proof is given by the combination of the local existence with a priori estimates. Since the local existence theorem is obtained in a standard way [11], we devote ourselves to the estimates under the a priori assumption

$$N_1(T) := \sup_{0 \leq t \leq T} \left\{ \sum_{k=0}^{3} (1+t)^k \|\partial_t^k \theta(t)\| + \sum_{k=0}^{2} (1+t)^{k+\frac{1}{2}} \|\partial_t^k q_x(t)\| \right\} \leq \varepsilon. \tag{3.8}$$

Estimate 1. Multiplying $(3.2)_1$ by $q(\theta, x)$ and integrating it over $[0,t] \times R$, we get

$$\int_R Q(\theta, x)dx + \int_0^t \int_R \{q(\theta,x)_x\}^2 dx d\tau$$
$$= \int_R Q(\theta,x)dx \bigg|_{t=0} \leq C\|(\theta_0, s_0 - \underline{s})\|^2, \tag{3.9}$$

where $Q(\theta, x) = \int_0^\theta q(\eta, x)d\eta$, which is equivalent to θ^2.

Estimate 2. Define $\bar{V} = q(\theta, x)_x$. Then

$$\bar{V}_t = q_{xt}. \tag{3.10}$$

Mutiplying (3.10) by $(1+t)\bar{V}$ and integrating it over R, we have

$$\frac{1}{2}\frac{d}{dt}\left\{(1+t)\int_R q_x^2 dx\right\} + (1+t)\int_R (-p_v)\theta_t^2 dx = \frac{1}{2}\int_R q_x^2 dx. \tag{3.11}$$

Integrating (3.11) and applying (3.9), we get

$$\begin{aligned}&(1+t)\int_R q_x^2 dx + \int_0^t (1+\tau)\int_R (-p_v)\theta_t^2 dx d\tau \\ &\le C\|(\theta_0, s_0 - \underline{s})\|_1^2.\end{aligned} \tag{3.12}$$

Estimate 3. Differentiate $(3.2)_1$ in t:

$$(\theta_t)_t = \{q(\theta, x)_t\}_{xx}. \tag{3.13}$$

Multiplying (3.13) by $(1+t)^2 q_t$ and integrating it over R, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{(1+t)^2\int_R (-p_v)\theta_t^2 dx\right\} + (1+t)^2\int_R q_{tx}^2 dx \\ &= (1+t)\int_R (-p_v)\theta_t^2 dx - (1+t)^2\int_R (-p_v)_t\theta_t^2 dx \\ &\le (1+t)\int_R (-p_v)\theta_t^2 dx + C(1+t)^2 \sup|\theta_t| \int_R \theta_t^2 dx\end{aligned} \tag{3.14}$$

and hence, by (3.12) and $N_1(T) \le \varepsilon$,

$$\begin{aligned}&(1+t)^2\int_R (-p_v)\theta_t^2 dx + \int_0^t (1+\tau)^2\int_R q_{tx}^2 dx d\tau \\ &\le C\|(\theta_0, s_0 - \underline{s})\|_2^2.\end{aligned} \tag{3.15}$$

Estimate 4. Mutiply (3.13) by $-p_v(\theta + \bar{v}, s)$ to obtain

$$q_{tt} = (-p_v)q_{txx} + (-p_v)_t\theta_t. \tag{3.16}$$

Multiplying (3.16) by $(1+t)^3(-q_{txx})$ yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{(1+t)^3\int_R q_{tx}^2 dx\right\} + (1+t)^3\int_R (-p_v)q_{txx}^2 dx \\ &= \frac{3}{2}(1+t)^2\int_R q_{tx}^2 dx + (1+t)^3\int_R q_{txx}(p_v)_t\theta_t dx\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}&(1+t)^3\int_R q_{tx}^2 dx + \int_0^t (1+\tau)^3\int_R (-p_v)q_{txx}^2 dx d\tau \\ &\le C\|(\theta_0, s_0 - \underline{s})\|_3^2,\end{aligned} \tag{3.18}$$

in a similar fashion to Estimate 3.

Estimate 5. Similar methods to Estimate 3 and 4 give the estimates:

$$(1+t)^4 \int_R (-p_v)\theta_{tt}^2 dx + \int_0^t (1+\tau)^4 \int_R q_{ttx}^2 dx d\tau \\ \leq C\|(\theta_0, s_0 - \underline{s})\|_4^2, \tag{3.19}$$

$$(1+t)^5 \int_R q_{ttx}^2 dx + \int_0^t (1+\tau)^5 \int_R (-p_v) q_{ttxx}^2 dx d\tau \\ \leq C\|(\theta_0, s_0 - \underline{s})\|_5^2, \tag{3.20}$$

and

$$(1+t)^6 \int_R (-p_v)\theta_{ttt}^2 dx + \int_0^t (1+\tau)^6 \int_R q_{tttx}^2 dx d\tau \\ \leq C\|(\theta_0, s_0 - \underline{s})\|_6^2. \tag{3.21}$$

Combining Estimates 1-5 completes the proof of Proposition 3.1. □

Remark 3. Since $q(\theta, x) = p(\bar{v}, s) - p(\theta + \bar{v}, s) = -p_v(\cdot, s)\theta$, the Sobolev inequality and (3.4) yield

$$\begin{aligned} \sup_R |\theta| &\leq C \sup_R |q| \\ &\leq C\|q(t)\|^{\frac{1}{2}} \|q_x(t)\|^{\frac{1}{2}} \\ &\leq C\|\theta(t)\|^{\frac{1}{2}} \|q_x(t)\|^{\frac{1}{2}} \leq C(1+t)^{-\frac{1}{4}}. \end{aligned} \tag{3.22}$$

Due to $q_t = -p_v(\theta + \bar{v}, s)\theta_t$, we have

$$\begin{aligned} \sup_R |\theta_t| &\leq C \sup_R |q_t| \\ &\leq C\|q_t(t)\|^{\frac{1}{2}} \|q_{tx}(t)\|^{\frac{1}{2}} \\ &\leq C\|\theta_t(t)\|^{\frac{1}{2}} \|q_{tx}(t)\|^{\frac{1}{2}} \leq C(1+t)^{-\frac{5}{4}}. \end{aligned} \tag{3.23}$$

Since $q_{tt} = -p_{vv}\theta_t^2 - p_v\theta_{tt}$,

$$\begin{aligned} \sup_R |q_{tt}| &\leq C\|q_{tt}(t)\|^{\frac{1}{2}} \|q_{ttx}(t)\|^{\frac{1}{2}} \\ &\leq C\left(\|\theta_t^2(t)\| + \|\theta_{tt}(t)\|\right)^{\frac{1}{2}} \|q_{ttx}(t)\|^{\frac{1}{2}} \\ &\leq C\left(\|\theta_t(t)\|_{L^\infty}\|\theta_t(t)\| + \|\theta_{tt}(t)\|\right)^{\frac{1}{2}} \|q_{ttx}(t)\|^{\frac{1}{2}} \\ &\leq C(1+t)^{-\frac{9}{4}}, \end{aligned} \tag{3.24}$$

and

$$\sup_R |\theta_{tt}| \le C \left(\sup_R |q_{tt}| + \sup_R |\theta_t(t)|^2 \right). \tag{3.25}$$

Therefore,

$$\sup_R |\theta_{tt}(t)| \le C(1+t)^{-\frac{9}{4}}. \tag{3.26}$$

Proof of Proposition 3.2.

First step.

We first investigate the Cauchy problem for the homogeneous linearlized equation to $(3.2)_1$:

$$\begin{cases} \bar{\theta}_t = (a(x)\bar{\theta})_{xx} \\ \bar{\theta}|_{t=0} = \theta_0(x), \end{cases} \tag{3.27}$$

where $a(x) \equiv -p_v(\bar{v}(x), s(x)) \to -p_v(\underline{v}, \underline{s}) \equiv \underline{a}$ as $x \to \pm\infty$. To obtain the precise decay estimates of $\bar{\theta}$, we here again combine the L^2-energy method with the explicit formula using the Green function. First, multiplying $(3.27)_1$ by $a(x)\bar{\theta}$ and integrating it over $[0,t] \times R$, we have

$$\frac{1}{2}\int_R a(x)\bar{\theta}^2 dx + \int_0^t \int_R \{(a(x)\bar{\theta})_x\}^2 dx d\tau = \frac{1}{2}\int_R a(x)\bar{\theta_0}^2 dx. \tag{3.28}$$

Next, multiplying $(3.27)_1$ by $a(x)$ and differentiating it in x, we obtain

$$(a(x)\bar{\theta})_{xt} = \{a(x)(a(x)\bar{\theta})_{xx}\}_x. \tag{3.29}$$

Multiplying (3.29) by $(1+t)(a(x)\bar{\theta})_x$ and integrating it over $[0,t] \times R$, we have

$$\begin{aligned} &\frac{1}{2}(1+t)\int_R \{(a(x)\bar{\theta})_x\}^2 dx + \int_0^t (1+\tau)\int_R a(x)\{(a(x)\bar{\theta})_{xx}\}^2 dx d\tau \\ &= \frac{1}{2}\int_R \{(a(x)\bar{\theta}_0)_x\}^2 dx + \int_0^t \int_R \{(a(x)\bar{\theta})_x\}^2 dx d\tau. \end{aligned} \tag{3.30}$$

By virtue of (3.28) and $(a(x)\bar{\theta})_{xx} = \bar{\theta}_t$, we obtain

$$\begin{aligned} &(1+t)\int_R \{(a(x)\bar{\theta})_x\}^2 dx + \int_0^t (1+\tau)\int_R a(x)\left[\{(a(x)\bar{\theta})_{xx}\}^2 + \bar{\theta}_t^2\right] dx d\tau \\ &\le C\|(\theta_0, s_0 - \underline{s})\|_1^2. \end{aligned} \tag{3.31}$$

Since $(\partial_t^k\bar{\theta})_t = (a(x)\partial_t^k\bar{\theta})_{xx}$, $k = 1,2,3$, same estimates as above give the following.

Lemma 3.1. *If $(\theta_0, s - \underline{s}) \in H^6(R) \times H^6(R)$, then it holds that*

$$\sum_{k=0}^{3} \left\{ (1+t)^{2k} \int_R a(x)(\partial_t^k \bar{\theta})^2 dx + (1+t)^{2k+1} \int_R \{(a(x)\partial_t^k \bar{\theta})_x\}^2 dx \right\}$$
$$+ \sum_{k=0}^{2} \int_0^t (1+\tau)^{2k} \int_R \{a(x)(\partial_t^k \bar{\theta})_x\}^2 dx d\tau$$
$$+ \sum_{k=0}^{2} \int_0^t (1+\tau)^{2k+1} \int_R a(x)[\{(a(x)\partial_t^k \bar{\theta})_{xx}\}^2 + (\partial_t^k \bar{\theta}_t)^2] dx d\tau$$
$$\leq C\|(\theta_0, s_0 - \underline{s})\|_6^2. \tag{3.32}$$

Remark 4. By Sobolev inequality and (3.32), we obtain

$$\begin{aligned} \sup_R |\bar{\theta}| &= \sup_R |a(x)^{-1}| \sup_R |a(x)\bar{\theta}| \\ &\leq C\|\sqrt{a(x)}\bar{\theta}(t)\|^{\frac{1}{2}} \|(a(x)\bar{\theta})_x(t)\|^{\frac{1}{2}} \\ &\leq C(1+t)^{-\frac{1}{4}}, \end{aligned} \tag{3.33}$$

$$\begin{aligned} \sup_R |\bar{\theta}_t| &\leq C\|\sqrt{a(x)}\bar{\theta}_t(t)\|^{\frac{1}{2}} \|(a(x)\bar{\theta}_t)_x(t)\|^{\frac{1}{2}} \\ &\leq C(1+t)^{-\frac{5}{4}}, \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} \sup_R |\bar{\theta}_{tt}| &\leq C\|\sqrt{a(x)}\bar{\theta}_{tt}(t)\|^{\frac{1}{2}} \|(a(x)\bar{\theta}_{tt})_x(t)\|^{\frac{1}{2}} \\ &\leq C(1+t)^{-\frac{9}{4}}. \end{aligned} \tag{3.35}$$

Second step.

Assuming that $\theta_0 \in L^1(R)$, we now obtain an asymptotic profile $\bar{\theta}_0$ of $\bar{\theta}$ defined in (2.6). Rewrite $(3.27)_1$ as

$$\bar{\theta}_t = \underline{a}\bar{\theta}_{xx} + \{(a(x) - \underline{a})\bar{\theta}\}_{xx},$$

to have the expression

$$\bar{\theta}(t,x) = \int_R G(t, x-y)\theta_0(y) dy + \int_0^t \int_R G(t-\tau, x-y)\{(a(y) - \underline{a})\bar{\theta}(\tau, y)\}_{yy} dy d\tau, \tag{3.36}$$

where

$$G(t,x) = \frac{1}{\sqrt{4\pi \underline{a} t}} \exp\left(-\frac{x^2}{4\underline{a}t}\right).$$

Integration by parts yields

$$
\begin{aligned}
&\int_{\frac{t}{2}}^{t}\int_{R} G\cdot\{(a(y)-\underline{a})\bar{\theta}(\tau,y)\}_{yy}dyd\tau\\
&\quad=-\frac{1}{\underline{a}}\int_{\frac{t}{2}}^{t}\int_{R} G_{\tau}\cdot(a(y)-\underline{a})\bar{\theta}(\tau,y)dyd\tau\\
&\quad=-\frac{1}{\underline{a}}\left[\int_{R} G\cdot(a(y)-\underline{a})\bar{\theta}(\tau,y)dy\right]_{\frac{t}{2}}^{t}+\frac{1}{\underline{a}}\int_{\frac{t}{2}}^{t}\int_{R} G\cdot(a(y)-\underline{a})\bar{\theta}_{\tau}(\tau,y)dyd\tau\\
&\quad=-\frac{1}{\underline{a}}(a(x)-\underline{a})\bar{\theta}(t,x)+\frac{1}{\underline{a}}\int_{R} G\left(\frac{t}{2},x-y\right)(a(y)-\underline{a})\bar{\theta}(\tau,y)dy\\
&\qquad+\frac{1}{\underline{a}}\int_{\frac{t}{2}}^{t}\int_{R} G\cdot(a(y)-\underline{a})\bar{\theta}_{\tau}(\tau,y)dyd\tau.
\end{aligned}
$$

Hence, by (3.36)

$$
\begin{aligned}
(\bar{\theta}-\bar{\theta}_0)(t,x)&=\frac{1}{a(x)}\int_{R} G\left(\frac{t}{2},x-y\right)(a(y)-\underline{a})\bar{\theta}\left(\frac{t}{2},y\right)dy\\
&\quad+\frac{\underline{a}}{a(x)}\int_{0}^{\frac{t}{2}}\int_{R} G_{yy}(t-\tau,x-y)(a(y)-\underline{a})\bar{\theta}(\tau,y)dyd\tau\\
&\quad+\frac{1}{a(x)}\int_{\frac{t}{2}}^{t}\int_{R} G(t-\tau,x-y)(a(y)-\underline{a})\bar{\theta}_{\tau}(\tau,y)dyd\tau\\
&\equiv I_1+I_2+I_3.
\end{aligned}
\tag{3.37}
$$

Here we have used $\bar{\theta}_0=\frac{\underline{a}}{a(x)}\int_R G\cdot\theta_0(y)dy$.

By $\theta_0\in L^1(R)$, it is easy to see

$$
|\bar{\theta}_0(t,x)|\le Ct^{-\frac{1}{2}}. \tag{3.38}
$$

Since $s_0-\underline{s}\in L^1(R)$, $a(y)-\underline{a}\in L^1(R)$. Hence

$$
\begin{aligned}
|I_1|&\le C\sup_R\left|G\left(\frac{t}{2},x-y\right)\right|\sup_R\left|\bar{\theta}\left(\frac{t}{2},y\right)\right|\|a(\cdot)-\underline{a}\|_{L^1}\\
&\le Ct^{-\frac{3}{4}},
\end{aligned}
\tag{3.39}
$$

$$
\begin{aligned}
|I_2|&\le C\int_0^{\frac{t}{2}}\sup_R|G_{yy}|\sup_R|\bar{\theta}(\tau,y)|\|a(\cdot)-\underline{a}\|_{L^1}d\tau\\
&\le C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{3}{2}}(1+\tau)^{-\frac{1}{4}}d\tau,\\
&\le Ct^{-\frac{3}{4}},
\end{aligned}
\tag{3.40}
$$

and

$$
\begin{aligned}
|I_3| &\le C\int_{\frac{t}{2}}^{t} \sup_R |G| \sup_R |\bar\theta_\tau(\tau,y)| \|a(\cdot)-\underline{a}\|_{L^1} d\tau \\
&\le C\int_0^{\frac{t}{2}} (t-\tau)^{-\frac12}(1+\tau)^{-\frac54} d\tau \\
&\le Ct^{-\frac34}.
\end{aligned} \tag{3.41}
$$

Here we have used (3.33) and (3.34). Combinig (3.37)-(3.41), we gives

$$
\sup_R |(\bar\theta-\bar\theta_0)(t,x)| \le Ct^{-\frac34}. \tag{3.42}
$$

Next, we estimate $\sup_R |(\bar\theta-\bar\theta_0)_t(t,x)|$. Differentiate (3.37) in t to have

$$
\begin{aligned}
(\bar\theta-\bar\theta_0)_t(t,x) &= \frac{1}{a(x)}\int_R G_t\left(\frac{t}{2}, x-y\right)(a(y)-\underline{a})\bar\theta\left(\frac{t}{2},y\right)dy \\
&+ \frac{1}{a(x)}\int_R G\left(\frac{t}{2}, x-y\right)(a(y)-\underline{a})\bar\theta_\tau\left(\frac{t}{2},y\right)dy \\
&+ \frac{\underline{a}}{a(x)}\int_0^{\frac{t}{2}}\int_R G_{yyt}\cdot(a(y)-\underline{a})\bar\theta(\tau,y)dyd\tau \\
&+ \frac{1}{a(x)}\int_{\frac{t}{2}}^{t}\int_R G\cdot(a(y)-\underline{a})\bar\theta_{\tau\tau}(\tau,y)dyd\tau.
\end{aligned} \tag{3.43}
$$

In a similar fashion to the previous estimates, we have

$$
\sup_R |(\bar\theta-\bar\theta_0)_t(t,x)| \le Ct^{-\frac74}. \tag{3.44}
$$

Differentiating (3.43) in t, we also obtain

$$
\sup_R |(\bar\theta-\bar\theta_0)_{tt}(t,x)| \le Ct^{-\frac{11}{4}}. \tag{3.45}
$$

Here, we go back to (3.37). By virtue of (3.42) and (3.44),

$$
\begin{cases}
\sup_R |\bar\theta(t,x)| \le \sup_R |\bar\theta_0(t,x)| + Ct^{-\frac34} \le Ct^{-\frac12}, \\
\sup_R |\bar\theta_t(t,x)| \le \sup_R |(\bar\theta_0)_t(t,x)| + Ct^{-\frac74} \le Ct^{-\frac32}.
\end{cases} \tag{3.46}
$$

Therefore, applying (3.46), instead of (3.33) and (3.34), to (3.37), we obtain

$$
\sup_R |\bar\theta(t,x)-\bar\theta_0(t,x)| \le Ct^{-1}. \tag{3.47}
$$

Similary, we have

$$
\sup_R |(\bar\theta-\bar\theta_0)_t(t,x)| \le Ct^{-2}. \tag{3.48}
$$

However, this method is not applicable to $(\bar\theta-\bar\theta_0)_{tt}$, because we have $\bar\theta_{ttt}$ in the expression of $(\bar\theta-\bar\theta_0)_{tt}$.

The L^2-estimates to $\bar\theta-\bar\theta_0$ are also obtained by applying the Hausdorff-Young inequality. Thus we have the following lemma.

Lemma 3.2. *In addition to the assumption in Lemma 3.1, if* $(\theta_0, s-\underline{s}) \in L^1(R)\times L^1(R)$, *then it holds that as* $t \longrightarrow \infty$,

$$\left\{\begin{array}{l} \|((\bar{\theta}-\bar{\theta}_0),(\bar{\theta}-\bar{\theta}_0)_t,(\bar{\theta}-\bar{\theta}_0)_{tt})(t,\cdot)\|_{L^\infty} = O(t^{-1},t^{-2},t^{-\frac{11}{4}}) \\ \|((\bar{\theta}-\bar{\theta}_0),(\bar{\theta}-\bar{\theta}_0)_t,(\bar{\theta}-\bar{\theta}_0)_{tt})(t,\cdot)\|_{L^2} = O(t^{-\frac{3}{4}},t^{-\frac{7}{4}},t^{-\frac{5}{2}}). \end{array}\right. \tag{3.49}$$

Third step.

We now turn to (3.2). The perturbation $\Theta := \theta - \bar{\theta}$ satisfies

$$\left\{\begin{array}{l} \Theta_t = (a(x)\Theta)_{xx} + \Phi(\theta,x)_{xx} \\ \Theta|_{t=0} = 0, \end{array}\right. \tag{3.50}$$

where

$$\Phi(\theta,x) = -\{p(\theta+\bar{v},s) - p(\bar{v},s) - p_v(\bar{v},s)\theta\}. \tag{3.51}$$

As same as (3.37), we have the expression

$$\begin{aligned} \Theta(t,x) &= \frac{1}{a(x)}\int_R G\left(\frac{t}{2},x-y\right)(a(y)-\underline{a})\Theta\left(\frac{t}{2},y\right)dy \\ &+ \frac{1}{a(x)}\int_0^{\frac{t}{2}}\int_R G_t\cdot(a(y)-\underline{a})\Theta(\tau,y)dyd\tau \\ &+ \frac{1}{a(x)}\int_{\frac{t}{2}}^t\int_R G\cdot(a(y)-\underline{a})\Theta_\tau(\tau,y)dyd\tau \\ &+ \frac{\underline{a}}{a(x)}\int_0^{\frac{t}{2}}\int_R G_{yy}\cdot\Phi dyd\tau + \frac{1}{a(x)}\int_{\frac{t}{2}}^t\int_R G_t\cdot\Phi(\tau,y)dyd\tau \\ &\equiv II_1+II_2+II_3+II_4+II_5. \end{aligned} \tag{3.52}$$

We estimate each term in (3.52). By virtue of (3.22),(3.23) and (3.46),

$$\begin{aligned} \|\Theta(t)\|_{L^\infty} &\le \|\theta(t)\|_{L^\infty} + \|\bar{\theta}(t)\|_{L^\infty} \\ &\le C(1+t)^{-\frac{1}{4}}, \end{aligned} \tag{3.53}$$

and

$$\begin{aligned} \|\Theta_t(t)\|_{L^\infty} &\le \|\theta_t(t)\|_{L^\infty} + \|\bar{\theta}_t(t)\|_{L^\infty} \\ &\le C(1+t)^{-\frac{5}{4}}. \end{aligned} \tag{3.54}$$

Hence, II_1-II_3 are easily estimated as

$$|II_1, II_2, II_3| \le Ct^{-\frac{3}{4}}. \tag{3.55}$$

Estimates of II_4 and II_5 are as follows:

$$\begin{aligned}|II_4| &\le C\int_0^{\frac{t}{2}} \sup_R |\Phi| \|G_{yy}(t-\tau)\|_{L^1} d\tau \\ &\le C\int_0^{\frac{t}{2}} \sup_R |\theta|^2 (t-\tau)^{-1} d\tau \\ &\le Ct^{-\frac{1}{2}},\end{aligned} \tag{3.56}$$

and

$$\begin{aligned}|II_5| &\le C\left| -\left[\int_R G\Phi(\tau,y)dy\right]_{\frac{t}{2}}^t + \int_{\frac{t}{2}}^t \int_R G\Phi_\tau(\tau,y)dyd\tau\right| \\ &\le C\Bigg\{ |\Phi(\theta,x)| + \left|\int_R G\left(\frac{t}{2}, x-y\right)\Phi dy\right| \\ &\qquad + \int_{\frac{t}{2}}^t \sup_R |\Phi_\tau(\tau,y)| \|G(t-\tau)\|_{L^1} d\tau \Bigg\} \\ &\le C\left\{ \sup_R |\theta|^2 + \int_{\frac{t}{2}}^t \sup_R |\theta| \sup_R |\theta_t| d\tau \right\} \\ &\le Ct^{-\frac{1}{2}}.\end{aligned} \tag{3.57}$$

Therefore, we obtain

$$\|\Theta(t)\|_{L^\infty} \le C(1+t)^{-\frac{1}{2}}, \tag{3.58}$$

and hence

$$\|\theta(t)\|_{L^\infty} \le \|\Theta(t)\|_{L^\infty} + \|\bar{\theta}(t)\|_{L^\infty} \le C(1+t)^{-\frac{1}{2}}. \tag{3.59}$$

Applying (3.54) and (3.58), instead of (3.53), to (3.52) again, we have

$$\|\Theta(t)\|_{L^\infty} \le C(1+t)^{-\frac{3}{4}}. \tag{3.60}$$

If we obtain faster decay of $\|\Theta_t(t)\|_{L^\infty}$ than (3.54), then $\|\Theta(t)\|_{L^\infty}$ will decay faster than (3.60). Hence we next estimate $\|\Theta_t(t)\|_{L^\infty}$ using explicit formula. Since Θ_t satisfies

$$\begin{cases} (\Theta_t)_t = (\underline{a}\Theta_t)_{xx} + \{(a(x)-\underline{a})\Theta_t\}_{xx} + (\Phi(\theta,x)_t)_{xx} \\ \Theta_t|_{t=0} = \Phi(\theta_0,x)_{xx}, \end{cases} \tag{3.61}$$

we also have the expression similar to (3.52)

$$\Theta_t(t,x) = \frac{\underline{a}}{a(x)} \int_R G\left(\frac{t}{2}, x-y\right)(a(y)-\underline{a})\Theta_t\left(\frac{t}{2}, y\right) dy$$
$$+ \frac{1}{a(x)} \int_0^{\frac{t}{2}} \int_R G_t \cdot (a(y)-\underline{a})\Theta_t(\tau) dy d\tau + \frac{1}{a(x)} \int_{\frac{t}{2}}^t \int_R G \cdot (a(y)-\underline{a})\Theta_{tt}(\tau) dy d\tau$$
$$+ \frac{1}{a(x)} \left\{ \int_R G_t \left(\frac{t}{2}, x-y\right) \Phi\left(\frac{t}{2}\right) dy + \int_0^{\frac{t}{2}} \int_R G_{tt} \cdot \Phi dy d\tau \right\}$$
$$+ \frac{1}{a(x)} \left\{ \int_{\frac{t}{2}}^t \int_R G \cdot \Phi_{tt}(\tau, y) dy d\tau + \int_R G\left(\frac{t}{2}, x-y\right) \Phi_t dy - \Phi_t(\theta, x) \right\}.$$
$$\equiv III_1 + III_2 + III_3 + III_4 + III_5. \tag{3.62}$$

Applying (3.54) and (3.59), we obtain

$$|III_1| \le C \sup_R \left|G\left(\frac{t}{2}\right)\right| \sup_R |\Theta_t| \|a(\cdot)-\underline{a}\|_{L^1}$$
$$\le Ct^{-\frac{7}{4}}, \tag{3.63}$$

$$|III_2| \le C\|a(\cdot)-\underline{a}\|_{L^1} \int_0^{\frac{t}{2}} \sup_R |G_t(t-\tau)| \sup_R |\Theta_t(\tau,y)| d\tau$$
$$\le C \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{3}{2}} (1+\tau)^{-\frac{5}{4}} d\tau$$
$$\le Ct^{-\frac{7}{4}}, \tag{3.64}$$

and

$$|III_4| \le C \left\{ \sup_R \left|G_t\left(\frac{t}{2}\right)\right| \sup_R \left|\Phi\left(\frac{t}{2}, y\right)\right| + \int_0^{\frac{t}{2}} \|G_{tt}(t-\tau)\|_{L^1} \sup_R |\Phi(\tau,y)| d\tau \right\}$$
$$\le C \left\{ (1+t)^{-1-1} + \int_0^{\frac{t}{2}} (t-\tau)^{-2} (1+\tau)^{-1} d\tau \right\}$$
$$\le C(1+t)^{-2} \log(2+t). \tag{3.65}$$

Since

$$\|\Theta_{tt}(t)\|_{L^\infty} \le \|\theta_{tt}(t)\|_{L^\infty} + \|\bar{\theta}_{tt}(t)\|_{L^\infty} \le C(1+t)^{-\frac{9}{4}}$$

by (3.26), (3.49) and $\|(\bar{\theta}_0)_{tt}(t)\|_{L^\infty} \le C(1+t)^{-\frac{5}{2}}$, we have

$$|III_3| \le C\|a(\cdot)-\underline{a}\|_{L^1} \int_{\frac{t}{2}}^t \sup_R |G(t-\tau)| \sup_R |\Theta_{tt}(\tau,y)| d\tau$$
$$\le C \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{9}{4}} d\tau \le Ct^{-\frac{7}{4}}, \tag{3.66}$$

and

$$
\begin{aligned}
|III_5| &\leq C\left\{\int_{\frac{t}{2}}^{t} \|G(t-\tau)\|_{L^1} \sup_R |\Phi_{tt}(\tau)|d\tau + \left\|G\left(\frac{t}{2}\right)\right\|_{L^1} \sup_R \left|\Phi_t\left(\frac{t}{2}\right)\right|\right. \\
&\qquad \left. + \sup_R |\Phi_y(t,y)|\right\} \\
&\leq C\left\{\int_{\frac{t}{2}}^{t} (\sup_R |\theta_t(\tau)|^2 + \sup_R |\theta(\tau)| \sup_R |\theta_{tt}(\tau)|)d\tau\right. \\
&\qquad \left. + \sup_R \left|\theta\left(\frac{t}{2}\right)\right| \sup_R \left|\theta_t\left(\frac{t}{2}\right)\right| + \sup_R |\theta(t)| \sup_R |\theta_t(t)|\right\} \\
&\leq C\left\{\int_{\frac{t}{2}}^{t} (1+\tau)^{-\frac{5}{2}} d\tau + (1+t)^{-\frac{7}{4}}\right\} \leq Ct^{-\frac{3}{2}}.
\end{aligned} \tag{3.67}
$$

Here we have used $|\partial_t \Phi(\theta, x)| \leq C|\theta||\theta_t|, |\partial_{tt}\Phi(\theta, x)| \leq C(|\theta_t|^2 + |\theta||\theta_{tt}|)$. Combining (3.62)-(3.67) we have

$$\|\Theta_t(t)\|_{L^\infty} \leq Ct^{-\frac{3}{2}} \tag{3.68}$$

and hence

$$\|\theta_t(t)\|_{L^\infty} \leq \|\Theta_t(t)\|_{L^\infty} + \|\bar{\theta}_t(t)\|_{L^\infty} \leq Ct^{-\frac{3}{2}}. \tag{3.69}$$

Applying (3.68), instead of (3.54), and (3.69) to (3.62) again, we have

$$\|\Theta_t(t)\|_{L^\infty} \leq C(1+t)^{-\frac{7}{4}}. \tag{3.70}$$

We now go back to the estimate of $\|\Theta(t)\|_{L^\infty}$. Applying (3.60) and (3.70) to (3.52), we obtain the sharper estimate

$$\|\Theta(t)\|_{L^\infty} \leq C(1+t)^{-1}\log(2+t). \tag{3.71}$$

By differentiating $(3.61)_1$ with respect to t, we have the explicit formula of Θ_{tt}, similar to (3.62). Estimating all terms, we obtain

$$\|\Theta_{tt}(t)\|_{L^\infty} \leq C(1+t)^{-\frac{11}{4}} \tag{3.72}$$

the details of which are omitted. If we apply (3.72) to (3.62), then we get

$$\|\Theta_t(t)\|_{L^\infty} \leq C(1+t)^{-2}\log(2+t) \tag{3.73}$$

The L^2-estimates of Θ are also obtained by the Hausdorff-Young inequality.

$$
\left\{
\begin{aligned}
&\|\Theta(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}}, \\
&\|\Theta_t(t)\|_{L^2} \leq C(1+t)^{-\frac{7}{4}}, \\
&\|\Theta_{tt}(t)\|_{L^2} \leq C(1+t)^{-\frac{5}{2}}.
\end{aligned}
\right. \tag{3.74}
$$

Once more again, applying (3.74) to (3.52) and (3.62), we obtain

$$
\left\{
\begin{aligned}
&\|\Theta(t)\|_{L^\infty} \leq C(1+t)^{-1}, \\
&\|\Theta_t(t)\|_{L^\infty} \leq C(1+t)^{-2}.
\end{aligned}
\right. \tag{3.75}
$$

Thus we obtain the following lemma.

Lemma 3.3. *In addition to the assumptions in Lemma 3.1, if* $(\theta_0, s-\underline{s}) \in L^1(R)\times L^1(R)$, *then it holds that as* $t \to \infty$,

$$\begin{cases} \|(\Theta, \Theta_t, \Theta_{tt})(t,\cdot)\|_{L^\infty} = O(t^{-1}, t^{-2}, t^{-\frac{11}{4}}), \\ \|(\Theta, \Theta_t, \Theta_{tt})(t,\cdot)\|_{L^2} = O(t^{-\frac{3}{4}}, t^{-\frac{7}{4}}, t^{-\frac{5}{2}}). \end{cases} \tag{3.76}$$

Combining Lemmas 3.2 and 3.3, we conclude the Proposition 3.2. □

Proof of Proposition 3.3.

Rewrite (3.6) as

$$q_t - \underline{a} q_{xx} = -\{p_v(\theta + \bar{v}, s_0) + \underline{a}\} q_{xx}, \tag{3.77}$$

to have the expression

$$q(t,x) = \int_R G(t, x-y) q_0(y) dy - \int_0^t \int_R G(t-\tau, x-y)\{p_v(\theta + \bar{v}, s_0) + \underline{a}\} q_{yy} dy d\tau. \tag{3.78}$$

Here we put $q(0,x) = q_0(x)$ and $\underline{a} = -p_v(\underline{v}, \underline{s})$. Integration by parts yields

$$\begin{aligned} & -\int_0^{\frac{t}{2}} \int_R G(t-\tau, x-y)\{p_v(\theta + \bar{v}, s_0) + \underline{a}\} q_{yy} dy d\tau \\ & = \int_0^{\frac{t}{2}} \int_R G(t-\tau, x-y)(q_\tau - \underline{a}\theta_\tau) dy d\tau \\ & = \left\{ \left[\int_R G(t-\tau, x-y)(q - \underline{a}\theta) dy \right]_0^{\frac{t}{2}} - \int_0^{\frac{t}{2}} \int_R G_\tau(t-\tau, x-y)(q - \underline{a}\theta) dy d\tau \right\}. \end{aligned} \tag{3.79}$$

Thus we obtain

$$\begin{aligned} (q - \bar{q}_0)(t,x) = & \int_R G \cdot (q - \underline{a}\theta) dy \Big|_{\tau = \frac{t}{2}} - \int_0^{\frac{t}{2}} \int_R G_\tau(t-\tau, x-y)(q - \underline{a}\theta) dy d\tau \\ & - \int_{\frac{t}{2}}^t \int_R G(t-\tau, x-y)\{p_v(\theta + \bar{v}, s_0) + \underline{a}\} q_{yy} dy d\tau \end{aligned} \tag{3.80}$$

and

$$\begin{aligned} (q - \bar{q}_0)_x(t,x) = & \int_R G_x \cdot (q - \underline{a}\theta) dy \Big|_{\tau = \frac{t}{2}} - \int_0^{\frac{t}{2}} \int_R G_{x\tau}(t-\tau, x-y)(q - \underline{a}\theta) dy d\tau \\ & - \int_{\frac{t}{2}}^t \int_R G_x(t-\tau, x-y)\{p_v(\theta + \bar{v}, s_0) + \underline{a}\} q_{yy} dy d\tau. \end{aligned} \tag{3.81}$$

Here we have used $\bar{q}_0(t,x) = \underline{a} \int_R G(t, x-y)\theta_0(y) dy$. Dividing the final term of (3.81) as

$$-\left(\int_{\frac{t}{2}}^{t-1} + \int_{t-1}^t \right) \int_R G_x \cdot \{p_v(\theta + \bar{v}, s_0) + \underline{a}\} q_{yy} dy d\tau = (i) + (ii),$$

we seek the L^∞-estimate of (i) and (ii). Noting that

$$\begin{cases} q - \underline{a}\theta = -\{p(\theta+\bar{v}, s_0) - p(\bar{v}, s_0) - p_v(\bar{v}, s_0)\theta + (p_v(\bar{v}, s_0) - p_v(\underline{v}, \underline{s}))\theta\}, \\ \qquad = O(|\theta|^2 + |a(x) - \underline{a}||\theta|), \\ p_v(\theta+\bar{v}, s_0) + \underline{a} = p_v(\theta+\bar{v}, s_0) - p_v(\bar{v}, s_0) + p_v(\bar{v}, s_0) - p_v(\underline{v}, \underline{s}), \\ \qquad\qquad = O(|\theta| + |a(x) - \underline{a}|), \end{cases}$$

we obtain

$$\begin{aligned} |(i)| &\le C \int_{\frac{t}{2}}^{t-1} \int_R |G|(|\theta| + |a(y) - \underline{a}|)|\theta_\tau| dy d\tau \\ &\le C \int_{\frac{t}{2}}^{t-1} \|G(t-\tau)\|_{L^\infty} \left(\|\theta(\tau)\|_{L^2} \|\theta_\tau(\tau)\|_{L^2} + \|a(y) - \underline{a}\|_{L^1} \|\theta_\tau(\tau)\|_{L^\infty}\right) d\tau \\ &\le C \int_{\frac{t}{2}}^{t-1} (t-\tau)^{-1}(1+\tau)^{-\frac{3}{2}} d\tau \le C t^{-\frac{3}{2}} \log(2+t), \end{aligned}$$

and

$$\begin{aligned} |(ii)| &\le C \int_{t-1}^{t} \|G(t-\tau)\|_{L^2} \|\theta_\tau(\tau)\|_{L^\infty} \left(\|\theta(\tau)\|_{L^2} + \|a(y) - \underline{a}\|_{L^2}\right) d\tau \\ &\le C \int_{t-1}^{t} (t-\tau)^{-\frac{3}{4}} (1+\tau)^{-\frac{3}{2}} d\tau \le C t^{-\frac{3}{2}}, \end{aligned}$$

The other terms are estimated in a similar fashion to Proposition 3.2. The details are omitted. □

4. Pointwise estimate by approximate Green function.

In this final section, we devote ourselves to the proof of Theorem 2. In an expression obtained by differentiating V of (2.13) in x, $\int_{\frac{t}{2}}^{t} \int_R G_x \cdot \{(p_v(\tilde{v}, s) + \underline{a}) V_y\}_y dy d\tau$ is rewritten as follows:

$$\begin{aligned} &- \int_{\frac{t}{2}}^{t} \int_R G_x \cdot \{(p_v(\tilde{v}, s) + \underline{a}) V_y\}_y dy d\tau \\ &= - \int_{\frac{t}{2}}^{t} \int_R G_{yy} \cdot (p_v(\tilde{v}, s) + \underline{a}) V_y dy d\tau \\ &= \frac{1}{\underline{a}} \int_{\frac{t}{2}}^{t} \int_R G_\tau \cdot (p_v(\tilde{v}, s) + \underline{a}) V_y dy d\tau \\ &= \frac{1}{\underline{a}} \left\{ \left[\int_R G \cdot (p_v(\tilde{v}, s) + \underline{a}) V_y dy \right]_{\frac{t}{2}}^{t} - \int_{\frac{t}{2}}^{t} \int_R G \cdot \{(p_v(\tilde{v}, s) + \underline{a}) V_y\}_\tau dy d\tau \right\} \\ &= \frac{(p_v(\tilde{v}, s) + \underline{a})}{\underline{a}} V_x(t, x) - \frac{1}{\underline{a}} \int_R G \cdot (p_v(\tilde{v}, s) + \underline{a}) V_y dy \Big|_{\tau = \frac{t}{2}} \\ &\quad - \frac{1}{\underline{a}} \int_{\frac{t}{2}}^{t} \int_R G \cdot \{(p_v(\tilde{v}, s) + \underline{a}) V_y\}_\tau dy d\tau, \end{aligned}$$

where G is defined in (2.6) and $\underline{a} = -p_v(\underline{v}, \underline{s})$. Hence,

$$
\begin{aligned}
V_x(t,x) &= \frac{\underline{a}}{-p_v(\tilde{v},s)} \int_R G_x(t,x-y)V_0(y)dy \\
&\quad - \frac{\underline{a}}{-p_v(\tilde{v},s)} \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\right) \int_R G_x\{F_y + V_{\tau\tau}\}dyd\tau \\
&\quad + \frac{\underline{a}}{-p_v(\tilde{v},s)} \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\right) \int_R G_x p(\tilde{v},s)_{y\tau}dyd\tau \\
&\quad - \frac{1}{-p_v(\tilde{v},s)} \int_R (p_v(\tilde{v},s)+\underline{a}))G\left(\frac{t}{2}\right)V_y\left(\frac{t}{2}\right)dy \\
&\quad - \frac{\underline{a}}{-p_v(\tilde{v},s)} \int_0^{\frac{t}{2}} \int_R (p_v(\tilde{v},s)+\underline{a}))\,G_{yy}V_ydyd\tau \\
&\quad - \frac{1}{-p_v(\tilde{v},s)} \int_{\frac{t}{2}}^t \int_R \{(p_v(\tilde{v},s)+\underline{a}))\,V_y\}_\tau\, Gdyd\tau \\
&= J_1 + (J_{21}+J_{22}) + (J_{31}+J_{32}) + J_4 + J_5 + J_6.
\end{aligned} \tag{4.1}
$$

By $V_0 \in L^1(R)$, it is easy to see

$$
|J_1| \le Ct^{-1}. \tag{4.2}
$$

To estimate all other terms, we use (2.5) in Proposition 1 and (2.11) in Theorem 1. Integration by parts in x yields

$$
\begin{aligned}
|J_{21}| &\le C\left|\int_0^{\frac{t}{2}}\left(\int_R G_{yy}Fdx + \int_R G_y z_\tau dy\right)d\tau\right| \\
&\le C\left[\int_0^{\frac{t}{2}}\int_R |G_{yy}||F|dyd\tau + \left|\left[\int_R G_y zdy\right]_0^{\frac{t}{2}} - \int_0^{\frac{t}{2}}\int_R G_{y\tau}zdyd\tau\right|\right] \\
&\le C\left[\int_0^{\frac{t}{2}} \sup_R |G_{yy}(t-\tau)|\|V_y(\tau)\|^2 d\tau + \left\|G\left(\frac{t}{2}\right)\right\|\left\|z_x\left(\frac{t}{2}\right)\right\|\right. \\
&\qquad \left. + \sup_R |G_y(t)|\|z(0)\|_{L^1} + \int_0^{\frac{t}{2}} \|G_\tau(t-\tau)\|\|z_y(\tau)\|d\tau\right] \\
&\le C\left[t^{-\frac{3}{2}}\int_0^{\frac{t}{2}}\|V_y(\tau)\|^2 d\tau + t^{-\frac{1}{4}-\frac{3}{2}} + t^{-1} + t^{-\frac{5}{4}}\int_0^{\frac{t}{2}}(1+\tau)^{-\frac{3}{2}}d\tau\right] \\
&\le Ct^{-1},
\end{aligned} \tag{4.3}
$$

and

$$
\begin{aligned}
|J_{22}| &\le C\left|\int_{\frac{t}{2}}^t\int_R G_{yy}Fdxd\tau + \int_{\frac{t}{2}}^t\int_R G_x z_\tau dxd\tau\right| \\
&\le C\left|\frac{1}{\underline{a}}\left(\left[\int_R GFdx\right]_{\frac{t}{2}}^t - \int_{\frac{t}{2}}^t\int_R GF_\tau dxd\tau\right) + \int_{\frac{t}{2}}^t\int_R G_x z_\tau dxd\tau\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\left\{\sup_R |F(t)| + \sup_R \left|F\left(\frac{t}{2}\right)\right| \left\|G\left(\frac{t}{2}\right)\right\|_{L^1} \right.\\
&\qquad \left. + \int_{\frac{t}{2}}^t \int_R (|V_y|^2|\tilde{v}_t| + |V_y||z_y|)|G| dx d\tau + \int_{\frac{t}{2}}^t \|G_x(t-\tau)\| \|z_\tau(\tau)\| d\tau \right\}\\
&\leq C\left\{\sup_R |V_x(t)|^2 + \int_{\frac{t}{2}}^t \sup_R |V_y|^2 \sup_R |\tilde{v}_t| \|G(t-\tau)\|_{L^1} d\tau \right.\\
&\qquad \left. + \int_{\frac{t}{2}}^t \sup_R |V_y| \|z_y(\tau)\| \|G(t-\tau)\| d\tau + \int_{\frac{t}{2}}^t \|G_x(t-\tau)\| \|z_\tau(\tau)\| d\tau \right\}\\
&\leq C\left\{t^{-\frac{3}{2}} + t^{-\frac{3}{2}-\frac{3}{2}+1} + t^{-\frac{3}{4}-\frac{3}{2}+\frac{3}{4}} + t^{-\frac{3}{2}+\frac{1}{4}}\right\} \leq Ct^{-\frac{5}{4}}.
\end{aligned} \tag{4.4}
$$

For J_3, we have

$$
\begin{aligned}
|J_{31}| &\leq C\left|\int_0^{\frac{t}{2}} \int_R G_{yy} p(\tilde{v}, s)_\tau dy d\tau\right|\\
&\leq C\int_0^{\frac{t}{2}} \|G_{yy}(t-\tau)\|_{L^1} \sup_R |\tilde{v}_\tau(\tau)| d\tau\\
&\leq C\int_0^{\frac{t}{2}} (t-\tau)^{-1}(1+\tau)^{-\frac{3}{2}} d\tau \leq Ct^{-1},
\end{aligned} \tag{4.5}
$$

and

$$
\begin{aligned}
|J_{32}| &\leq C\left|\int_{\frac{t}{2}}^t \int_R G_{yy} p(\tilde{v}, s)_\tau dy d\tau\right|\\
&\leq C\left|\int_{\frac{t}{2}}^t \int_R G_\tau p(\tilde{v}, s)_\tau dy d\tau\right|\\
&= C\left|\left[\int_R G p(\tilde{v}, s)_\tau dy\right]_{\frac{t}{2}}^t - \int_{\frac{t}{2}}^t \int_R G p(\tilde{v}, s)_{\tau\tau} dy d\tau\right|\\
&\leq C\left\{\sup_R |\tilde{v}_\tau(t)| + \sup_R \left|\tilde{v}_\tau\left(\frac{t}{2}\right)\right| \left\|G\left(\frac{t}{2}\right)\right\|_{L^1} \right.\\
&\qquad \left. + \int_{\frac{t}{2}}^t \left(\sup_R |\tilde{v}_\tau|^2 + \sup_R |\tilde{v}_{\tau\tau}|\right) \|G(t-\tau)\|_{L^1} d\tau \right\}\\
&\leq C\left(t^{-\frac{3}{2}} + t^{-\frac{9}{4}+1}\right) \leq Ct^{-\frac{5}{4}}.
\end{aligned} \tag{4.6}
$$

For last three terms, we have

$$
\begin{aligned}
|J_4| &\leq C\left(\sup_R |V_y(t)| \|\tilde{v}(t,\cdot) - \bar{v}(\cdot)\| \left\|G\left(\frac{t}{2}\right)\right\| + \|s(\cdot) - \underline{s}\|_{L^1} \sup_R \left|G\left(\frac{t}{2}\right)\right| \sup_R |V_y(t)|\right)\\
&\leq C\left(t^{-\frac{3}{4}-\frac{1}{4}-\frac{1}{4}} + t^{-\frac{1}{2}-\frac{3}{4}}\right) \leq Ct^{-\frac{5}{4}},
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
|J_5| \le & C\left(\int_0^{\frac{t}{2}} \sup_R |V_y(\tau)| \|\tilde{v}(t,\cdot)-\bar{v}(\cdot)\| \|G_{yy}(t-\tau)\| d\tau\right.\\
& \left. + \|s(\cdot)-\underline{s}\|_{L^1} \int_0^{\frac{t}{2}} \sup_R |G_{yy}(t-\tau)| \sup_R |V_y(\tau)| d\tau \right)\\
\le & C\left(\int_0^{\frac{t}{2}} (1+\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{1}{4}}(t-\tau)^{-\frac{5}{4}} d\tau + \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{3}{2}}(1+\tau)^{-\frac{3}{4}} d\tau\right)\\
\le & C\left\{t^{-\frac{5}{4}}\log(2+t) + t^{-\frac{3}{2}+\frac{1}{4}}\right\} \le Ct^{-\frac{5}{4}}\log(2+t),
\end{aligned} \tag{4.8}
$$

and

$$
\begin{aligned}
|J_6| \le & C\left(\int_{\frac{t}{2}}^{t} \sup_R |z_y(\tau)| \|G(t-\tau)\| \|\tilde{v}(t,\cdot)-\bar{v}(\cdot)\| d\tau\right.\\
& + \|s(\cdot)-\underline{s}\|_{L^1} \int_{\frac{t}{2}}^{t} \sup_R |z_y(\tau)| \sup_R |G(t-\tau)| d\tau\\
& \left. + C\int_{\frac{t}{2}}^{t} \sup_R |\tilde{v}_t(\tau)| \sup_R |V_y(\tau)| \|G(t-\tau)\|_{L^1} d\tau \right)\\
\le & C\left(t^{-\frac{3}{2}-\frac{1}{4}+\frac{3}{4}} + t^{-\frac{3}{2}+\frac{1}{2}} + t^{-\frac{3}{2}-\frac{3}{4}+1}\right) \le Ct^{-1}.
\end{aligned} \tag{4.9}
$$

Combining (4.2)-(4.9) we have the desired rate

$$
\sup_R |V_x(t,x)| \le Ct^{-1}. \tag{4.10}
$$

Next, we estimate $\sup_x |z_x(t,x)|$ in a similar way to above. Differentiating (2.13) in t, we have

$$
\begin{aligned}
z(t,x) = & \int_R G_t(t,x-y)V_0(y)dy + \int_R G\cdot p(\tilde{v},s)_{y\tau} dy\Big|_{\tau=\frac{t}{2}}\\
& - \left\{\int_0^{\frac{t}{2}}\int_R G_\tau \cdot p(\tilde{v},s)_{y\tau} dy d\tau - \int_{\frac{t}{2}}^{t}\int_R G\cdot p(\tilde{v},s)_{y\tau\tau} dy d\tau\right\}\\
& - \int_R G\cdot\{F_y + V_{\tau\tau}\} dy\Big|_{\tau=\frac{t}{2}}\\
& + \left\{\int_0^{\frac{t}{2}}\int_R G_t\{F_y+V_{\tau\tau}\} dy d\tau + \int_{\frac{t}{2}}^{t}\int_R G\cdot\{F_{yt}+V_{\tau\tau\tau}\} dy d\tau\right\}\\
& - \int_R \left\{(p_v(\tilde{v},s)+\underline{a})\, G_y\right\}_y V dy\Big|_{\tau=\frac{t}{2}}\\
& - \int_0^{\frac{t}{2}}\int_R \left\{(p_v(\tilde{v},s)+\underline{a})\, G_y\right\}_{yt} V dy d\tau\\
& - \int_{\frac{t}{2}}^{t}\int_R \left\{(p_v(\tilde{v},s)+\underline{a})\, V_y\right\}_\tau G_y dy d\tau.
\end{aligned} \tag{4.11}
$$

Here we have used the integration by parts to gather the derivatives of G in the part on the domain $\left[0, \frac{t}{2}\right]$, while, to gather those in the other part on the domain $\left[\frac{t}{2}, t\right]$. By the almost same calculations as the estimate of $\sup_R |V_x(t,x)|$, we have the desired rate

$$\sup_R |z(t,x)| \leq Ct^{-\frac{3}{2}}. \tag{4.12}$$

Applying the Hausdorff-Young inequality, we also have L^2-estimate:

$$\|(V_x, z)(t, \cdot)\|_{L^2} = O(t^{-\frac{3}{4}}, t^{-\frac{5}{4}}),$$

which completes the proof of Theorem 2. $\square$

Remark 5. We can also get

$$\|V(t,\cdot)\|_{L^\infty} \leq Ct^{-\frac{1}{2}}, \|V(t,\cdot)\|_{L^2} \leq Ct^{-\frac{1}{4}}.$$

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Large Time Behavior of Solutions to the Cauchy Problem for one-dimensional thermoelastic system with dissipation

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Abstract

In this paper we investigate the large time behavior of solutions to the Cauchy problem on R for a one-dimensional thermoelastic system with dissipation. When the initial data is suitably small, [S. Zheng, Chin. Ann. Math. 8B(1987), 142-155] established the global existence and the decay properties of the solution. Our aim is to improve the results and to obtain the sharper decay properties, which seems to be optimal. The proof is given by the energy method and the Green function method.

Key words and phrases: Thermoelastic system, dissipation, decay rate, Green function.
AMS subject classifications: 35B40, 35L60, 35L70, 76R50

1 Introduction

In this paper we investigate the large time behavior of solutions to the Cauchy problem for a one-dimensional thermoelastic system with dissipation on $\mathbf{R} \times (0, \infty)$:

$$\begin{cases} w_{tt} - a(w_x, \theta)w_{xx} + b(w_x, \theta)\theta_x + \alpha w_t = 0 \\ c(w_x, \theta)\theta_t + b(w_x, \theta)w_{xt} - d(\theta, \theta_x)\theta_{xx} = 0 \\ w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), \quad \theta(x, 0) = \theta_0(x), \end{cases} \tag{1.1}$$

where α is a positive constant, and smooth functions a, b, c and d satisfy

$$b \neq 0, \quad \text{and} \quad a, c, d \geq \delta_0 > 0 \quad \text{under considerations.} \tag{1.2}$$

For the derivation of this system refer to [9], [1]. In [9] Slemrod also showed the global existence theorem for the system (1.1) with $\alpha = 0$ on the interval $[0, 1]$. Damping mechanism was discussed

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in [1]. Nevertheless, for lack of the Poincaré type inequality our problem (1.1) is not necessarily clear. Instead of this system, by introducing new unknown functions

$$w_x = v, \quad w_t = v, \quad \theta = \theta, \tag{1.3}$$

Zheng [10] considered the corresponding system

$$\begin{cases} v_t - u_x = 0 \\ u_t - a(v,\theta)v_x + b(v,\theta)\theta_x + \alpha u = 0 \\ c(v,\theta)\theta_t + b(v,\theta)u_x - d(\theta,\theta_x)\theta_{xx} = 0 \end{cases} \tag{1.4}$$

with

$$(v,u,\theta)|_{t=0} = (v_0,u_0,\theta_0)(x). \tag{1.5}$$

In [10] he established the global existence of the solution of (1.4), (1.5) together with its decay order, when the initial data (v_0, u_0, θ_0) in $H^3(\mathbf{R})$ are suitably small.

Our main purpose is to observe the large time behavior of the solution of (1.1). However, instead of treating (1.1) directly, we first consider (1.4), (1.5) using L^2-energy method, which improves the result in [10]. Faster decay estimates of $\partial_{x,t}^k u_t$ obtained here play an important role in the next process. That is, regarding u_t in (1.4) as an inhomogeneous term, we have a parabolic system of (v,θ) and hence the "explicit" formula of (v,θ) using the Green functions $G_1(x,t)$, $G_2(x,t)$, which will give sharper estimates of (v,u,θ) if $(v_0,u_0,\theta_0) \in L^1(\mathbf{R})$. This method has been developed by the second author [5,6]. See also [7]. Finally, define a solution $(w,\theta)(x,t)$ of (1.1) by $w(x,t) = \int_{-\infty}^{x} v(y,t)dy$, where (v,u,θ) is a solution of (1.4) with its initial data

$$v_0 = w_{0x},\ u_0 = w_1,\ \theta_0 = \theta_0. \tag{1.6}$$

Thus we obtain a solution to the original Cauchy problem (1.1). Below, we sketch this procedure and state theorems.

First, linearize (1.2) around $(v,u,\theta) = (0,0,0)$:

$$\begin{cases} v_t - u_x = 0 \\ u_t - v_x + b_0\theta_x + u = g_2 \\ \theta_t + b_0 u_x - \theta_{xx} = g_3, \end{cases} \tag{1.7}$$

where we have normalized as

$$\alpha = 1,\ a(0,0) = d(0,0) = 1,\ b(0,0) = b_0 \tag{1.8}$$

and set

$$\begin{cases} g_2 = (a(v,\theta) - 1)v_x - (b(v,\theta) - b_0)\theta_x \\ g_3 = \frac{1}{c(v,\theta)}((b_0 - b(v,\theta))u_x + (d(\theta,\theta_x) - 1)\theta_{xx}). \end{cases} \tag{1.9}$$

By denoting the Lebesgue space(resp. the Sobolev space) by $L^p = L^p(\mathbf{R})$ with its norm $\|\cdot\|_{L^p}$(resp. $H^m = H^m(\mathbf{R})$ with its norm $\|\cdot\|_m$), especially $\|\cdot\|_{L^2} = \|\cdot\|_0 := \|\cdot\|$, our first theorem based on the L^2-energy method is the following.

Theorem 1 *Suppose that* $(v_0, u_0, \theta_0) \in H^4(\mathbf{R})$ *is suitably small. Then, the Cauchy problem (1.4), (1.5) has a unique global solution* $(v, u, \theta) \in C([0, \infty]; H^4(\mathbf{R}))$, *which satisfies*

$$
\begin{aligned}
&E(t; v, u, \theta) := E_1(t; v, u, \theta) + \int_0^t E_2(\tau; v, u, \theta) d\tau \\
&= \|(v, \theta)(t)\|^2 + (1+t)\|(v_x, u, \theta_x)(t)\|^2 + (1+t)^2\|\partial_x(v_x, u, \theta_x), \partial_t(v, \theta)(t)\|^2 \\
&+(1+t)^3\|\partial_x^2(v_x, u, \theta_x), \partial_t(v_x, u, \theta_x)(t)\|^2 \\
&+(1+t)^4\|\partial_x^3(v_x, u, \theta_x), \partial_x^4 u, \partial_t\partial_x(v_x, u, \theta_x), \partial_t u_{xx}, \partial_t^2(v, u, \theta)(t)\|^2 \\
&+\int_0^t \{\|(v_x, u, \theta_x)(\tau)\|^2 + (1+\tau)\|\partial_x(v_x, u, \theta_x), \partial_t(v, \theta)(\tau)\|^2 \\
&\quad +(1+\tau)^2\|\partial_x^2(v_x, u, \theta_x), \partial_t(v_x, u, \theta_x)(\tau)\|^2 \\
&\quad +(1+\tau)^3\|\partial_x^3(v_x, u, \theta_x), \partial_x\partial_t(v_x, u, \theta_x), \partial_t^2(v, \theta)(\tau)\|^2 \\
&\quad +(1+\tau)^4\|\partial_x^4 u, \partial_t^2(v_x, u, \theta_x), \partial_x\partial_t^2(v, u, \theta), \partial_t\partial_x^3(v, \theta), \partial_x^5\theta(\tau)\|^2\} d\tau \\
&\le C\|v_0, u_0, \theta_0\|_4^2.
\end{aligned}
\tag{1.10}
$$

In the next step we first obtain "explicit" formula of (v, θ). From the decay orders obtained in Theorem 1, the term u_t in the left-hand side of $(1.7)_2$(the second equation of (1.7)) decays faster than the other terms. Hence, differentiating $(1.7)_2$ once in x and using $(1.7)_1$, we regard (1.7) as a parabolic system of (v, u):

$$
\begin{cases}
v_t - v_{xx} + b_0\theta_{xx} = -u_{xt} + g_{2x} \\
b_0 v_t + \theta_t - \theta_{xx} = g_3,
\end{cases}
\tag{1.11}
$$

or

$$
A\begin{pmatrix} v \\ \theta \end{pmatrix}_t - B\begin{pmatrix} v \\ \theta \end{pmatrix}_{xx} = \begin{pmatrix} lu_{xt} + g_{2x} \\ g_3 \end{pmatrix} := \mathbf{F}, \tag{1.12}
$$

where

$$
A = \begin{pmatrix} 1 & 0 \\ b_0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -b_0 \\ 0 & 1 \end{pmatrix}. \tag{1.13}
$$

Setting

$$
\begin{pmatrix} v \\ \theta \end{pmatrix} = P\begin{pmatrix} V \\ \Theta \end{pmatrix} \tag{1.14}
$$

for a regular constant matrix P, we have

$$
\begin{pmatrix} V \\ \Theta \end{pmatrix}_t - P^{-1}A^{-1}BP\begin{pmatrix} V \\ \Theta \end{pmatrix}_{xx} = P^{-1}A^{-1}\mathbf{F}. \tag{1.15}
$$

The eigenvalues k_1, k_2 of $A^{-1}B = \begin{pmatrix} 1 & -b_0 \\ -b_0 & b_0^2+1 \end{pmatrix}$ are

$$0 < k_1 = \frac{b_0^2 + 2 - \sqrt{(b_0^2+2)^2 - 4}}{2} < k_2 = \frac{b_0^2 + 2 + \sqrt{(b_0^2+2)^2 - 4}}{2}, \tag{1.16}$$

and corresponding unit vectors are

$$\frac{1}{\sqrt{b_0^2 + (k_1-1)^2}} \begin{pmatrix} lb_0 \\ k_1 - 1 \end{pmatrix} := \begin{pmatrix} p_{11} \\ p_{21} \end{pmatrix}, \quad \frac{1}{\sqrt{b_0^2 + (k_2-1)^2}} \begin{pmatrix} lb_0 \\ k_2 - 1 \end{pmatrix} := \begin{pmatrix} p_{12} \\ p_{22} \end{pmatrix}. \tag{1.17}$$

Hence, a matrix $P := \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$ gives the diagonalized system

$$\begin{pmatrix} V \\ \Theta \end{pmatrix}_t - \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} V \\ \Theta \end{pmatrix}_{xx} = P^{-1}A^{-1}\mathbf{F}, \tag{1.18}$$

and hence the "explicit" formula is

$$\begin{pmatrix} V \\ \Theta \end{pmatrix}(x,t) = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}(\cdot, t) * \begin{pmatrix} V_0 \\ \Theta_0 \end{pmatrix} + \int_0^t \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}(\cdot, t-\tau) * P^{-1}A\mathbf{F}(\cdot, \tau)d\tau \tag{1.19}$$

where $\begin{pmatrix} V_0 \\ \Theta_0 \end{pmatrix} = P^{-1}\begin{pmatrix} v_0 \\ \theta_0 \end{pmatrix}$,

$$G_i(x,t) = \frac{1}{\sqrt{4\pi k_i t}} \exp\left(-\frac{x^2}{4k_i t}\right), \quad i = 1,2 \tag{1.20}$$

and $*$ means the convolution in x. Note that, since $A^{-1}B$ is a real symmetric matrix, P and ${}^tP$ are orthogonal matrices and

$$\begin{cases} \sum_{i=1}^2 p_{ij}^2 = \sum_{i=1}^2 p_{ji}^2 = 1, & j = 1,2 \\ \sum_{i=1}^2 p_{ij}p_{ik} = \sum_{i=1}^2 p_{ji}p_{ki} = 0, & j \neq k. \end{cases} \tag{1.21}$$

By (1.19), $\begin{pmatrix} V \\ \Theta \end{pmatrix} = P^{-1}\begin{pmatrix} v \\ \theta \end{pmatrix}$ gives

$$\begin{aligned} \begin{pmatrix} v \\ \theta \end{pmatrix}(x,t) &= P\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}P^{-1} * \begin{pmatrix} v_0 \\ \theta_0 \end{pmatrix} \\ &+ \int_0^t P\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}P^{-1} * A^{-1}\begin{pmatrix} lu_{xt} + g_{2x} \\ g_3 \end{pmatrix} d\tau. \end{aligned} \tag{1.22}$$

From (1.17) and (1.21)

$$\begin{aligned} P\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}P^{-1} &= \begin{pmatrix} p_{11}^2 G_1 + p_{12}^2 G_2 & p_{11}p_{21}G_1 + p_{12}p_{22}G_2 \\ p_{11}p_{21}G_1 + p_{12}p_{22}G_2 & p_{21}^2 G_1 + p_{22}^2 G_2 \end{pmatrix} \\ &:= \begin{pmatrix} (1-\alpha)G_1 + \alpha G_2 & \gamma(G_1 - G_2) \\ \gamma(G_1 - G_2) & \beta G_1 + (1-\beta)G_2 \end{pmatrix} \end{aligned} \tag{1.23}$$

with $0<\alpha,\beta,|\gamma|<1$. Thus, we have a "explicit" formula of (v,θ):

$$\begin{pmatrix} v \\ \theta \end{pmatrix}(x,t)=\begin{pmatrix} (1-\alpha)G_1+\alpha G_2 & \gamma(G_1-G_2) \\ \gamma(G_1-G_2) & \beta G_1+(1-\beta)G_2 \end{pmatrix}(\cdot,t)*\begin{pmatrix} v_0 \\ \theta_0 \end{pmatrix}$$

$$+\int_0^t \begin{pmatrix} (1-\alpha)G_1+\alpha G_2 & \gamma(G_1-G_2) \\ \gamma(G_1-G_2) & \beta G_1+(1-\beta)G_2 \end{pmatrix}(\cdot,t-\tau)*\begin{pmatrix} lu_{xt}+g_{2x} \\ b_0(u_{xt}-g_{2x})+g_3 \end{pmatrix}(\cdot,\tau)d\tau, \quad (1.24)$$

which is "explicit" in the sense that several kinds of information about u_{xt}, g_2, g_3 are already known. From $(1.7)_2$, u has the form

$$u(x,t)=v_x-b_0\theta_x-u_t+g_2. \tag{1.25}$$

From (1.24) and (1.25), $(v_x,\ u,\ \theta_x)$ instead of (v_x,u_x,θ_x) have same decay order if u_t and g_2 decay faster. From this point of view the decay orders obtained in Theorem 1 seem to be reasonable. Compare this to the result of Zheng [10]. See also [2, 4].

Further, if the initial data (v_0,θ_0) is in $L^1(\mathbf{R})$, then these decay orders are improved. In fact, we have the following second main theorem.

Theorem 2 *In addition to the assumptions in Theorem 1, suppose that (v_0,θ_0) is in $L^1(\mathbf{R})$. Then, the solution (v,u,θ) of (1.4), (1.5) satisfies the decay estimates*

$$\begin{aligned}
&(1+t)^{1/4}\|(v,\theta)(t)\|+(1+t)^{1/2}\|(v,\theta)(t)\|_{L^\infty}\\
&+(1+t)^{3/4}\|(v_x,u,\theta_x)(t)\|+(1+t)\|(v_x,u,\theta_x)(t)\|_{L^\infty}\\
&+(1+t)^{5/4}\|(v_{xx},u_x,\theta_{xx})(t)\|+(1+t)^{3/2}\|(v_{xx},u_x,\theta_{xx})(t)\|_{L^\infty}\\
&\le C(\|v_0,u_0,\theta_0\|_4+\|v_0,\theta_0\|_{L^1}).
\end{aligned} \tag{1.26}$$

Remark. 1 In this stage the assumption $u_0\in L^1$ is not necessary.

Finally, consider the Cauchy problem to the original system (1.1). Taking (1.3) and the first component of (1.24)(denote by $(1.24)_1$) into consideration, we assume $w_{0x}=v_0$ with $w_0\in H^5(\mathbf{R})\cap L^1(\mathbf{R})$, and set

$$w(x,t)=\int_{-\infty}^x v(y,t)dy. \tag{1.27}$$

By $(1.7)_1$, $w_t(x,t)=\int_{-\infty}^x v_t(y,t)dy=\int_{-\infty}^x u_x(y,t)dy=u(x,t)$. Hence, (w,θ) satisfies (1.1). Estimating $(1.24)_2$ and (1.27) with $(1.24)_1$, we have the following theorem.

Theorem 3 *Suppose that $(w_0,w_1,\theta_0)\in H^5(\mathbf{R})\times H^4(\mathbf{R})\times H^4(\mathbf{R})$ is suitably small and w_0, w_{0x}, w_1, θ_0 are in $L^1(\mathbf{R})$, and that (v,u,θ) is a solution of (1.4) with $(v,u,\theta)|_{t=0}=(w_{0x},w_1,\theta_0)$ obtained in*

Theorem 2. Then, (w, θ) *defined by (1.27) and* $(1.24)_2$ *is a solution of (1.1), which satisfies*

$$\begin{aligned}
&(1+t)^{-1/4}\|w(t)\| + \|w(t)\|_{L^\infty} \\
&+(1+t)^{1/4}\|(w_x,\theta)(t)\| + (1+t)^{1/2}\|(w_x,\theta)(t)\|_{L^\infty} \\
&+(1+t)^{3/4}\|(w_{xx},w_t,\theta_x)(t)\| + (1+t)\|(w_{xx},w_t,\theta_x)\|_{L^\infty} \\
&+(1+t)^{5/4}\|(w_{xxx},w_{tx},\theta_t,\theta_{xx})(t)\| + (1+t)^{3/2}\|(w_{xxx},w_{tx},\theta_{xx})(t)\|_{L^\infty} \\
&\le C(\|w_0\|_5 + \|w_1,\theta_0\|_4 + \|w_0,w_{0x},w_1,\theta_0\|_{L^1}).
\end{aligned} \tag{1.28}$$

2 L^2-Energy Estimates

In this section we prove Theorem 1 employing the L^2-energy method. Our present concern is the the Cauchy problem to the system of equations (1.4) with the initial data (1.5).

The global existence of the solution is given by the combination of the local existence (Proposition 2.1) and the a priori estimates (Proposition 2.2). This observation immediately gives the proof of Theorem 1.

By multiplying $(1.4)_1$ by $a(v,\theta)$, the resultant system becomes the symmetric hyperbolic-parabolic system. Thus, the local existence theorem below immediately follows from the general theory constructed in Kawashima [3]. The readers are referred to [8], too.

Proposition 2. 1 (Local Existence) *Let* $s \ge 3$ *be an integer. Suppose that* $(v_0,u_0,\theta_0) \in H^s(\mathbf{R})$. *Then, there exists a positive constant* T_0*, depending only on* $\|(v_0,u_0,\theta_0)\|_s$*, such that the initial value problem* (1.4) *and* (1.5) *has a unique solution* (v,u,θ) *satisfying that*

$$\begin{aligned}
(u,v) &\in C^0([0,T_0];H^s(\mathbf{R})) \cap C^1([0,T_0];H^{s-1}(\mathbf{R})), \\
\theta &\in C^0([0,T_0];H^s(\mathbf{R})) \cap C^1([0,T_0];H^{s-2}(\mathbf{R})) \cap L^2([0,T_0];H^{s+1}(\mathbf{R})).
\end{aligned}$$

Our theory concerning the asymptotic states requires the solutions (v,u,θ) to be in the space $H^4(\mathbf{R})$ in the spatial variable x. Thus, we fix $s=4$ hereafter. Then, we introduce the solution space

$$X(0,T) := \{(v,u,\theta)\,|E(t;v,u,\theta) < \infty\}$$

Also, we use the supremum of $E(t;v,u,\theta) = E_1(t;v,u,\theta) + \int_0^t E_2(\tau;v,u,\theta)d\tau$:

$$N(T)^2 := N(T;v,u,\theta)^2 = \sup_{0\le t\le T} E(t;v,u,\theta).$$

Apparently, it holds that

$$\|(v,u,\theta)(t)\|_4 \le E(t;v,u,\theta).$$

Thus, we can combine the following a priori estimates with the local existence theorem.

Proposition 2. 2 (A Priori Estimates) *Let* $(v, u, \theta) \in X(0, T)$ *be a solution of (1.7), (1.5) satisfying* $N(T) \le 1$. *Then, there exists a positive constant* ε_2 *such that if* $\|v_0, u_0, \theta_0\|_4 \le \varepsilon_2$, *then* (v, u, θ) *satisfies (1.10) for* $0 \le t \le T$.

We now devote ourselves to the proof of Proposition 2.2, which will be done in several steps.

Step. 1 We first multiply $(1.7)_1$, $(1.7)_2$, $(1.7)_3$ by v, u, θ, respectively, to have

$$(\tfrac{1}{2}\int v^2 dx)_t + \int u v_x dx = 0$$

$$(\tfrac{1}{2}\int u^2 dx)_t + \int (-u v_x + b_0 u \theta_x + u^2) dx = \int g_2 \cdot u dx$$

$$(\tfrac{1}{2}\int \theta^2 dx)_t + \int (-b_0 u \theta_x + \theta_x^2) dx = \int g_3 \cdot \theta dx.$$

Here and hereafter, the integrand **R** is often abbreviated. Adding three equations, we have

$$\frac{1}{2}\frac{d}{dt}\|(v, u, \theta)(t)\|^2 + \|(u, \theta_x)(t)\|^2 = \int (g_2 \cdot u + g_3 \cdot \theta) dx := F_1^{(0)}(t; g). \tag{2.1$_0$}$$

Integrating $(2.1)_0$ over $[0, t], t \le T$, we have first lemma.

Lemma 2. 1 *For some constant* C *independent of* t *it holds that*

$$\|(v, u, \theta)(t)\|^2 + \int_0^t \|(u, \theta_x)(\tau)\|^2 d\tau \le C(\|v_0, u_0, \theta_0\|^2 + \int_0^t F_1^{(0)}(\tau; g) d\tau). \tag{2.2}$$

Step. 2 Multiplying $(1.7)_1$, $(1.7)_2$, $(1.7)_3$ by $-\partial_x^2 v, -\partial_x^2 u, -\partial_x^2 \theta$, respectively, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(v_x, u_x, \theta_x)(t)\|^2 + \|(u_x, \theta_{xx})(t)\|^2 \\ &= \int (\partial_x g_2 \cdot \partial_x u + \partial_x g_3 \cdot \partial_x \theta) dx := F_1^{(1)}(t; g). \end{aligned} \tag{2.1$_1$}$$

We also multiply $(1.7)_2$, $(1.7)_3$ by u_t, θ_t, respectively, and add the resultant equations to have

$$\begin{aligned} &\frac{d}{dt}[\frac{1}{2}\|(u, \theta_x)(t)\|^2 + \int (b_0 \theta_x - v_x) u dx] + \int (u_t^2 + \theta_t^2 - u_x^2 + 2b_0 \theta_t u_x) dx \\ &= \int (g_2 \cdot u_t + g_3 \cdot \theta_t) dx. \end{aligned} \tag{2.3}$$

Calculating $(2.1)_1$ + $(2.3)\times\lambda$ for a small positive constant λ, we have

$$\begin{aligned} &\frac{d}{dt}[\frac{1}{2}\|(v_x, u_x)(t)\|^2 + \frac{1+\lambda}{2}\|\theta_x(t)\|^2 + \frac{\lambda}{2}\|u(t)\|^2] + \int \lambda(b_0\theta_x - v_x) u dx \\ &+(1-\lambda)\|u_x(t)\|^2 + \lambda\|(u_t, \theta_t)(t)\|^2 + \|\theta_{xx}(t)\|^2 + \int 2\lambda b_0 u_x \cdot \theta_t dx \\ &= \int (\partial_x g_2 \cdot \partial_x u + \lambda \partial_x g_2 \cdot u_t + \partial_x g_3 \cdot \partial_x \theta + \lambda g_3 \cdot \theta_t) dx := F_2^{(1)}(t; g). \end{aligned} \tag{2.4$_1$}$$

and hence

$$\|(v_x,u,u_x,\theta_x)(t)\|^2+\int_0^t\|(u_x,u_t,\theta_t,\theta_{xx})(\tau)\|^2d\tau \\ \le C(\|v_0,u_0,\theta_0\|_1^2+\int_0^t F_2^{(1)}(\tau;g)d\tau). \tag{2.5}$$

Moreover, differentiating $(1.7)_2$ with respect to x and using $(1.7)_1$, we have

$$v_t-v_{xx}+u_{tx}+b_0\theta_{xx}=g_{2x},$$

and, by multiplying this by v,

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2+\|v_x(t)\|^2-\int(u_t+b_0\theta_x)v_xdx \\ =-\int g_2\cdot v_xdx:=F_3^{(1)}(t;g). \tag*{$(2.6)_1$}$$

By (2.2), (2.5) and the Schwarz inequality

$$\|v(t)\|^2+\int_0^t\|v_x(\tau)\|^2d\tau \\ \le C(\|v_0,u_0,\theta_0\|_1^2+\int_0^t(F_1^{(0)}+F_2^{(1)}+F_3^{(1)})(\tau;g)d\tau). \tag{2.7}$$

We now have had the integrability of $\|v_x(\tau)\|^2$ on $[0,t]$. Hence we turn back to $(2.4)_1$ and multiply $(2.4)_1$ by $(1+t)$ to obtain

$$(1+t)\|(v_x,u,u_x,\theta_x)(t)\|^2+\int_0^t(1+\tau)\|(v_x,u_t,\theta_t,\theta_{xx})(\tau)\|^2d\tau \\ \le C(\|v_0,u_0,\theta_0\|_1^2+\int_0^t(F_1^{(0)}(\tau;g)+(1+\tau)F_2^{(1)}(\tau;g)+F_3^{(1)}(\tau;g))d\tau) \\ =C(\|v_0,u_0,\theta_0\|_1^2+\int_0^t H_1(\tau;g)d\tau). \tag{2.8}$$

Combining (2.8) and (2.7) we have the second lemma.

Lemma 2. 2 *It holds that*

$$(1+t)\|(v_x,u,u_x,\theta_x)(t)\|^2+\int_0^t(\|v_x(\tau)\|^2+(1+\tau)\|(u_x,u_t,\theta_t,\theta_{xx})(\tau)\|^2)d\tau \\ \le C(\|v_0,u_0,\theta_0\|_1^2+\int_0^t H_1(\tau;g)d\tau). \tag{2.9}$$

Step. 3 Estimates of higher order derivatives corresponding to $(2.1)_1$, $(2.4)_1$, $(2.6)_1$, respectively, become

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^k(v,u,\theta)(t)\|^2+\|\partial_x^k(u,\theta_x)(t)\|^2 \\ =\int(\partial_x^k g_2\cdot\partial_x^k u+\partial_x^k g_3\cdot\partial_x^k\theta)dx:=F_1^{(k)}(t;g), \tag*{$(2.1)_k$}$$

$$\frac{d}{dt}[\frac{1}{2}\|\partial_x^k(v,u)(t)\|^2 + \frac{1+\lambda}{2}\|\partial_x^k\theta(t)\|^2 + \frac{\lambda}{2}\|\partial_x^{k-1}u(t)\|^2]$$
$$+\int \lambda(b_0\partial_x^k\theta - \partial_x^k v)\partial_x^{k-1}u dx + (1-\lambda)\|\partial_x^k u(t)\|^2$$
$$+\lambda\|\partial_x^{k-1}(u_t,\theta_t)(t)\|^2 + \|\partial_x^k\theta_x(t)\|^2 + \int 2\lambda b_0\partial_x^k u \cdot \partial_x^{k-1}\theta_t dx \tag{2.4$_k$}$$
$$= \int (\partial_x^k g_2 \cdot \partial_x^k u + \lambda\partial_x^{k-1}g_2 \cdot \partial_x^{k-1}u_t + \partial_x^k g_3 \cdot \partial_x^k\theta)$$
$$+\lambda\partial_x^{k-1}g_3 \cdot \partial_x^{k-1}\theta_t dx := F_2^{(k)}(t;g),$$

and

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^{k-1}v(t)\|^2 + \|\partial_x^k v(t)\|^2 - \int(\partial_x^{k-1}u_t + b_0\partial_x^k\theta)\partial_x^k v dx$$
$$= -\int \partial_x^{k-1}g_2 \cdot \partial_x^k v dx := F_3^{(k)}(t;g) \tag{2.6$_k$}$$

for $k = 2,3,4$. Same method as that of obtaining Lemmas 2.1-2.2 yields the third lemma.

Lemma 2. 3 *It holds that*

$$(1+t)^2\|\partial_x(v_x,u,u_x,\theta_x)(t)\|^2$$
$$+\int_0^t [(1+\tau)\|\partial_x^2 v(\tau)\|^2 + (1+\tau)^2\|\partial_x(u_x,u_t,\theta_t,\theta_{xx})(\tau)\|^2]d\tau \tag{2.10}$$
$$\leq C(\|v_0,u_0,\theta_0\|_2^2 + \int_0^t H_2(\tau;g)d\tau),$$

$$(1+t)^3\|\partial_x^2(v_x,u,u_x,\theta_x)(t)\|^2$$
$$+\int_0^t [(1+\tau)^2\|\partial_x^3 v(\tau)\|^2 + (1+\tau)^3\|\partial_x^2(u_x,u,\theta_t,\theta_{xx})(\tau)\|^2]d\tau \tag{2.11}$$
$$\leq C(\|v_0,u_0,\theta_0\|_3^2 + \int_0^t H_3(\tau;g)d\tau),$$

and

$$(1+t)^4\|\partial_x^3(v_x,u,u_x,\theta_x)(t)\|^2$$
$$+\int_0^t [(1+\tau)^3\|\partial_x^4 v(\tau)\|^2 + (1+\tau)^4\|\partial_x^3(u_x,u,\theta_t,\theta_{xx})(\tau)\|^2]d\tau \tag{2.12}$$
$$\leq C(\|v_0,u_0,\theta_0\|_4^2 + \int_0^t H_4(\tau;g)d\tau),$$

where

$$H_m(\tau;g) = \sum_{k=1}^{m}\{(1+\tau)^{k-1}F_1^{(k-1)}(\tau;g) + (1+\tau)^k F_2^{(k)}(\tau;g) + (1+\tau)^{k-1}F_3^{(k)}(\tau;g)\}. \tag{2.13}$$

Step. 4 We next estimate the derivatives of (v,u,θ) with respect to t. Differentiate (1.7) in t once to have

$$\begin{cases} (v_t)_t-(u_t)_x=0 \\ (u_t)_t-(v_t)_x+b_0(\theta_t)_x+u_t=g_{2t} \\ (\theta_t)_t+b_0(u_t)_x-(\theta_t)_{xx}=g_{3t}. \end{cases} \tag{2.14}$$

Since $\|(v_t,u_t,\theta_t)|_{t=0}\|\le C(\|v_0,u_0\|_1+\|\theta_0\|_2)$ and that $(1+\tau)\|(v_t=u_x,u_t,\theta_t)(\tau)\|^2$ is integrable on $[0,t]$ by Lemma 2.2, same way as in Lemma 2.3 yields the following lemma.

Lemma 2. 4 *It holds that*

$$\begin{aligned} &(1+t)^2\|\partial_t(v,u,\theta)(t)\|^2+\int_0^t(1+\tau)^2\|\partial_t(u,\theta_x)(\tau)\|^2d\tau \\ &\le C(\|v_0,u_0\|_1^2+\|\theta_0\|_2^2+\int_0^t[H_1(\tau;g)+(1+\tau)^2F_1^{(0)}(\tau;g_t)]d\tau, \end{aligned} \tag{2.15}$$

$$\begin{aligned} &(1+t)^3\|\partial_t(v_x,u,u_x,\theta_x)(t)\|^2+\int_0^t[(1+\tau)^2\|v_{tx}(\tau)\|^2] \\ &+(1+\tau)^3\|\partial_t(u_x,u_t,\theta_t,\theta_{xx})(\tau)\|^2d\tau \\ &\le C(\|v_0,u_0\|_2^2+\|\theta_0\|_3^2+\int_0^t[H_1(\tau;g)+(1+\tau)^2H_1(\tau;g_t)]d\tau \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} &(1+t)^4\|\partial_x\partial_t(v_x,u,u_x,\theta_x)(t)\|^2+\int_0^t\{(1+\tau)^3\|v_{txx}(\tau)\|^2 \\ &+(1+\tau)^4\|\partial_x\partial_t(u_x,u_t,\theta_t,\theta_{xx}(\tau)\|^2)d\tau \\ &\le C(\|v_0,u_0\|_3^2+\|\theta_0\|_4^2+\int_0^t\{H_1(\tau;g)+(1+\tau)^2H_2(\tau;g_t)\}d\tau. \end{aligned} \tag{2.17}$$

Step. 5 Differentiating (2.14) in t once more, we have

Lemma 2. 5 *It holds that*

$$\begin{aligned} &(1+t)^4\|\partial_t^2(v,u,\theta)(t)\|^2+\int_0^t(1+\tau)^4\|\partial_t^2(u,\theta_x)(\tau)\|^2d\tau \\ &\le C(\|v_0,u_0,\theta_0\|_4^2+\int_0^t[H_1(\tau;g)+(1+\tau)^2H_1(\tau;g_t)+(1+\tau)^4F_1^{(0)}(\tau;g_{tt})]d\tau. \end{aligned} \tag{2.18}$$

Step. 6 Adding all inequalities obtained in Lemmas 2.1-2.5, we have

$$\begin{aligned} &E_1(t;v,u,\theta)+\int_0^tE_2(\tau;v,u,\theta)d\tau \\ &\le C(\|v_0,u_0,\theta_0\|^2+\int_0^t[H_4(\tau;g)+(1+\tau)^2H_2(\tau;g_t)+(1+\tau)^4F_1^{(0)}(\tau;g_{tt})d\tau. \end{aligned} \tag{2.19}$$

Here we have used $F_1^{(0)}(t;g) \ll H_1(t;g) \ll H_2(t;g) \ll H_3(t;g) \ll H_4(t;g)$, where $F \ll H$ means that all terms of F are included in H.

The last term of (2.19) has higher orders of (v,u,θ) and estimated as follows.

Lemma 2. 6 *For small positive constant ν it holds that*

$$C\int_0^t [H_4(\tau;g) + (1+\tau)^2 H_2(\tau;g_t) + (1+\tau)^4 F_1^{(0)}(\tau;g_{tt})d\tau$$

$$\leq C\|v_0,u_0,\theta_0\|_4^2 + \nu\int_0^t E_2(\tau;v,u,\theta)d\tau + CN(T)^{3/2}.$$

The proof of lemma 2.6 is not difficult, but many and tedious calculations are necessary. So, we only show a few terms. For example, $\int_0^t H_4(\tau;g)d\tau$ includes

$$J_1 := \int_0^t \int \frac{1}{c(v,\theta)}(b_0 - b(v,\theta))u_x \cdot \theta dx d\tau,$$

$$J_2 := \int_0^t (1+\tau)^4 \int (a(v,\theta)-1)v_{xxxx}u_{xxxt}dx d\tau,$$

the latter of which is in $\int_0^t (1+\tau)^4 \int \partial_x^3 g_2 \cdot \partial_x^3 u_t dx d\tau$. J_1 is estimated as follows:

$$\int_0^t \int [-(\frac{\theta}{c(v,\theta)})_x (b_0 - b(v,\theta))u + \frac{\theta}{c(v,\theta)} b(v,\theta)_x u] dx d\tau$$

$$\leq CN(T)^{1/2}\int_0^t \int (v_x^2 + u^2 + \theta_x^2) dx d\tau \leq CN(T)^{3/2}.$$

Since $v_t = u_x$,

$$J_2 = \int_0^t (1+\tau)^4 [\frac{d}{d\tau}\int (a(v,\theta)-1)v_{xxxx}u_{xxx}dx]$$

$$-\int (a(v,\theta)_\tau v_{xxxx}u_{xxx} - (a(v,\theta)-1)v_{txxxx}u_{xxx})dx d\tau$$

$$= (1+\tau)^4 \int (a(v,\theta)-1)v_{xxxx}u_{xxx}dx|_{\tau=0}^{\tau=t} - 4\int_0^t (1+\tau)^3 \int (a(v,\theta)-1)v_{xxxx}u_{xxx}dx d\tau$$

$$-\int_0^t (1+\tau)^4 \int (a(v,\theta)_\tau v_{xxxx}u_{xxx} + a(v,\theta)_x u_{xxxx}u_{xxx} + (a(v,\theta)-1)u_{xxxx}^2) dx d\tau$$

$$\leq CN(T)^{3/2} + C\|v_0,u_0,\theta_0\|_4^2 + \nu\int_0^t (1+\tau)^4 \|u_{xxxx}(\tau)\|^2 d\tau$$

$$+CN(T)^{1/2}\int_0^t [(1+\tau)^3 \|(v_{xxxx},u_{xxx})(\tau)\|^2 + (1+\tau)^4 \|u_{xxxx}(\tau)\|^2] d\tau$$

$$\leq C\|v_0,u_0,\theta_0\|_4^2 + \nu\int_0^t (1+\tau)^4 \|u_{xxxx}(\tau)\|^2 d\tau + CN(T)^{3/2}.$$

The other terms are omitted. We now have reached to the inequality

$$N(T) \leq C(\|v_0, u_0, \theta_0\|_4^2 + N(T)^{3/2}),$$

and hence

$$N(T) \leq C\|v_0, u_0, \theta_0\|_4^2$$

provided that $\|v_0, u_0, \theta_0\|_4$ is suitably small. Thus, we have completed the proof of Proposition 2.2.

3 Estimates in L^1-Framework

In this section we prove Theorem 2. Assuming $(v_0, \theta_0) \in L^1$ in addition to the assumptions in Theorem 1, we remind the "explicit" formula (1.24) of (v, θ). In order to obtain the estimates of (v, θ), it is enough to estimate $I_1 := G * v_0$, $I_2 := G * \theta_0$, $II := \int_0^t G * u_{xt}$, $III := \int_0^t G * g_{2x} d\tau$ and $IV := \int_0^t G * g_3 d\tau$, where $G = G_1$ or G_2, and g_2, g_3, G_1, G_2 are, respectively, given by (1.9), (1.20).

First, we seek for the L^∞-norm of v, θ. Since $\|G(t)\|_{L^\infty} \leq O(t^{-1/2})$, it is easily seen that

$$|I_1| + |I_2| \leq Ct^{-1/2} \tag{3.1}$$

(From now on we denote a constant depending on $\|v_0, u_0, \theta_0\|_4 + \|v_0, \theta_0\|_{L^1}$ simply by C). Dividing the integrand $(0, t)$ into $(0, t/2) \cup (t/2, t)$ and using the Hausdorff-Young inequality, we have

$$\begin{aligned} |II| &\leq \int_0^{t/2} \|G_x(t-\tau)\| \|u_t(\tau)\| d\tau + \int_{t/2}^t \|G(t-\tau)\| \|u_{xt}(\tau)\| d\tau \\ &\leq C \int_0^{t/2} (t-\tau)^{-3/4}(1+\tau)^{-3/2} d\tau + C \int_{t/2}^t (t-\tau)^{-1/4}(1+\tau)^{-2} d\tau \\ &\leq Ct^{-3/4}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} |III| &\leq \int_0^{t/2} \|G_x(t-\tau)\|_{L^\infty} \|g_2\|_{L^1} d\tau + \int_{t/2}^t \|G(t-\tau)\|_{L^\infty} \|g_{2x}\| d\tau \\ &\leq C \int_0^{t/2} (t-\tau)^{-3/2} \|(v, \theta)(\tau)\| \|(v_x, \theta_x)(\tau)\| d\tau \\ &\quad + C \int_{t/2}^t (t-\tau)^{-1/2} (\|(v_x, \theta_x)(\tau)\|^2 + \|(v, \theta)(\tau)\| \|(v_{xx}, \theta_{xx})(\tau)\| d\tau) \\ &\leq Ct^{-3/2} \int_0^{t/2} 1 \cdot (1+\tau)^{-1/2} d\tau + C(1+t)^{-1} \int_{t/2}^t (t-\tau)^{-1/2} d\tau \leq Ct^{-1/2} \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} |IV| &\leq \int_0^t \|G(t-\tau)\|_{L^\infty} \|g_3(\tau)\|_{L^1} d\tau \\ &\leq C \int_0^t (t-\tau)^{-1/2} \|(v, \theta)(\tau)\| \|(u_x, \theta_{xx})(\tau)\| d\tau \\ &\leq C(\int_0^{t/2} + \int_{t/2}^t)(t-\tau)^{-1/2} \cdot 1 \cdot (1+\tau)^{-1/2} d\tau \leq Ct^{-1} \ln(2+t). \end{aligned} \tag{3.4}$$

Hence, together with $\|(v,\theta)(t)\|_{L^\infty} \le C$, (3.2) - (3.4) and (1.24) give

$$\|(v,\theta)(t)\|_{L^\infty} \le C(1+t)^{-1/2}\ln(2+t), \tag{3.5}$$

which will be improved soon after getting the estimates of $\|(v,\theta)(t)\|$.

Next, we seek for $\|(v,\theta)(t)\|$ in a similar fashion to the above:

$$\|I_1\| + \|I_2\| \le \|G(t)\|(\|v_0\|_{L^1} + \|\theta_0\|_{L^1}) \le Ct^{-1/4}, \tag{3.6}$$

$$\begin{aligned} \|II\| + \|III\| &\le \int_0^{t/2} (\|G_x\|_{L^1}\|u_t\| + \|G_x\|\|g_2\|_{L^1})d\tau \\ &+ \int_{t/2}^t (\|G\|_{L^1}\|u_{xt}\| + \|G\|\|g_{2x}\|_{L^1})d\tau \le Ct^{-1/4}, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \|IV\| &\le \int_0^t \|G\|\|g_3\|_{L^1}d\tau \\ &\le C(\int_0^{t/2} + \int_{t/2}^t)(t-\tau)^{-1/4}\|(v,\theta)(\tau)\|\|(u_x,\theta_{xx})(\tau)\|d\tau \\ &\le Ct^{-1/4}\ln(2+t). \end{aligned} \tag{3.8}$$

Hence

$$\|(v,\theta)(t)\| \le C(1+t)^{-1/4}\ln(2+t). \tag{3.9}$$

Applying (3.9), just obtained, to (3.4) and (3.7) we have

$$\|IV\|_{L^\infty} \le C(1+t)^{-1/2}, \quad \|IV\| \le Ct^{-1/4}$$

from which we obtain the desired estimate

$$(1+t)^{1/2}\|(v,\theta)(t)\|_{L^\infty} + (1+t)^{1/4}\|(v,\theta)(t)\| \le C. \tag{3.10}$$

By (1.24) the estimates of $I_{1x}, \cdots, IV_x$ yield

$$(1+t)\|(v_x,\theta_x)(t)\|_{L^\infty} + (1+t)^{3/4}\|(v_x,\theta_x)(t)\| \le C. \tag{3.11}$$

From (1.25), (3.11) and the Sobolev inequality

$$\begin{aligned} \|u(t)\|_{L^\infty} &\le C(\|(v_x,\theta_x)(t)\|_{L^\infty} \\ &+ \|u_t(t)\|_{L^\infty} + \|(v,\theta)(t)\|_{L^\infty}\|(v_x,\theta_x)(t)\|_{L^\infty} \le C(1+t)^{-1} \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \|u(t)\| &\le C(\|(v_x,\theta_x)(t)\| + \|u_t(t)\| + \|(v,\theta)(t)\|_{L^\infty}\|(v_x,\theta_x)(t)\|) \\ &\le C(1+t)^{-3/4}. \end{aligned} \tag{3.13}$$

Similarly, we have

$$\begin{aligned} &\|(v_{xx},\theta_{xx},u_x)(t)\|_{L^\infty} \le C(1+t)^{-3/2} \\ &\|(v_{xx},\theta_{xx},u_x)(t)\| \le C(1+t)^{-5/4}. \end{aligned} \tag{3.14}$$

By $(1.7)_1$ and $(1.7)_3$ v_t and θ_t have same decay orders as (3.14). Eqs. (3.10) - (3.14) yield the desired estimate (1.26). Here, we note that the assumption $u_0 \in L^1$ is not necessary till now.

4 Thermoelastic System of Second Order

In the final section we consider the original second order thermoelastic system (1.1) with dissipation, and prove Theorem 3.

For the solution (v, u, θ) of (1.4) with the initial data $(v_0, u_0, \theta_0) = (w_{0x}, w_1, \theta_0)$ obtained in Theorems 1 - 2, the equations (1.24), (1.27) give the solution (w, θ) of (1.1) by

$$\begin{aligned} w(x,t) &= (G_{11} * w_0)(x,t) + \int_{-\infty}^{x} (G_{12} * \theta_0)(\xi, t) d\xi \\ &+ \int_0^t [(G_{11} + b_0 G_{12})(\cdot, t-\tau) * (-u_t + g_2)(\cdot, \tau)](x) d\tau \\ &+ \int_{-\infty}^{x} \int_0^t [G_{12}(\cdot, t-\tau) * g_3(\cdot, \tau)](\xi) d\tau d\xi \\ &= (1) + (2) + (3) + (4) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \theta(x,t) &= (G_{12} * w_{0x})(x,t) + (G_{22} * \theta_0)(x,t) \\ &\int_0^t [G_{12}(\cdot, t-\tau) * (-u_{xt} + g_{2x})(\cdot, \tau) + G_{22}(\cdot, t-\tau) * (b_0(-u_{xt} + g_{2x}) + g_3)(\cdot, \tau)] d\tau, \end{aligned} \tag{4.2}$$

where

$$\begin{cases} G_{11} = \alpha G_1 + (1-\alpha) G_2, \quad G_{12} = \gamma (G_1 - G_2) \\ G_{22} = \beta G_1 + (1-\beta) G_2. \end{cases} \tag{4.3}$$

First, note that, for any $f \in L^1 \cap L^2$,

$$\int_{-\infty}^{x} (G_{12} * f)(\xi) d\xi = \int_{-\infty}^{x} [(G_1 - G_2) * f](\xi) d\xi$$

$$= \int_{-\infty}^{x} [\int_{-\infty}^{\infty} \frac{1}{\sqrt{4k_1 \pi t}} e^{-\frac{(\xi-y)^2}{4k_1 t}} f(y) dy - \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k_2 \pi t}} e^{-\frac{(\xi-y)^2}{4k_2 t}} f(y) dy] d\xi$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{\eta^2}{4t}} (f(\xi + \sqrt{k_1}\eta) - f(\xi + \sqrt{k_2}\eta)) d\eta d\xi$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{\infty} \frac{(\sqrt{k_1} - \sqrt{k_2})\eta}{\sqrt{4\pi t}} e^{-\frac{\eta^2}{4t}} \int_0^1 f'(\xi + \sqrt{k_1}\eta + \lambda(\sqrt{k_2} - \sqrt{k_1})\eta) d\lambda d\eta d\xi$$

$$= (\sqrt{k_1} - \sqrt{k_2}) \int_{-\infty}^{\infty} \frac{\eta}{\sqrt{4\pi t}} e^{-\frac{\eta^2}{4t}} \int_0^1 f(x + (\sqrt{k_1} + \lambda(\sqrt{k_2} - \sqrt{k_1}))\eta) d\lambda d\eta$$

$$= (\sqrt{k_1} - \sqrt{k_2}) \int_{-\infty}^{\infty} \eta \cdot G_0(\eta, t) \int_0^1 f(x + (\sqrt{k_1} + \lambda(\sqrt{k_2} - \sqrt{k_1}))\eta) d\lambda d\eta.$$

Hence,

$$|\int_{-\infty}^{x} [(G_1 - G_2) * f](\xi) d\xi| \leq C \sup_{\mathbf{R}} |\eta \cdot G_0(\eta, t)| \cdot \|f\|_{L^1} \leq C \|f\|_{L^1} \tag{4.4}$$

and

$$\| \int_{-\infty}^{x} [(G_1 - G_2) * f](\xi) d\xi \| \leq C \|\eta \cdot G_0(\eta, t)\| \|f\| \leq C t^{1/4} \|f\|. \tag{4.5}$$

Using (4.4), (4.5) we estimate each term of (4.1). First two terms are easily estimated as

$$|(1)| \leq C(1+t)^{-1/2}, \quad \|(1)\| \leq C(1+t)^{-1/4} \tag{4.6}$$

and

$$|(2)| \leq C, \quad \|(2)\| \leq C(1+t)^{1/4} \tag{4.7}$$

if $\theta_0 \in L^1$. In this section, only by C denote a constant depending on $\|w_0\|_5 + \|w_1, \theta_0\|_4 + \|w_0, w_{0x}, w_1, \theta_0\|_{L^1}$. For (3) it is enough to estimate $(3)_1 := \int_0^t G * u_t d\tau$ and $(3)_2 := \int_0^t G * g_2 d\tau$, where $G = G_1$ or G_2. By the integration by parts in τ,

$$(3)_1 = [G(t-\tau) * u(\tau)]|_{\tau=0}^{\tau=t/2} + \int_0^{t/2} G_t(t-\tau) * u(\tau) d\tau + \int_{t/2}^{t} G(t-\tau) * u_t(\tau) d\tau$$

and hence, from Theorems 1-2,

$$\begin{aligned} |(3)_1| &\leq \|G(t/2)\| \|u(t/2)\| + \|G(t)\|_{L^\infty} \|w_1\|_{L^1} \\ &+ \int_0^{t/2} \|G_t(t-\tau)\| \|u(\tau)\| d\tau + \int_{t/2}^{t} \|G(t-\tau)\| \|u_t(\tau)\| d\tau \\ &\leq C(t^{-1} + t^{-1/2} + \int_0^{t/2} (t-\tau)^{-5/4} (1+\tau)^{-3/4} d\tau + \int_{t/2}^{t} (t-\tau)^{-1/4} (1+\tau)^{-3/2} d\tau) \\ &\leq C t^{-1/2} \end{aligned} \tag{4.8}$$

and

$$
\begin{aligned}
\|(3)_1\| &\le \|G(t/2)\|_{L^1}\|u(t/2)\| + \|G(t)\|\|w_1\|_{L^1} \\
&+\int_0^{t/2} \|G_t(t-\tau)\|_{L^1}\|u(\tau)\|u(\tau)\|d\tau + \int_{t/2}^t \|G(t-\tau)\|_{L^1}\|u_t(\tau)\|d\tau \\
&\le C(t^{-3/4}+t^{-1/4}+\int_0^{t/2}(t-\tau)^{-1}(1+\tau)^{-3/4}d\tau + \int_{t/2}^t 1\cdot(1+\tau)^{-3/2}d\tau) \\
&\le Ct^{-1/4}.
\end{aligned} \tag{4.9}
$$

Since

$$
\|g_2(t)\|_{L^1} \le C\|(v,\theta)(t)\|\|(v_x,\theta_x)(t)\| \le C(1+t)^{-1}, \tag{4.10}
$$

it holds that

$$
\begin{aligned}
|(3)_2| &\le \int_0^t \|G(t-\tau)\|_{L^\infty}\|g_2(\tau)\|_{L^1}d\tau \\
&\le C(\int_0^{t/2}+\int_{t/2}^t)(t-\tau)^{-1/2}(1+\tau)^{-1}d\tau \le C(1+t)^{-1/2}\ln(2+t),
\end{aligned} \tag{4.11}
$$

and that

$$
\|(3)_2\| \le \int_0^t \|G(t-\tau)\|\|g_2(\tau)\|_{L^1}d\tau \le C(1+t)^{-1/4}\ln(2+t). \tag{4.12}
$$

Estimates of the final term (4) are as follows:

$$
\begin{aligned}
|(4)| &\le C\int_0^t \|g_3\|_{L^1}d\tau \\
&\le C\int_0^t (\|(v,\theta)(\tau)\|\|u_x(\tau)\| + \|(\theta,\theta_x)(\tau)\|\|\theta_{xx}\|)d\tau \\
&\le C\int_0^t (1+\tau)^{-1/4-5/4}d\tau \le C
\end{aligned} \tag{4.13}
$$

and

$$
\begin{aligned}
\|(4)\| &\le C\int_0^t (t-\tau)^{1/4}\|g_3(\tau)\|d\tau \\
&\le C\int_0^t (t-\tau)^{-1/4}(1+\tau)^{-1/2-5/4}d\tau \le Ct^{1/4}.
\end{aligned} \tag{4.14}
$$

Combining (4.6)-(4.14) we obtain

$$
(1+t)^{-1/4}\|w(t)\| + \|w(t)\|_{L^\infty} \le C.
$$

The other terms $w_x = v, w_t = u, \theta$ etc. are same as the orders in Theorem 2.

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