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「粘性保存則系の非線型波の安定性と
派生する消散型波動方程式の拡散現象」

研究成果報告書

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研究代表者　西原　健二

(早稲田大学 政治経済学術院 教授)

まえがき

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「粘性保存則系の非線型波の安定性と
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の研究成果報告書である。本研究は、

研究代表者	西原 健二	(早稲田大学政治経済学術院教授)
研究分担者	松村 昭孝	(大阪大学大学院情報科学研究科教授)

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さて, 双曲型保存則の方程式系は, 非線型波として, 衝撃波，希薄波及び接触不連続波をもつ．実在の物理現象では多くの場合, 何らかの粘性効果が働き，粘性的双曲型保存則系となる．この場合には，粘性効果により平滑化された粘性衝撃波，希薄波，粘性的接触波及び散逸波を持つ．この研究では, 通常の Newton 粘性によるものと, Porous media 中の流れに現れる摩擦効果による項を持つ粘性的双曲型保存則系について考察した．さらに, Porous media 中の流れはダルシーの法則によって対応する放物型方程式の解に漸近するすることが予想され，いろいろなケースにそのことが証明されてきた．その事実から派生して，消散型波動方程式が時間発展と共に対応する放物型方程式に漸近することが予想され，実際，平成 13-15 年度の研究（基盤研究 (C)(2) 「粘性的双曲型保存則系の非線型波

の安定性」研究代表者 西原健二）において，その一部が示された．本研究では，それらを発展させた研究が主に研究代表者によってなされ，分担者は主として，衝撃波が平滑化された粘性衝撃波, 希薄波，粘性的接触波および散逸波などの非線型波の安定性, 或いはCauchy 問題, 半空間における初期値境界値問題の解の漸近挙動が研究された．

第 1 章では, 本研究の概要について述べる．まず, 代表者の研究の概要と分担者の研究の概要を述べる．発表論文リスト，シンポジウム等での講演リストも添付する．さらに，保存則や流体の研究を主題として含む大きな国際研究集会 " Hyperbolic Problems: Theory, Numerics and Applications " が大阪において開催された．特記事項として，この集会について述べておく．

第 2 章では，3 篇の論文と，投稿予定の「論説」を主要論文としてその全文を示す．3 篇の論文はすでに学術誌に掲載されている．

最後に，本研究の遂行には, 多くの方々との研究交流, ご協力を頂きましたことによっている．ここに深く感謝いたします。

2008(平成 20) 年 3 月
研究代表者　西原健二

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第1章　研究概要

1.1　研究代表者による研究の概要

研究代表者（西原 健二）は，Porous media 中の圧縮性流の方程式系に対するコーシー問題の解の漸近挙動の考察から派生して得られた，2 階消散型波動方程式の解の拡散現象についての研究を主として行った．2 階消散型波動方程式の解の拡散現象については，平成 13-15 年度の研究（基盤研究 (C)(2) 研究課題「粘性的双曲型保存則系の非線型波の安定性」研究代表者 西原健二）において，特に，論文“K. Nishihara, L^p-L^q estimates of solutions to the damped wave equation in 3-dimensional space and their application, Math. Z. 244, 631-649(2003)”において考察が深められ，本研究においてもいろいろな状況に応じた発展がなされた．そこで使われた手法の一部は他の方程式系にも応用された．また，粘性的双曲型保存則系の非線型波である希薄波の安定性については，その典型例である圧縮性粘性流の等エントロピーモデル，等温モデルにおいて考察された ([1],[2])．これらの研究については，研究論文として執筆され，学術雑誌に掲載されている．かぎカッコ内の番号が本節末の「研究論文リスト」の番号に対応している．また，それらの内容はいろいろなシンポジウムなど学術的会合において発表された．それらは本節末に「講演リスト」としてまとめられている．

空間 N 次元半線型消散型波動方程式

$$(DW) \qquad u_{tt} - \Delta u + u_t = f(u), \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N$$

の半線形項 $f(u)$ は

$$f(u) = |u|^{\rho-1}u \ \text{ または } \ |u|^\rho, \quad \rho > 1$$

のときは湧き出し項として働き，

$$f(u) = -|u|^{\rho-1}u \quad \rho > 1$$

のときは吸収項として働く．(DW) のコーシー問題を考えると，対応する半線型熱方程式に対する結果から，半線形項 f が湧き出しのときは，臨界指数（藤田指数と呼ぶ）

$$\rho_F(N) = 1 + \frac{2}{N}$$

があって，

期待 I

$$\rho > \rho_F(N)\ (\text{優臨界}) \quad \Rightarrow \quad \begin{cases} \text{小さな } L^1 \text{ データに対し一意的大域解があって，} \\ \text{その漸近形はガウス核の定数倍である} \end{cases}$$

$$\left.\begin{array}{r} \rho = \rho_F(N)\ (\text{臨界}) \\ 1 < \rho < \rho_F(N)\ (\text{劣臨界}) \end{array}\right\} \quad \Rightarrow \quad \begin{cases} \text{適当なデータに対し，その小ささに拘わらず，} \\ \text{有限時間内に解は爆発し，その爆発時間も評価} \\ \text{される} \end{cases}$$

が期待され，適当な状況の下では肯定的に結果が示されている．一方，f が吸収項として働くときは，大きいデータに対しても大域解が存在し，$t \to \infty$ のとき，その漸近形は

期待 II

$$\begin{array}{rcl} \rho > \rho_F(N)\ (\text{優臨界}) & \Rightarrow & \text{漸近形はガウス核の定数倍となる} \\ \rho = \rho_F(N)\ (\text{臨界}) & \Rightarrow & \text{漸近形は漸近的ガウス核の定数倍となる} \\ 0 < \rho < \rho_F(N)\ (\text{劣臨界}) & \Rightarrow & \begin{cases} \text{漸近形は対応する半線型熱方程式の定常解} \\ \text{となる} \end{cases} \end{array}$$

となることが期待される．

半線形項が湧き出し項として働くとき，半直線上の初期値境界値問題に対しては，線型問題の解の減衰が速いため，新しい臨界指数が現われ，優臨界，臨界，劣臨界の場合に，上記の「期待 I」が正しいことが示され，爆発時間の評価なども得られた ([3])．

吸収項として働く場合の半線型消散型波動方程式のコーシー問題に対し，論文"G. Todorova and B. Yordanov, Critical exponent for a nonlinear wave equation with damping, J. Differential Equations **174**(2001), 464-489"において開発された重み付きエネルギー評価法を応用することによって，劣臨界指数の場合に，大域解が半線型熱方程式の定常解と同じ減衰率で減衰することが，[5] において示された．ただし，初期データの空間方向の減衰の仮定が強く，不満が残るものであったが，エネルギー法によるシャープな減衰率が得られたのは評価できるものと考えている．

さらに，初期データの空間方向の減衰の仮定は，定常解の空間方向の減衰率の観点から見て合理的な場合に改良された ([6])．それは，Todorova-Yordanov の方法を改良したもので，劣臨界指数の場合の解の減衰については最適の結果となって，定常解への漸近が残る課題となった．得られた減衰率は優臨界，臨界指数の場合にも有効であるが，予想される漸近形であるガウス核あるいは漸近的ガウス核の定数倍から見ると減衰率が不足したものである．しかしながら，優臨界指数の場合には，この不足した減衰率を足がかりにして，

線形消散型波動方程式に対する解表示を用いて改良し，結果として，ガウス核の定数倍が解の漸近形となることを，4 次元までの空間における問題に対して証明した ([6],[8]).

Todorova-Yordanov による重み付きエネルギー評価の改良は，[4] において応用された. ガウス核の定数倍が漸近形となることを示す手法は，[7] で楕円性を持つ非線形消散系に応用された.

これまでの結果の多くは可積分なデータに対する結果であったが，[9] において，可積分とは限らないデータに対して，半線形項が湧き出し項として働く場合に，消散型波動方程式のコーシー問題を考察した. 対応する半線形熱方程式では，" T-Y. Lee, W-M. Ni, Global existence, large time behavior and life span of solutions of a semilinear parabolic Cauchy problem, Trans. Amer. Math. Soc. 333 (1992) 365–378 " の結果から，新しい臨界指数が現れる. [9] では優臨界指数の場合に，半線形熱方程式のコーシー問題の解の漸近形を求めると同時に，Lee-Ni の証明の別証明を与え，その方法は空間 3 次元までにおける消散型波動方程式のコーシー問題にも応用可能なものであった. 臨界や劣臨界指数の場合や高次元空間における問題の場合にも研究が継続中である.

以上のような消散型波動方程式に対する結果のうち，キーとなった論文 [5],[9]，応用例となる [7] を主要論文として，第 2 章でそれらの全文を与える. また，日本数学会発行の雑誌「数学」に投稿予定の論説「消散型波動方程式のコーシー問題の解の拡散現象」も全文を与えた.

研究代表者（西原 健二）による研究論文リスト

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研究代表者（西原 健二）による講演リスト

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4. 吸収項を持つ消散型波動方程式の解の大域的挙動, 熊本大学応用解析セミナー (熊本大学), 2005 年 2 月.
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1.2 研究分担者による研究の概要

研究分担者（松村 昭孝）は1次元圧縮性粘性流の方程式系に対する非線型波—粘性衝撃波，希薄波，接触不連続波に対応する粘性接触波および散逸波—の安定性に関する研究を深めた．また，半導体に関する関するモデル方程式系は圧縮性粘性流の方程式系と強く関連性を持つものとなる．これについても定常解の安定性についての研究を行った．これらの研究については，研究論文として執筆され，学術雑誌に掲載されている．また，それらの内容はいろいろなシンポジウムなど学術的会合において発表された．それらは本節末に,「研究論文リスト」,「講演リスト」としてまとめられている．

さて，1次元単独バーガース方程式に対して，流速 (flux) が凸関数でない場合に，コーシー問題では粘性衝撃波と希薄波の重ね合わせに漸近することが予想されているが，未だ未解決である．しかしながら，半直線上の初期値境界値問題に対しては，流速が凸の場合には，Liu-松村（研究分担者）-西原（研究代表者）による研究“Asymptotic behaviors of solutions for the Burgers equation with boundary corresponding to rarefaction wave, SIAM J. Math. Anal. 29(2)(1998 Mar.), 293-308”があり，流速が凸でない場合が未解決として残されていた．ここでは，単独の一次元粘性保存則の方程式の半空間上での初期値境界値問題を考察し、これまであまり考察されていなかった流束が凸関数でない場合につぃも，定常境界層解と希薄波の重ね合せ（合成波）が漸近安定であることを希薄波が適当に小である条件下で示した ([8]).

1次元圧縮性粘性流の等エントロピーモデルは，保存則系の粘性モデルの典型的な 2×2 システムとなるが，この 2×2 システムに対する半空間上での自由境界値問題，特に，境界上で相転移等により流れ込みが有る場合，適当な条件下で解は境界層解と粘性衝撃波との重ね合わせに漸近することを示した ([1]).

一般の空間1次元の圧縮性粘性流のモデル方程式系は，3×3 システムで表されるが，対応する非粘性 3×3 システムには 2×2 システムでは現れない非線型波である接触不連続波が現れる．圧縮性粘性流の理想気体モデルは典型的な 3×3 システムとなるが，このシステムに対する半空間上の初期値境界値問題を考察し，接触不連続波に対応する粘性的接触波の漸近安定性を自由境界の境界条件の下で示すことに成功した（[2]). 圧縮性粘性流体の空間一次元理想気体モデル（3×3 システム）に対するコーシー問題は一般に初期値境界値問題よりも難解であるが，コーシー問題についても考察し,接触不連続波に対応する粘性的接触波の漸近安定性を初期擾乱の平均がゼロの条件の下で示すことにも成功した ([4]). 複数の非線型波の重ねあわせに関する安定性は一層難解な問題となるが，おなじく，圧縮性粘性流の理想気体モデルの、粘性的接触波と希薄波の一次結合の重ね合わせ（合成波）がその波の強さが適当に小さければ漸近安定であることを証明し，それについても論文を準備中である．

最後に，半導体をモデルとする方程式系は，緩和項が付加されたオイラー方程式とポアソン方程式の連立系にさらに量子効果を現わすボームポテンシャル項を加えた連立系となる．この系に対する定常解の存在と安定性の問題を全空間上で考察した．1次元モデルでは，無限遠方での状態が亜音速のみならず，超音速になっても量子効果により定常解が存在して，この定常解は漸近安定であることを示した([5])．また，緩和時間零極限においては，解は移流・拡散モデルの解に漸近することを示すことに成功した([7])．これらの結果はドーピングプロファイルと呼ばれる外部電荷密度が小さいという条件の下のものである．しかし，量子効果を考慮しないモデルを有界区間上で周期境界条件で考察する初期・境界値問題では，任意に大きなドーピングに対しても対応する定常解が漸近安定であることを示した([6])．

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10. 粘性および熱伝導性を持つ圧縮性理想気体の方程式系の解のある長時間挙動について；非線形解析セミナー, 慶応義塾大学理工学部数理科学科, ２００６年１１月１日.
11. Asymptotic stability of a composite wave of two viscous shock waves for the equations of one-dimensional motion of the viscous and heat-conductive gas ; Workshop on Mathematical Analysis on Nonlinear Phenomenain honor of Professor Atusi Tani on the occasion of his 60th birthday, 慶應義塾大学, ２００６年１２月２０日.
12. 単独粘性保存則に対する半直線上のある初期値境界値問題について, 橋本伊都子との共同講演 2007 年度　日本数学会　秋期総合分科会 9 月 21 日～24 日, 講演は 9 月 23 日 (日).
13. 一次元粘性保存則系の解の長時間挙動粘性気体の方程式系を軸に., I, II 〈Survey Lecture〉 [Large-time behavior of solution for one-dimensional system of viscous conservation laws (in particular for viscous gas), I,II] 研究集会「微分方程式の総合的研究」, 東京大学数理科学研究科, 2007, 12.14 ～12.15

1.3 国際研究集会 “Hyperbolic Problems: Theory, Numerics and Applications” の開催

第10回双曲型問題に関する国際会議（Tenth International Conference on Hyperbolic Problems）を，HOTEL HANKYU EXPO PARK, Osaka において，2004年9月13日より9月17日の5日間に渡って開催した．この国際会議は，双曲型偏微分方程式の数学理論・計算・応用すべての側面について，それらに携わる研究者たちが一堂に集まり，最新の研究成果を発表し，討議し，さらに将来の展望を得ることを目的とし，1986年以来隔年ごとに世界各地で開催されてきたものである．第9回の会議で，2004年には日本で開催することが決定し，組織委員会 (Local Organizing Committee) が組織された．大阪大学教授 松村 昭孝（本研究分担者）と九州大学教授 川島 秀一を共同議長とし，東京工業大学助教授 西畑 伸也 を事務局長，及び，大阪電気通信大学教授 浅倉 史興，宇宙開発研究機構 主任研究官 相曽 秀昭，早稲田大学教授 西原 健二（本研究代表者）を組織委員とした．科学研究費のほか，日本万国博覧会記念協会等の資金援助の下，また多くの方々の協力の下，国内外から200名近い参加者を得て開催された．海外からの参加者も120名ほどで，海外からの参加者の方が多いと言うまさに国際集会となった．西原は組織委員として，日本入国に際し，査証の必要な中国を初めとするロシア，ブラジル等からの参加者およそ30名の関係書類を会議直前まで整え，幸い全員が事故も無く入国し，会議に参加でき，講演，討論に参加できた．

大阪において本国際会議を主催することにより，国内外より200名近い参加者（海外から120名程）を得たことで，純粋数学，応用数学の両面から本分野の重要性を日本の数学界に示すことができたと考えている．

また，会議の終了した後，プロシーディングスも，浅倉史興をチーフエディター，組織委員をエディターとして編集され，2006年に，2巻の書物として発行された:

Hyperbolic Problems: Theory, Numerics and Applications -I-,

Hyperbolic Problems: Theory, Numerics and Applications -II-,

edited by F. Asakura(Chief Editor), H. Aiso, S. Kawashima, A. Matsumura, S. Nishibata and K. Nishihara, Yokohama Publishers, 2006.

第2章　主要研究論文

3 篇の論文と論説について全文を与える．論文は学術雑誌の巻，号，ページ等を示した．

1. K. Nishihara and H. J. Zhao, Decay properties of solutions to the Cauchy problem for the damped wave equation with absorption, J. Math. Anal. Appl. 313 (2006) 598 - 610.

2. T. Narazaki and K. Nishihara, Asymptotic behavior of solutions for the damped wave equation with slowly decaying data, J. Math. Anal. Appl. 338 (2008) 803 - 819.

3. K. Nishihara, Asymptotic profile of solutions to nonlinear dissipative evolution system with ellipticity, Z. angew. Math. Phys. 57(2006), 604-614.

4. 西原 健二，消散型波動方程式のコーシー問題の解の拡散現象，投稿予定.

2.1 Decay Properties of Solutions to the Cauchy Problem for the Damped Wave Equation with Absorption

Kenji Nishihara
School of Political Science and Economics,
Waseda University, Tokyo, 169-8050 Japan
(e-mail: kenji@waseda.jp)

Huijiang Zhao
School of Mathematics and Statistics, Wuhan University
Wuhan 430072, P. R. China
(e-mail: hhjjzhao@hotmail.com.)

Abstract

We consider the Cauchy problem for the damped wave equation with absorption

$$u_{tt} - \Delta u + u_t + |u|^{p-1}u = 0, \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N.$$

The behavior of u as $t \to \infty$ is expected to be same as that for the corresponding heat equation

$$\phi_t - \Delta\phi + |\phi|^{p-1}\phi = 0, \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N,$$

which has the similarity solution $w_a(t,x)$ with the form $t^{-1/(p-1)}f(x/\sqrt{t})$ depending on $a = \lim_{|x|\to\infty} |x|^{2/(p-1)}f(x) \geq 0$ provided that p is less than the Fujita exponent $p_c(N) := 1 + 2/N$. In this paper, as a first step, if $1 < p < p_c(N)$ and the data $(u_0, u_1)(x)$ decays exponentially as $|x| \to \infty$ without smallness condition, the solution is shown to decay with orders as $t \to \infty$

$$(\|u(t)\|_{L^2},\ \|u(t)\|_{L^{p+1}},\ \|\nabla u(t)\|_{L^2}) = O\left(t^{-\frac{1}{p-1}+\frac{N}{4}},\ t^{-\frac{1}{p-1}+\frac{N}{2(p+1)}},\ t^{-\frac{1}{p-1}-\frac{1}{2}+\frac{N}{4}}\right), \qquad (*)$$

those of which seem to be reasonable, because the similarity solution $w_a(t,x)$ have the same decay rates as $(*)$. For the proof, the weighted L^2-energy method will be employed with suitable weight, similar to that in Todorova and Yordanov [32].

1 Introduction

We consider the Cauchy problem for the semilinear damped wave equation with absorption:

$$u_{tt} - \Delta u + u_t + |u|^{p-1}u = 0, \quad (t,x) \in (0,\infty) \times \mathbf{R}^N, \tag{1.1}$$

$$(u, u_t)(0,x) = (u_0, u_1)(x), \quad x \in \mathbf{R}^N, \tag{1.2}$$

where $p > 1$. When $(u_0, u_1) \in H^1 \times L^2$ and

$$1 < p < \frac{N+2}{N-2}\ (N \geq 3), \quad 1 < p < \infty\ (N = 1, 2), \tag{1.3}$$

there exists a unique solution $u \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$ (Strauss [31], Ginibre and Velo [9], Brenner [1], Matsumura [24], Kawashima, Nakao and Ono [21] etc.). In [21] it is shown that, for $1 \le N \le 3$, if

$$1+\frac{4}{N} < p < \frac{N+2}{N-2} \ (N \ge 3), \quad 1+\frac{4}{N} < p < \infty \ (N=1,2),$$

then the solution $u(t,x)$ decays as

$$\|u(t,\cdot)\|_{L^2} = O\left(t^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{2})}\right), \quad \text{when} \quad (u_0,u_1) \in H^1(\mathbf{R}^N) \times (L^2 \cap L^r)(\mathbf{R}^N) \ (1 \le r \le 2)$$

(for more details refer [21]), whose rate is same as that of solutions to the linear heat equation. Based on [21], Karch [20] showed that the Gauss kernel is an asymptotic profile when $p > 1+4/N$ with $1 \le N \le 3$. Very recently, in Hayashi, Kaikina and Naumkin [11] the asymptotic profile of u for $p > 1+2/N$ with $N=1$ has been shown to be the Gauss kernel.

On the other hand, the global existence and blow-up of small weak solutions to the damped wave equation with the forcing term

$$u_{tt} - \Delta u + u_t = |u|^p \tag{1.4}$$

with (1.2) have been also investigated. Todorova and Yordanov [32] have shown that

$$p_c(N) := 1 + \frac{2}{N} \tag{1.5}$$

is the critical exponent, which is called the Fujita exponent named after Fujita [6], in any dimensional space. Refer to Zhang [33] for the blow-up in the critical case, and Li and Zhou [22], Nishihara [27] for the blow-up time. See also Ikehata, Miyaoka and Nakatake [18], Ikehata and Tanizawa [17], Ono [29, 30], Galley [7, 8], Karch [20], and references therein for the global existence and its profile. Recently, the first author has shown in [26] that the linear damped wave equation is approximated by the corresponding heat equation in 3-dimensional space. He has precisely derived the L^p-L^q estimate on the difference of each solution. See also Marcati and Nishihara [23] in 1-dimensional space, Hosono and Ogawa [15] in 2-dimensional space and Narazaki [25] in general space dimension and Ikehata [16], Ikehata and Nishihara [19], Chill and Haraux [3] in the abstract setting. These are applied to the semilinear problem (1.4) with (1.2). Note that the basic estimates on the solution to the linear damped wave equation were obtained by Matsumura [24].

Summing up these results, the solution to the damped wave equation (1.4) is expected to have the similar behavior as $t \to \infty$ to that to the corresponding heat equation

$$\phi_t - \Delta\phi = |\phi|^p. \tag{1.6}$$

In this paper, we consider decay properties of solutions $u(t,x)$ to (1.1)-(1.2) when

$$1 < p < p_c(N) = 1 + \frac{2}{N}, \tag{1.7}$$

whose decay rates should be related to the Cauchy problem for the semilinear heat equation

$$\phi_t - \Delta\phi + |\phi|^{p-1}\phi = 0, \quad (t,x) \in (0,\infty) \times \mathbf{R}^N, \tag{1.8}$$

$$\phi(0,x) = \phi_0(x), \quad x \in \mathbf{R}^N. \tag{1.9}$$

For any $p > 1$, (1.8) has a solution $w^*(t,x) := ((p-1)t)^{-1/(p-1)}$. For p satisfying (1.7), it was proved by Brezis, Peletier and Terman [2] that there exists a family of positive self-similar solutions $w_a(t,x)$ such that

$$\lim_{|x|\to\infty} |x|^{2/(p-1)} w_a(t,x) =: a \geq 0$$

exists. The solution $w_a(t,x)$ has the form

$$w_a(t,x) = t^{-\frac{1}{p-1}} f\left(\frac{x}{\sqrt{t}}\right) \tag{1.10}$$

with

$$-\Delta f - \frac{y \cdot \nabla f}{2} + |f|^{p-1} f = \frac{1}{p-1} f. \tag{1.11}$$

We recall some results on the asymptotic behavior of solutions to (1.8)-(1.9) with (1.7). Gmira and Véron [10] showed that, if $\phi_0 \geq 0$, $\phi_0 \in L^1(\mathbf{R}^N)$ and $\lim_{|x|\to\infty} |x|^{2/(p-1)} \phi_0(x) = +\infty$, then

$$\lim_{t\to\infty} t^{\frac{1}{p-1}} \Big(\phi(t,\cdot) - w^*(t,\cdot)\Big) = 0, \quad \text{uniformly on } \ \{x \in \mathbf{R}^N;\, |x| \leq C\sqrt{t}\}.$$

Escobedo and Kavian [4] proved that, if

$$\phi_0 \not\equiv 0,\ 0 \leq \phi_0(x) \leq Ce^{-\beta|x|^2} \ \text{ for some } \ \beta > 0,\ C > 0, \tag{1.12}$$

then

$$\lim_{t\to\infty} t^{\frac{1}{p-1}} \|\phi(t,\cdot) - w_0(t,\cdot)\|_{L^\infty} = 0. \tag{1.13}$$

Note that $w_0(t,x)$ decays exponentially as $|x| \to +\infty$. When

$$\phi_0 \in L^1(\mathbf{R}^N), \quad \phi_0 \not\equiv 0, \quad \lim_{|x|\to\infty} |x|^{\frac{2}{p-1}} \phi_0(x) =: a \geq 0, \tag{1.14}$$

it has been proved by Escobedo, Kavian and Matano [5] that, depending on $a \geq 0$, the positive similarity solution $w_a(t,x)$ is uniquely determined and

$$\lim_{t\to\infty} t^{\frac{1}{p-1}} \|\phi(t,\cdot) - w_a(t,\cdot)\|_{L^\infty} = 0. \tag{1.15}$$

From the observation in the above, our conjecture is that the solution $u(t,x)$ to (1.1)-(1.2) also satisfies (1.15) if the data (u_0, u_1) satisfy the condition corresponding to (1.14).

In this paper, corresponding to Escobedo and Kavian [4], we show the decay properties of the solution u to (1.1)-(1.2), provided that

$$|u_0(x)|,\ |u_1(x)| \leq Ce^{-\beta|x|^2} \ \text{ for } \ \beta > 0,\ C > 0, \tag{1.16}$$

but no smallness condition is assumed. To apply the weighted L^2-energy method, we assume that

$$I_0^2 := \int_{\mathbf{R}^N} e^{\beta|x|^2} \left(u_1^2 + |\nabla u_0|^2 + u_0^2\right)(x)\, dx < +\infty \ \text{ for some } \ \beta > 0 \tag{1.17}$$

in stead of (1.16). Denoting the solution space $X(0,T)$ by

$$X(0,T) = C\left([0,T);\, H^1(\mathbf{R}^N)\right) \cap C^1\left([0,T);\, L^2(\mathbf{R}^N)\right),$$

we have our main theorem.

Theorem 1.1 *Assume that* $1 < p < 1 + 2/(N-2)\,(N \geq 3)$, $1 < p < \infty\,(N = 1,2)$ *and that* $(u_0, u_1) \in H^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$ *with (1.17). Then the solution* $u(t,x) \in X(0,\infty)$ *to (1.1)-(1.2) uniquely exists, which satisfies for* $t \geq 0$

$$\|u(t,\cdot)\|_{L^2} \leq CI_0(1+t)^{-\frac{1}{p-1}+\frac{N}{4}}, \quad \|u(t,\cdot)\|_{L^{p+1}} \leq CI_0(1+t)^{-\frac{1}{p-1}+\frac{N}{2(p+1)}}, \tag{1.18}$$

$$\|\nabla u(t,\cdot)\|_{L^2} + \|u_t(t,\cdot)\|_{L^2} \leq CI_0(1+t)^{-\frac{1}{p-1}-\frac{1}{2}+\frac{N}{4}} \tag{1.19}$$

for some positive constant C *provided that* $1 < p \leq 1 + 4/N$.

Remark 1.1. In the supercritical case $p > p_c(N)$, the asymptotic profile of the solution u is expected to be the Gauss kernel $G(t,x)$, whose L^r-norm $(1 < r \leq \infty)$ decays as $\|G(t,\cdot)\|_{L^r} = O\left(t^{-\frac{N}{2}(1-\frac{1}{r})}\right)$. Hence, the decay rates (1.18)-(1.19) are less sharp, and so the subcritical case (1.7) is mainly kept in mind.

Remark 1.2. The L^r-norm $(1 \leq r \leq \infty)$ of the similarity solution w_a decays as

$$\|w_a(t,\cdot)\|_{L^r} = t^{-\frac{1}{p-1}}\left(\int_{\mathbf{R}^N} t^{N/2}\cdot\left|f\left(\frac{x}{\sqrt{t}}\right)\right|^r \frac{dx}{t^{N/2}}\right)^{1/r} = Ct^{-\frac{1}{p-1}+\frac{N}{2r}}. \tag{1.20}$$

Hence, in the subcritical case the decay rates (1.18)-(1.19) are sharp in the L^2-sense. However, compared to our goal (1.15), the results are dissatisfactory. We have

$$\|u(t,\cdot)\|_{L^s} = O\left(t^{-\frac{1}{p-1}+\frac{1}{s}}\right) \quad \begin{cases} 1 \leq s \leq \infty & \text{if} \quad N = 1, \\ 1 \leq s < \infty & \text{if} \quad N = 2, \\ 1 \leq s \leq \frac{2N}{N-2} & \text{if} \quad N \geq 3. \end{cases} \tag{1.21}$$

applying the Sobolev inequality $(N = 1)$ and the Gagliardo and Nirenberg inequality$(N \geq 2)$ to (1.18)-(1.19) and (2.9) in the next section, which will be derived after stating Theorem 2.1. Note that Hayashi, Kaikina and Naumkin [13] have recently obtained (1.15) for $p_c(N) - \varepsilon < p < p_c(N)$ (ε is a small positive constant) with $N = 1$ and the small data with suitable positivity (see Hayashi, Kaikina and Naumkin [12] in the critical case). See also their quite recent paper [14] for large data.

Notations. By C_i, $c_i (i = 0, 1, 2, \cdots)$ or simply C we denote several generic constants. The constant depending on $a, b, \cdots$ is denoted by $C(a, b, \cdots)$. The Lebesgue space $L^q(\mathbf{R}^N)$(resp. Sobolev space $H^m(\mathbf{R}^N)$) were already used with its norm

$$\|f\|_{L^q(\mathbf{R}^N)} = \left(\int_{\mathbf{R}^N} |f(x)|^q\,dx\right)^{1/q} \quad \left(\text{resp.} \quad \|f\|_m = \left(\sum_{k=0}^{m} \|\partial_x^k f\|_{L^2(\mathbf{R}^N)}\right)^{1/2}\right).$$

In particular, $\|f\| := \|f\|_{L^2(\mathbf{R}^N)} = \|f\|_0$. The space $\mathbf{R}^N$ of $L^q(\mathbf{R}^N)$ or the integrand $\mathbf{R}^N$ will be often abbreviated. For brevity, $\|f(t,\cdot)\|_{L^q} = (\int |f(t,x)|^q\,dx)^{1/q}$ will be written simply by $\|f(t)\|_{L^q}$ etc.

2 Proof of Theorem 1.1

In this section we shall prove Theorem 1.1 applying the weighted L^2-energy method. The weight function is chosen as

$$e^{\psi(t,x)}, \quad \text{with} \quad \psi(t,x) = \frac{a|x|^2}{4(t+t_0)}, \quad (0 < a < 1,\ t_0 \geq 1) \tag{2.1}$$

(later a is determined as $1/4$), which is a modification of the weight introduced in Todorova and Yordanov [32]. See also Ikehata and Tanizawa [17]. The weight function ψ satisfies

$$\begin{cases} \nabla\psi = \dfrac{ax}{2(t+t_0)}, \quad |\nabla\psi|^2 = \dfrac{a^2|x|^2}{4(t+t_0)^2} \\ \psi_t = -\dfrac{a|x|^2}{4(t+t_0)^2} < 0, \quad \dfrac{|\nabla\psi|^2}{\psi_t} = -a. \end{cases} \tag{2.2}$$

For the interval $I = [\tau, \tau + t_1]$, $t_1 > 0$ and any fixed $M > 0$, we adopt the solution space

$$X_M(I) = \left\{ \begin{array}{c} u \in C(I;\, H^1) \cap C^1(I;\, L^2),\ e^{\psi(t,\cdot)}(u_t, \nabla u, u)(t,\cdot) \in L^2 \\ \text{with } \sup_{t\in I} E_\psi(t;u)^{1/2} \le M \end{array} \right\} \tag{2.3}$$

where

$$E_\psi(t;u) = \int e^{2\psi(t,x)} \left(|u_t|^2 + |\nabla u|^2 + u^2 \right)(t,x)\, dx. \tag{2.4}$$

Also, denote

$$E_\psi(\tau; u_0^\tau, u_1^\tau) = \int e^{2\psi(\tau,x)} \left(|u_1^\tau|^2 + |\nabla u_0^\tau|^2 + |u_0^\tau|^2 \right)(x)\, dx. \tag{2.5}$$

Clearly, $E_\psi(0;u_0,u_1) < M$ for suitable $t_1 \ge 1$ and $M > 0$ by (1.17). The global existence theorem for (1.1)-(1.2) is well-known. However, we need that the solution u remains in $X_M(I)$ provided that $E_\psi(0;u_0,u_1) < M$. Hence we prepare the local existence theorem in $X_M(I)$ for

$$\begin{cases} u_{tt} - \Delta u + u_t + |u|^{p-1}u = 0, \quad t > \tau,\ x \in \mathbf{R}^N \\ (u, u_t)(\tau, x) = (u_0^\tau, u_1^\tau)(x), \quad x \in \mathbf{R}^N. \end{cases} \tag*{$(2.6)_\tau$}$$

Proposition 2.1 *Let $N \ge 1$ and $1 < p < N/(N-2)\,(N \ge 3)$, $1 < p < \infty\,(N = 1, 2)$. For any $M > 0$ and some constant $C_1 > 0$, if $(u_0^\tau, u_1^\tau) \in H^1 \times L^2$ satisfies $E_\psi(\tau; u_0^\tau, u_1^\tau)^{1/2} \le M$, then there exists a time $t_1 = t_1(M)$ depending only on M such that the Cauchy problem $(2.6)_\tau$ has a unique solution $u(t,x)$ in $X_{2C_1M}(\tau, \tau + t_1)$.*

The sketch of the proof will be given in the Appendix. The local solution $u(t,x) \in X_M([0,T])$ satisfies the following a priori estimates.

Proposition 2.2 *Let p satisfy the conditions in Proposition 2.1 and*

$$\alpha(p) := \frac{1}{p-1} - \frac{N}{4} > 0. \tag{2.7}$$

Then the solution $u(t,x) \in X_M([0,T])$ to (1.1)-(1.2) satisfies the estimates:

$$\begin{aligned} &\int_{\mathbf{R}^N} e^{2\psi(t,x)} \left(|u_t|^2 + |\nabla u|^2 + u^2 + |u|^{p+1} \right)(t,x)\, dx \\ &+ \int_0^t \int_{\mathbf{R}^N} e^{2\psi(\tau,x)} \left(|u_t|^2 + |\nabla u|^2 + |\nabla\psi|^2 u^2 + |u|^{p+1} \right)(\tau,x)\, dx\, d\tau \\ &\le C_0 \int_{\mathbf{R}^N} e^{2\psi(0,x)} \left(|u_1|^2 + |\nabla u_0|^2 + u_0^2 + |u_0|^{p+1} \right)(x)\, dx =: C_0 \bar{E}_\psi(0; u_0, u_1), \end{aligned} \tag{2.8}$$

$$(t+t_0)^{2\alpha(p)}\int_{\mathbf{R}^N} e^{2\psi(t,x)}\left(|u_t|^2+|\nabla u|^2+u^2+|u|^{p+1}\right)(t,x)\,dx$$
$$+(t+t_0)^{-\varepsilon}\int_0^t(\tau+t_0)^{2\alpha(p)+\varepsilon}\int_{\mathbf{R}^N} e^{2\psi(\tau,x)}\left(|u_t|^2+|\nabla u|^2+|u|^{p+1}\right)(\tau,x)\,dx\,d\tau \tag{2.9}$$
$$\leq C_\varepsilon\left(1+\bar{E}_\psi(0;u_0,u_1)\right)$$

and

$$(t+t_0)^{2\alpha(p)+1}\int_{\mathbf{R}^N} e^{2\psi(t,x)}\left(|u_t|^2+|\nabla u|^2+|u|^{p+1}\right)(t,x)\,dx$$
$$+(t+t_0)^{-\varepsilon}\int_0^t(\tau+t_0)^{2\alpha(p)+1+\varepsilon}\int_{\mathbf{R}^N} e^{2\psi(\tau,x)}|u_t|^2(\tau,x)\,dx\,d\tau \tag{2.10}$$
$$\leq C_\varepsilon\left(1+\bar{E}_\psi(0;u_0,u_1)\right)$$

for some $t_0\geq 1$ *and any fixed* $\varepsilon>0$ *with* $C_\varepsilon\to\infty$ *as* $t\to\infty$.

Propositions 2.1-2.2 imply the global existence theorem in $X_M(0,\infty)$. In particular, the estimate (2.8) and the Gagliardo and Nirenberg inequality play a role to extend the local solution to the global one.

Lemma 2.1 *Let the exponents* s,q,r $(1\leq s,q,r\leq\infty)$ *and* $\sigma\in[0,1]$ *satisfy*

$$\frac{1}{s}=\sigma\left(\frac{1}{r}-\frac{1}{N}\right)+(1-\sigma)\frac{1}{q},$$

with $r\leq N$ *except for the case* $(s,r)=(\infty,N)$ *when* $N\geq 2$. *Then it holds that*

$$\|u\|_{L^s}\leq C\|u\|_{L^q}^{1-\sigma}\|\nabla u\|_{L^r}^{\sigma},\quad u\in L^q,\ \nabla u\in L^r$$

for $C=C(s,q,r,N)$.

Applying Lemma 2.1 to the local solution on $[0,T]$, we have

$$\left(\int e^{2\psi(t,x)}|u(t,x)|^{p+1}\,dx\right)^{1/(p+1)}$$
$$\leq C\left(\int e^{\frac{2}{p+1}\psi(t,x)}u(t,x)^2\,dx\right)^{\frac{1}{2}\cdot(1-\sigma)}\left(\int e^{2\psi(t,x)}(|\nabla u|^2+u^2)(t,x)\,dx\right)^{\frac{1}{2}\cdot\sigma}. \tag{2.11}$$

for $\sigma=N(\frac{1}{2}-\frac{1}{p+1})$ (≤ 1 when $p\leq 1+\frac{4}{N-2}$). In fact, since $f:=e^{2\psi/(p+1)}u$ satisfies

$$\nabla f=e^{\frac{2}{p+1}\psi}\left(\nabla u+\frac{1}{p+1}\frac{x}{t+t_0}u\right),$$

the inequality

$$\|f\|_{L^{p+1}}\leq C\|f\|^{1-\sigma}\|\nabla f\|^{\sigma}$$

implies (2.11). From (2.12) and (2.8),

$$E_\psi(t;u)\leq\bar{E}_\psi(t;u)\leq C_1^2\bar{E}_\psi(0;u_0,u_1)\leq C_1^2\left(E_\psi(0;u_0,u_1)+CE_\psi(0;u_0,u_1)^{\frac{p+1}{2}}\right),$$

where

$$\bar{E}_\psi(t;u)=\int e^{2\psi(t,x)}\left(|u_t|^2+|\nabla u|^2+u^2+|u|^{p+1}\right)(t,x)\,dx. \tag{2.12}$$

Hence, for a given data (u_0, u_1) take $M > 0$ so that

$$C_1^2 \left(E_\psi(0; u_0, u_1) + CE_\psi(0; u_0, u_1)^{\frac{p+1}{2}}\right) < M^2,$$

then $E_\psi(T; u) < M^2$, which allows the local solution to extend beyond the time T.

Theorem 2.1 *Let p satisfy the condition in Proposition 2.2. If $E_\psi(0; u_0, u_1) < +\infty$, then the Cauchy problem (1.1)-(1.2) has a unique global solution satisfying (2.8)-(2.10) for any $t \geq 0$.*

Theorem 1.1 is a direct consequence of Theorem 2.1, and (1.21) is derived as follows. By (2.9) the L^1-norm of $u(t,x)$ is estimated as

$$\|u(t)\|_{L^1} \leq \left(\int e^{-2\psi(t,x)} dx\right)^{1/2} \left(\int e^{-2\psi(t,x)} u(t,x)^2\, dx\right)^{1/2} \leq C(1+t)^{-\frac{1}{p-1}+\frac{N}{2}}.$$

Also, the L^∞-norm for $N = 1$ follows from the Sobolev inequality and (1.18)-(1.19):

$$\|u(t,\cdot)\|_{L^\infty} \leq \|u(t,\cdot)\|^{1/2} \|\nabla u(t,\cdot)\|^{1/2} \leq C(t+t_0)^{-\frac{1}{p-1}}.$$

In Lemma 2.1, if $(s,q,r,N) = (s,2,2,2)$, $2 < s < \infty$, then $\sigma = 1 - 2/s < 1$ and

$$\|u(t,\cdot)\|_{L^s} \leq C(t+t_0)^{(-\frac{1}{p-1}+\frac{N}{4})\cdot\frac{2}{s}+(-\frac{1}{p-1}-\frac{1}{2}+\frac{N}{4})\cdot(1-\frac{2}{s})} = C(t+t_0)^{-\frac{1}{p-1}+\frac{1}{s}}.$$

If $(q,r) = (2,2)$ and $s \leq \frac{2N}{N-2}$ for $N \geq 3$, then $\sigma \leq 1$ and the same estimate as above holds. Thus (1.21) is completed.

Proof of Proposition 2.2. Multiplying (1.1) by $e^{2\psi}u_t$ and $e^{2\psi}u$, we have

$$\begin{aligned} 0 = {} & e^{2\psi}u_t \left(u_{tt} - \Delta u + u_t + |u|^{p-1}u\right) \\ = {} & \frac{d}{dt}\left[e^{2\psi}\left(\frac{1}{2}(|u_t|^2 + |\nabla u|^2) + \frac{1}{p+1}|u|^{p+1}\right)\right] \\ & + e^{2\psi}\left[\left(\left(1 + \frac{|\nabla\psi|^2}{\psi_t}\right) - \psi_t\right)|u_t|^2 + \frac{-2\psi_t}{p+1}|u|^{p+1}\right] \\ & - \nabla\cdot(e^{2\psi}u_t\nabla u) + \frac{1}{-\psi_t}e^{2\psi}|\psi_t\nabla u - u_t\nabla\psi|^2, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} 0 = {} & e^{2\psi}u\left(u_{tt} - \Delta u + u_t + |u|^{p-1}u\right) \\ = {} & \frac{d}{dt}\left[e^{2\psi}\left(uu_t + \frac{1}{2}u^2\right)\right] \\ & + \left[e^{2\psi}\left(|\nabla u|^2 - \psi_t u^2 + |u|^{p+1}\right)\right] \\ & + \left[e^{2\psi}\left(-2\psi_t uu_t - |u_t|^2 + 2u\nabla\psi\cdot\nabla u\right)\right] - \nabla\cdot(e^{2\psi}u\nabla u). \end{aligned} \tag{2.14}$$

Since we choose ψ in (2.1) with (2.2), integrating (2.13) and (2.14) over $\mathbf{R}^N$, we respectively get

$$\begin{aligned} & \frac{d}{dt}\int e^{2\psi}\left(\frac{1}{2}\left(|u_t|^2 + |\nabla u|^2\right) + \frac{1}{p+1}|u|^{p+1}\right) dx \\ & + \int e^{2\psi}\left(\left(1 - a + \frac{1}{a}|\nabla\psi|^2\right)|u_t|^2 + \frac{2}{a(p+1)}|\nabla\psi|^2|u|^{p+1}\right) dx \\ & \leq 0, \end{aligned} \tag{2.15}$$

and

$$\begin{aligned}
&\frac{d}{dt}\int e^{2\psi}\left(uu_t+\frac{1}{2}u^2\right)\,dx\\
&+\int e^{2\psi}\left(|\nabla u|^2+\frac{1}{a}|\nabla\psi|^2u^2+|u|^{p+1}\right)\,dx-\int e^{2\psi}|u_t|^2\,dx\\
&\le\int e^{2\psi}\left(\frac{2}{a}|\nabla\psi|^2|uu_t|+2|\nabla\psi||u||\nabla u|\right)\,dx\\
&\le\int e^{2\psi}\left(\frac{4}{a}|\nabla\psi|^2|u_t|^2+2a|\nabla u|^2+\frac{3}{4a}|\nabla\psi|^2u^2\right)\,dx.
\end{aligned}\tag{2.16}$$

Here (2.16) is rewritten by

$$\begin{aligned}
&\frac{d}{dt}\int e^{2\psi}\left(uu_t+\frac{1}{2}u^2\right)\,dx\\
&+\int e^{2\psi}\left((1-2a)|\nabla u|^2+\frac{1}{4a}|\nabla\psi|^2u^2+|u|^{p+1}\right)\,dx\\
&-\int e^{2\psi}\left(1+\frac{4}{a}|\nabla\psi|^2\right)|u_t|^2\,dx\le 0.
\end{aligned}\tag{2.17}$$

Adding (2.15) to (2.17) multiplied by $\nu\,(0<\nu<1)$, we get

$$\begin{aligned}
&\frac{d}{dt}\int e^{2\psi}\left\{\frac{1}{2}\left(|u_t|^2+2\nu uu_t+\nu u^2\right)+\frac{1}{2}|\nabla u|^2+\frac{1}{p+1}|u|^{p+1}\right\}\,dx\\
&+\int e^{2\psi}\left\{\left(1-a-\nu+\frac{1-4\nu}{a}|\nabla\psi|^2\right)|u_t|^2+\nu(1-2a)|\nabla u|^2\right.\\
&\quad\left.+\frac{\nu}{4a}|\nabla\psi|^2u^2+\left(\nu+\frac{2}{a(p+1)}|\nabla\psi|^2\right)|u|^{p+1}\right\}\,dx\le 0.
\end{aligned}\tag{2.18}$$

We determine $a=\nu=1/4$. Then, (2.18) yields

$$\begin{aligned}
&\frac{d}{dt}\tilde{E}_\psi(t;u)+H_\psi(t;u)\\
&:=\frac{1}{2}\frac{d}{dt}\int e^{2\psi}\left(|u_t|^2+uu_t+\frac{1}{2}u^2+|\nabla u|^2+\frac{2}{p+1}|u|^{p+1}\right)\,dx\\
&\quad+\frac{1}{4}\int e^{2\psi}\left(2|u_t|^2+\frac{1}{2}|\nabla u|^2+|\nabla\psi|^2u^2+|u|^{p+1}\right)\,dx\\
&\le 0,
\end{aligned}\tag{2.19}$$

which is the key inequality in the proof. Note that

$$\frac{1}{2}\bar{E}_\psi(t;u)\ge\tilde{E}_\psi(t;u)\ge c\bar{E}_\psi(t;u),\tag{2.20}$$

where $\bar{E}_\psi(t;u)$ is defined in (2.12). Integrating (2.19) over $[0,t]$ and using (2.20), we have (2.8).

Further, multiply (2.19) by $(t+t_0)^{2\alpha(p)+\varepsilon}(0<\varepsilon<1)$, then

$$\frac{d}{dt}\left[(t+t_0)^{2\alpha(p)+\varepsilon}\tilde{E}_\psi(t;u)\right]+(t+t_0)^{2\alpha(p)+\varepsilon}\left[H_\psi(t;u)-\frac{2\alpha(p)+\varepsilon}{t+t_0}\tilde{E}_\psi(t;u)\right]\le 0.\tag{2.21}$$

Making use of (2.20), we have

$$
\begin{aligned}
&H_\psi(t;u) - \frac{2\alpha(p)+\varepsilon}{t+t_0}\tilde{E}_\psi(t;u) \\
&\geq \left[\frac{1}{8}\int e^{2\psi}\left(|u_t|^2+|\nabla u|^2+|u|^{p+1}\right)dx - \frac{2\alpha(p)+1}{2t_0}\int e^{2\psi}\left(|u_t|^2+|\nabla u|^2+|u|^{p+1}\right)dx\right] \qquad (2.22)\\
&+\left[\frac{1}{8}\int e^{2\psi}\left(|\nabla\psi|^2u^2+|u|^{p+1}\right)dx - \frac{2\alpha(p)+1}{2(t+t_0)}\int e^{2\psi}u^2\,dx\right].
\end{aligned}
$$

Second to the last term in (2.22) is estimated from below by

$$
\frac{1}{16}\hat{E}_\psi(t;u) := \frac{1}{16}\int e^{2\psi}\left(|u_t|^2+|\nabla u|^2+|u|^{p+1}\right)dx \quad \text{if} \quad t_0 \geq 8(2\alpha(p)+1). \qquad (2.23)
$$

The last term is estimated from below by

$$
-C(t+t_0)^{-\frac{p+1}{p-1}+\frac{N}{2}}. \qquad (2.24)
$$

In fact, by denoting

$$
\frac{2\alpha(p)+1}{2(t+t_0)}\int e^{2\psi}u^2\,dx = \int_{\kappa|x|\geq\sqrt{t+t_0}} + \int_{\kappa|x|\leq\sqrt{t+t_0}} =: I_1 + I_2
$$

with $\kappa = 1/16\sqrt{(2\alpha(p)+1)}$, each term is estimated as follows:

$$
\begin{aligned}
I_1 \leq\ & \frac{2\alpha(p)+1}{2(t+t_0)}\int_{\kappa|x|\geq\sqrt{t+t_0}}\frac{\kappa^2|x|^2}{t+t_0}e^{2\psi}u^2\,dx \\
\leq\ & \frac{2\kappa^2(2\alpha(p)+1)}{a^2}\int_{\mathbf{R}^N}\frac{a^2|x|^2}{4(t+t_0)^2}e^{2\psi}u^2\,dx = \frac{1}{8}\int_{\mathbf{R}^N}e^{2\psi}|\nabla\psi|^2u^2\,dx
\end{aligned}
$$

since $a = 1/4$, and

$$
\begin{aligned}
I_2 &\leq \int_{\kappa|x|\leq\sqrt{t+t_0}}\frac{2\alpha(p)+1}{2(t+t_0)}e^{2\psi\cdot\frac{p-1}{p+1}}\cdot e^{2\psi\cdot\frac{2}{p+1}}u^2\,dx \\
&\leq \frac{1}{8}\int_{\mathbf{R}^N}e^{2\psi}|u|^{p+1}\,dx + C\int_{\kappa|x|\leq\sqrt{t+t_0}}(t+t_0)^{-\frac{p+1}{p-1}}e^{\frac{|x|^2}{8(t+t_0)}}\,dx \\
&= \frac{1}{8}\int_{\mathbf{R}^N}e^{2\psi}|u|^{p+1}\,dx + C(t+t_0)^{-\frac{p+1}{p-1}+\frac{N}{2}}
\end{aligned}
$$

by the Young inequality with $\frac{p-1}{p+1}+\frac{2}{p+1}=1$. Combining (2.21) with (2.22)-(2.24), we get

$$
\begin{aligned}
&\frac{d}{dt}\left[(t+t_0)^{2\alpha(p)+\varepsilon}\tilde{E}_\psi(t;u)\right] + \frac{1}{16}(t+t_0)^{2\alpha(p)+\varepsilon}\hat{E}_\psi(t;u) \\
&\leq C(t+t_0)^{2\alpha(p)+\varepsilon}\cdot(t+t_0)^{-\frac{p+1}{p-1}+\frac{N}{2}} = C(t+t_0)^{-1+\varepsilon}.
\end{aligned} \qquad (2.25)
$$

Hence, integrating (2.25) over $(0,t)$, $t \leq T$ and using (2.20), we obtain

$$
\begin{aligned}
&(t+t_0)^{2\alpha(p)+\varepsilon}\bar{E}_\psi(t;u) + \frac{1}{16}\int_0^t(\tau+t_0)^{2\alpha(p)+\varepsilon}\hat{E}_\psi(\tau;u)\,d\tau \\
&\leq t_0^{2\alpha(p)+\varepsilon}\bar{E}_\psi(0;u) + C\int_0^t(\tau+t_0)^{-1+\varepsilon}\,d\tau \\
&\leq Ct_0^{2\alpha(p)+\varepsilon}\bar{E}_\psi(0;u_0,u_1) + C_\varepsilon(t+t_0)^\varepsilon.
\end{aligned} \qquad (2.26)
$$

Dividing (2.26) by $(t+t_0)^\varepsilon$, we reach the second desired estimate (2.9). Concerning the weight about t we note that, if we take $\varepsilon = 0$, then the term $\log(t+t_0)$ comes out and the result become less sharp. The method to adopt $\varepsilon > 0$ instead of $\varepsilon < 0$ is seen in Nishikawa [28].

To obtain the third estimate (2.10), multiplying (2.15) with $a = 1/4$ by $(t+t_0)^{\alpha(p)+1+\varepsilon}$, we have

$$\begin{aligned}
&\frac{d}{dt}\left[(t+t_0)^{2\alpha(p)+1+\varepsilon}\int e^{2\psi}\left\{\frac{1}{2}\left(|u_t|^2+|\nabla u|^2\right)+\frac{1}{p+1}|u|^{p+1}\right\}dx\right]\\
&+\frac{3}{4}(t+t_0)^{2\alpha(p)+1+\varepsilon}\int e^{2\psi}|u_t|^2\,dx \qquad (2.27)\\
&\le C(\alpha(p)+1)(t+t_0)^{2\alpha(p)+\varepsilon}\int e^{2\psi}\left(|u_t|^2+|\nabla u|^2+|u|^{p+1}\right)dx.
\end{aligned}$$

Integrating (2.27) over $[0,t]$ and using (2.9) just obtained, we easily show (2.10).

Thus we have completed the proof of Proposition 2.2.

Appendix

We sketch the proof of Proposition 2.1. The proof of similar local existence theorem is seen in Ikehata and Tanizawa [17]. Since $\psi(t,x)$ is decreasing in t, it is enough to show the case $\tau = 0$ in $(2.7)_\tau$. So, our problem is

$$\begin{cases} u_{tt}-\Delta u+u_t = -|u|^{p-1}u, & (t,x)\in[0,\infty)\times\mathbf{R}^N,\\ (u,u_t)(0,x) = (u_0,u_1)(x). \end{cases} \qquad (L)$$

We show that, if $E_\psi(0;u_0,u_1)\le M^2$, then there exists $t_1 = t_1(M) > 0$ such that (L) has a unique solution $u(t,x)$ in $X_{2C_1M}([0,t_1])$, where C_1 is some constant determined later. We construct an approximate sequence $\{u^{(n)}(t,x)\}$ as follows:

The first function $u^{(0)}(t,x)$ is a solution to

$$\begin{cases} u_{tt}-\Delta u+u_t = 0,\\ (u,u_t)(0,x) = (u_0,u_1)(x), \end{cases} \qquad (A1)$$

and, iteratively, $u^{(n+1)}(t,x)$, $n = 0,1,2,\cdots$, is a solution to

$$\begin{cases} u_{tt}-\Delta u+u_t = -|u^{(n)}|^{p-1}u^{(n)},\\ (u,u_t)(0,x) = (u_0,u_1)(x). \end{cases} \qquad (A2)$$

It is enough to assert the following three claims:

(i) For any $t\ge 0$, $E_\psi(t;u^{(0)})^{1/2}\le C_1M$.

(ii) For some $t = t_1(M) > 0$, if $u^{(n)}\in X_{2C_1M}([0,t_1])$, then $u^{(n+1)}\in X_{2C_1M}([0,t_1])$.

(iii) For some $t_1 = t_1(M) > 0$ taken to be smaller if necessary,

$$\sup_{0\le t\le t_1} E_\psi(t;u^{(n+1)}-u^{(n)}) \le \frac{1}{4}\sup_{0\le t\le t_1} E_\psi(t;u^{(n)}-u^{(n-1)}).$$

Since we have $u \in C([0,\infty);\, H^1) \cap C^1([0,\infty);\, L^2)$, multiplying (A1) by $e^{2\psi}(u_t + \frac{1}{4}u)$, we have

$$\frac{1}{2}\frac{d}{dt}\int e^{2\psi(t,x)}\left(|u_t|^2 + uu_t + \frac{1}{2}u^2 + |\nabla u|^2\right) dx \leq 0.$$

Hence, by

$$\frac{1}{2}E_\psi(t;u) \geq \int e^{2\psi(t,x)}\left(|u_t|^2 + uu_t + \frac{1}{2}u^2 + |\nabla u|^2\right) dx \geq cE_\psi(t;u),$$

for some constant $C_1 > 0$

$$E_\psi(t;u) \leq C_1^2 E_\psi(0;u_0,u_1) \leq (C_1M)^2,$$

which means (i). Next, multiplying (A2) by $e^{2\psi}(u_t + \frac{1}{4}u)$, we have

$$\begin{aligned}
E_\psi(t;u) &\leq (C_1M)^2 + C\int_0^t \int e^{2\psi}|u^{(n)}|^p(|u| + |u_t|)\, dx d\tau \\
&\leq (C_1M)^2 + \int_0^t \int e^{2\psi}\left(|u^{(n)}|^{2p} + u^2 + |u_t|^2\right) dx d\tau \\
&\leq (C_1M)^2 + C\int_0^{t_1} E_\psi(\tau;u^{(n)})^p\, d\tau + C\int_0^t E_\psi(\tau;u)\, d\tau, \quad t \leq t_1,
\end{aligned}$$

since $\int e^{2\psi}|u^{(n)}|^{2p}\, dx \leq CE_\psi(t;u^{(n)})^p$, which is obtained by similar way to (2.11). Hence the Gronwall inequality implies

$$E_\psi(t;u) \leq \left((C_1M)^2 + C(2C_1M)^{2p}t_1\right)e^{Ct_1} \leq (2C_1M)^2$$

if $0 < t_1 \ll 1$, which gives (ii). The assertion (iii) is followed from the similar way to (ii).

Thus we have completed the proof of Proposition 2.1.

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2.2 Asymptotic behavior of solutions for the damped wave equation with slowly decaying data

Takashi Narazaki
Department of Mathematical Sciences, Faculty of Science,
Tokai University, Hiratsuka 259-1292, Japan
(e-mail: narazaki@ss.u-tokai.ac.jp)

Kenji Nishihara
Faculty of Political Science and Economics,
Waseda Unviversity, Tokyo, 169-8050 Japan
(e-mail: kenji@waseda.jp)

Abstract

We consider the Cauchy problem for the damped wave equation

$$u_{tt} - \Delta u + u_t = |u|^{\rho-1}u, \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N$$

and the heat equation

$$\phi_t - \Delta\phi = |\phi|^{\rho-1}\phi, \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N.$$

If the data is small and slowly decay likely $c_1(1+|x|)^{-kN}$, $0 < k \le 1$, then the critical exponent is $\rho_c(k) = 1 + \frac{2}{kN}$ for the semilinear heat equation. In this paper it is shown that in the supercritical case there exists a unique time global solution to the Cauchy problem for the semilinear heat equation in any dimensional space $\mathbf{R}^N$, whose asymptotic profile is given by

$$\Phi_0(t,x) = \int_{\mathbf{R}^N} \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{N/2}} \frac{c_1}{(1+|y|^2)^{kN/2}} dy$$

provided that the data ϕ_0 satisfies $\lim_{|x|\to\infty}\langle x\rangle^{kN}\phi_0(x) = c_1(\neq 0)$. Even in the semilinear damped wave equation in the supercritical case a time global solution u with the data $(u,u_t)(0,x) = (u_0,u_1)(x)$ is shown in low dimensional spaces $\mathbf{R}^N$, $N = 1,2,3$, to have the same asymptotic profile $\Phi_0(t,x)$ provided that $\lim_{|x|\to\infty}\langle x\rangle^{kN}(u_0+u_1)(x) = c_1(\neq 0)$. Those proofs are given by elementary estimates on the explicit formulas of solutions.

1 Introduction

In this paper we have concerned with asymptotic behavior of solutions to the Cauchy problem for the semilinear damped wave equation

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^{\rho-1}u, & (t,x) \in (0,\infty) \times \mathbf{R}^N, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^N, \end{cases} \tag{1.1}$$

with $\rho > 1$. It has been recognized that the solution to (1.1) behaves as that to the corresponding semilinear heat equation

$$\begin{cases} \phi_t - \Delta\phi = |\phi|^{\rho-1}\phi, & (t,x) \in (0,\infty) \times \mathbf{R}^N, \\ \phi(0,x) = \phi_0(x), & x \in \mathbf{R}^N, \end{cases} \tag{1.2}$$

For small data ϕ_0 in $L^1(\mathbf{R}^N)$, global existence, asymptotic profile of solutions and blow-up within a finite time, estimates on the blow-up time for (1.2) are well-known in Fujita [4], Hayakawa [5] and the survey papers Levine [15], Deng and Levine [3] and references therein. Those for (1.1) are also known in [6, 7, 8, 12, 13, 18, 20, 21, 24, 25] etc. and references therein. Roughly speaking, if ρ is bigger than the Fujita exponent $\rho_F := 1 + 2/N$, then a time-global solution exists and behaves as $\theta_0 G(t,x)$ (θ_0 : constant) with the Gauss kernel

$$G(t,x) = (4\pi t)^{-N/2} e^{-\frac{|x|^2}{4t}}, \quad |x|^2 = x_1^2 + \cdots + x_N^2.$$

While, if $\rho \leq \rho_F$, then the solution blows up within a finite time.

In this paper, we consider the initial data not necessarily in $L^1(\mathbf{R}^N)$, that is,

$$u_0(x), u_1(x), \phi_0(x) = O(\langle x \rangle^{-kN}) \quad \text{as } |x| \to \infty \tag{1.3}$$

with

$$0 < k \leq 1, \tag{1.4}$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$. If $k = 0$ formally, then the data is a constant and the equation become the ordinary differential equation. Hence the solution blows up, and we can expect that the critical exponent depends on k. In fact, for (1.2) with (1.3), (1.4) Lee and Ni [14] showed that the critical exponent $\rho_c(k)$ is

$$\rho_c(k) = 1 + \frac{2}{kN} \tag{1.5}$$

and that, if $\rho < \rho_c(k)$ or $\rho = \rho_c(1)$, and $\phi_0(x) \geq 0$, then the solution $\phi(t,x)$ blows up within a finite time and the life-span is also estimated, while, if $\rho \geq \rho_c(k)$ except for $\rho = \rho_c(1)$ and $\phi_0(x)$ is suitably small, then the solution globaly exists. The asymptotic profile is not given. We note that if $k > 1$, then the data is in $L^1(\mathbf{R}^N)$ and the new exponent $\rho_c(k)$ is continuated to the Fujita exponent at $k = 1$. The proof in [14] is by the comparison principle, which may not be available for the damped wave equation. See also Cazenave and Weissler [1].

Thus, our first aim is to obtain the asymptotic profile of the solution $\phi(t,x)$ in the supercritical case

$$\rho > \rho_c(k), \tag{1.6}$$

whose proof is not by the comparison principle. The solution ϕ to (1.2) is defined by that of the integral equation

$$\begin{aligned} \phi(t,x) &= \int_{\mathbf{R}^N} G(t,x-y)\phi_0(y)\,dy + \int_0^t \int_{\mathbf{R}^N} G(t-\tau,x-y)|\phi|^{\rho-1}\phi(\tau,y)\,dy\,d\tau \\ &=: (P_N(t)\phi_0)(x) + \int_0^t (P_N(t-\tau)|\phi|^{\rho-1}\phi(\tau,\cdot))(x)\,d\tau \end{aligned} \tag{1.7}$$

and the solution space Y^k_∞ is defined by

$$\begin{aligned} Y^k_\infty &= \{\phi \in C([0,\infty);\ \mathsf{B}^{0,kN}); \\ &\quad \|\phi\|_{Y^k_\infty} := \sup_{[0,T)}\{a_k(t)\|\phi(t,\cdot)\|_{\mathsf{B}^0} + b_k(t)\|\phi(t,\cdot)\|_{\mathsf{B}^{0,kN}}\} < \infty\}, \end{aligned} \tag{1.8}$$

where

$$\begin{aligned} \mathsf{B}^m &= \{f \in C^m; \partial_x^\alpha f \in L^\infty\,(0 \le |\alpha| \le m)\}, \\ \mathsf{B}^{m,l} &= \{f \in \mathsf{B}^m; \langle x\rangle^l|\partial_x^\alpha f| < \infty\,(0 \le |\alpha| \le m)\} \end{aligned}$$

and

$$a_k(t) = \begin{cases} (1+t)^{kN/2} & (0 < k < 1) \\ (1+t)^{N/2}/\log(2+t) & (k=1), \end{cases} \quad b_k(t) = \begin{cases} 1 & (0 < k < 1) \\ 1/\log(2+t) & (k=1). \end{cases} \tag{1.9}$$

Then our first theorem is the following.

Theorem 1.1 *(i)* (Global Existence) *Suppose that the data $\phi_0 \in \mathsf{B}^{0,kN}(0 < k \le 1)$ is sufficiently small. Then there exists a unique solution ϕ to (1.2) in Y^k_∞.*
(ii) The function

$$\Phi_0(t,x) = \int_{\mathbf{R}^N} G(t,x-y)\frac{c_1}{\langle y\rangle^{kN}}dy, \quad c_1 \ne 0 \tag{1.10}$$

satisfies for some positive constant C_1 independent of t

$$C_1^{-1} \le a_k(t)\|\Phi_0(t,\cdot)\|_{\mathsf{B}^0} \le C_1. \tag{1.11}$$

(iii) (Asymptotic Profile) *If the data ϕ_0 in (i) satisfies*

$$\lim_{|x|\to\infty}\langle x\rangle^{kN}\phi_0(x) = c_1(\ne 0), \tag{1.12}$$

the solution obtained in (i) satisfies

$$\lim_{t\to\infty} a_k(t)\|\phi(t,\cdot) - \Phi_0(t,\cdot)\|_{\mathsf{B}^0} = 0. \tag{1.13}$$

From Theorem 1.1 $\Phi_0(t,x)$ is an asymptotic profile of the time global solution $\phi(t,x)$. We note that the profile is depending only on the data, not on the semilinear term. It is interesting to compare with the profile $\theta_0 G(t,x)$ for L^1-data, where

$$\theta_0 = \int_{\mathbf{R}^N}\phi_0(x)\,dx + \int_0^\infty\int_{\mathbf{R}^N}|\phi|^{p-1}\phi(t,x)\,dx\,dt. \tag{1.14}$$

The proof is given by elementary estimates on the explicit formula (1.7), not by the comparison principle. Hence, similar way will be available for the damped wave equation. By $S_N(t)g$, denote a solution to the Cauchy problem (1.1) with data $(u_0,u_1)(x) = (0,g)(x)$. Then the solution u to (1.1) satisfies the integral equation

$$u(t,\cdot) = S_N(t)(u_0+u_1) + \partial_t(S_N(t)u_0) + \int_0^t S_N(t-\tau)|u|^{p-1}u(\tau,\cdot)\,d\tau. \tag{1.15}$$

$S_N(t)g$ is decomposed to the sum of wave part and parabolic part of the form

$$S_N(t)g = e^{-t/2}W_N(t)g + J_{0N}(t)g, \tag{1.16}$$

where

$$
\begin{aligned}
W_1(t)g &= \frac{1}{2}\int_{|z|\le t} g(x+z)\,dz && \text{(D'Alembert formula)}\\
W_2(t)g &= \frac{1}{2\pi}\int_{|z|\le t} \frac{g(x+z)}{\sqrt{t^2-|z|^2}}dz && \text{(Poisson formula)}\\
W_3(t)g &= \frac{t}{4\pi}\int_{S^2} g(x+t\omega)\,d\omega && \text{(Kirchhoff formula)}
\end{aligned} \tag{1.17}
$$

and

$$
\begin{aligned}
J_{01}(t)g &= \frac{e^{-t/2}}{2}\int_{|z|\le t}(I_0(\frac{\sqrt{t^2-|z|^2}}{2})-1)g(x+z)\,dz\\
J_{02}(t)g &= \frac{e^{-t/2}}{2\pi}\int_{|z|\le t}\frac{\cosh\frac{\sqrt{t^2-|z|^2}}{2}-1}{\sqrt{t^2-|z|^2}}g(x+z)\,dz\\
J_{03}(t)g &= \frac{e^{-t/2}}{4\pi}\int_{|z|\le t}\frac{I_1(\frac{\sqrt{t^2-|z|^2}}{2})}{2\sqrt{t^2-|z|^2}}g(x+z)\,dz.
\end{aligned} \tag{1.18}
$$

Here, I_ν is the modified Bessel function of order ν given by

$$
I_\nu(y)=\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+\nu+1)}(\frac{y}{2})^{2m+\nu} \tag{1.19}
$$

with the Gamma function Γ, and $S^2=\{\omega=(\omega_1,\omega_2,\omega_3);\ |\omega|=\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}=1\}$ with its surface element $d\omega$. The formula of $S_N(t)g$ is referred in Courant and Hilbert [2]. The decompositions are proposed in Marcati and Nishihara [16]($N=1$), Ikehata, Nishihara and Zhao [10]($N=2$) and Nishihara [20]($N=3$). In the case $N=2$ see Hosono and Ogawa [9]. In general dimensional space see Narazaki [18]. L^p-L^q estimate($1\le q\le p\le\infty$) on $J_{0N}(t)g$ with $N=1,2,3$ were given by

$$
\begin{aligned}
\|J_{0N}(t)g\|_{L^p} &\le C(1+t)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|g\|_{L^q} && (t\ge 0),\\
\|(J_{0N}(t)-P_N(t))g\|_{L^p} &\le Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-1}\|g\|_{L^q} && (t>0).
\end{aligned} \tag{1.20}
$$

For the basic estimate on $S_N(t)g$ see Matsumura [17].

If we define a weak solution to (1.1) by the solution to (1.15) with (1.16), then we can easily conjectured that similar result for (1.1) to Theorem 1.1 holds, since the parabolic part is near to the solution of heat equation and wave part decays fast. In fact, our second theorem is the following.

Theorem 1.2 *Let $N=1,2$ or 3. Suppose that*

$$
u_0\in \mathsf{B}^{[\frac{N}{2}],kN},\quad u_1\in\mathsf{B}^{0,kN} \tag{1.21}
$$

are small. Then a unique weak solution $u\in Y_\infty^k$ $(0<k\le 1)$ to (1.1) exists. Moreover, if

$$
\lim_{|x|\to\infty}\langle x\rangle^{kN}(u_0+u_1)(x)=c_1(\neq 0), \tag{1.22}
$$

then it follows that

$$
\lim_{t\to\infty}a_k(t)\|u(t,\cdot)-\Phi_0(t,\cdot)\|_{\mathsf{B}^0}=0, \tag{1.23}
$$

where $\Phi_0(t,x)$ is defined by (1.10).

It will be worth why we have treated only the cases $N=1,2,3$. Different from the parabolic equation, we have not the smoothing effect for the damped wave equation. The regularity problem may happen in the wave part. In fact, when $N\geq 4$, $W_N(t)g$ includes the derivatives of g. For example,

$$\begin{aligned} S_4(t)g &= \frac{e^{-t/2}}{4\pi^2 t}\partial_t \int_0^t \frac{r^3\,dr}{\sqrt{t^2-r^2}}\int_{S^3} g(x+r\omega)\,d\omega \\ &\quad + \frac{e^{-t/2}}{4\pi^2 t}\partial_t \int_0^t \frac{(\cosh\frac{\sqrt{t^2-r^2}}{2}-1)r^3\,dr}{\sqrt{t^2-r^2}}\int_{S^3} g(x+r\omega)\,d\omega \\ &= e^{-t/2}W_4(t)g + J_{04}(t)g \end{aligned}$$

and

$$W_4(t)g = \frac{1}{4\pi^2}\int_0^t \frac{r\,dr}{\sqrt{t^2-r^2}}\int_{S^3}(2g(x+r\omega)+r\nabla g(x+r\omega)\cdot\omega)\,d\omega$$

(cf. [10, 22, 23]). Hence, to obtain a solution to (1.15) we need more consideration, in particular, on the solution space. Our problem in higher dimensional space will be considered in the forthcoming paper. The semilinear damped wave equation in the critical and subcritical cases should be investigated, which will be also considered in the forthcoming papers. Here we refer Ikehata and Ohta [11], where they showed that for L^m-data$(1\leq m\leq 2)$, not necessarily L^1-data, the critical exponent is $1+\frac{2m}{N}$ for (DW).

The content of this paper is as follows. The basic estimates for the heat equation and Theorem 1.1 is proved in Section 2, which becomes a study on the damped wave equation. In Section 3 Theorem 1.2 will be shown by assuming the basic estimates(Lemma 3.1) on $W_N(t)g, J_{0N}(t)$ etc. In the final section Lemma 3.1 will be proved using the explicit formulas (1.17)-(1.18).

Here and after, by $C(a,b,\cdots), C_{a,b,\cdots}$ or $c(a,b,\cdots), c_{a,b,\cdots}$ we denote several positive constants depending on $a,b,\cdots$. Without confusions, we denote them simply by C, c, whose quantities are changed line to line. Also, the integrand $\mathbf{R}^N$ is often abbreviated.

2 Semilinear heat equation

The solution ϕ to (1.2) is defined by (1.7). The linear part

$$(P_N(t)\phi_0)(x) = \int_{\mathbf{R}^N} G(t,x-y)\phi_0(y)\,dy \tag{2.1}$$

is estimated as follows.

Lemma 2.1 *(i) When $\phi_0\in \mathsf{B}^{0,kN}(0<k\leq 1)$, it follows that for any $x\in\mathbf{R}^N$ and $t>0$*

$$(a_k(t)+b_k(t)\langle x\rangle^{kN})|(P_N(t)\phi_0)(x)|\leq C\|\phi_0\|_{\mathsf{B}^{0,kN}}. \tag{2.2}$$

Both a_k and b_k are defined in (1.9).
(ii) Moreover, if ϕ_0 satisfies $\phi_0(x)\geq c_0|x|^{-kN}$, $|x|\geq R$, for some positive constants c_0, R, then for any fixed constant $A>0$ there exist some constants $t_0>0$ and $C>0$ such that for $|x|\leq A$, $t\geq t_0$

$$\langle x\rangle^{kN}(P_N(t)\phi_0)(x)\geq C^{-1}c_0 a_k(t)^{-1}. \tag{2.3a}$$

In other words, for $t\geq t_0$

$$\|P_N(t)\phi_0\|_{\mathsf{B}^0}\geq C^{-1}c_0 a_k(t)^{-1}. \tag{2.3b}$$

Proof. For $0 < t \le 1$ clearly $|(P_N(t)\phi_0)(x)| \le \|\phi_0\|_{\mathbf{B}^0} \le \|\phi_0\|_{\mathbf{B}^{0,kN}}$. For $t \ge 1$

$$\begin{aligned}
|(P_N(t)\phi_0)(x)| &\le (4\pi t)^{-\frac{N}{2}} \int_{\mathbf{R}^N} e^{-\frac{|x-y|^2}{4t}} \cdot \frac{\langle y\rangle^{kN}|\phi_0(y)|}{\langle y\rangle^{kN}}\, dy \\
&\le C\|\phi_0\|_{\mathbf{B}^{0,kN}} t^{-\frac{kN}{2}} \left(\int_{|\frac{x}{\sqrt{t}}-z|\le 1} + \int_{|\frac{x}{\sqrt{t}}-z|\ge 1}\right) e^{-\frac{|z|^2}{4}} \left(\frac{1}{t} + |\frac{x}{\sqrt{t}} - z|^2\right)^{-\frac{kN}{2}} dz \\
&\le C\|\phi_0\|_{\mathbf{B}^{0,kN}} t^{-\frac{kN}{2}} \left(\int_{|\frac{x}{\sqrt{t}}-z|\le 1} \left(\frac{1}{t} + |\frac{x}{\sqrt{t}} - z|^2\right)^{-\frac{kN}{2}} dz + \int_{\mathbf{R}^N} e^{-\frac{|z|^2}{4}} dz\right) \\
&\le C\|\phi_0\|_{\mathbf{B}^{0,kN}} t^{-\frac{kN}{2}} \left(\int_0^1 (\frac{1}{t} + r^2)^{-\frac{kN}{2}} r^{N-1} dr + 1\right).
\end{aligned}$$

When $k < 1$, the last integral is clearly bounded. When $k = 1$, it is estimated by

$$\int_0^1 (\frac{1}{t} + r^2)^{-\frac{1}{2}} dr = \left[\log (r + \sqrt{\frac{1}{t} + r^2})\right]_0^1 \le C \log (t+2).$$

Hence we have

$$a_k(t)|(P_N(t)\phi_0)(x)| \le C\|\phi_0\|_{\mathbf{B}^{0,kN}}. \tag{2.4}$$

Since $\langle x\rangle \le C(\langle x-y\rangle + \langle y\rangle)$,

$$\begin{aligned}
\langle x\rangle^{kN}|(P_N(t)\phi_0)(x)| &\le Ct^{-\frac{N}{2}} \int_{\mathbf{R}^N} e^{-\frac{|x-y|^2}{4t}} \left(\langle x-y\rangle^{kN} + \langle y\rangle^{kN}\right) |\phi_0(y)|\, dy \\
&\le C\|\phi_0\|_{\mathbf{B}^{0,kN}} \left(\int_{\mathbf{R}^N} e^{-\frac{|z|^2}{4}} \frac{\langle\sqrt{t}z\rangle^{kN}}{\langle x-\sqrt{t}z\rangle^{kN}} dz + 1\right).
\end{aligned}$$

When $0 < t \le 1$, the last integral is estimated by $\int e^{-\frac{|z|^2}{4}} \langle z\rangle^{kN} dz \le C$. When $t \ge 1$, since $\langle\sqrt{t}z\rangle \le \sqrt{t}\langle z\rangle$ and $\langle x - \sqrt{t}z\rangle = \sqrt{t}\sqrt{\frac{1}{t} + |\frac{x}{\sqrt{t}} - z|^2}$, the last integral is estimated by

$$\begin{aligned}
&\left(\int_{|\frac{x}{\sqrt{t}}-z|\le 1} + \int_{|\frac{x}{\sqrt{t}}-z|\ge 1}\right) \left(\frac{\sqrt{1+|z|^2}}{\sqrt{\frac{1}{t} + |\frac{x}{\sqrt{t}} - z|^2}}\right)^{kN} dz \\
\le\ & C\int_0^1 \sup_{|z|\ge 0} \left(e^{-\frac{|z|^2}{4}} \langle z\rangle^{kN}\right) \cdot (\frac{1}{t} + r^2)^{-\frac{kN}{2}} r^{N-1} dr + \int_{\mathbf{R}^N} e^{-\frac{|z|^2}{4}} |z|^{kN} dz \\
\le\ & Cb_k(t)^{-1}.
\end{aligned}$$

Hence we have

$$b_k(t)\langle x\rangle^{kN}|(P_N(t)\phi_0)(x)| \le C\|\phi_0\|_{\mathbf{B}^{0,kN}}. \tag{2.5}$$

By (2.4)-(2.5) we have shown (i).

(ii) We set

$$\begin{aligned}
\langle x\rangle^{kN}(P_N(t)\phi_0)(x) &= \langle x\rangle^{kN} \left(\int_{|y|\le R} + \int_{|y|\ge R}\right) G(t, x-y)\phi_0(y)\, dy \\
&=: \phi_1(t,x) + \phi_2(t,x).
\end{aligned}$$

For ϕ_1 it is easy to show

$$|\phi_1(t,x)| \le C(A, R, \phi_0)(1+t)^{-\frac{N}{2}} \ (t \ge 0).$$

For ϕ_2, since $C\langle x\rangle \geq \langle y\rangle/\langle x-y\rangle$,

$$\begin{aligned}
\phi_2(t,x) &\geq C^{-1}c_0\int_{|y|\geq R} G(t,x-y)\langle x-y\rangle^{-kN}dy \\
&\geq C^{-1}c_0 t^{-\frac{kN}{2}}\int_{|\frac{x}{\sqrt{t}}-z|\geq\frac{R}{\sqrt{t}}} e^{-\frac{|z|^2}{4}}\frac{dz}{(\frac{1}{t}+|z|^2)^{kN/2}} \\
&\to C^{-1}c_0 a_k(t)^{-1} \quad \text{as } t\to\infty.
\end{aligned}$$

Since ϕ_1 decays faster than $a_k(t)^{-1}$, (2.3) holds when $t \geq t_0$ for some constant $t_0 > 0$. □

To show the small data global existence of solutions to (1.7) we use the following lemma.

Lemma 2.2 *If $\phi \in Y^k_\infty (0 < k \leq 1)$, then it follows that*

$$\|\phi_N\|_{Y^k_\infty} \leq C\|\phi\|^\rho_{Y^k_\infty}, \tag{2.6}$$

where ϕ_N is the semilinear part of (1.7)

$$\phi_N(t,x) = \int_0^t (P_N(t-\tau)|\phi|^{\rho-1}\phi(\tau,\cdot))(x)\,d\tau. \tag{2.7}$$

Proof. First let $0 < k < 1$. When $\rho \geq 2$, by Lemma 2.1,

$$\begin{aligned}
|\phi_N(t,x)| &\leq C\int_0^t (1+t-\tau)^{-\frac{kN}{2}}\||\phi|^\rho(\tau,\cdot)\|_{\mathrm{B}^{0,kN}}d\tau \\
&\leq C\|\phi\|^\rho_{Y^k_\infty}\int_0^t (1+t-\tau)^{-\frac{kN}{2}}(1+\tau)^{-\frac{(\rho-1)kN}{2}}d\tau \\
&\leq C(1+t)^{-\frac{kN}{2}}\|\phi\|^\rho_{Y^k_\infty} \quad \text{by } \rho > \rho_c(k),
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
\langle x\rangle^{kN}|\phi_N(t,x)| &\leq C\int_0^t \||\phi|^\rho(\tau,\cdot)\|_{\mathrm{B}^{0,kN}}d\tau \\
&\leq C\|\phi\|^\rho_{Y^k_\infty}\int_0^t (1+\tau)^{-\frac{(\rho-1)kN}{2}}d\tau \leq C\|\phi\|^\rho_{Y^k_\infty}.
\end{aligned} \tag{2.9}$$

Hence we have (2.6). When $1 < \rho < 2$, the estimate are delicate a little bit:

$$\begin{aligned}
|\phi_N(t,x)| &\leq C\left(\int_0^{t/2}(1+t-\tau)^{-\frac{kN}{2}}\||\phi|^\rho(\tau,\cdot)\|_{\mathrm{B}^{0,kN}}d\tau\right. \\
&\quad \left.+\int_{t/2}^t (1+t-\tau)^{-\frac{(\rho-1)kN}{2}}\||\phi|^\rho(\tau,\cdot)\|_{\mathrm{B}^{0,(\rho-1)kN}}d\tau\right) \\
&\leq C\|\phi\|^\rho_{Y^k_\infty}\left(\int_0^{t/2}(1+t-\tau)^{-\frac{kN}{2}}(1+\tau)^{-\frac{(\rho-1)kN}{2}}d\tau\right. \\
&\quad \left.+\int_{t/2}^t (1+t-\tau)^{-\frac{(\rho-1)kN}{2}}(1+\tau)^{-\frac{kN}{2}}d\tau\right) \\
&\leq C(1+t)^{-\frac{kN}{2}}\|\phi\|^\rho_{Y^k_\infty}.
\end{aligned} \tag{2.10}$$

The estimate (2.9) is available for $1 < \rho < 2$. Hence we also have (2.6).

Next let $k=1$. When $\rho \geq 2$,

$$\begin{aligned}
&|\phi_N(t,x)| \\
\leq\ & C\int_0^t (1+t-\tau)^{-\frac{N}{2}} \log(2+t-\tau) \| |\phi|^\rho(\tau,\cdot)\|_{\mathsf{B}^{0,N}} d\tau \\
\leq\ & C\|\phi\|_{Y^1_\infty}^\rho \int_0^t (1+t-\tau)^{-\frac{N}{2}} \log(2+t-\tau)\cdot(1+\tau)^{-\frac{(\rho-1)N}{2}} (\log(2+\tau))^\rho d\tau \\
\leq\ & C\|\phi\|_{Y^1_\infty}^\rho (1+t)^{-\frac{N}{2}} \log(2+t) \quad \text{by} \quad \rho > \rho_c(1),
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
&\langle x\rangle^N |\phi_N(t,x)| \\
\leq\ & C\int_0^t \log(2+t-\tau) \| |\phi|^\rho(\tau,\cdot)\|_{\mathsf{B}^{0,N}} d\tau \\
\leq\ & C\|\phi\|_{Y^1_\infty}^\rho \int_0^t \log(2+t-\tau)\cdot(1+\tau)^{-\frac{(\rho-1)N}{2}} (\log(2+\tau))^\rho d\tau \\
\leq\ & C\|\phi\|_{Y^1_\infty}^\rho \log(2+t).
\end{aligned} \tag{2.12}$$

Thus (2.11) and (2.12) imply (2.6). When $1<\rho<2$, estimates are similar to (2.9)-(2.12) and omitted. □

Proof of Theorem 1.1. Once we have Lemma 2.2, then the iteration $\{\phi^{(n)}\}_{n=0}^\infty \subset Y^k_\infty$ defined by

$$\phi^{(0)}(t,\cdot) = P_N(t)\phi_0, \quad \phi^{(n+1)}(t,\cdot) = \phi^{(0)}(t,\cdot) + \int_0^t P_N(t-\tau)|\phi^{(n)}|^{\rho-1}\phi^{(n)}(\tau,\cdot)\, d\tau$$

is shown to be a Cauchy sequence by a standard way if $\phi_0 \in \mathsf{B}^{0,kN}$ is small. This proves (i). Part (ii) follows from Lemma 2.1 (ii).

Finally, when $\phi \in Y^k_\infty$ is a small solution, we claim

$$\lim_{t\to\infty} a_k(t)\|\phi_N(t,\cdot)\|_{\mathsf{B}^0} = 0, \tag{2.13}$$

$$\lim_{t\to\infty} a_k(t)\|P_N(t)\phi_0 - \Phi_0(t,\cdot)\|_{\mathsf{B}^0} = 0, \tag{2.14}$$

which show (iii).

It is similar to (2.8)-(2.12) how to prove (2.13). First, let $0<k<1$. When $\rho>2$, we choose a small constant $\delta>0$ as

$$(1+\delta)kN < N, \quad (\rho-1-\delta)kN > (1+\delta)kN, \quad \rho > 1 + \frac{2}{kN} + \delta. \tag{2.15}$$

Then we have

$$\begin{aligned}
|\phi_N(t,x)| \ \leq\ & C\int_0^t (1+t-\tau)^{-\frac{(1+\delta)kN}{2}} \| |\phi|^\rho(\tau,\cdot)\|_{\mathsf{B}^{0,(1+\delta)kN}} d\tau \\
\leq\ & C\|\phi\|_{Y^k_\infty}^\rho \int_0^t (1+t-\tau)^{-\frac{(1+\delta)kN}{2}} (1+\tau)^{-\frac{(\rho-1-\delta)kN}{2}} d\tau \\
\leq\ & C(1+t)^{-\frac{(1+\delta)kN}{2}} \quad \text{by} \quad (2.15).
\end{aligned} \tag{2.16}$$

When $1<\rho\leq 2$, we choose $\delta>0$ as

$$(1+\delta)kN < N, \quad \rho > 1 + \frac{2}{kN} + \delta, \tag{2.17}$$

then, since $(\rho-1-\delta)kN < kN$,

$$
\begin{aligned}
|\phi_N(t,x)| &\le C\left(\int_0^{t/2}(1+t-\tau)^{-\frac{(1+\delta)kN}{2}}\| |\phi|^\rho(\tau,\cdot)\|_{\mathrm{B}^{0,(1+\delta)kN}}\,d\tau\right.\\
&\quad \left.+\int_{t/2}^{t}(1+t-\tau)^{-\frac{(\rho-1-\delta)kN}{2}}\| |\phi|^\rho(\tau,\cdot)\|_{\mathrm{B}^{0,(\rho-1-\delta)kN}}\,d\tau\right)\\
&\le C\|\phi\|_{Y_\infty^k}^{\rho}\left(\int_0^{t/2}(1+t-\tau)^{-\frac{(1+\delta)kN}{2}}(1+\tau)^{-\frac{(\rho-1-\delta)kN}{2}}\,d\tau\right.\\
&\quad \left.+\int_{t/2}^{t}(1+t-\tau)^{-\frac{(\rho-1-\delta)kN}{2}}(1+\tau)^{-\frac{(1+\delta)kN}{2}}\,d\tau\right)\\
&\le C(1+t)^{-\frac{(1+\delta)kN}{2}} \quad \text{by} \quad (2.17).
\end{aligned}
\tag{2.18}
$$

Next, let $k=1$. When $\rho>2$, we choose $\delta>0$ as

$$
\rho > 1+\frac{2}{N}+2\delta, \quad (\rho-1-2\delta)N > N, \tag{2.19}
$$

then

$$
\begin{aligned}
\langle y\rangle^{(1+\delta)N}|\phi|^\rho(\tau,y) &\le (\langle y\rangle^N|\phi|)^{1+\delta}|\phi|^{\rho-1-\delta}\\
&\le C(1+\tau)^{-\frac{(\rho-1-\delta)N}{2}}(\log(2+\tau))^\rho\|\phi\|_{Y_\infty^1}^{\rho}\\
&\le C(1+\tau)^{-\frac{(\rho-1-2\delta)N}{2}}
\end{aligned}
$$

and $\langle y\rangle^{-(1+\delta)N}\in L^1\cap L^\infty(\mathbf{R}_y^N)$. Hence

$$
\begin{aligned}
|\phi_N(t,x)| &\le C\int_0^t P_N(t-\tau)\left(\langle\cdot\rangle^{-(1+\delta)N}\cdot\langle\cdot\rangle^{(1+\delta)N}|\phi|^\rho(\tau,\cdot)\right)d\tau\\
&\le C\int_0^t(1+\tau)^{-\frac{(\rho-1-2\delta)N}{2}}P_N(t-\tau)\langle\cdot\rangle^{-(1+\delta)N}\,d\tau\\
&\le C\int_0^t(1+t-\tau)^{-\frac{N}{2}}(1+\tau)^{-\frac{(\rho-1-2\delta)N}{2}}\,d\tau\\
&\le C(1+t)^{-\frac{(1+\delta)N}{2}},
\end{aligned}
\tag{2.20}
$$

because $P_N(t)$ is linear, its kernel is positive, and $|P_N(t)g|\le C(1+t)^{\frac{N}{2}}\|g\|_{L^1\cap L^\infty}$. When $1<\rho\le 2$, we choose $\delta>0$ as

$$
\rho > 1+\frac{2}{N}+2\delta. \tag{2.21}
$$

Since $(\rho-1-\delta)N<N$,

$$
\begin{aligned}
|\phi_N(t,x)| &\le C\left(\int_0^{t/2}(1+\tau)^{-\frac{(\rho-1-2\delta)N}{2}}P_N(t-\tau)\langle\cdot\rangle^{-(1+\delta)N}\,d\tau\right.\\
&\quad \left.+\int_{t/2}^{t}(1+t-\tau)^{-\frac{(\rho-1-\delta)N}{2}}\| |\phi|^\rho(\tau,\cdot)\|_{\mathrm{B}^{0,(\rho-1-\delta)N}}\,d\tau\right)\\
&\le C\left(\int_0^{t/2}(1+t-\tau)^{-\frac{N}{2}}(1+\tau)^{-\frac{(\rho-1-2\delta)kN}{2}}\,d\tau\right.\\
&\quad \left.+\int_{t/2}^{t}(1+t-\tau)^{-\frac{(\rho-1-\delta)N}{2}}(1+\tau)^{-\frac{N}{2}}\,d\tau\right)\\
&\le C(1+t)^{-\frac{N}{2}} \quad \text{by} \quad (2.21).
\end{aligned}
\tag{2.22}
$$

Thus, by (2.15)-(2.22) we have (2.13).

Next, we show (2.14). By (1.12), for any $\varepsilon > 0$ there is a constant $R = R(\varepsilon) > 0$ such that

$$|\langle y\rangle^{kN}\phi_0(y) - c_1| \leq \varepsilon \quad \text{or} \quad |\phi_0(y) - c_1\langle y\rangle^{-kN}| \leq \frac{\varepsilon}{\langle y\rangle^{kN}} \quad \text{if } |x| \geq R.$$

Hence

$$\begin{aligned} & |\phi^{(0)}(t,x) - \Phi_0(t,x)| \\ \leq\ & \int_{|y|\leq R} G(t,x-y)|\phi_0(y) - c_1\langle y\rangle^{-kN}|\,dy + \int_{|y|\geq R} G(t,x-y)\frac{\varepsilon\,dy}{\langle y\rangle^{kN}} \\ \leq\ & C(R)(1+t)^{-N/2} + \varepsilon C a_k(t)^{-1}. \end{aligned}$$

Since $\lim_{t\to\infty} a_k(t)(1+t)^{-N/2} = 0$,

$$\limsup_{t\to\infty} a_k(t)|\phi^{(0)}(t,x) - \Phi_0(t,x)| \leq \varepsilon C.$$

Therefore we have (2.14) because ε can be chosen arbitrarily small.

Thus we have completed Theorem 1.1. □

3 Damped wave equation

A weak solution to (1.1) is defined by the solution of the integral equation

$$u(t,\cdot) = S_N(t)(u_0 + u_1) + \partial_t(S_N(t)u_0) + \int_0^t S_N(t-\tau)|u|^{\rho-1}u(\tau,\cdot)\,d\tau \tag{3.1}$$

with (1.16). The t-derivative of $S_N(t)$ is

$$\partial_t(S_N(t)g) = e^{-t/2}(-\frac{1}{2}W_N(t)g + \partial_t(W_N(t)g)) + \partial_t(J_{0N}(t)g).$$

When $N = 1, 2$, we put $J_{1N}(t) = \partial_t(J_{0N}(t)g)$. When $N = 3$, since $I_1(y)/y|_{y=0} = \frac{1}{2}$

$$\begin{aligned} \partial_t(J_{03}(t)g) &= e^{-t/2}\frac{t}{8}W_3(t)g + \int_{|z|\leq t} \partial_t\left[\frac{e^{-t/2}I_1(\frac{\sqrt{t^2-|z|^2}}{2})}{8\pi\sqrt{t^2-|z|^2}}\right] g(x+z)\,dz \\ &=: e^{-t/2}\frac{t}{8}W_3(t)g + J_{13}(t)g. \end{aligned}$$

Hence

$$\partial_t(S_N(t)g) = e^{-t/2}\tilde{W}_N(t)g + J_{1N}(t)g, \tag{3.2}$$

where

$$\tilde{W}_N(t)g = \begin{cases} -\frac{1}{2}W_N(t)g + \partial_t(W_N(t)g) & (N=1,2) \\ (-\frac{1}{2}+\frac{t}{8})W_N(t)g + \partial_t(W_N(t)g) & (N=3). \end{cases} \tag{3.3}$$

In the result, (3.1) is rewritten as

$$\begin{aligned} u(t,\cdot) &= (e^{-t/2}W_N(t) + J_{0N}(t))(u_0+u_1) + (e^{-t/2}\tilde{W}_N(t) + J_{1N}(t))u_0 \\ &\quad + \int_0^t \underbrace{(e^{-(t-\tau)/2}W_N(t-\tau) + J_{0N}(t-\tau))}_{S_N(t-\tau)}|u|^{\rho-1}u(\tau,\cdot)\,d\tau. \end{aligned} \tag{3.4}$$

Thus we need similar estimates to (2.2) or (1.19), which play a key role in the proof for the damped wave equation.

Lemma 3.1 *Let $N = 1,2,3$. Assume $g \in \mathsf{B}^{0,kN}$ or $g \in \mathsf{B}^{[\frac{N}{2}],kN}$ $(0 < k \leq 1)$. Then the following estimates hold:*

$$(a_k(t) + b_k(t)\langle x\rangle^{kN})|e^{-t/2}(W_N(t)g)(x)| \leq Ce^{-t/4}\|g\|_{\mathsf{B}^{0,kN}}, \tag{3.5a}$$

$$(a_k(t) + b_k(t)\langle x\rangle^{kN})|e^{-t/2}(\tilde{W}_N(t)g)(x)| \leq Ce^{-t/4}\|g\|_{\mathsf{B}^{[\frac{N}{2}],kN}}, \tag{3.5b}$$

$$(a_k(t) + b_k(t)\langle x\rangle^{kN})|(J_{0N}(t)g - P_N(t)g)(x)| \leq C(1+t)^{-1}\|g\|_{\mathsf{B}^{0,kN}}, \tag{3.6a}$$

$$(a_k(t) + b_k(t)\langle x\rangle^{kN})|(J_{1N}(t)g)(x)| \leq C(1+t)^{-1}\|g\|_{\mathsf{B}^{0,kN}}, \tag{3.6b}$$

where a_k and b_k are defined in (1.9).

Once we get Lemma 3.1, we have the basic estimate on $S_N(t)$.

Lemma 3.2 *Let $N = 1,2,3$.*
(0) Assume $(u_0, u_1) \in \mathsf{B}^{[\frac{N}{2}],kN} \times \mathsf{B}^{0,kN}$, then

$$u^{(0)}(t,x) := (S_N(t)(u_0 + u_1))(x) + \partial_t(S_N(t)u_0)(x)$$

satisfies

$$(a_k(t) + b_k(t)\langle x\rangle^{kN})|u^{(0)}(t,x)| \leq C\|u_0, u_1\|_{\mathsf{B}^{[\frac{N}{2}],kN}\times\mathsf{B}^{0,kN}}. \tag{3.7}$$

(i) Assume $g \in \mathsf{B}^{0,kN}$, then

$$(a_k(t) + b_k(t)\langle x\rangle^{kN})|(S_N(t)g)(x)| \leq C\|g\|_{\mathsf{B}^{0,kN}}. \tag{3.8}$$

(ii) Moreover, if g satisfies $g(x) \geq c_0|x|^{-kN}$, $|x| \geq R$ for some positive constant c_0, R, then for fixed constant $A > 0$ there exist some constants $t_0 > 0$ and $C > 0$ such that for $|x| \leq A$, $t \geq t_0$,

$$\langle x\rangle^{kN}|(S_N(t)g)(x)| \geq C^{-1}c_0a_k(t)^{-1}. \tag{3.9a}$$

In other words, for $t \geq t_0$

$$\|S_N(t)g\|_{\mathsf{B}^0} \geq C^{-1}c_0a_k(t)^{-1}. \tag{3.9b}$$

Proof of Lemma 3.2. (0) By (3.6a) and (2.2), it is clear that

$$(a_k(t) + b_k(t)\langle x\rangle^{kN})|(J_{0N}(t)g)(x)| \leq C\|g\|_{\mathsf{B}^{0,kN}}, \tag{3.10}$$

which together with (3.5) and (3.6b) shows (3.7).
(i) Both (3.5a) and (3.10) yield (3.8).
(ii) By (3.6a) and (2.3), for $|x| \leq A$

$$\begin{aligned}\langle x\rangle^{kN}J_{0N}(t)g &\geq \langle x\rangle^{kN}P_N(t)g - C_Aa_k(t)^{-1}(1+t)^{-1}\|g\|_{\mathsf{B}^0} \\ &\geq C^{-1}c_0a_k(t)^{-1}\ (t \geq t_0).\end{aligned} \tag{3.11}$$

Since $e^{-t/2}W_N(t)g$ decays much faster, we have (3.9). □

Our main theorem 1.2 is now proved rather easily.

Proof of Theorem 1.2. We define the iteration $\{u^{(n)}(t,x)\}$ by

$$\begin{aligned} u^{(0)}(t,\cdot) &= S_N(t)(u_0+u_1)+\partial_t(S(t)u_0) \\ u^{(n+1)}(t,\cdot) &= u^{(0)}(t,\cdot)+\int_0^t S_N(t-\tau)|u^{(n)}|^{\rho-1}u^{(n)}(\tau,\cdot)\,d\tau. \end{aligned}$$

By (3.7), $u^{(0)} \in Y_\infty^k (0<k\le 1)$. Since $S_N(t)$ satisfies the estimate (3.8), same as (2.2) for $P_N(t)$, we have

$$\|u_N\|_{Y_\infty^k} \le C\|u\|_{Y_\infty^k}^\rho,$$

similarly to Lemma 2.2, where $u_N(t,\cdot)=\int_0^t S_N(t-\tau)|u|^{\rho-1}u(\tau,\cdot)\,d\tau$. Hence, by a standard way we easily show that $\{u^{(n)}\}$ is a Cauchy sequence in Y_∞^k if $\|u_0,u_1\|_{\mathrm{B}^{[\frac{N}{2}],kN}\times \mathrm{B}^{0,kN}}$ is small. Thus we have a time-global weak solution to (1.1).

By (3.6a) and (2.14)

$$\begin{aligned} & |(J_{0N}(t)(u_0+u_1))(x)-\Phi_0(t,x)| \\ \le\ & |(J_{0N}(t)-P_N(t))(u_0+u_1)(x)|+|(P_N(t)(u_0+u_1))(x)-\Phi_0(t,x)| \\ =\ & o(a_k(t)^{-1}), \end{aligned}$$

and $|e^{-t/2}W_N(t)(u_0+u_1)|$, $|e^{-t/2}\tilde{W}_N(t)u_0|$ and $|J_{1N}(t)u_0|$ are also $o(a_k(t)^{-1})$ as $t\to\infty$ by (3.5a), (3.5b), (3.6b). The nonlinear part $u_N(t,x)$ is shown to decay faster than $a_k(t)^{-1}$ by the same method as that in (2.15)-(2.22). Because $S_N(t)$ is linear, its kernel is positive for $N=1,2,3$, and $S_N(t)g$ satisfies (3.8), same as (2.2).

Thus we obtain the asymptotic profile of the solution u. □

4 Proof of Lemma 3.1

In $N=1,3$ we need basic properties on the modified Bessel functions.

Lemma 4.1 *The modified Bessel function I_ν of order ν satisfies*

$$I_0(0)=1,\quad I_1(y)/y|_{y=0}=\frac{1}{2}, \tag{4.1}$$

$$I_0'(y)=I_1(y),\quad I_1'(y)=I_0(y)-\frac{1}{y}I_1(y), \tag{4.2}$$

and, moreover, as $y\to\infty$,

$$\begin{aligned} I_\nu(y) = \sqrt{\tfrac{1}{2\pi y}}e^y\Big(1 &- \tfrac{(\nu-1/2)(\nu+1/2)}{2y}+\tfrac{(\nu-1/2)(\nu-3/2)(\nu+3/2)(\nu+1/2)}{2!2^2y^2} \\ &-\cdots+(-1)^k\tfrac{(\nu-1/2)\cdots(\nu-(k-1/2))(\nu+(k-1/2))\cdots(\nu+1/2)}{k!2^ky^k}+O(y^{-k-1})\Big). \end{aligned} \tag{4.3}$$

For the proof, see e.g. [19].

We prove Lemma 3.1 only in the case of $N=3$ and $N=2$. The case $N=1$ is similar to $N=3$, and omitted.

(I) $N=3$. The estimate (3.5a) is clear by

$$|(W_3(t)g)(x)|\le t\|g\|_{\mathrm{B}^0}$$

and

$$\begin{aligned}\langle x\rangle^{kN}|(W_3(t)g)(x)| &\le Ct\int_{S^2}(\langle x+t\omega\rangle^{kN}+\langle t\omega\rangle^{kN})|g(x+t\omega)|\,d\omega\\ &\le Ct\|g\|_{\mathsf{B}^{0,kN}}+Ct\langle t\rangle^{kN}\|g\|_{\mathsf{B}^0}.\end{aligned}$$

Since

$$\partial_t(W_3(t)g)(x)=\frac{1}{4\pi}\int_{S^2}g(x+t\omega)\,d\omega+\frac{t}{4\pi}\int_{S^2}\nabla g(x+t\omega)\cdot\omega\,d\omega,$$

(3.5b) is also clear. For (3.6a) we express $J_{03}(t)g$ and $P_3(t)g$ by polar coordinates, and decompose the integrand

$$\begin{aligned}&J_{03}(t)g-P_3(t)g\\ =\;&\frac{1}{4\pi t}\left(\int_0^{t^{2/3}}+\int_{t^{2/3}}^t\right)\int_{S^2}\left(\frac{e^{-\frac{t}{2}}I_1(\frac{\sqrt{t^2-r^2}}{2})t}{2\sqrt{t^2-r^2}}-\frac{e^{-\frac{r^2}{4t}}}{\sqrt{4\pi t}}\right)g(x+r\omega)r^2\,dr\,d\omega\\ &-\int_t^\infty\int_{S^2}\frac{e^{-r^2/4t}}{(4\pi t)^{3/2}}g(x+r\omega)r^2\,dr\,d\omega\\ =:\;&(X_1+X_2)+X_3.\end{aligned}$$

For $0<t\le 2$, clearly

$$\|X_i\|_{\mathsf{B}^{0,kN}}\le C\|g\|_{\mathsf{B}^{0,kN}}\ (i=1,2,3).\tag{4.4}$$

For $t>2$, X_3 is estimated as

$$|X_3|\le C\|g\|_{\mathsf{B}^0}e^{-t/8}\int_t^\infty e^{-r^2/8t}t^{-3/2}r^2\,dr\le Ce^{-\beta t}\|g\|_{\mathsf{B}^0}\tag{4.5}$$

for some constant $\beta>0$ and

$$\begin{aligned}&\langle x\rangle^{kN}|X_3|\\ \le\;&C\int_t^\infty\int_{S^2}e^{-\frac{r^2}{4t}}(\langle x+r\omega\rangle^{kN}+\langle r\omega\rangle^{kN})|g(x+r\omega)|t^{-3/2}r^2\,dr\,d\omega\\ \le\;&C\int_t^\infty e^{-\frac{r^2}{4t}}(\|g\|_{\mathsf{B}^{0,kN}}+\langle r\rangle^{kN}\|g\|_{\mathsf{B}^0})t^{-3/2}r^2\,dr\\ \le\;&Ce^{-\beta t}\|g\|_{\mathsf{B}^{0,kN}}.\end{aligned}\tag{4.6}$$

Since $I_\nu(y)\,(\nu\ge 0)$ is increasing, by (4.3) for $t^{2/3}\le r\le t$, $t>2$, $\nu=1,2$,

$$\begin{aligned}&e^{-t/2}I_\nu(\frac{\sqrt{t^2-r^2}}{2})\le e^{-t/2}I_\nu(\frac{\sqrt{t^2-t^{4/3}}}{2})\\ \le\;&\sqrt{\frac{1}{\pi\sqrt{t^2-t^{4/3}}}}e^{-\frac{t}{2}+\frac{\sqrt{t^2-t^{4/3}}}{2}}(1+O(t^{-1}))\le Ce^{-\beta t^{1/3}}\end{aligned}$$

and $\dfrac{e^{-r^2/4t}}{(4\pi t)^{3/2}}\le Ce^{-\beta t^{1/3}}$. Hence we have

$$(1+\langle x\rangle^{kN})|X_2|\le Ce^{-\beta t^{1/3}}\|g\|_{\mathsf{B}^{0,kN}}.\tag{4.7}$$

Main part is X_1. Since $(\frac{r^2}{t^2})^m\le\frac{r^2}{t^2}\ (m\ge 1)$ for $r\le t^{2/3}$, by (4.3)

$$\begin{aligned}&\left(\frac{e^{-r^2/4t}}{\sqrt{4\pi t}}\right)^{-1}\cdot\frac{1}{2}e^{-t/2}I_\nu(\frac{\sqrt{t^2-r^2}}{2})\\ =\;&\sqrt{4\pi t}\cdot\frac{1}{2}\sqrt{\frac{1}{\pi\sqrt{t^2-r^2}}}e^{\frac{r^2}{4t}-\frac{t}{2}+\frac{\sqrt{t^2-r^2}}{2}}(1+O(t^{-1}))=1+\frac{1}{t}O(1+\frac{r^2}{t}),\end{aligned}\tag{4.8}$$

because

$$e^{\frac{r^2}{4t}-\frac{t}{2}+\frac{\sqrt{t^2-r^2}}{2}} = e^{-\frac{r^4}{4t^3}\cdot\frac{1}{(1+\sqrt{1-r^2/t^2})^2}} = 1+\frac{1}{t}O(\frac{r^2}{t}).$$

Further, since

$$\frac{t}{\sqrt{t^2-r^2}} = (1-\frac{r^2}{t^2})^{-1/2} = 1+\frac{1}{t}O(\frac{r^2}{t}),$$

we have

$$\begin{aligned}
&|X_1| \\
\leq\ & \frac{C}{t}\int_0^{t^{2/3}}\int_{S^2}\frac{e^{-r^2/4t}}{\sqrt{4\pi t}}\left|\left(\frac{e^{-r^2/4t}}{\sqrt{4\pi t}}\right)^{-1}\cdot\frac{1}{2}e^{-t/2}I_\nu(\frac{\sqrt{t^2-r^2}}{2})\cdot(1-\frac{r^2}{t^2})^{-1/2}-1\right| \\
&\cdot|g(x+r\omega)|r^2\,dr\,d\omega \\
\leq\ & \frac{C}{t^2}\int_0^{t^{2/3}}\int_{S^2}\frac{e^{-r^2/4t}}{\sqrt{4\pi t}}(1+\frac{r^2}{t})\frac{\langle x+r\omega\rangle^{kN}|g(x+r\omega)|}{\langle x+r\omega\rangle^{kN}}r^2\,dr\,d\omega \\
\leq\ & \frac{C\|g\|_{\mathrm{B}^{0,kN}}}{t}\int_{R^3}e^{-|z|^2/4}(1+|z|^2)\frac{dz}{\langle x-\sqrt{t}z\rangle^{kN}}\ (r\omega=-\sqrt{t}z).
\end{aligned}$$

Further, similar to deriving (2.4),

$$\begin{aligned}
&|X_1| \\
\leq\ & C\|g\|_{\mathrm{B}^{0,kN}}t^{-1-\frac{kN}{2}}\left(\int_{|\frac{x}{\sqrt{t}}-z|\leq 1}+\int_{|\frac{x}{\sqrt{t}}-z|>1}\right)e^{-\frac{|z|^2}{4}}(1+|z|^2)(\frac{1}{t}+|\frac{x}{\sqrt{t}}-z|^2)^{-\frac{kN}{2}}\,dz \\
\leq\ & C\|g\|_{\mathrm{B}^{0,kN}}t^{-1-\frac{kN}{2}}\left(\int_{|\frac{x}{\sqrt{t}}-z|\leq 1}(\frac{1}{t}+|\frac{x}{\sqrt{t}}-z|^2)^{-\frac{kN}{2}}\,dz+\int_{R^N}e^{-\frac{|z|^2}{4}}(1+|z|^2)\,dz\right) \\
\leq\ & C\|g\|_{\mathrm{B}^{0,kN}}t^{-1}a_k(t)^{-1}.
\end{aligned} \tag{4.9}$$

By a similar fashion we have

$$\langle x\rangle^{kN}|X_1| \leq C\|g\|_{\mathrm{B}^{0,kN}}t^{-1}b_k(t)^{-1}. \tag{4.10}$$

Combining (4.4) with (4.5)-(4.10), we have (3.6a). Finally we estimate $J_{13}(t)g$. By (4.2)

$$\begin{aligned}
&J_{13}(t)g \\
=\ & \frac{1}{4\pi t}\int_0^t\int_{S^2}\frac{e^{-t/2}}{2}\left[I_0(\frac{\sqrt{t^2-r^2}}{2})\frac{t^2}{2(t^2-r^2)}\right. \\
&\left.-I_1(\frac{\sqrt{t^2-r^2}}{2})(\frac{t}{2\sqrt{t^2-r^2}}+\frac{2t^2}{(t^2-r^2)\sqrt{t^2-r^2}})\right]\cdot g(x+r\omega)r^2\,dr\,d\omega \\
=\ & \int_0^{t^{2/3}}+\int_{t^{2/3}}^t =: X_4+X_5.
\end{aligned}$$

Clearly $(1+\langle x\rangle^{kN})|X_i| \leq C\|g\|_{\mathrm{B}^{0,kN}}$ for $0<t\leq 2$. When $t>2$, it is clear that $(1+\langle x\rangle^{kN})|X_5| \leq Ce^{-\beta t^{1/3}}\|g\|_{\mathrm{B}^{0,kN}}$. Since

$$\frac{t^2}{2(t^2-r^2)} = \frac{1}{2}+\frac{1}{t}O(\frac{r^2}{t}),\quad \frac{t}{2\sqrt{t^2-r^2}}+\frac{2t^2}{(t^2-r^2)\sqrt{t^2-r^2}} = \frac{1}{2}+\frac{1}{t}O(\frac{r^2}{t})$$

for $r < t^{2/3}$, $t > 2$ together with (4.8), we have

$$|X_4| \le Ct^{-2} \int_0^{t^{2/3}} \int_{S^2} \frac{e^{-r^2/4t}}{\sqrt{4\pi t}} (1 + \frac{r^2}{t}) |g(x + r\omega)| r^2 \, dr \, d\omega.$$

Therefore, X_4 is estimated similar to X_1. Thus we obtain (3.8b), and complete the proof in $N = 3$.

(II) $N = 2$. Since

$$|(W_2(t)g)(x)| \le \|g\|_{B^0} \int_0^t \frac{r\, dr}{\sqrt{t^2 - r^2}} \le t\|g\|_{B^0}$$

and

$$\begin{aligned} \langle x\rangle^{kN} |(W_2(t)g)(x)| &\le C \int_{|z|\le t} \frac{(\langle x+z\rangle^{kN} + \langle z\rangle^{kN}) |g(x+z)|}{\sqrt{t^2 - |z|^2}} dz \\ &\le Ct(\|g\|_{B^{0,kN}} + \langle t\rangle^{kN} \|g\|_{B^0}), \end{aligned}$$

we have (3.5a). Since

$$\partial_t (W_2(t)g) = \frac{t}{2\pi} \int_0^t \int_{S^1} \frac{\nabla g(x + r\omega)\cdot\omega}{\sqrt{t^2 - r^2}} \, dr\, d\omega,$$

we easily have (3.5b). For (3.6a)

$$\begin{aligned} & J_{02}(t)g - P_2(t)g \\ = \; & \frac{1}{4\pi t} \left(\int_0^{t^{2/3}} + \int_{t^{2/3}}^t \right) \int_{S^1} \left(\frac{2te^{-\frac{t}{2}}(\cosh\frac{\sqrt{t^2-r^2}}{2} - 1)}{\sqrt{t^2 - r^2}} - e^{-\frac{r^2}{4t}} \right) \cdot g(x + r\omega) r\, dr\, d\omega \\ & - \int_t^\infty \int_{S^1} \frac{e^{-r^2/4t}}{4\pi t} g(x + r\omega) r \, dr\, d\omega \\ =: \; & (X_1 + X_2) + X_3. \end{aligned}$$

When $0 < t \le 2$, $(1 + \langle x\rangle^{kN} |X_i| \le C\|g\|_{B^{0,kN}}$ $(i = 1,2,3)$, and both X_2 and X_3 decay exponentially similar to those in $N = 3$. Moreover, when $r \le t^{2/3}, t > 2$, for some positive constant $\beta < 1/4$

$$\begin{aligned} 2e^{-\frac{t}{2}}(\cosh\frac{\sqrt{t^2 - r^2}}{2} - 1) &= e^{-\frac{r^2}{4t}} (e^{\frac{r^2}{4t} - \frac{t}{2} + \frac{\sqrt{t^2-r^2}}{2}} + O(e^{-\beta t})) \\ &= e^{-\frac{r^2}{4t}} (1 + \frac{1}{t} O(1 + \frac{r^2}{t^2})), \end{aligned} \tag{4.11}$$

and $\frac{t}{\sqrt{t^2-r^2}} = 1 + \frac{1}{t}O(\frac{r^2}{t})$. Hence

$$\begin{aligned} |X_1| &\le \frac{C}{t} \int_0^{t^{2/3}} \int_{S^1} e^{-\frac{r^2}{4t}} (1 + \frac{1}{t} + \frac{r^2}{t^2} - 1) |g(x + r\omega)| r\, dr\, d\omega \\ &\le Ct^{-2} \int_0^{t^{2/3}} \int_{S^1} e^{-\frac{r^2}{4t}} (1 + \frac{r^2}{t}) |g(x + r\omega)| r \, dr\, d\omega. \end{aligned}$$

This is estimated similarly to the case in $N=3$, which shows (3.6a). Finally,

$$\begin{aligned} J_{12}(t)g &= \partial_t(J_{02}(t)g) \\ &= \frac{e^{-\frac{t}{2}}}{4\pi}\int_0^t\int_{S^1}\left(\frac{t\sinh\frac{\sqrt{t^2-r^2}}{2}}{2(t^2-r^2)} - \frac{2(\cosh\frac{\sqrt{t^2-r^2}}{2}-1)}{\sqrt{t^2-r^2}}(\frac{1}{2}+\frac{t}{t^2-r^2})\right) g(x+r\omega)r\,dr\,d\omega \\ &= \int_0^{t^{2/3}} + \int_{t^{2/3}}^t =: X_4 + X_5. \end{aligned}$$

X_5 decays exponentially. When $t>1$, since

$$2e^{-\frac{t}{2}}\sinh\frac{\sqrt{t^2-r^2}}{2} = e^{-\frac{r^2}{4t}}(1+\frac{1}{t}O(1+\frac{r^2}{t^2}))$$

with (4.11), and

$$\frac{t}{2(t^2-r^2)} = \frac{1}{2t}(1+\frac{1}{t}O(\frac{r^2}{t})),$$

$$-\frac{1}{\sqrt{t^2-r^2}}(\frac{1}{2}+\frac{t}{t^2-r^2}) = -\frac{1}{2t}(1+\frac{1}{t}O(1+\frac{r^2}{t^2})),$$

we have

$$|X_4| \le Ct^{-2}\int_0^{t^{2/3}}\int_{S^1} e^{-\frac{r^2}{4t}}(1+\frac{r^2}{t})|g(x+r\omega)|r\,dr\,d\omega,$$

which derives (3.6b).

Thus we have completed the proof of Lemma 3.1.

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2.3 Asymptotic profile of solutions to nonlinear dissipative evolution system with ellipticity

Kenji Nishihara

Abstract. We consider the Cauchy problem for the nonlinear dissipative evolution system with ellipticity on one dimensional space

$$\begin{cases} \psi_t = -(1-\alpha)\psi - \theta_x + \alpha\psi_{xx}, & (t,x) \in (0,\infty)\times \mathbf{R} \\ \theta_t = -(1-\alpha)\theta + \nu^2\psi_x + \alpha\theta_{xx} + 2\psi\theta_x, \end{cases}$$

with $0 < \nu^2 < 4\alpha(1-\alpha)$, $0 < \alpha < 1$. S. Q. Tang and H. Zhao [4] have considered the problem and obtained the optimal decay property for suitably small data. In this paper we derive the asymptotic profile using the Gauss kernel $G(t,x)$, which show the prcise behavior of solution as time tends to infinity. In fact, we will show that the asymptotic formula

$$\left\| \begin{pmatrix} \psi \\ \theta \end{pmatrix}(t,x) - D_0 e^{-(1-\alpha-\frac{\nu^2}{4\alpha})t} G(t,x) \begin{pmatrix} \cos(\frac{\nu}{2\alpha}x + \frac{\pi}{4} + \beta_0) \\ -\nu\sin(\frac{\nu}{2\alpha}x + \frac{\pi}{4} + \beta_0) \end{pmatrix} \right\|_{L^p(\mathbf{R}_x)}$$
$$= e^{-(1-\alpha-\frac{\nu^2}{4\alpha})t} o(t^{-\frac{1}{2}(1-\frac{1}{p})}),$$

holds, where D_0, β_0 are determined by the data. It is the key point to reformulate the system to the nonlinear parabolic one by suitable changing variables.

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Key words. Evolution system with ellipticity, parabolic system, asymptotic profile.

1 Introduction

We consider the Cauchy problem for the nonlinear dissipative evolution system with ellipticity on one dimensional space

$$\begin{cases} \psi_t = -(1-\alpha)\psi - \theta_x + \alpha\psi_{xx}, & (t,x) \in (0,\infty)\times \mathbf{R} \\ \theta_t = -(1-\alpha)\theta + \nu^2\psi_x + \alpha\theta_{xx} + 2\psi\theta_x, \end{cases} \tag{1.1}$$

$$(\psi,\theta)(0,x) = (\psi_0,\theta_0)(x), \quad x \in \mathbf{R}. \tag{1.2}$$

The parameters $\alpha, \nu(>0)$ are assumed to satisfy

$$0 < \nu^2 < 4\alpha(1-\alpha), \quad 0 < \alpha < 1. \tag{1.3}$$

The system (1.1) is a special case of the simplified system

$$\begin{cases} \psi_t = -(\sigma-\alpha)\psi - \sigma\theta_x + \alpha\psi_{xx}, \\ \theta_t = -(1-\beta)\theta + \nu^2\psi_x + \beta\theta_{xx} + 2\psi\theta_x. \end{cases} \tag{1.4}$$

The system (1.4) with $\alpha = 0$, together with the system having the conservative nonlinear term $(\psi\theta)_x$ instead of $2\psi\theta_x$, was originally introduced in Hsieh [1] to observe the nonlinear interaction between ellipticity and dissipation, and (1.4) is found in Prof. S.Q.Tang's Ph.D. thesis(private communication) and Hsieh, Tang and Wang [2]. As written in Tang and Zhao [4], if we ignore the damping and diffusion in the moment, then the linearized system of (1.4)

$$\begin{cases} \psi_t = -\sigma\theta_x \\ \theta_t = \nu^2\psi_x \end{cases}$$

is elliptic, and the equilibrium $(\psi, \theta) \equiv 0$ is unstable. By taking the nonlinear term $2\psi\theta_x$ into consideration, it is still unstable when $|\psi| \ll 1$, and it switch to be stable when $|\psi| \gg 1$ because the system become hyperbolic. In their paper [4] they have obtained the decay estimates of the solution to (1.1)-(1.2), whose system has the damping and diffusion effects, provided that the data is small.

In this paper we improve their result to obtain the asymptotic profile of the solution using the Gauss kernel

$$G(t,x) = \frac{1}{\sqrt{4\pi\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right). \tag{1.5}$$

To do so, we change (ϕ, θ) to the new variables (u, v)

$$\begin{pmatrix} \psi \\ \theta \end{pmatrix} = 2e^{-(1-\alpha-\frac{\nu^2}{4\alpha})t} \begin{pmatrix} uT_-(x) - vT_+(x) \\ -\nu(uT_+(x) + vT_-(x)) \end{pmatrix} \tag{1.6}$$

or

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{4} e^{(1-\alpha-\frac{\nu^2}{4\alpha})t} \begin{pmatrix} \psi T_-(x) - \frac{\theta}{\nu}T_+(x) \\ -(\psi T_+(x) + \frac{\theta}{\nu}T_-(x)) \end{pmatrix}, \tag{1.7}$$

so that the Cauchy problem (1.1)-(1.2) is reformulated to the problem for a new parabolic system

$$\begin{cases} u_t = \alpha u_{xx} + F_+ \\ v_t = \alpha v_{xx} + F_- \end{cases} \tag{1.8}$$

with the initial data

$$\begin{pmatrix} u \\ v \end{pmatrix}(0,x) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}(x) := \frac{1}{4} \begin{pmatrix} \psi_0(x)T_-(x) - \frac{\theta_0(x)}{\nu}T_+(x) \\ -(\psi_0(x)T_+(x) + \frac{\theta_0(x)}{\nu}T_-(x)) \end{pmatrix}. \tag{1.9}$$

Here

$$\begin{aligned} F_\pm &= F_\pm(u, v, u_x, v_x) \\ &= e^{-(1-\alpha-\frac{\nu^2}{4\alpha})t} T_\pm(x)(uT_-(x) - vT_+(x))\{\tfrac{\nu}{\alpha}(uT_-(x) - vT_+(x)) + 2(u_xT_+(x) + v_xT_-(x))\} \end{aligned} \tag{1.10}$$

and

$$T_\pm(x) = \cos\frac{\nu}{2\alpha}x \pm \sin\frac{\nu}{2\alpha}x = \sqrt{2}\cos\left(\frac{\nu}{2\alpha}x \mp \frac{\pi}{4}\right). \tag{1.11}$$

The solution (u, v) to (1.8)-(1.9) is given by that to the integral equation

$$\begin{cases} u(t,x) = \int_{\mathbf{R}} G(t, x-y)u_0(y)\, dy + \int_0^t \int_{\mathbf{R}} G(t-\tau, x-y)F_+(\tau, y)\, dy\, d\tau \\ v(t,x) = \int_{\mathbf{R}} G(t, x-y)v_0(y)\, dy + \int_0^t \int_{\mathbf{R}} G(t-\tau, x-y)F_-(\tau, y)\, dy\, d\tau, \end{cases} \tag{1.12}$$

and, if $(u_0, v_0) \in L^1(\mathbf{R})$, then it is expected that

$$|(u - \delta_+ G, v - \delta_- G)(t,\cdot)| = o(t^{-\frac{1}{2}}), \quad t \to \infty, \tag{1.13}$$

where

$$\begin{aligned} \delta_+ &= \int_{\mathbf{R}} u_0(x)\,dx + \int_0^\infty \int_{\mathbf{R}} F_+(\tau, y)\,dy\,d\tau \\ \delta_- &= \int_{\mathbf{R}} v_0(x)\,dx + \int_0^\infty \int_{\mathbf{R}} F_-(\tau, y)\,dy\,d\tau. \end{aligned} \tag{1.14}$$

Hence, by (1.6) we expect (ψ, θ) to behave as

$$2e^{-(1-\alpha-\frac{\nu^2}{4\alpha})t} \begin{pmatrix} \delta_+ T_-(x) - \delta_- T_+(x) \\ -\nu(\delta_+ T_+(x) + \delta_- T_-(x) \end{pmatrix} G(t,x) \tag{1.15}$$

as $t \to \infty$. Moreover, since

$$T_+(x) = \sqrt{2}\cos(\frac{\nu}{2\alpha}x - \frac{\pi}{4}) = \sqrt{2}\cos(\frac{\nu}{2\alpha}x + \frac{\pi}{4} - \frac{\pi}{2}) = \sqrt{2}\sin(\frac{\nu}{2\alpha}x + \frac{\pi}{4})$$

by (1.11),

$$\delta_+ T_-(x) - \delta_- T_+(x) = \sqrt{2(\delta_+^2 + \delta_-^2)}\cos(\frac{\nu}{2\alpha}x + \frac{\pi}{4} + \beta_0), \tag{1.16}$$

with

$$\cos\beta_0 = \frac{\delta_+}{\sqrt{\delta_+^2 + \delta_-^2}}, \quad \sin\beta_0 = \frac{\delta_-}{\sqrt{\delta_+^2 + \delta_-^2}}. \tag{1.17}$$

Similarly,

$$\delta_+ T_+(x) + \delta_- T_-(x) = \sqrt{2(\delta_+^2 + \delta_-^2)}\sin(\frac{\nu}{2\alpha}x + \frac{\pi}{4} + \beta_0). \tag{1.18}$$

Substituting (1.16), (1.18) to (1.13), we expect that

$$\begin{pmatrix} \psi \\ \theta \end{pmatrix} - 2\sqrt{2(\delta_+^2 + \delta_-^2)}e^{-(1-\alpha-\frac{\nu^2}{4\alpha})t}G(t,x) \begin{pmatrix} \cos(\frac{\nu}{2\alpha}x + \frac{\pi}{4} + \beta_0) \\ -\nu\sin(\frac{\nu}{2\alpha}x + \frac{\pi}{4} + \beta_0) \end{pmatrix} = o(t^{-\frac{1}{2}}). \tag{1.19}$$

That is, the solution (ψ, θ) behaves like the Gauss kernel vibrating with the period $\frac{4\alpha\pi}{\nu}$ in x-direction, whose phases have $\frac{\pi}{2}$ gap and whose magnitudes have the ratio $1 : \nu$, and decays exponentially in t-direction.

Even when

$$\nu^2 \geq 4\alpha(1-\alpha)$$

in stead of (1.3), the system (1.1) can be formally reformulated to (1.8). But, the order of nonlinear term seems to be too small to prove the global existence. Also, when $(\psi_0, \theta_0) \notin L^1(\mathbf{R})$, especially

$$\lim_{x \to \pm\infty} (\psi_0, \theta_0)(x) = (\psi_\pm, \theta_\pm), \quad (\psi_+, \theta_+) \neq (\psi_-, \theta_-),$$

see Zhu and Wang [5]. For the related results see the references in [4, 5]

2 Reformulation of the problem

We derive (1.8) from (1.1). Multiplying (1.1) by e^{At}, $A := 1 - \alpha - \frac{\nu^2}{4\alpha}$, we have

$$(e^{At}\psi)_t = -\frac{\nu^2}{4\alpha}e^{At}\psi - (e^{At}\theta)_x + \alpha(e^{At}\psi)_{xx}, \tag{2.1}$$

$$(e^{At}\theta)_t = -\frac{\nu^2}{4\alpha}e^{At}\theta + \nu^2(e^{At}\psi)_x + \alpha(e^{At}\theta)_{xx} + e^{At}f, \tag{2.2}$$

where

$$f = 2\psi\theta_x. \tag{2.3}$$

Moreover, noting that

$$T_-'(x) = -\frac{\nu}{2\alpha}T_+(x), \quad T_+'(x) = \frac{\nu}{2\alpha}T_-(x), \quad T_\pm''(x) = -(\frac{\nu}{2\alpha})^2 T_\pm(x), \tag{2.4}$$

we calculate (2.1) $\times\, T_-(x)$ and (2.2) $\times\, T_+(x)$:

$$\begin{aligned}(e^{At}\psi T_-(x))_t &= -\tfrac{\nu^2}{4\alpha}e^{At}\psi T_-(x) - (e^{At}\theta)_x T_-(x) + \alpha(e^{At}\psi T_-(x))_{xx}\\ &\quad -\alpha\left(-(\tfrac{\nu}{2\alpha})^2 e^{At}\psi T_-(x) - 2(e^{At}\psi)_x\tfrac{\nu}{2\alpha}T_+(x)\right)\\ &= \nu(e^{At}\psi)_x T_+(x) - (e^{At}\theta)_x T_-(x) + \alpha(e^{At}\psi T_-(x))_{xx}\end{aligned}$$

and

$$\begin{aligned}(e^{At}\theta T_+(x))_t &= -\tfrac{\nu^2}{4\alpha}e^{At}\theta T_+(x) + \nu^2(e^{At}\psi)_x T_+(x) + \alpha(e^{At}\theta T_+(x))_{xx}\\ &\quad -\alpha\left(-(\tfrac{\nu}{2\alpha})^2 e^{At}\theta T_+(x) + 2(e^{At}\theta)_x\tfrac{\nu}{2\alpha}T_-(x)\right) + e^{At}fT_+(x)\\ &= \nu^2(e^{At}\psi)_x T_+(x) - \nu(e^{At}\theta)_x T_-(x) + \alpha(e^{At}\theta T_+(x))_{xx} + e^{At}fT_+(x).\end{aligned}$$

Hence, $\frac{1}{4}\{(2.1) \times T_-(x) - \frac{1}{\nu}(2.2) \times T_+(x)\}$ with (1.7) implies

$$u_t = \alpha u_{xx} - \frac{1}{4\nu}e^{At}fT_+(x). \tag{2.5}$$

Similarly, $-\frac{1}{4}\{(2.1) \times T_+(x) + \frac{1}{\nu}(2.2) \times T_-(x)\}$ gives

$$v_t = \alpha v_{xx} - \frac{1}{4\nu}e^{At}fT_-(x). \tag{2.6}$$

From (1.7) and

$$T_+(x)^2 + T_-(x)^2 = 2, \tag{2.7}$$

(1.6) easily follows. From the second equation in (1.6)

$$\theta_x = -2\nu e^{-At}\{(u_xT_+(x) + v_xT_-(x)) + \frac{\nu}{2\alpha}(uT_-(x) - vT_+(x))\},$$

which yields

$$\begin{aligned}F_\pm &= -\tfrac{1}{4\nu}e^{At}fT_\pm(x)\\ &= e^{-At}T_\pm(x)(uT_-(x) - vT_+(x))\{\tfrac{\nu}{\alpha}(uT_-(x) - vT_+(x)) + 2(u_xT_+(x) + v_xT_-(x))\}.\end{aligned}$$

Thus we have obtained (1.8) with (1.10).

3 Global existence and asymptotic profile

First, we solve the reformulated problem (1.8)-(1.9) by solving the integral equation (1.12). The order of the nonlinear term is 2, which is less than the Fujita exponent $1+\frac{2}{N}$, $N=1$ and small for the global existence, generally. However, e^{-At} in $F_\pm$ helps us to obtain the global existence theorem and the asymptotic profile.

The solution space is defined by

$$X(0,\infty)=\{(u,v)\in C([0,\infty);\, L^1\cap L^\infty(\mathbf{R}));\ (u,v)_x\in C([0,\infty);\, L^1(\mathbf{R}))\}$$

with the norm

$$\|u,v\|_X=\sup_{0\le t<\infty}\{\|(u,v)(t)\|_{L^1}+(1+t)^{\frac{1}{2}}(\|(u,v)(t)\|_{L^\infty}+\|(u,v)_x(t)\|_{L^1})\}. \tag{3.1}$$

For the initial data we assume that

$$\begin{aligned}&(u_0,v_0)\in L^1\cap L^\infty(\mathbf{R}),\quad (u_0,v_0)_x\in L^1(\mathbf{R}),\\ &\|u_0,v_0\|_{L^1\cap L^\infty}+\|(u_0,v_0)_x\|_{L^1}\le\varepsilon(<1).\end{aligned} \tag{3.2}$$

As usual, define the iteration $\{(u,v)^{(n)}\}$ by

$$(u,v)^{(0)}(t,x)=\int_{\mathbf{R}}G(t,x-y)(u_0,v_0)(y)\,dy \tag{3.3}$$

$$(u,v)^{(n)}(t,x)=(u,v)^{(0)}(t,x)+\int_0^t\int_{\mathbf{R}}G(t-\tau,x-y)(F_+,F_-)^{(n-1)}(\tau,y)\,dy\,d\tau, \tag{3.4}$$

where

$$F_\pm^{(n)}(t,x)=F_\pm(u^{(n)},v^{(n)},u_x^{(n)},v_x^{(n)})(t,x). \tag{3.5}$$

It is easy to show

$$\|(u,v)^{(0)}\|_X\le C_0(\|u_0,v_0\|_{L^1\cap L^\infty}+\|(u_0,v_0)_x\|_{L^1}\le C_0\varepsilon. \tag{3.6}$$

Moreover, we show for some positive constant C_0

(i) when $\varepsilon\ll 1$,

$$\|(u,v)^{(n-1)}\|_X\le 2C_0\varepsilon\ \Rightarrow\ \|(u,v)^{(n)}\|_X\le 2C_0\varepsilon,$$

(ii) for less constant ε if necessary,

$$\|(u,v)^{(n+1)}-(u,v)^{(n)}\|_X\le\frac{1}{2}\|(u,v)^{(n)}-(u,v)^{(n-1)}\|_X.$$

Then we get the solution (u,v) to (1.12) in $X(0,\infty)$. Denoting

$$U=(u,v),\ U_0=(u_0,v_0)\ \text{ with }\ |U|=|u|+|v|,\ |U_0|=|u_0|+|v_0|, \tag{3.7}$$

noting that

$$\begin{aligned}&|F_\pm(U_1,U_{1x})-F_\pm(U_2,U_{2x})|\\ &\le Ce^{-At}(|U_1,U_2,U_{1x},U_{2x}||U_1-U_2|+|U_1,U_2||U_{1x}-U_{2x}|).\end{aligned} \tag{3.8}$$

we prove (i). For X-norm of u,

$$
\begin{aligned}
&\|u^{(n)}(t)\|_{L^1}\\
&\le \|u^{(0)}\|_{L^1} + \int_0^t \|G(t-\tau,\cdot)\|_{L^1}\|F_+^{(n-1)}(\tau)\|_{L^1}\,d\tau\\
&\le \|u^{(0)}\|_{L^1}\\
&\quad + \int_0^t Ce^{-A\tau}(\|U^{(n-1)}(\tau)\|_{L^\infty}\|U^{(n-1)}(\tau)\|_{L^1} + \|U_{1x}^{(n-1)}(\tau)\|_{L^1}\|U^{(n-1)}(\tau)\|_{L^\infty})\,d\tau\\
&\le \|u^{(0)}\|_{L^1} + \int_0^t Ce^{-A\tau}(1+\tau)^{-\frac{1}{2}}\|U^{(n-1)}\|_X\,d\tau\\
&\le \|u^{(0)}\|_{L^1} + C(2C_0\varepsilon)^2,
\end{aligned}
$$

$$
\begin{aligned}
\|u^{(n)}(t)\|_{L^\infty} &\le \|u^{(0)}\|_{L^\infty} + \int_0^t \|G(t-\tau,\cdot)\|_{L^\infty}\|F_+^{(n-1)}(\tau)\|_{L^1}\,d\tau\\
&\le \|u^{(0)}\|_{L^\infty} + \int_0^t C(t-\tau)^{-\frac{1}{2}}e^{-A\tau}(1+\tau)^{-\frac{1}{2}}\|U^{(n-1)}\|_X\,d\tau\\
&\le \|u^{(0)}\|_{L^\infty} + (1+t)^{-\frac{1}{2}}\cdot C(2C_0\varepsilon)^2
\end{aligned}
$$

and

$$
\begin{aligned}
\|u_x^{(n)}(t)\|_{L^1} &\le \|u_x^{(0)}\|_{L^1} + \int_0^t \|G_x(t-\tau,\cdot)\|_{L^1}\|F_+^{(n-1)}(\tau)\|_{L^1}\,d\tau\\
&\le \|u_x^{(0)}\|_{L^1} + (1+t)^{-\frac{1}{2}}\cdot C(2C_0\varepsilon)^2.
\end{aligned}
$$

Hereandafter C denotes several positive constants. Hence

$$\|u^{(n)}\|_X \le \|u^{(0)}\|_X + C(2C_0\varepsilon)^2. \tag{3.9}$$

In a similar fashion to the above, we have

$$\|v^{(n)}\|_X \le \|v^{(0)}\|_X + C(2C_0\varepsilon)^2. \tag{3.10}$$

Adding (3.9) and (3.10), we have

$$
\begin{aligned}
\|U^{(n)}\|_X &\le \|U^{(0)}\|_X + C(2C_0\varepsilon)^2\\
&\le C_0\varepsilon(1+4CC_0\varepsilon)\\
&\le 2C_0\varepsilon \quad \text{if } \varepsilon \le 1/(4CC_0).
\end{aligned}
\tag{3.11}
$$

The proof of (ii) is almost same as (i) and omitted.

Thus we reach to the first theorem.

Theorem 3.1 (Global existence) *There exists a sufficiently small constant $\varepsilon > 0$ such that, if (3.3) holds, then the solution $(u,v) \in X(0,\infty)$ to (1.12), and hence to the Cauchy problem (1.8)-(1.9), uniquely exists, which satisfies*

$$\|(u,v)(t,\cdot)\|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}, \quad 1 \le p \le \infty, \tag{3.12}$$

$$\|(u_x,v_x)(t,\cdot)\|_{L^1} \le C(1+t)^{-\frac{1}{2}}. \tag{3.13}$$

Further, consider the asymptotic profile of the solution obtained in Theorem 3.1. Then $F_\pm$ is integrable over $[0,\infty)\times\mathbf{R}$ and $\delta_\pm$ are well-defined by (1.14). The main theorem is given by the following.

Theorem 3.2 (Asymptotic profile) *The solution (u,v) obtained in Theorem 3.1 satisfies as $t \to \infty$*

$$\|(u-\delta_+ G, v-\delta_- G)(t,\cdot)\|_{L^p} = o(t^{-\frac{1}{2}(1-\frac{1}{p})}). \tag{3.14}$$

Proof. The solution (u,v) satisfies (1.12). Hence, setting

$$\delta_+^I = \int_{\mathbf{R}} u_0(x)\,dx, \quad \delta_+^F = \int_0^\infty \int_{\mathbf{R}} F_+(\tau,y)\,dy\,d\tau, \tag{3.15}$$

we estimate as follows. For the initial data,

$$\begin{aligned} &\int_{\mathbf{R}} G(t,x-y)u_0(y)\,dy - \delta_+^I G(t,x) \\ &= \int_{\mathbf{R}} (G(t,x-y)-G(t,x))u_0(y)\,dy \\ &= \int_{\mathbf{R}}\int_0^1 G_x(t,x-\lambda y)\,d\lambda \cdot (-y)u_0(y)\,dy \end{aligned} \tag{3.16}$$

and, if $(1+|\cdot|)u_0 \in L^1(\mathbf{R})$ for a moment, then

$$\|\int_{\mathbf{R}} G(t,x-y)u_0(y)\,dy - \delta_+^I G(t,x)\|_{L^p} \le Ct^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}.$$

When $u_0 \in L^1(\mathbf{R})$ only, for any small constant $\eta >$, there exists the constant $M = M(\eta) > 0$ such that

$$\int_{|y|\ge M} |u_0(y)|\,dy \le \eta.$$

Hence

$$\begin{aligned} &t^{\frac{1}{2}(1-\frac{1}{p})}\|\int_{|y|\ge M}(G(t,x-y)-G(t,x))u_0(y)\,dy\|_{Lp} \\ &\le t^{\frac{1}{2}(1-\frac{1}{p})}\|\int_{|y|\ge M} 2\|G(t,\cdot)\|_{L^p}|u_0(y)|\,dy \\ &\le C\int_{|y|\ge M}|u_0(y)|\,dy \le C\eta \end{aligned}$$

and

$$\begin{aligned} &t^{\frac{1}{2}(1-\frac{1}{p})}\|\int_{|y|\le M}(G(t,x-y)-G(t,x))u_0(y)\,dy\|_{Lp} \\ &\le C\int_{|y|\le M}\int_0^1 d\lambda\cdot\frac{|y|}{\sqrt{t}}|u_0(y)|\,dy \\ &\le \frac{CM}{\sqrt{t}}\int_{\mathbf{R}}|u_0(y)|\,dy \to 0 \quad \text{as} \quad t\to\infty. \end{aligned}$$

Therefore, by (3.16),

$$\overline{\lim_{t\to\infty}}\, t^{\frac{1}{2}(1-\frac{1}{p})}\|\int_{\mathbf{R}} G(t,x-y)u_0(y)\,dy - \delta_+^I G(t,x)\|_{L^p} \le C\eta,$$

which implies

$$\|\int_{\mathbf{R}} G(t,x-y)u_0(y)\,dy - \delta_+^I G(t,x)\|_{L^p} = o(t^{-\frac{1}{2}(1-\frac{1}{p})}). \tag{3.17}$$

For the "forcing term" we sketch the estimates, which are similar to those in Karch [3]. We devide the integrand of the difference into three parts:

$$
\begin{aligned}
&\int_0^t \int_{\mathbf{R}} G(t-\tau, x-y) F_+(\tau, y)\, dy\, d\tau - \delta_+^F G(t,x) \\
&= \int_{t/2}^t \int_{\mathbf{R}} G(t-\tau, x-y) F_+(\tau, y)\, dy\, d\tau - G(t,x) \int_{t/2}^{\infty} \int_{\mathbf{R}} F_+(\tau, y)\, dy\, d\tau \\
&\quad + \int_0^{t/2} \int_{\mathbf{R}} (G(t-\tau, x-y) - G(t,x)) F_+(\tau, y)\, dy\, d\tau \\
&=: K_1 + K_2 + K_3.
\end{aligned}
\tag{3.18}
$$

Since

$$
\begin{aligned}
\|K_1\|_{L^1} &\le \int_{t/2}^t \|G(t-\tau, \cdot)\|_{L^1} \|F_+(\tau)\|_{L^1}\, d\tau \\
&\le \int_{t/2}^t Ce^{-A\tau} \|U(\tau)\|_{L^\infty} (\|U(\tau)\|_{L^1} + \|U_x(\tau)\|_{L^1})\, d\tau, \\
&= o(1)
\end{aligned}
$$

and

$$
\begin{aligned}
\|K_1\|_{L^\infty} &\le \int_{t/2}^t C(t-\tau)^{-\frac{1}{2}} e^{-A\tau} \|U(\tau)\|_{L^\infty} (\|U(\tau)\|_{L^1} + \|U_x(\tau)\|_{L^1})\, d\tau, \\
&= o(t^{-\frac{1}{2}}).
\end{aligned}
$$

Combining them, we have

$$
\|K_1\|_{L^p} = o(t^{-\frac{1}{2}(1-\frac{1}{p})}). \tag{3.19}
$$

For K_2 it is easy to show

$$
\|K_2\|_{L^p} = o(t^{-\frac{1}{2}(1-\frac{1}{p})}). \tag{3.20}
$$

Following [3], for small constant $\mu (0 < \mu < \frac{1}{2})$ we devide K_3 into two integrands:

$$
K_3 = \left(\int_{\Omega_1(t)} + \int_{\Omega_2(t)} \right) (G(t-\tau, x-y) - G(t,x)) F_+(\tau, y)\, dy\, d\tau, \tag{3.21}
$$

with

$$
\Omega_1(t) = [0, \mu t] \times \{y \in \mathbf{R};\ |y| \le \mu t^{1/2}\}, \quad \Omega_2(t) = ([0, \frac{t}{2}] \times \mathbf{R}) \backslash \Omega_1(t). \tag{3.22}
$$

Since

$$
t^{\frac{1}{2}(1-\frac{1}{p})} \| \int_{\Omega_1(t)} \|_{L^p} \le t^{\frac{1}{2}(1-\frac{1}{p})} \int_0^{\mu t} \int_{|y| \le \mu\sqrt{t}} \|G(t-\tau, \cdot - y) - G(t, \cdot)\|_{L^p} |F_+(\tau, y)|\, dy\, d\tau,
$$

the changing variables $\frac{\cdot}{\sqrt{t}} = \xi$, $\frac{y}{\sqrt{t}} = z$, $\tau = st$ yield

$$
\begin{aligned}
&t^{\frac{1}{2}(1-\frac{1}{p})} \| \int_{\Omega_1(t)} \|_{L^p} \\
&\le \int_0^{\mu} \int_{|z| \le \mu} \left(\int_{\mathbf{R}} \left| \frac{\exp\left(-\frac{(\xi - z)^2}{4\alpha(1-s)}\right)}{\sqrt{4\pi\alpha(1-s)}} - \frac{\exp\left(-\frac{\xi^2}{4\alpha}\right)}{\sqrt{4\pi\alpha}} \right|^p d\xi \right)^{1/p} |F_+(st, \sqrt{t}z)| \sqrt{t}dz \cdot tds.
\end{aligned}
$$

Here, samely as above, for any small constant $\eta > 0$, there exists the constant $\mu = \mu(\eta) > 0$ such that, if $|s| \le \mu$, $|z| \le \mu$, then

$$\left(\int_{\mathbf{R}} \left| \frac{\exp\left(-\frac{(\xi-z)^2}{4\alpha(1-s)}\right)}{\sqrt{4\pi\alpha(1-s)}} - \frac{\exp\left(-\frac{\xi^2}{4\alpha}\right)}{\sqrt{4\pi\alpha}} \right|^p d\xi \right)^{1/p} \le \eta$$

Hence

$$t^{\frac{1}{2}(1-\frac{1}{p})} \| \int_{\Omega_1(t)} \|_{L^p} \le \eta \int_{\Omega_1(t)} |F_+(\tau, y)| \, dy \, d\tau \le C\eta. \tag{3.23}$$

Using this η, we derive

$$\begin{aligned} \| \int_{\Omega_2(t)} \|_{L^p} &\le \int_{\Omega_2(t)} (\|G(t-\tau, \cdot)\|_{L^p} + \|G(t, \cdot)\|_{L^p}) |F_+(\tau, y)| \, dy \, d\tau \\ &\le Ct^{-\frac{1}{2}(1-\frac{1}{p})} \int_{\Omega_2(t)} |F_+(\tau, y)| \, dy \, d\tau = o(t^{-\frac{1}{2}(1-\frac{1}{p})}). \end{aligned} \tag{3.24}$$

Combining (3.21)-(3.24), we have

$$\overline{\lim_{t\to\infty}} t^{\frac{1}{2}(1-\frac{1}{p})} \|K_3\|_{L^p} \le C\eta,$$

which implies

$$\|K_3\|_{L^p} = o(t^{-\frac{1}{2}(1-\frac{1}{p})}). \tag{3.25}$$

From (3.17)-(3.20) and (3.25), the desired estimate (3.15) on u has been obtained. The estimates on v are completely same as those on u, which completes the proof. □

Theorems 3.1-3.2 on (u, v) are returned back to the theorem on the original variables (ϕ, θ) by (1.6).

Theorem 3.3 *Let $(\psi_0, \theta_0) \in L^1 \cap L^\infty(\mathbf{R})$, $(\psi_0, \theta_0)_x \in L^1(\mathbf{R})$ be sufficiently small. Then the Cauchy problem (1.1)-(1.2) has a unique solution $(\phi, \theta) \in C([0, \infty); L^1 \cap L^\infty(\mathbf{R}))$ with $(\psi, \theta)_x \in C([0, \infty); L^1(\mathbf{R}))$, which satisfying the decay estimates*

$$\|(\psi, \theta)(t, \cdot)\|_{L^p} \le Ce^{-(1-\alpha-\frac{\nu^2}{4\alpha})t}(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}, \tag{3.26}$$

$$\|(\psi, \theta)_x(t, \cdot)\|_{L^1} \le Ce^{-(1-\alpha-\frac{\nu^2}{4\alpha})t}. \tag{3.27}$$

Moreover, the asymptotic formula

$$\left\| \begin{pmatrix} \psi \\ \theta \end{pmatrix} (t, x) - 2\sqrt{2(\delta_+^2 + \delta_-^2)} e^{-(1-\alpha-\frac{\nu^2}{4\alpha})t} G(t, x) \begin{pmatrix} \cos\left(\frac{\nu}{2\alpha}x + \frac{\pi}{4} + \beta_0\right) \\ -\nu \sin\left(\frac{\nu}{2\alpha}x + \frac{\pi}{4} + \beta_0\right) \end{pmatrix} \right\|_{L^p(\mathbf{R}_x)}$$
$$= e^{-(1-\alpha-\frac{\nu^2}{4\alpha})t} o(t^{-\frac{1}{2}(1-\frac{1}{p})})$$

holds, where $\delta_\pm, \beta_0$ are given by (1.14), (1.17), respectively.

The asymptotic formula in Theorem 3.3 is derived as the procedure (1.13)-(1.19).

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Kenji Nishihara
School of Political Science and Economics
Waseda Unviversity
Tokyo, 169-8050 Japan
(e-mail: kenji@waseda.jp)

2.4 消散型波動方程式のコーシー問題の解の拡散現象

西原 健二（早稲田大学 政治経済学術院）

1 序

本稿では，消散型波動方程式(Damped wave equation, Wave equation with dissipation)のコーシー問題

$$(DW) \qquad \begin{cases} u_{tt} - \Delta u + u_t = f(u), & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (u, u_t)(0,x) = (u_0, u_1)(x), & x \in \mathbf{R}^N \end{cases}$$

を考える．$f(u) \equiv 0$ のときはもちろん線形の消散型波動方程式であり，半線形項 $f(u)$ を考察するときは指数 $\rho > 1$ を持つ

$$(1.1) \qquad f(u) = |u|^{\rho-1}u, \ \pm|u|^{\rho} \ \text{または} \ f(u) = -|u|^{\rho-1}u$$

等を考える．多くの数学者によって，

“ 消散型波動方程式の解は $t \to \infty$ とともに拡散現象を持つ ”

ことが認識されてきた．すなわち，(DW) の解 $u(t,x)$ は，$t \to \infty$ のとき，対応する熱方程式(Heat equation) または拡散方程式(Diffusive equation) のコーシー問題

$$(H) \qquad \begin{cases} \phi_t - \Delta\phi = f(\phi), & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ \phi(0,x) = \phi_0(x), & x \in \mathbf{R}^N \end{cases}$$

の解 $\phi(t,x)$ に漸近すると．筆者はここ数年この問題やその周辺を考察してきた．それらについて論じて行きたい．

まず，拡散現象を示すモデルを 2 つ提示する．それらが考察の動機でもあり，消散型波動方程式の拡散現象が容易に想像できると思われるからである．

モデル 1.1 (熱伝導を表す方程式 (Li [32])) 時刻 t において，空間 1 次元的に無限に伸びる針金の点 x における熱流を $q(t,x)$ とする．熱は x 方向のみに伝わるものとすると，$[a,b]$

間の温度 ϕ の変化は $\frac{d}{dt}\int_a^b \phi(t,x)\,dx$ で，$x = a, b$ における熱の出入り $q(t,a) - q(t,b) = -\int_a^b q_x(t,x)\,dx$ に等しくなるので，

$$(1.2)\qquad \frac{d}{dt}\int_a^b \phi(t,x)\,dx = -\int_a^b q_x(t,x)\,dx \quad \therefore \quad \phi_t + q_x = 0$$

を得る．フーリエの法則によって，熱流 q は温度変化に比例する（比例定数はマイナス）ので，

$$(1.3)\qquad q(t,x) = -\kappa\phi_x(t,x) \quad (\kappa > 0: \text{熱伝導係数}).$$

(1.3) を (1.2) に代入して，空間 1 次元の**線形熱方程式**

$$(1.4)\qquad \phi_t - \kappa\phi_{xx} = 0$$

を得る．(1.3) が通常のフーリエの法則であるが，時間遅れのフーリエの法則

$$(1.5)\qquad q(t+\tau, x) = -\kappa\phi_x(t,x) \quad (0 < \tau \ll 1)$$

を仮定すると，$q(t+\tau, x)$ を時刻 t でテーラー展開し，2 次以降の項を無視すると，

$$(1.6)\qquad q(t,x) + \tau q_t(t,x) = -\kappa\phi_x(t,x)$$

となる．これを x で微分して (1.2) を使えば，空間 1 次元の**消散型波動方程式**

$$(1.7)\qquad \tau\phi_{tt} + \phi_t - \kappa\phi_{xx} = 0$$

を得る．$\tau \to 0+$ とすれば，その導き方からも，特異極限としても，消散型波動方程式 (1.7) が熱方程式 (1.4) に漸近することが予想される．

モデル 1.2 (多孔質中の圧縮性流の方程式系) 多孔質中の 1 次元圧縮性流の方程式は Lagrange 座標系を用いると

$$(1.8)\qquad \begin{cases} v_t - u_x = 0, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^1 \\ u_t + p_x = -\alpha u, & (\alpha > 0: \text{定数}) \end{cases}$$

と表される．ここに，$v(>0)$ は比体積 ($= 1/\rho$, ρ : 密度)，u は速度，p は圧力で，バロトロピー流 $p = f(v)$ を仮定する．典型例は $p(v) = v^{-\gamma}$ ($\gamma \ge 1$) である．$\gamma = 1$ のときは等温流，$\gamma > 1$ のときは等エントロピー流である．第 1 式は質量保存を表し，第 2 式は運動量保存則（と第 1 式）から得られる．(1.8) の第 2 式は，通常のニュートン粘性の場合は $-\alpha u$ の代わりに，$\nu(\frac{u_x}{v})_x$（$\nu > 0$: 粘性係数）となる．多孔質中では，速度に比例する摩

擦による抵抗が大きく，$-\alpha u$ となっている．さて，(1.8) の解の存在については西田 [45] を参照．解の挙動は，$t \to \infty$ のときダルシーの法則 (Darcy's law) によって，

$$(1.9) \qquad \begin{cases} \bar{v}_t - \bar{u}_x &= 0, \\ p(\bar{v})_x &= -\alpha \bar{u}, \end{cases}$$

の解に漸近することが予想されていた．(1.8) が初期データ

$$(1.10) \qquad (v,u)(0,x) = (v_0,u_0)(x) \to (v_\pm, u_\pm) \quad (v_\pm,\, u_\pm : \text{定数}, v_\pm > 0)$$

を持つとき，Hsiao-Liu [17] は，適当な補正関数 $(\hat{v},\hat{u})(t,x)$ を導入し，$V(t,x) := \int_{-\infty}^{x} (v - \bar{v} - \hat{v})(t,y)\, dy$ の 2 階消散型波動方程式のコーシー問題に定式化をし，適当な正則性とデータが小さいという条件の下で

$$(1.11) \qquad \|(v-\bar{v})(t,\cdot)\|_{L^2 \cap L^\infty} = O(t^{-1/2}), \quad t \to \infty$$

を得た．簡単に言って，(1.8) は $p'(v) < 0$ より，v の消散型波動方程式，(1.9) は $\bar{v}$ の拡散方程式で，(1.11) は消散型波動方程式の拡散現象を表している．この事実から，(1.11) で，$\|(v-\bar{v})(t,\cdot)\|_{L^\infty}$ は $\|(v-\bar{v})(t,\cdot)\|_{L^2}$ より減衰が速いはずではないだろうか？ 1 次元の拡散方程式の解の L^p-ノルムは $O(t^{-\frac{1}{2}(1-\frac{1}{p})})$ で減衰するはずであるから．このことが筆者の考察の直接の動機となった．

次節では，(DW) が線型，すなわち，$f(u) \equiv 0$ の場合に，解表示を用いて，対応する (H) の解とどの様に近いか，或いは (DW) は波動方程式と熱方程式の中間にあって，解がどの様に波動的性質と拡散（熱）的性質を持つかを考察する．第 3 節では，第 2 節の線型の場合の結果を半線形の問題に応用する．$f(u) = -|u|^{\rho-1}u$ のときは半線形項は吸収項として，$f(u) = |u|^{\rho-1}u,\ \pm|u|^\rho$ のときは湧出し項として働き，それぞれ事情が異なる．最後の節では関連する話題について議論して締めくくりたい．

2 線型消散型波動方程式

線型消散型波動方程式のコーシー問題

$$(DW)_L \qquad \begin{cases} u_{tt} - \Delta u + u_t = 0, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^N \end{cases}$$

を，対応する線形拡散（熱）方程式のコーシー問題

$$(H)_L \qquad \begin{cases} \phi_t - \Delta\phi = 0, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ \phi(0,x) = \phi_0(x), & x \in \mathbf{R}^N \end{cases}$$

と関連させて，それらの解の“差”の評価を試みる．$(\mathrm{H})_L$ の解 $\phi(t,x)$ を

$$(2.1)\qquad (P_N(t)\phi_0)(x)=\int_{\mathbf{R}^N} G_N(t,x-y)\phi_0(y)\,dy,\quad G_N(t,x)=(4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}$$

と表すと，よく知られているように，$\phi_0\in L^q$, $1\le q\le p\le\infty$ に対して，L^p-L^q 評価

$$(2.2)\qquad \|\partial_t^\alpha\partial_x^\beta P_N(t)\phi_0\|_{L^p}\le C\|\phi_0\|_{L^q}t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-\alpha-\frac{|\beta|}{2}},\quad t>0,$$

が成立する．ここに，$\alpha\in\mathbf{N}_0:=\{0,1,2,\ldots\}$, $\beta=(\beta_1,\ldots,\beta_N)\in\mathbf{N}_0\times\cdots\times\mathbf{N}_0$, $|\beta|=\beta_1+\cdots+\beta_N$ で，$\partial_t=\partial/\partial t$, $\partial_x^\beta=\partial^{|\beta|}/\partial_{x_1}^{\beta_1}\cdots\partial_{x_N}^{\beta_N}$ である．C は正定数を表す．$(\mathrm{DW})_L$ に対しては，(u_0,u_1) の代わりに，初期値 $(0,g)(x)$ を持つコーシー問題

$$(2.3)\qquad \begin{cases} v_{tt}-\Delta v+v_t=0, & (t,x)\in\mathbf{R}_+\times\mathbf{R}^N,\\ (v,v_t)(0,x)=(0,g)(x), & x\in\mathbf{R}^N\end{cases}$$

の解を $v(t,x)=(S_N(t)g)(x)$ と表すと，$(\mathrm{DW})_L$ の解 $u(t,x)$ は

$$(2.4)\qquad u(t,x)=(S_N(t)(u_0+u_1))(x)+\partial_t(S_N(t)u_0)(x)$$

と表すことができる．そこで，$S_N(t)g$ について，(2.1) のように解表示を求め，それを解析して (2.2) に関連した評価を求める．以下，$N=3$ のときを中心にして議論を進める．

2.1 線型消散型波動方程式の解表示

$N=1,2,3$ のときの $S_N(t)g$ の解表示は次のように与えられる (Courant and Hilbert [2]):

$$(2.5)_1\qquad (S_1(t)g)(x)=\frac{e^{-t/2}}{2}\int_{|z|\le t} I_0\left(\frac{1}{2}\sqrt{t^2-|z|^2}\right)g(x+z)\,dz$$

$$(2.5)_2\qquad (S_2(t)g)(x)=\frac{e^{-t/2}}{2\pi}\int_{|z|\le t}\frac{\cosh\frac{1}{2}\sqrt{t^2-|z|^2}}{\sqrt{t^2-|z|^2}}g(x+z)\,dz$$

$$(2.5)_3\qquad (S_3(t)g)(x)=\frac{e^{-t/2}}{4\pi t}\partial_t\int_{|z|\le t} I_0\left(\frac{1}{2}\sqrt{t^2-|z|^2}\right)g(x+z)\,dz.$$

ここに，$I_\nu(y)$ は変形ベッセル関数で，級数では

$$(2.6)\qquad I_\nu(y)=\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+\nu+1)}\left(\frac{y}{2}\right)^{2m}\quad (\Gamma:\text{ガンマ関数})$$

で与えられ，ベッセル関数 $J_\nu(y)$ を使うと $I_\nu(y)=i^{-\nu}J_\nu(iy)$ $(\nu\in\mathbf{N}_0)$, または，2 階常微分方程式 $I_\nu''+\frac{1}{y}I_\nu'-\left(1+\frac{\nu^2}{y^2}\right)I_\nu=0$ の解でもある．解表示 (2.5) の導出は，N 次元波動方程式のコーシー問題

$$\begin{cases} \bar{v}_{tt}-\Delta\bar{v}=0, & (t,x)\in\mathbf{R}_+\times\mathbf{R}^N,\\ (\bar{v},\bar{v}_t)(0,x)=(0,\bar{g})(x), & x\in\mathbf{R}^N\end{cases}$$

の解表示に次元低下法と適当な変数変換を応用して得られる．$B_r^N(x)$, $\partial B_r^N(x)$ を中心 x, 半径 r の N 次元球，N 次元球面として，

$$
\begin{aligned}
\bar{v}(t,x) &=: (W_N(t)\bar{g})(x) \\
(2.7)_N \qquad &= \begin{cases}
\dfrac{1}{\gamma_N}\left(\dfrac{1}{t}\dfrac{\partial}{\partial t}\right)^{\frac{N-3}{2}}\left(\dfrac{t^{N-2}}{|\partial B_t|}\displaystyle\int_{\partial B_t(x)}\bar{g}(y)\,dS_y\right) & (N=\text{奇数}) \\
\quad (\gamma_N = 1\cdot 3\cdots(N-2),\ |\partial B_t| = \frac{2\sqrt{\pi^N}}{\Gamma(\frac{N}{2})}t^{N-1}) & \\
\dfrac{1}{\gamma_N}\left(\dfrac{1}{t}\dfrac{\partial}{\partial t}\right)^{\frac{N-2}{2}}\left(\dfrac{t^{N}}{|\partial B_t|}\displaystyle\int_{B_t(x)}\dfrac{\bar{g}(y)\,dy}{\sqrt{t^2-|y-x|^2}}\right) & (N=\text{偶数}) \\
\quad (\gamma_N = 2\cdot 4\cdots N,\ |B_t| = \frac{2\sqrt{\pi^N}}{N\Gamma(\frac{N}{2})}t^{N}) &
\end{cases}
\end{aligned}
$$

であることを既知とすると，$\boldsymbol{x}_{N+1} = (x_1,\ldots,x_N,x_{N+1}) = (\boldsymbol{x}_N, x_{N+1})$ と表して，$(N+1)$ 次元波動方程式のコーシー問題 $(2.7)_{N+1}$ で，$\bar{g}(\boldsymbol{x}_{N+1}) = g(\boldsymbol{x}_N)e^{cx_{N+1}}$ ととると，解 $\bar{v}$ は $\bar{v}(t,\boldsymbol{x}_{N+1}) = \tilde{v}(t,\boldsymbol{x}_N)\cdot e^{cx_{N+1}}$ と表せ，$\tilde{v}$ は，N 次元のコーシー問題

$$
\begin{cases}
\tilde{v}_{tt} - \Delta\tilde{v} - c^2\tilde{v} = 0, & (t,x)\in\mathbf{R}_+\times\mathbf{R}^N, \\
(\tilde{v},\tilde{v}_t)(0,x) = (0,g)(x), & x = \boldsymbol{x}_N \in \mathbf{R}^N
\end{cases}
$$

の解である（次元低下法）．さらに，変数変換 $v(t,x) = e^{-ct}\tilde{v}(t,x)$, $c=1/2$ をとれば，v は N 次元消散型波動方程式のコーシー問題 (2.3) を満たす．

従って，3 次元の解表現 $(S_3(t)g)(x)$ を導くには，$(2.7)_{N=4}$ の解表現

$$
(W_4(t)\bar{g})(\boldsymbol{x}_4) = \frac{1}{4\pi^2 t}\frac{\partial}{\partial t}\int_{B_t^4(0)}\frac{\bar{g}(\boldsymbol{x}_4+\boldsymbol{y}_4)}{\sqrt{t^2-|\boldsymbol{y}_4|^2}}\,d\boldsymbol{y}_4
$$

に，$\bar{g}(\boldsymbol{x}_4) = g(\boldsymbol{x}_3)e^{\frac{1}{2}x_4}$ を代入し，$e^{-t/2}$ を掛けて得られる：

$$
\begin{aligned}
(S_3(t)g)(\boldsymbol{x}_3) &= e^{-\frac{t}{2}}\cdot e^{-\frac{1}{2}x_4}\cdot\frac{1}{4\pi^2 t}\frac{\partial}{\partial t}\int_{B_t^4(0)}\frac{g(\boldsymbol{x}_3+\boldsymbol{y}_3)e^{\frac{1}{2}(x_4+y_4)}}{\sqrt{t^2-|\boldsymbol{y}_3|^2-y_4^2}}\,d\boldsymbol{y}_3 dy_4 \\
&= \frac{e^{-t/2}}{4\pi^2 t}\frac{\partial}{\partial t}\int_{B_t^3(0)}g(\boldsymbol{x}_3+\boldsymbol{y}_3)\,d\boldsymbol{y}_3\int_{-\sqrt{t^2-|\boldsymbol{y}_3|^2}}^{\sqrt{t^2-|\boldsymbol{y}_3|^2}}\frac{e^{\frac{1}{2}y_4}\,dy_4}{\sqrt{(t^2-|\boldsymbol{y}_3|^2)-y_4^2}} \\
&= \frac{e^{-t/2}}{4\pi^2 t}\frac{\partial}{\partial t}\int_{B_t^3(0)}I_0\left(\frac{1}{2}\sqrt{t^2-|\boldsymbol{y}_3|^2}\right)g(\boldsymbol{x}_3+\boldsymbol{y}_3)\,d\boldsymbol{y}_3
\end{aligned}
$$

を得る．ここで，等式 $\int_{-a}^{a}\frac{e^{cy}\,dy}{\sqrt{a^2-y^2}} = \pi I_0(ca)$ を使った．$S_1(t)g$ についても同様．

$(S_2(t)g)(\boldsymbol{x}_2)$ は，$W_3(t)g$ を使って，$S^2 = \partial B_1^3(0)$ と書けば，

$$
\begin{aligned}
(S_2(t)g)(\boldsymbol{x}_2) &= e^{-\frac{t}{2}}\cdot e^{-\frac{1}{2}x_3}\frac{t}{4\pi}\left(\int_{S^2,\omega_3\ge 0}+\int_{S^2,\omega_3<0}\right)g(x_1+t\omega_1,x_2+t\omega_2)e^{\frac{1}{2}(x_3+t\omega_3)}\,d\omega \\
&= e^{-\frac{t}{2}}\frac{1}{4\pi}\int_{B_t^2(0)}g(\boldsymbol{x}_2+\boldsymbol{y}_2)\left(e^{\frac{1}{2}\sqrt{t^2-|\boldsymbol{y}_2|^2}}+e^{-\frac{1}{2}\sqrt{t^2-|\boldsymbol{y}_2|^2}}\right)\frac{dy}{\sqrt{t^2-|\boldsymbol{y}_2|^2}} \\
&= \frac{e^{-t/2}}{2\pi}\int_{|\boldsymbol{y}_2|\le t}g(\boldsymbol{x}_2+\boldsymbol{y}_2)\frac{\cos\frac{1}{2}\sqrt{t^2-|\boldsymbol{y}_2|^2}}{\sqrt{t^2-|\boldsymbol{y}_2|^2}}\,d\boldsymbol{y}_2.
\end{aligned}
$$

2.2 解表示 $S_N(t)g$ の分解と L^p-L^q 評価

変形ベッセル関数 I_ν の性質も用いて $S_N(t)g\,(N=1,2,3)$ を解析する．

補題 2.1 変形ベッセル関数 $I_\nu\,(\nu\in\mathbf{N}_0)$ は

$$I_0(0)=1,\ I_1(y)/y\Big|_{y=0}=1/2,\ (I_0(y)-\frac{2}{y}I_1(y))/y^2\Big|_{y=0}=1/8, \tag{2.8}$$

$$I_0'(y)=I_1(y),\quad I_1'(y)=I_0(y)-I_1(y)/y \tag{2.9}$$

を満たし，さらに，$y\to\infty$ のとき次の展開公式を満たす:

$$\begin{aligned}I_\nu(y)=\tfrac{e^y}{\sqrt{2\pi y}}\Big(1-\tfrac{(\nu-1/2)(\nu+1/2)}{2y}+\tfrac{(\nu-1/2)(\nu-3/2)(\nu+3/2)(\nu+1/2)}{2!2^2y^2}\\ -\cdots+(-1)^k\tfrac{(\nu-1/2)\cdots(\nu-(k-1/2))(\nu+(k-1/2))\cdots(\nu+1/2)}{k!2^ky^k}+O(y^{-k-1})\Big).\end{aligned} \tag{2.10}$$

$S_3(t)g$ を極座標表現し，(2.8),(2.9) を用いて変形すると，

$$\begin{aligned}&(S_3(t)g)(x)\\ &=\frac{e^{-t/2}}{4\pi t}\partial_t\int_0^t\iint_{S^2}I_0(\frac{1}{2}\sqrt{t^2-r^2})g(x+r\omega)r^2\,d\omega dr\\ &=e^{-t/2}\frac{t}{4\pi}\iint_{S^2}g(x+t\omega)\,d\omega+\frac{e^{-t/2}}{8\pi}\int_0^t\iint_{S^2}I_1(\frac{1}{2}\sqrt{t^2-r^2})\frac{g(x+r\omega)r^2\,d\omega dr}{\sqrt{t^2-r^2}}.\end{aligned} \tag{2.11}$$

ここで，$\frac{t}{4\pi}\iint_{S^2}g(x+t\omega)\,d\omega$ が $(2.7)_{N=3}$ の 3 次元波動方程式の解 $W_3(t)g$（キルヒホフの公式）であることに注意して，

$$S_N(t)g=e^{-t/2}W_N(t)g+J_{0N}(t)g\quad(N=3) \tag{2.12$_3$}$$

と表現しよう．$N=2,1$ のときも $(2.12)_3$ と同様に

$$\begin{aligned}&(S_2(t)g)(x)\\ &=\frac{e^{-t/2}}{2\pi}\int_{|z|\le t}\frac{g(x+z)\,dz}{\sqrt{t^2-|z|^2}}+\frac{1}{2\pi}\int_{|z|\le t}\frac{\cosh(\frac{1}{2}\sqrt{t^2-|z|^2})-1}{\sqrt{t^2-|z|^2}}g(x+z)\,dz\\ &=:e^{-t/2}(W_2(t)g)(x)+(J_{02}(t)g)(x),\end{aligned} \tag{2.12$_2$}$$

$$\begin{aligned}&(S_1(t)g)(x)\\ &=\frac{e^{-t/2}}{2}\int_{|z|\le t}g(x+z)\,dz+\frac{1}{2}\int_{|z|\le t}\Big(I_0(\frac{1}{2}\sqrt{t^2-|z|^2})-1\Big)g(x+z)\,dz\\ &=:e^{-t/2}(W_1(t)g)(x)+(J_{01}(t)g)(x).\end{aligned} \tag{2.12$_1$}$$

と表現できる．よく知られているように，$W_2(t)g$, $W_1(t)g$ はポアッソンの公式とダランベールの公式である．

さらに，$S_N(t)g$ の t-微分も計算すると，

$$\begin{aligned}\partial_t(S_3(t)g)(x) &= e^{-t/2}\{(-\frac{1}{2}+\frac{t}{8})W_3(t)g+\partial_t(W_3(t)g)\} \\ &\quad + \int_0^t\iint_{S^2}\partial_t\Big[\frac{e^{-t/2}I_1(\frac{1}{2}\sqrt{t^2-r^2})}{8\pi\sqrt{t^2-r^2}}\Big]g(x+r\omega)r^2\,d\omega dr \\ &=: e^{-t/2}(\tilde{W}_3(t)g)(x)+J_{13}(t)g\end{aligned} \tag{2.13$_3$}$$

で，$N=1,2$ のときは

$$\begin{aligned}\partial_t(S_N(t)g) &= e^{-t/2}\{-\frac{1}{2}W_N(t)g+\partial_t(W_N(t)g)\}+\partial_t(J_{0N}(t)g) \\ &=: e^{-t/2}(\tilde{W}_N(t)g)(x)+J_{1N}(t)g\end{aligned} \tag{2.13$_{1,2}$}$$

と書ける．このように分解して書くことの効果は次の評価が成立するからである．

命題 2.1 任意の $p,q(1\le q\le p\le\infty)$ と $g\in L^q$ に対し，$N=1,2,3$ のとき，次の評価が成立する：

$$\|J_{0N}(t)g\|_{L^p}\le C\|g\|_{L^q}(1+t)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})},\quad t\ge 0, \tag{2.14}$$

$$\|(J_{0N}(t)-P_N(t))g\|_{L^p}\le C\|g\|_{L^q}t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-1},\quad t>0, \tag{2.15}$$

$$\|J_{1N}(t)g\|_{L^p}\le C\|g\|_{L^q}(1+t)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-1},\quad t\ge 0. \tag{2.16}$$

この命題から，$J_{0N}(t)g$ は $t\to\infty$ のとき $P_N(t)g$ と同様の挙動をするので，(2.12) は速く減衰する波動部分と拡散部分に分解された:

$$S_N(t)g=\underbrace{e^{-t/2}\cdot W_N(t)g}_{\text{波動部分}}+\underbrace{J_{0N}(t)g}_{\text{拡散部分}}. \tag{2.17}$$

$(\mathrm{DW})_L$ の解 $u(t,x)$ に対しては，(2.4) と (2.12),(2.13) より，

$$\begin{aligned}&u(t,x)\\&=S_N(t)(u_0+u_1)(x)+\partial_t(S_N(t)u_0)(x)\\&=\underbrace{e^{-t/2}\{W_N(t)(u_0+u_1)+\tilde{W}_N(t)u_0\}(x)}_{\text{波動部分}}+\underbrace{J_{0N}(t)(u_0+u_1)(x)+J_{1N}(t)u_0(x)}_{\text{拡散部分}}.\end{aligned} \tag{2.18}$$

と書くことが出来る．命題 2.1 から，$(\mathrm{DW})_L$ と $(\mathrm{H})_L$ の解の“差”は次のように評価できる．

定理 2.1 $N=1,2,3$ かつ $1\le q\le p\le\infty$ とし，$u_0,u_1\in L^q$ とする．もし，

$$\phi_0(x)=(u_0+u_1)(x) \tag{2.19}$$

ならば，$(DW)_L$ の解 u と $(H)_L$ の解 ϕ の差は $t>0$ のとき次の形で評価される：

$$\|(u-\phi)(t,\cdot)-e^{-t/2}\{W_N(t)(u_0+u_1)+\tilde{W}_N(t)u_0\}\|_{L^p}\le C\|u_0,u_1\|_{L^q}t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-1}. \tag{2.20}$$

注意 2.1 (a) 一般次元における $(DW)_L$ の解の評価は松村 [36] によって与えられた：

$$\|u(t,x)\|_{L^p} \le C(1+t)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}(\|u_0,u_1\|_{L^q}+\|u_0\|_{W^{[\frac{N}{2}]+1,2}}+\|u_1\|_{W^{[\frac{N}{2}],2}}), \tag{2.21}$$

ここに，$1 \le q \le 2 \le p \le \infty$ で，$f \in W^{m,q}$ とは $\partial_x^\beta f \in L^q\,(|\beta| \le m)$ である．
(b) Gallay-Raugel [8] は，拡散方程式の解を用いて (DW) の解の漸近展開を得ている．これも (DW) の解の拡散現象を表しているといえる．導出はスケール変換の方法による．

よく知られているように，拡散方程式は平滑化効果(smoothing effect) を持つが，有限伝播性(finite propagation property) は持たない．波動方程式は有限伝播性を持つが，平滑化効果は持たず，微分の損失 (derivative loss) すら起きる．したがって，定理 2.1 は次のように解釈すると面白いかもしれない：

(1) 解の形 (2.5) から消散型波動方程式は 有限伝播性 を持つが，平滑化効果 は持たない．むしろ，解に 特異性(singularity) を生じる．実際，$\tilde{W}_N(t)g$ は t-微分を含み，次元が高ければ，$W_N(t)g$ 自身も t-微分を含むから．しかし，特異性を含めば，波動の性質から錐 (cone) に沿って伝播するが，拡散効果から生じる $e^{-t/2}$ が掛けられているのでその強さは指数的減衰する．

(2) そこで，(2.20) は，特異性を除けば，消散型波動方程式の解の漸近形 (asymptotic profile) は初期値 $\phi_0 = u_0 + u_1$ を持つ拡散方程式の解であることを示している．別の言葉で言えば，初期値が十分な正則性 (regularity) を持てば，消散型波動方程式は $t \to \infty$ のとき拡散方程式と同様の挙動をする．

非線形の問題を扱うには，$t=0$ の近傍の $e^{-t/2}W_N(t)g$ 等の評価も必要である．初期値に適当な正則性があれば簡単に次の定理 2.1 の評価を得る．注意 2.1(a) の松村による評価 (2.21) における正則性の必要性も分かる．松村の証明はフーリエ変換による．フーリエ変換したとき，高周波部分（波動部分に対応する）の評価には初期値の正則性を要するが指数的減衰をし，低周波部分が拡散部分に対応している．

命題 2.2 $N=1,2,3$ とし，$1 \le q \le p \le \infty$ とする．$t \ge 0$ に対し，次の評価が成立する：

$$\|e^{-t/2}(W_N(t)g)(\cdot)\|_{L^p} \le Ce^{-t/4}\|g\|_{L^q},$$

$$\|e^{-t/2}\partial_t(W_N(t)g)(\cdot)\|_{L^p} \le Ce^{-t/4}\|g\|_{W^{[\frac{N}{2}],q}}.$$

本節の内容は，[35, 46, 47, 50, 23] の線形に関する結果をまとめなおしたものである．波動部分と拡散部分に分割する考え方は細野-小川 [16]($N=2$), 楢崎 [42]（一般次元）も詳しい．解表現を用いた L^1-評価については小野 [51, 52, 53] を参照．

$N=3$ のときの命題 2.1 の証明は，$0 < t \le t_0$ のときは容易で，$t \ge t_0$ のときは，(2.15) を示せば，$P_N(t)g$ の評価 (2.2) を使って (2.14) が示される．(2.15) のために，差の積分領

域を3つに分ける:

$$\begin{aligned}&(J_{03}(t)-P_3(t))g\\&=\frac{1}{4\pi t}\left(\int_0^{t^{\frac{1+\varepsilon}{2}}}+\int_{t^{\frac{1+\varepsilon}{2}}}^{t}\right)\iint_{S^2}\left(\frac{e^{-t/2}}{2}I_1(\frac{\sqrt{t^2-r^2}}{2})\frac{t}{\sqrt{t^2-r^2}}-\frac{e^{-r^2/4t}}{\sqrt{4\pi t}}\right)g(x+r\omega)r^2\,dr\,d\omega\\&\quad-\frac{1}{4\pi t}\int_t^{\infty}\iint_{S^2}\frac{e^{-r^2/4t}}{\sqrt{4\pi t}}g(x+r\omega)r^2dr\,d\omega\\&=:(X_1+X_2)+X_3,\quad(0<\varepsilon<1).\end{aligned}$$

X_2, X_3 は指数的減衰が得られ，主要部分である X_1 : $0<r<t^{\frac{1+\varepsilon}{2}}$ $(t\ge t_0)$ のとき，補題 2.1 の展開公式 (2.10) より，

$$\begin{aligned}&\left(\frac{e^{-r^2/4t}}{\sqrt{4\pi t}}\right)^{-1}\cdot\frac{e^{-t/2}}{2}I_1(\frac{\sqrt{t^2-r^2}}{2})\\&=\frac{\sqrt{4\pi t}}{2}\sqrt{\frac{1}{2\pi\cdot\frac{1}{2}\sqrt{t^2-r^2}}}e^{\frac{r^2}{4t}-\frac{t}{2}+\frac{\sqrt{t^2-r^2}}{2}}(1+\frac{1}{t}O(1+\frac{r^2}{t}))\\&=1+\frac{1}{t}O(1+\frac{r^2}{t})\end{aligned}$$

が成立することによる．$(2.13)_3$ で与えられる $J_{13}(t)g$ についても t-微分をして整理し，やはり展開公式 (2.10) を使って (2.16) が得られる．

3 半線形消散型波動方程式

本節では半線形消散型波動方程式のコーシー問題 (DW) を，半線形拡散方程式 (H) と関連させて考える．デュアメルの原理によって，(DW) の解は，積分方程式

$$u(t,\cdot)=S_N(t)(u_0+u_1)+\partial_t(S_N(t)u_0)+\int_0^t S_N(t-\tau)f(u)(\tau,\cdot)\,d\tau \tag{3.1}$$

の解で定義する．解の大域存在と漸近挙動，或いは解の有限時間内の爆発と爆発時間の評価などに関心がある．3.1 節では半線形項が 湧き出し項(sourcing term) として働く

$$f_1(u)=|u|^{\rho-1}u \quad \text{または} \quad f_2(u)=|u|^{\rho},\quad \rho>1 \tag{3.2}$$

を扱い，3.2 節では 吸収項(absorbing term) として働く半線形項

$$f(u)=-|u|^{\rho-1}u,\ \rho>1 \tag{3.3}$$

を扱う．それぞれの場合で解の状況が違ってくる．(H) に対してはそれぞれの場合にかなり詳しい結果も得られており，現在も精力的に研究がなされている．第2節の考察から，(H) に対する結果と同様の結果が (DW) に対しても期待される．しかしながら，(DW) に

対しては満足すべき結果が多く得られているわけではない．よく知られているように，放物型方程式に対しては解の平滑化効果だけでなく，強力な証明道具として最大値原理或いは比較原理が成立する．それに反して，定理 2.1 の解釈でも述べたように，消散型波動方程式には平滑化効果は無く，むしろ，解の特異性が生ずる．そして，最大値原理も波動方程式に対しては一般には成立しない．そこで，(DW) に対する結果の証明を試みる場合，解の正則性に注意しながら，最大値原理に依らない，例えば，L^2-エネルギー法などを活用することになる．

3.1 半線形項が涌出し項として働く場合

まず，涌出し項 (sourcing term) として働く半線型項 $f_1(\phi) = |\phi|^{\rho-1}\phi$ を持つ問題

$$(H)_S \qquad \begin{cases} \phi_t - \Delta\phi = |\phi|^{\rho-1}\phi, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N \\ \phi(0,x) = \phi_0(x), & x \in \mathbf{R}^N \end{cases}$$

に対する結果を思い出そう．藤田 [6] による先駆的な仕事によって，臨界指数

$$(3.4) \qquad \rho_F(N) = 1 + \frac{2}{N} \quad (\text{藤田指数と呼ぶ})$$

があって，指数 ρ の大きさによって解の状況は次のように分類される:

(i) $\rho > \rho_F(N)$（優臨界）のとき，ϕ_0 が小さいならば，$(\mathrm{H})_S$ の一意的時間大域解 $\phi(t,x)$ が存在し，その漸近形は，$\theta_0 G_N(t,x)$ で与えられる．ただし，

$$(3.5) \qquad \theta_0 = \int_{\mathbf{R}^N} \phi_0(x)\,dx + \int_0^t \int_{\mathbf{R}^N} f(\phi)(t,x)\,dx\,dt$$

(ii) $\rho = \rho_F(N)$（臨界）のとき，$\phi_0 \geq 0$, $\phi_0 \neq 0$ ならば，正値時間局所解 $\phi(t,x)$, $t \in [0,T]$ は有限時間内に爆発する，すなわち，ある $T_\infty < \infty$ があって，$T \to T_\infty$ のとき $\phi(T,x) \nearrow \infty$．さらに，$\phi_0$ の代わりに，$\varepsilon\phi_0\,(0 < \varepsilon \ll 1)$ をとるとき，解の存在時間(lifespan)$T_\varepsilon = \sup\{T;\ [0,T]$ で解$\phi(t,x)$ が存在する $\}$ とすると，T_ε は

$$(3.6) \qquad T_\varepsilon \sim e^{C\varepsilon^{-(\rho-1)}} \quad (\varepsilon \to 0+)$$

と評価される．ただし，(3.6) の意味は正定数 C_1, C_2 と ε_1 があって，$e^{C_1 e^{-(\rho-1)}} \leq T_\varepsilon \leq e^{C_2 e^{-(\rho-1)}}\,(0 < \varepsilon \leq \varepsilon_1)$ となること．

(iii) $\rho > \rho_F(N)$（劣臨界）のとき，$\phi_0 \geq 0$, $\phi_0 \neq 0$ ならば，正値時間局所解 $\phi(t,x)$, $t \in [0,T]$ は有限時間内に爆発する．さらに，ϕ_0 の代わりに，$\varepsilon\phi_0\,(0 < \varepsilon \ll 1)$ をとるとき，解の存在時間 T_ε は

$$(3.7) \qquad T_\varepsilon \sim C\varepsilon^{-\frac{1}{\frac{1}{\rho-1}-\frac{N}{2}}} \quad (\varepsilon \to 0+)$$

と評価される．

[6] では劣臨界の場合の爆発が示され，臨界の場合の爆発は早川 [10] による．Weissler [57], Lee-Ni [30] やサーベイ論文 Levine [31], Deng-Levine [3] 等も参照．$f_1(\phi)$ の代わりに $f_2(\phi)=|\rho|^\rho$ に対しても上記の結果は成立するが $f_1(\phi)$ に対して述べられている．最近は爆発の形状なども含めて $(\mathrm{H})_S$ の研究が進展しているがここでは触れず，石毛-溝口による論説 [26] のみを挙げておく．

さて，$f(u)=f_1(u)$ または $f_2(u)$ の場合の半線型消散型波動方程式のコーシー問題

$$(DW)_S \qquad \begin{cases} u_{tt}-\Delta u+u_t=f(u), & (t,x)\in\mathbf{R}_+\times\mathbf{R}^N \ (f=f_i(u),\ i=1\,\mathrm{or}\,2) \\ (u,u_t)(0,x)=(u_0,u_1)(x), & x\in\mathbf{R}^N \end{cases}$$

に対しても，$(\mathrm{H})_S$ に対する上記の結果 (i)-(iii) が成立することが予想される．大域存在は池畠-宮岡-中竹 [25] 等に依るが，$N\geq 3$ では藤田指数とのギャップがあった．それは L^1 ではなく L^2 における評価に依存していたことによる．筆者は [25] に動機付けられ，前節の結果を用いて，優臨界の場合に次の定理を得た．

定理 3.1 (優臨界，大域存在，$N=3$([46])) $f=f_1(u)$ または $f_2(u)$ とする．$\rho>\rho_F(3)$ のとき，$(u_0,u_1)\in Z_0:=(W^{1,1}\cap W^{1,\infty})\times(L^1\cap L^\infty)$ が小さいならば，$(\mathrm{DW})_S$ の弱解すなわち積分方程式 (3.1) の解が一意大域的に存在し，

$$\|u(t,\cdot)\|_{L^p}\leq C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}\|u_0,u_1\|_{Z_0} \quad (1\leq p\leq\infty) \tag{3.9}$$

をみたす．さらに，$1\leq p\leq\infty$ のとき，

$$\|u(t,\cdot)-\theta_0 G_3(t,\cdot)\|_{L^p}=o(t^{-\frac{3}{2}(1-\frac{1}{p})}),\quad t\to\infty \tag{3.9}$$

を満たし，$\theta_0 G_3(t,x)$ が解の漸近形である．ここに，θ_0 は (3.5) の代わりに，次で与えられる：

$$\theta_0=\int_{\mathbf{R}^3}(u_0+u_1)(x)\,dx+\int_0^t\int_{\mathbf{R}^3}f(u)(t,x)\,dxdt. \tag{3.10}$$

一方，臨界，劣臨界の場合の爆発に関しては，$f_2(u)=|u|^\rho$, $N=1,2$ のとき解表示を用いた Li-Zhou による優れた結果 [33] がある．彼らの方法を踏襲して，$N=3$ のときも解は爆発する．

定理 3.2 (臨界と劣臨界，爆発，$N=3$([47])) $\rho\leq\rho_F(3)$ とし，$0\leq\varepsilon\ll 1$ をとり，

$$\begin{aligned} f=f_1(u) &\ \Rightarrow\ \text{初期値は } (0,\varepsilon u_1)(x),\ u_1(x)\geq 0,\ \textstyle\int_{\mathbf{R}^3}u_1(x)\,dx>0 \\ f=f_2(u) &\ \Rightarrow\ \text{初期値は } (\varepsilon u_0,\varepsilon u_1)(x),\ \textstyle\int_{\mathbf{R}^3}(u_0+u_1)(x)\,dx>0 \end{aligned} \tag{3.11}$$

を仮定する．このとき，$(\mathrm{DW})_S$ の時間局所的弱解 $u(t,x)$ は有限時間内に爆発し，解の存在時間 T_ε は $\varepsilon\to 0$ のとき次のように評価される：

$$T_\varepsilon\sim\begin{cases} \exp C\varepsilon^{-(\rho-1)} & \rho=\rho_F(3) \\ C\varepsilon^{-1/(\frac{1}{\rho-1}-\frac{N}{2})} & \rho<\rho_F(3). \end{cases} \tag{3.12}$$

一般次元については，$f_2(u) = |u|^\rho$ に対して，Todorova-Yordanov による優れた結果 [54] がある．$\rho_F(N) < \rho < \frac{N}{N-2}(N = 1, 2$ のときは $\rho_F(N) < \rho < \infty)$ のとき，コンパクトな台を持つ小さなデータ $(u_0, u_1) \in W^{1,2} \times L^2$ に対し時間大域解が存在し，一方，$1 < \rho < \rho_F(N)$ のときは，$\int_{\mathbf{R}^N} u_i(x)\, dx > 0\, (i = 0, 1)$ ならば解は有限時間内に爆発することを示した．この論文では $e^{\psi(t,x)}$ の形の重み関数が有効に使われた．臨界指数の場合の爆発は Zhang [62] によって示されている．優臨界の場合の漸近形については Karch [27], 林-Kaikina-Naumkin [11] を参照．高次元空間における臨界，劣臨界の場合の解の存在時間の評価は未解決．

よく知られているように，積分方程式の解を得るには適当な完備な解空間を導入し，その中で逐次近似列を作り，その列がコーシー列であることを示すのが一般的な方法である．コーシー列であることを示すのに L^p-L^q 評価（命題 2.1-2.2）が使われる．定理 3.1 の場合は解空間 $X_T(T \le \infty)$ を

$$
\begin{aligned}
X_T = \{&u \in C([0,T]; L^1 \cap L^\infty);\\
&\|u\|_{X_T} := \sup_{[0,T]}(\|u(t,\cdot)\|_{L^1} + (1+t)^{3/2}\|u(t,\cdot)\|_{L^\infty}) < \infty\}
\end{aligned}
$$

と定義し，$X = X_\infty$ の中で逐次近似列 $\{u^{(n)}\}$ を

$$
\begin{cases}
u^{(0)}(t,\cdot) = S_3(t)(u_0 + u_1) + \partial_t(S_3(t)u_0)\\
u^{(n+1)}(t,\cdot) = u^{(0)}(t,\cdot) + \int_0^t S_3(t-\tau) f(u^{(n)})(\tau,\cdot)\, d\tau
\end{cases}
$$

で定義する．

(i) $\|u^{(0)}\|_X \le \|u_0, u_1\|_{Z_0}$

(ii) $\|u_0, u_1\|_{Z_0}$が小さいとき，ある $C_0 > 0$ に対し

$$\|u^{(n)}\|_X \le 2C_0\|u_0, u_1\|_{Z_0} \;\Rightarrow\; \|u^{(n+1)}\|_X \le 2C_0$$

(iii) $\|u_0, u_1\|_{Z_0}$が（必要ならばさらに）小さいとき，$\|u^{(n+1)} - u^{(n)}\|_X \le \frac{1}{2}\|u^{(n)} - u^{(n-1)}\|_X$

を示して，X のコーシー列 $\{u^{(n)}\}$ の極限として一意解の存在と解の減衰を得，定理 3.1 が示される．

解の爆発を示すにはいろいろな方法があるが，t に関する下からの常微分不等式を使うのがひとつの方法である．例えば，$\alpha > 0$ として

$$I'(t) \ge cI^{1+\alpha}(t), \quad I(0) = \varepsilon > 0$$

ならば，これを解いて，$I(t) \ge (I^{-\alpha}(0) - \alpha ct)^\alpha$ を得ることから，存在時間の上からの評価 $T_\varepsilon \le \frac{1}{\alpha c}\varepsilon^{-\alpha}$ を得る．しかしながら，我々の問題 $(\mathrm{DW})_S$ は 2 階の方程式なので一般には解は振動し，評価を得るのは簡単ではない．Li-Zhou [33] は有効な一連の補題を示し，$N = 1, 2,\ f_2 = |u|^\rho$ の場合に結果を得た．

補題 3.1 定数 $\alpha, \beta\, (\alpha > 0,\ 0 \le \beta \le 1)$ に対し，関数 $I(t),\ t \ge 0$ が微分不等式

$$I''(t) + I'(t) \ge c_0 \frac{I^{1+\alpha}}{(1+t)^\beta}, \quad I(0) \ge \varepsilon > 0,\ I'(0) \ge 0 \tag{3.13}$$

を満たすならば，$I(t)$ は有限時間内に爆発し，存在時間 T_ε は次のように評価される:

$$T_\varepsilon \le \exp\left(C\varepsilon^{-\alpha}\right)\ (\beta = 1),\quad T_\varepsilon \le C\varepsilon^{-\frac{\alpha}{1-\beta}}\ (0 \le \beta \le 1). \tag{3.14}$$

補題 3.2 補題 3.1 の α, β に対し，関数 $G(t)$ が $t \ge t_0 + 2$, $t_0 \gg 1$ のとき積分不等式

$$G(t) \ge C_1\varepsilon + C_2\left(\int_{t-2}^{t}(t-\tau)\frac{G(\tau)^{1+\alpha}}{\tau^\beta}d\tau + 2\int_{t_0}^{t-2}\frac{G(\tau)^{1+\alpha}}{\tau^\beta}d\tau\right) \tag{3.15}$$

を満たすならば，$G(t)$ は有限時間内に爆発し，存在時間は上から (3.14) で評価される．

u を $(\mathrm{DW})_S$ の局所解とすると，(3.11) から $f_1(u)$ のときは u は正値解であることに注意して，$G(t) = \inf_{|x|\le\sqrt{t}}\{t^{N/2}|u(t,x)|\}$ とおけば $N = 3$ であっても，$\alpha = \rho - 1, \beta = (\rho-1)N/2\,(\le 1)$ に対し (3.15) が得られて定理 3.2 が得られる．

最後に，可積分とは限らない初期データに関する結果に言及をしておく．$(\mathrm{H})_S$ に対しては Lee-Ni [30] による結果から，$\phi_0(x) \sim (1+|x|)^{-kN}\,(0 < k \le 1)$ のとき，新しい臨界指数

$$p_c(k;N) = 1 + \frac{2}{kN}$$

が得られ，優臨界，臨界，劣臨界指数の場合にそれぞれ小さいデータに対する大域存在，爆発と存在時間の評価が得られている．$(\mathrm{DW})_S$ に対しては，池畠-太田 [24], 楢崎-西原 [44] も含めて研究が継続中である．

3.2 半線形項が吸収項として働く場合

半線形項が吸収項 (absorbing term) として働く $f(\phi) = -|\phi|^{\rho-1}\phi$ のとき，

$$(H)_A \qquad \begin{cases} \phi_t - \Delta\phi + |\phi|^{\rho-1}\phi = 0, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N \\ \phi(0,x) = \phi_0(x), & x \in \mathbf{R}^N \end{cases}$$

については多くの仕事がなされてきており，大雑把に言って，任意に大きな非負値データ $(\in L^1)$ に対し，時間大域解を持ち，$t \to \infty$ のときの漸近形は次のように分類される：

(iv) $\rho > \rho_F(N)$ (優臨界) $\Rightarrow$ $\phi(t,x) \sim \theta_0 G(t,x)$
(v) $\rho = \rho_F(N)$ (臨界) $\Rightarrow$ $\phi(t,x) \sim \theta_0(\log t)^{-N/2}G(t,x)$
(vi) $\rho < \rho_F(N)$ (劣臨界) $\Rightarrow$ $\phi(t,x) \sim w_b(t,x) := t^{-\frac{1}{\rho-1}}f_b(t,x)$,

ここに，$w_b(b \ge 0)$ はプロファイル f_b を持つ自己相似解で，f_b は微分方程式

$$-f'' - \left(\frac{r}{2} + \frac{N-1}{r}\right)f' + |f|^{\rho-1}f = \frac{1}{\rho-1}f,\quad \lim_{r\to\infty} r^{\frac{2}{\rho-1}}f(r) = b \tag{3.16}$$

の解である ([9, 28, 4, 5, 7] 等を参照).

同様の結果が

$$(DW)_A \qquad \begin{cases} u_{tt} - \Delta u + u_t + |u|^{\rho-1}u = 0, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N \\ (u, u_t)(0,x) = (u_0, u_1)(x), & x \in \mathbf{R}^N \end{cases}$$

に対しても期待される. 川島-中尾-小野 [29] は, $(u_0, u_1) \in W^{1,2} \times L^2$ に対し, $\rho < 1 + \frac{4}{N-2}$($N = 1,2$ のときは $\rho < \infty$) のとき, 大域解 $u \in C([0,\infty); W^{1,2}) \cap C^1([0,\infty); L^2)$ の存在を示した. さらに $(u_0, u_1) \in L^1$ も仮定すると, $\rho > 1 + \frac{4}{N}$ ならば, 解の減衰 $\|u(t)\|_{L^2} \le C(1+t)^{-N/4}(1 \le N \le 4)$ も示した. この減衰率はガウス核 $G(t,x)$ の L^2-減衰率と同じなので, 優臨界の場合の期待 **(iv)** の正しいことを示唆している. 実際, Karch [27] は $\|(u - \theta_0 G)(t)\|_{L^2} = o(t^{-N/4})$ を示した. 一方, 劣臨界指数の場合の減衰は, Todorova-Yordanov [54] によって開発された重み関数 $e^{\psi(t,x)}$ を応用して, 西原-Zhao [50] は初期データの x 方向の減衰が速い $e^{\beta|x|^2}u_i \in L^2\ (i = 0,1)$ とき解 u は

$$(3.17) \quad \|u(t)\|_{L^p} \le C(1+t)^{-\frac{1}{\rho-1}+\frac{N}{2p}}, \quad 1 < \rho < 1 + \frac{2}{N-2}, \ 1 \le p \begin{cases} \le \infty & (N=1) \\ < \infty & (N=2) \\ \le \frac{2N}{N-2} & (N \ge 3) \end{cases}$$

を得た. 自己相似解 w_b も同じ減衰率 $\|w_b(t)\|_{L^p} = O(t^{-\frac{1}{\rho-1}+\frac{N}{2p}})$ となるので劣臨界の場合は最適の減衰率と思われる. 臨界, 優臨界指数のときも (3.17) は成立するが, $\|G(t)\|_{L^p} = O(t^{-\frac{N}{2}(1-\frac{1}{p})})$ より, 上記の期待 **(iv)**,**(v)** から見れば減衰率が不足している. $w_b(b > 0)$ の x 方向の減衰は $|x|^{2/(\rho-1)}$ なのでこの改良も望まれた. これらは次のように改良されている.

定理 3.3 ([23, 48]) (I) $1 < \rho < 1 + \frac{2}{N-2}(N \le 3)$, $1 < \rho < \infty(N = 1,2)$ かつ $\rho \le 1 + \frac{4}{N}$ とし, $(u_0, u_1) \in H^1 \times L^2$ が $(u_0, \nabla u_0, u_1, |u_0|^{\frac{\rho+1}{2}}) \in L^{2,m}$ をみたすとする. ここに, $m = \frac{2}{\rho-1} - \frac{N-\delta}{2} > 0(\delta > 0)$ であり, $g \in L^{2,m}$ とは $(1+|x|^m)g \in L^2$ のことである. このとき, 大域解 u は (3.17) の p に対し

$$(3.18) \qquad \|u(t)\|_{L^p} \le C(1+t)^{-\frac{1}{\rho-1}+\frac{N}{2p}}$$

をみたす. ただし, $\rho > \rho_F(N)$ のときは $2m > N$ も必要.

(II) さらに, $N \le 3$(resp. $N = 4$) のとき, $(u_0, u_1) \in H^2 \times H^1$, $(\Delta u_0, \nabla u_1) \in L^{2,m}$ ならば, $1 + \frac{2}{N} < \rho \le 1 + \frac{4}{N}$(resp. $1 + \frac{2}{N} < \rho < 1 + \frac{4}{N}$) のとき,

$$(3.19) \qquad \|u(t) - \theta_0 G(t)\|_{L^p} = o(t^{-\frac{N}{2}(1-\frac{1}{p})}), \quad 1 \le p \le \infty$$

が成立する. θ_0 は (3.10) で与えられる.

この定理によって, 自己相似解 w_b から見てほぼ合理的な重みのもとで劣臨界の場合の解の最適な減衰が得られ, 優臨界の場合は, $N \le 4$ では解の漸近形がガウス核の定数倍

となることが得られた．臨界指数の場合の減衰や劣臨界指数の場合の漸近形については何も得られていないが，林-Kaikina-Naumkin による一連の仕事があって，臨界指数や臨界に近い劣臨界指数の場合に漸近形まで含めた結果がある ([12, 13, 14] および [15])．期待 **(iv)**–**(vi)** とのギャップは依然としてあり，それらは未解決として残されている．

定理 3.3 の (I) については，[54] のように，重み関数 $2e^{2\psi(t,x)}u_t$ を $(\mathrm{DW})_A$ に掛けて，発散形

$$\frac{\partial}{\partial t}\Big[e^{2\psi}\left((u_t^2+|\nabla u|^2)+\frac{2}{\rho+1}|u|^{\rho+1}\right)\Big]-\nabla\cdot\left(2e^{2\psi}u_t\nabla\psi\right)$$
$$+2e^{2\psi}\Big[(1-\psi_t+\frac{|\nabla\psi|^2}{\psi_t})u_t^2+\frac{1}{-\psi_t}|\psi_t\nabla u-u_t\nabla\psi|^2+\frac{2(-\psi_t)}{\rho+1}|u|^{\rho+1}\Big]=0$$

にする．このとき最終項の変形がうまく，u_t^2 の係数を見て，

$$\psi_t=\psi_t^2-|\nabla\psi|^2,\quad -\psi_t>0,$$

あるいはもっと単純に，$-\psi_t>0$ かつ $|\nabla\psi|^2/(-\psi_t)\ll 1$ となる ψ が取れれば重み付き L^2-評価が得られる．実際，[54] では $\psi(t,x)\geq|x|^2/4(t+K)$ と選んでいる．ψ の選択には自由性もあるので工夫をすれば改良が可能であろう．[50] では $\psi(t,x)=a|x|^2/4(t+t_0)$ $(0<a\ll 1)$ と選んで，吸収項から出る good term を利用して重み付き L^2-評価の減衰まで得た．さらに，[23, 48] では $\psi=\frac{m}{2}\ln\left(1+\frac{a|x|^2}{t+t_0}\right)$ と選んで減衰を得た．初期データの x 方向の重みが改良されている．(II) の優臨界の場合には，$\|G(t)\|_{L^1}=1$ から $\|u(t)\|_{L^1}$ の有界性を得るのが必要である．$(\mathrm{DW})_A$ で速く減衰するはずの u_{tt} を外力とみなすと，u の正則性と $\|u_{tt}(t)\|_{L^1}$ の可積分性を要する．林らの仕事においてもこのことが大切となっている．そのために，不足する評価 (3.18) からスタートして，解表示 (3.1) に L^p-L^q 評価 (命題 2.1-2.2) を応用し $\|u(t)\|_{L^\infty}$ の評価を求め，高階のエネルギー評価に適用して改良を繰り返す．$\|u(t)\|_{L^\infty}$ の評価は拡散方程式と違って簡単には行かず，$N=3$ を越えて $N=4$ までは適用できるが，$N\geq 5$ のときは難しそうである．

4 関連の問題

関連の問題を述べてこの小論を終わりたい．本稿では解表示を主に用いてきたので変数係数には対応できない．消散項に t の関数が係数としてつく

$$u_{tt}-\Delta u+a(t)u_t=0$$

場合には，フーリエ変換が有効で [58, 59, 60] を参照．抽象的な問題設定では [19, 22, 1]$(a(t)=1)$ や [61] の結果がある．もちろん $a(t)$ の代わりに，$a(x),a(t,x)$ の場合も問題である．また，非線形の消散項を持つ

$$u_{tt}-\Delta u+a(t,x)|u_t|^\sigma u_t=0$$

の場合，解の存在も問題でさらにσの大きさによって拡散現象が生じるかどうかも問題となろう ([37, 55, 40, 38, 21] 等)．これらについて，半空間や外部領域における初期値境界値問題の解の時間大域的挙動も興味のある問題となろう（[21, 39, 41, 20, 43, 49] 等を参照）．

筆者の動機となったモデル1.2に関連して，Jin-Xin モデル（最も簡単な緩和モデルの例）

$$v_t + u_x = 0, \quad u_t + v_x = f(v) - u$$

に対しては，$\bar{v}_t + \bar{u}_x = 0$, $\bar{v}_x = f(\bar{v}) - \bar{u}$ の解に漸近することが予想される．これは移流項を持つ1次元消散型波動方程式

$$v_{tt} - v_{xx} + f(v)_x + v_t = 0$$

の解が対応するバーガース方程式 $\bar{v}_t + f(\bar{v})_x = \bar{v}_{xx}$ の解に漸近することに対応する（[56] 等を参照)．最後に，多孔質中の圧縮性流について，モデル1.2では真空を考えていない．真空を考えるとオイラー座標でモデル方程式

$$\rho_t + (\rho u)_x = 0, \quad (\rho u)_t + (\rho u^2)_x + p(\rho)_x = -\alpha \rho u, \quad (\rho, u)(0, x) = (\rho_0, u_0)(x), \ \rho_0 \geq 0$$

を考える必要がある．$p(\rho) = \rho^\gamma$ が典型例で，$t \to \infty$ のときはやはりダルシーの法則から，$\bar{\rho}_t - \frac{1}{\alpha}(\bar{\rho}^\gamma)_{xx} = 0$, $p(\bar{\rho})_x = -\alpha \bar{\rho} \bar{u}$ と言うまさにポーラスメディア方程式が出て，Barenblatt 解に漸近するかという問題となる ([34, 18] 等を参照）．

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