Dividing cooperative surplus: axiomatic and non-cooperative approaches

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The list of notations

N: a set of players

v: a characteristic function

(N, v): a cooperative game

G: the set of all the games

θ: a permutation on N or $S \subseteq N$.

φ: a generic symbol for a solution or a value of cooperative games

Sh: the Shapley value

$\mathcal{C} = \{C_1, \ldots, C_m\}$: coalition structure

M: the set of indices in the coalition in $\mathcal{C}$.

$(N, v, \mathcal{C})$: a cooperative game with a coalition structure

GC: the set of all the games with coalition structures

$(M, v_{\mathcal{C}})$: the external game of $(N, v, \mathcal{C})$

ψ: a generic symbol for a solution of cooperative games with coalition structures

CV: the coalitional value

Sh^w: the weighted Shapley value

CV^ω: the weighed coalitional value

EV: the Egalitarian value

EV^w: the weighted Egalitarian value

ψ^δ: the Shapley-Egalitarian solution

$(N, v^{\mathcal{C}})$: the $\mathcal{C}$-communication restricted game of $(N, v, \mathcal{C})$

ψ^γ: the collective value

P: the Hart and Mas-Colell potential function

P^w: the Hart and Mas-Colell weighted potential function

$X = \{x, y, z, \ldots\}$: the set of social alternatives

$\mathcal{H} = (H_1, \ldots, H_\ell)$: a hierarchic structure

$(N, v, \mathcal{H})$: a game with a hierarchic structure

HV^w: the weighted value for a game with a hierarchic structure

$\mathcal{M} = (\mathcal{C}^1, \mathcal{C}^2, \ldots, \mathcal{C}^\ell)$: a social structure

$(N, v, \mathcal{M})$: a game with a social structure

Υ^{ω}: the weighted value for a game with a social structure

SBM^{ω}: the weighted social bidding mechanism

CBM^{δ}: the δ-coalitional bidding mechanism

CBM^{γ}: the γ-coalitional bidding mechanism

Contents

Chapter 1

Introduction and overview

1.1 Introduction to this thesis

One of the fundamental problems of social economy is how members in a society distribute surplus obtained by cooperation among themselves. For example, consider a firm as a central agent of the economic activity. On the one hand, the profit sharing between employees and employers in the firms are major concerns for both parties. On the other hand, when multiple firms jointly invest to collaborative project, the agreement on how they divide benefit of the project and share the cost of project is absolutely imperative to the achievement of the project.

In their seminal work, von Neumann and Morgenstern (1953) analyze this problem from a strategic viewpoint according to which members in a society arrive at an agreement regarding the distribution of their surplus by considering the possibilities available to submembers of the society. Since von Neumann and Morgenstern, this topic has been studied by researchers in cooperative game theory. In cooperative game theory, a member who has a concern in the distribution of surplus is called a player, and a rule that prescribes how players distribute their cooperative surplus is called a solution concept. In the last fifty years, several solution concepts, such as the Shapley value, the nucleolus, the core, the (pre)kernel, the bargaining set, and so on, have been proposed and investigated, and this field of research is still growing.

Besides their definitions, there are two major ways that support and justify solution concepts, and these two are two dominant streams in a solution theory. One is the axiomatic characterization of solutions. This approach has its origin in mathematics, which often uses it to characterize mathematical objects or concepts by several properties and make known a relationship among these concepts. In this approach, a list of appealing properties, called axioms, which a desirable solution is expected to satisfy, is proposed. Then, it is shown that the solution that is currently being dealt with is a unique solution that satisfies these properties. For example, the Shapley value is axiomatized by Shapley himself (Shapley 1953b) and several other authors; the nucleolus by Sobolev (1975) (see also Peleg and Sudhölter 2003); the core by Peleg (1986) and Tadenuma (1992); the prekernel by Peleg (1986); and so on.

The other approach is to explore non-cooperative foundations of solutions or implementations of solutions through a non-cooperative game. Specifically, this approach considers a bargaining model that is described by a non-cooperative game, and rational players obtain the payoff from this game which is the same as the one prescribed by the solution, i.e., in a subgame perfect equilibrium (SPE) of this game, each player obtains the payoff prescribed by the solution. This approach was first conducted by Nash (1953), who adopted it in order to justify his bargaining solution in a two-person bargaining problem, and now it is called (in a broad sense) a

Nash program.

By a detailed classification, the research pursuing the second approach can be subdivided into two categories. One is a body of research that considers natural or reasonable bargaining processes, where rational players who follow these bargaining processes arrive at the payoff distribution prescribed by solutions. Another is a body of research that considers decentralized mechanisms that implement solutions for a social planner who would like to impose on players the payoff distribution prescribed by solutions but does not have enough information to calculate it. The former is seen as a non-cooperative foundation of solutions, or a Nash program in a narrow sense, and Nash's original work fits into this category. The latter is seen as an implementation of solutions. To objectify this classification, let us consider the works that study the Shapley value from the second approach. Much of the research of the 90s such as Gul (1989), Hart and Mas-Colell (1992, 1996), Winter (1994), Evans (1996) and Dasgupta and Chiu (1998) belongs to the non-cooperative foundations of the Shapley value not the implementation of the Shapley value. This is because these works consider natural and reasonable bargaining processes that lead to the Shapley value. However, there is a reason that their bargaining models cannot be seen as a mechanism that implements the Shapley value. Their models have a few but critical insufficiencies in the mechanism that implements the Shapley value. First, there bargaining models work only in very restricted domains. For example, the model of Gul (1989) requires value-additivity of a characteristic function, and Winter (1994) requires that a characteristic function is convex. Second, their models achieve the Shapley value only in some SPE, not in all SPE of the games. Third, their models achieve the Shapley value only in an expected value, not in the realized value. This implies that a social planner recognizes that before the beginning of the bargaining, these bargaining models lead to the Shapley value payoff but after the game is over the payoffs of the players are very different from the one that the social planner would impose. Of course, the desired mechanism must overcome this critique. A seminal work by Pérez-Castrillo and Wettstein (2001) introduces a mechanism that implements the Shapley value. Their mechanism works in a large domain (any zero-monotonic environment), leads to the Shapley value payoff in all SPE, and achieve the Shapley value as the realized value.

In this thesis, a general framework—a cooperative game with a cooperation structure—is investigated. A cooperation structure may represent trade unions of countries, business combination of firms, parties in the political arena, or friendship of members, and is mathematically represented by a partition of players. In this framework, the study on the solution provides us information on how such structure affects on the payoff distribution of the players. Moreover, the study on the solutions can establish the basis on the study of what structure of coalitions can be realized after coalition formation process of players. In this thesis, the latter point is not seriously analyzed, but I believe that my finding in the thesis will help to obtain some insight on the cooperation structure resulting after endogenous coalition formation.

A cooperative game with a coalition structure introduced by Aumann and Dreze (1974) is a major class of this framework. In a cooperative game with a coalition structure, a solution concept is defined in the same manner as a cooperative game. Thus, solution concepts in a cooperative game are extended to a cooperative game with a coalition structure. There are two famous extensions of the Shapley value: the Aumann-Dreze value (Aumann and Dreze 1974) and the Owen's coalitional value (Owen 1977). Solution concepts in a cooperative game with a coalition structure are also studied from two major approaches discussed before.

In this thesis, we provide several contributions to a solution theory in a cooperative game or a cooperative game with a cooperation structure. These contributions are summarized as follows and each contribution corresponds to one chapter of the thesis:

- The Shapley value and other solutions in a cooperative game are axiomatized in a systematic way (Chapter 2);
- New solution concepts in a cooperative game with a coalition structure are introduced and characterized in several ways (Chapters 3 and 4);
- The mechanism considered by Pérez-Castrillo and Wettstein (2001) is applied to a collective choice problem in a quasi-utility environment and its fundamental properties are analyzed (Chapter 5);
- A cooperative game with a social structure is introduced. The mechanism considered by Pérez-Castrillo and Wettstein (2001) is applied to this class of games (Chapter 6); and
- New solution concepts introduced in Chapter 3 and 4 are implemented (Chapter 7).

An overview of each section of the thesis is explained in the next sections.
Finally, we refer to the papers that constitute this thesis.

Chapter 2, "Axiomatization of the values using the balanced cycle contributions property," mimeo, (with Takumi Kongo), 2008.

Chapter 3, "A two step Shapley value of games with coalition structures," to be appeared in International Game Theory Review.

Chapter 4, "The collective value: new solution concept of games with coalition structure," mimeo, 2008.

Chapter 5, "Bidding for social alternatives: a simple one-shot mechanism and its extension," mimeo, (with Kohei Kamaga), 2007.

Chapter 6, "Implementation of weighed values in hierarchical and horizontal cooperation structures," Mathematical Social Sciences, 56, 336-349, 2008.
"An implementation of the Owen's coalitional value: Another approach," Waseda Economics Studies, 2007 (in Japanese).

Chapter 7, "Implementation of the Shapley value of games with coalition structures," The Waseda Journal of Political Science and Economics 363, 105-125, 2006.

1.2 Overview of Part I

Part I of this thesis is devoted to axiomatic characterizations of solution concepts in a cooperative game theory. In Chapter 2, new axiomatization of solutions in a cooperative game are provided. To obtain the axiomatization results, we use one key axiom on a fairness criterion in a centralized society in order to characterize solutions. In Chapters 3 and 4, a cooperative game with a coalition structure is investigated. Two new solution concepts in a cooperative game with a coalition structure are introduced. These two solutions can be seen as extensions of the Shapley value to a cooperative game with a coalition structure and are essentially different from the Aumann-Dreze value and the Owen's coalitional value.

In Chapter 2, we provide an axiomatization of the Shapley value that is different from several other works on the axiomatization of the Shapley value (for example, Shapley 1953b, Myerson

1980, Young 1985, Hart and Mas-Colell 1989). The key axiom is the balanced cycle contributions property (BCC), which is a weaker condition than the balanced contributions property (BC) introduced by Myerson (1980). The BC requires that, for any pair of players, the claim from one player against another is balanced with the counter claim of the second player against the first. In contrast, the BCC requires not that claims between two players cannot be balanced, but that claims among all players are balanced in a cyclical manner, i.e. for any order of players, the sum of the claims from each player against his predecessor is balanced with the sum of the claims from each player against his successor. We show that the BCC together with the efficiency and axiom related to a null player axiomatize the Shapley value. One advantage of the BCC over the BC is that several solutions other than the Shapley value such as the Egalitarian value, the CIS value, the ENSC value, and their convex combination satisfies the BCC since the BCC is weaker than the BC. So we show that the Egalitarian value and the CIS value are also axiomatized by the BCC and the other axioms.

In Chapters 3 and 4, solutions in a cooperative game with a coalition structure are studied. Two new solution concepts in cooperative games with coalition structures are introduced and axiomatized. In Chapter 3, we first provide new axioms on a null player and symmetric players in a cooperative game with a coalition structure. These two new axioms are introduced in the spirit that, once a coalition is formed, this coalition has some property related to mutual aid among the members in the coalition. A weaker version of the null player axiom requires that even a null player can obtain some portion of a bargaining surplus if a coalition that he belongs to generates it. The new axiom on symmetric players is related to an *equity* criterion applied to members in the coalitions. This requires that two distinct players should be treated equally, i.e., receive the same amount, if these two are judged to be in an equal position in their *internal* coalition. Neither of the two traditional solutions, the Aumann-Dreze value and the Owen's coalitional value, satisfy these two axioms. We show that these two axioms with the usual three axioms (Efficiency, Additivity, and Coalitional Symmetry) lead to a unique solution concept which is also considered to be an extension of the Shapley value in a cooperative game with a coalition structure. This solution, named the Shapley-Egalitarian solution, is interpreted as an allocation of the cooperative surplus by using the Shapley value in two-step bargaining process: inter-coalition bargaining and intra-coalition bargaining. Moreover, the bargaining surplus of the coalition is allocated among the intra-coalition members in an egalitarian way. In the first step, each coalition obtains its Shapley value applied for a game among coalitions. The pure surplus of a coalition in the first-step bargaining (its Shapley value obtained from the first step minus the worth of the coalition) is divided equally among the players in the coalition. In the second step, players in the coalition receive their Shapley value applied to their own internal game. Thus, the Shapley-Egalitarian solution gives the sum of the payoffs in the first and the second steps to each player.

In Chapter 4, a new solution that has a similar formula to the Shapley-Egalitarian solution is considered. A critical difference is that in this solution, the asymmetric sizes of coalitions is seen as a factor affecting the bargaining outcome. The definition of our new solution concept, named the *collective value*, is also established by relying on a two-step bargaining process among players. In the first step, each coalition obtains its *weighted* Shapley value applied for a game among coalitions. The pure surplus of a coalition in the first-step bargaining (its weighted Shapley value obtained from the first step minus the worth of the coalition) is divided equally among players in the coalition. In the second step, players in the coalition receive their Shapley value applied for their own internal game. Thus, the collective value gives the sum of the payoffs in the first and the second steps to each player. On the surface, our solution concept appears to lie in a very different line of research from existing studies. However, the collective value matches endogenous and exogenous interpretations of coalition structures. Further, we explore a potential function for a cooperative game with a coalition structure, which is quite different from the one of Winter (1992). The collective value is expressed as the marginal contribution relative to this potential function. The potential function behind the solution concept inspires one of its properties similar to the balanced contributions of the Shapley value. We show that this property, called the *collective balanced contributions*, with some moderate additional conditions characterizes our solution. An axiomatization by the *additivity* axiom is also presented.

1.3 Overview of Part II

In Part II of this thesis, non-cooperative foundations or implementations of the solutions considered in Part I are explored. All the mechanism introduced in Part II are based on the bidding mechanism of Pérez-Castrillo and Wettstein (2001) because it has useful property explained earlier.

The bidding mechanism of Pérez-Castrillo and Wettstein (2001) (hereafter, we call it PW-bidding mechanism) is a non-cooperative game that consists of finitely repetition of the following three bargaining stages. In the first stage of the mechanism, all players participate in the bidding game. Each player simultaneously reveals payable bids to each of the other players in exchange for becoming a proposer in the subsequent stages 2 and 3. Then, the net bid (the sum of the bids the player pays to the other individuals minus the sum of the bids paid to the individual) is calculated for each player and a player with the highest net bid is the winner of the bidding game and becomes the proposer in the next stages with actual payment of his bids to the other players. In stage 2, the proposer makes an offer $x_j \in \mathbb{R}$ to any other j, and the responders sequentially decide to accept or reject the offer. In the case of acceptance by all the responders, the proposer actually pays her/his offer to the responders in return for obtaining the worth of total cooperation. On the other hand, if any j rejects the offer, the proposer leaves the bargaining with the worth of her/his stand-alone coalition, and the other players continue the same bargaining process for $n-1$ players. In a setting where there exists only one player, he obtains the worth of his stand-alone coalition. The PW-bidding mechanism is well-defined because the number of players is finite.

In this thesis, a non-cooperative game that consists of several stages and in the first stage of which players participate in the (weighted) bidding game is generically called a bidding mechanism. In Chapter 5, we consider a bidding mechanism applied to the collective choice problem and show that it has a useful property for selecting a socially efficient alternative. In Chapter 6, a bidding mechanism is considered in a situation where players are subdivided horizontally and vertically into coalitions. In Chapter 7, two bidding mechanism for a cooperative game with a coalition structure are introduced, and it is shown that these two implement the Shapley-Egalitarian solution and the collective value, respectively.

The purpose of Chapter 5 is to examine possible applications of the bidding mechanism in the context of the collective choice problem of social alternatives. Examples include the location of public facilities such as a public school, a disposal center, and nuclear-related equipment. We consider a standard quasi-linear environment with finite social alternatives that affect all of individuals in a society. We propose two different bidding mechanisms: a simple one-stage mechanism and a modified multi-stage mechanism. In a simple one-stage mechanism, all the individuals in the society participate in the bidding game. After an appropriate transfer of the bids, the winner of the bidding game chooses his most preferred social alternative. Our first result shows that this mechanism works only in limited situations where the individuals' most favorite alternatives generate the same sums of individuals' utilities. Moreover, we show that, even in cases where the mechanism can work successfully, it may fail to realize a socially efficient alternative as an equilibrium outcome. Next, we provide a modified three-stage mechanism. The first stage of this modified mechanism is the same as the simple one-stage mechanism. In contrast to the one-stage mechanism, after a proposer is determined, the modified mechanism leads to the second stage where the proposer offers a social alternative that he wants to realize and monetary transfers to the other individuals. Then, in the last stage, individuals other than the proposer sequentially reply "to accept the offer" or "to reject it." If the offer is unanimously accepted, the monetary transfers are carried out and the proposed social alternative is realized. In the case where the offer is rejected by at least one individual, the monetary transfers are not carried out and the proposer's most favorite alternative is realized as a social decision. We prove that this multi-stage bidding mechanism can work in any situation and always realizes a socially efficient alternative.

We obtain from the discussion in Chapter 5 the fundamental property of the bidding mechanism. The necessary and sufficient condition for the existence of equilibrium in the bidding game is that the sum of the payoffs of all the individuals obtained from the subgame after the bidding game is irrelevant to who is the winner of the bidding game. This condition seems at first glance to be hard to be satisfied but if we think back on the suggestion of the Coase Theorem, it is not difficult for us to consider a situation that satisfies this condition, because we only consider any type of bargaining after the determination of the winner of the bidding game. Moreover, we also demonstrate that the final payoff of each player in the bidding mechanism is the same as his expected payoff in a non-cooperative game where only the bidding game in the bidding mechanism is replaced by a stage in which one player is randomly selected as the winner and other stages are unchanged. Moreover, if we consider the weighted bidding game, any expected payoff obtained in any random selection of the winner is achieved as the actual value in the bidding game. This result is significant in the following two ways. First, it suggests that any mechanism that has a stage where one player is chosen by some random procedure can be replaced by a mechanism where the random selection stage is now replaced by the corresponding weighted bidding game. This bidding mechanism achieves the same SPE payoff as the former mechanism as a realized value. Thus, if we find a non-cooperative game that achieves some desirable payoff or solution at an expected value (and randomness is used to select one player from all the players), we can construct the corresponding bidding mechanism that implements the desirable payoff at a realized value. Second, assuming that the random selection of one player from all the players is considered to be more natural and reasonable than the bidding game, any bidding mechanism can be replaced by the non-cooperative game in which the (weighted) bidding game is replaced by random selection of the winner. Thus, if we find the bidding mechanism that implements some solution, we also find the non-cooperative foundation of the solution. Therefore, finding a bidding mechanism that leads to the payoff prescribed by some desirable solution is enough for both the implementation and the non-cooperative foundation of the solution.

In Chapter 6, a bidding mechanism is applied to a cooperative game with a social structure. A *cooperative game with a social structure* is a unified model, in which there exist both the hierarchical and horizontal coalition structures. We define a weighted value for these games and this value is a generalization of the Shapley value to such a game, and thus it coincides with the Shapley value, the weighted Shapley value with hierarchic structure, the coalitional value, and the weighted coalitional value in some special cases. Then, in order to achieve this value through the decentralized decision-making process, we consider a non-cooperative bargaining model. We propose a bargaining model called the *weighted social bidding mechanism*. In this bargaining, only the players in the highest ranked coalitions participate in the bidding stage to choose a proposer in the next stage. Then, the proposer determined in the previous stage offers a payoff distribution among all the players, and the players other than the proposer sequentially decide to either accept or reject the offer. In the case of acceptance by all players, the bargaining is over and their payoff is distributed according to the proposal. On the other hand, when there is a player who rejects the offer, the proposer leaves the bargaining and the remaining players continue the same bargaining process with the proposing coalition, i.e., the coalition that the proposer belongs to, retaining the right to choose a new proposer. We demonstrate that our mechanism works in any *zero-monotonic* environment. That is, if the underlying cooperative game is zero-monotonic, in any SPE of the weighted social bidding mechanism, the equilibrium payoff vector coincides with the value defined in this chapter. Our mechanism implements the Shapley value, the weighted Shapley value of Kalai and Samet (1987), the coalitional value, and the weighted coalitional value in some special cases in any zero monotonic environment.

In Chapter 7, two types of extension of the PW-bidding mechanism to a cooperative game with a coalition structure are introduced. These two games are different in the fists stage of the mechanism. In one mechanism (γ-coalitional bidding mechanism), each player's bids are treated equally but in another mechanism (δ-coalitional bidding mechanism), the player's bids are asymmetrically treated with respect to the size of the coalition that they belong to. In the first stage of the mechanisms, all the players participate in the bidding game. In the next stage, the proposer makes an offer to all the players, and the other players respond to the offer sequentially. In the case of acceptance by all players, the proposer pays her/his offer to any player in return for obtaining the value of their cooperation and the bargaining is over. On the other hand, when some player rejects the offer, (i) players in the proposing coalition which is a coalition that the proposer belongs to, participate in the bidding mechanism of PW-bidding mechanism for themselves, and (ii) the remaining players except for the members in the proposing coalition continue the same bargaining. So, in contrast with the social bidding mechanism in Chapter 6, the proposing coalition does not retain the right to choose a new proposer. Rather, the proposing coalition has risks being separated from players in other coalitions in exchange for a player in this coalition being a proposer. We show that the δ-coalitional bidding mechanism implements the Shapley-Egalitarian solution in any superadditive game and the γ-coalitional bidding mechanism implements the collective value in any superadditive game

1.4 Preliminary

1.4.1 A cooperative game and the Shapley value

A *cooperative game* or a simply *game* is a pair (N, v) where N is a finite set of n players and $v : 2^N \to \mathbb{R}$ is a characteristic function with $v(\emptyset) = 0$. A subset S of N is called a *coalition* and $v(S)$ is the *worth* of coalition S. The set of all the games is denoted by **G**. We use the short-cut notations of $S - i$ and $S \cup i$ instead of $S \setminus \{i\}$ and $S \cup \{i\}$ respectively for convenience. Given $(N, v) \in$ **G** and a coalition S, we denote the subgame of (N, v) to S by (S, v) if there is no risk of confusion.

A game (N, v) is *zero-monotonic* if for any player i and for any coalition $S \subseteq N - i$, $v(S \cup i) \geqq v(S) + v(\{i\})$, and is strictly zero-monotonic if the inequality holds in a strict manner. A game (N, v) is *superadditive* if for any two coalitions S and T with $S \cap T = \emptyset$, $v(S \cup T) \geqq v(S) + v(T)$, and is strictly superadditive if the inequality holds in a strict manner. A superadditive game is, of course, zero-monotonic, but the inverse is not true in general. A game (N, v) is *convex* if for any two coalitions S and T, $v(S \cup T) + v(S \cap T) \geqq v(S) + v(T)$, and is strictly convex if the inequality holds in a strict manner. The sets of all the zero-monotonic games, all the superadditive games, and all the convex games are denoted by $\mathbf{G}^M$, $\mathbf{G}^S$, and $\mathbf{G}^C$, respectively.

Player $i \in N$ is a *null player* if $v(S \cup i) = v(S)$ for any $S \subseteq N - i$ and a *dummy player* if $v(S \cup i) = v(S) + v(\{i\})$ for any $S \subseteq N - i$. Clearly a null player is also dummy but the converse does not hold. It is said that $i \in N$ and $j \in N$ are *symmetric* in (N, v) if $v(S \cup i) = v(S \cup j)$ for any $S \subseteq N \setminus \{i, j\}$ and $i \in N$ and $j \in$ are symmetric in (T, v), $T \subseteq N$, if $v(S \cup i) = v(S \cup j)$ for any $S \subseteq T \setminus \{i, j\}$.

Assuming that the grand coalition N will be formed, the question arises how to divide the worth $v(N)$ among the players. Thus, a solution of a game, which is also called a value of a game, is a function φ which assigns to every game $(N, v) \in \mathbf{G}$ a payoff vector $\varphi(N, v) = (\varphi_i(N, v))_{i \in N} \in \mathbb{R}^N$ that satisfies $\sum_{i \in N} \varphi_i(N, v) \leq v(N)$. If φ always distributes just $v(N)$ to the players, it is called an efficient solution.

A well-known solution was presented by Shapley (1953b). Let $\theta : N \to N$ denote a permutation on N and $\Theta(N)$ denote a set of all the permutations on N. A permutation θ is identified as an order $(i_1, \ldots, i_n)$ on N if $\theta(j) = k$ implies $i_k = j$, and *vice versa*. A set of players preceding to i at order θ is $A_i^\theta = \{j \in N \,:\, \theta(j) < \theta(i)\}$. A marginal contribution of player i at order θ in (N, v) is defined by $m_i^\theta(N, v) = v(A_i^\theta \cup i) - v(A_i^\theta)$. The Shapley value Sh of (N, v) is defined as follows:

$$\mathrm{Sh}_i(N, v) = \frac{1}{|\Theta(N)|} \sum_{\theta \in \Theta(N)} m_i^\theta(N, v), \text{ for all } i \in N,$$

where $|\cdot|$ represents the cardinality of the set. Thus, the Shapley value is an average of marginal contribution vectors where each order $\theta \in \Theta(N)$ occurs in an equal probability, that is, $1/|\Theta(N)|$.

The Shapley value is characterized by the four properties: (i) efficiency, (ii) additivity, (iii) symmetry and (iv) null player. Let φ be a solution on **G**. The efficiency requires that the solution distributes the worth of the grand coalition to the players. The additivity is that for any two games (N, v) and (N, v'), $\varphi(N, v + v') = \varphi(N, v) + \varphi(N, v')$ holds where the additive game $v + v'$ is defined by $(v + v')(S) = v(S) + v'(S)$ for any $S \subseteq N$. The symmetry says that two symmetric players in (N, v) receive the equal payoffs, thus, $\varphi_i(N, v) = \varphi_j(N, v)$ holds whenever i and j are symmetric in (N, v). The null players axiom is that the null player always obtains nothing.

1.4.2 A cooperative game with a coalition structure and the coalitional value

In various applications of cooperative games, it seems to be natural that players partitions themselves into some 'coalitions' such as labor union, syndicate of firms, customs unions in international economics, and so on. Such coalitions form a coalition structure $\mathcal{C} = \{C_1, \ldots, C_m\}$, which is partition of N, *i.e.*, it holds that $C_k \cap C_h = \emptyset$ for any k and any h with $k \neq h$ and $\bigcup_{k=1}^m C_k = N$. Such a situation, called a cooperative game with a coalition structure, is first systematically considered by Aumann and Dreze (1974) and developed by a number of authors. A counterpart of the Shapley value for such games was defined by Owen (1977) and given the axiomatic foundation from the viewpoint of coalition formation by Hart and Kurz (1983).

A *game with a coalition structure* is a triple $(N, v, \mathcal{C})$ where (N, v) is a game and $\mathcal{C} = \{C_1, \ldots, C_m\}$ is a coalition structure. We usually use notation $M = \{1, \ldots, m\}$ to denote the set of coalitional indices in $\mathcal{C}$. The set of all the games with coalition structures is denoted by **GC**. An order $\theta \in \Theta(N)$ is *consistent* with $\mathcal{C}$ if for any $i \in C_h \in \mathcal{C}$ and $j \in C_h \in \mathcal{C}$ and

$k \in N$, $\theta(i) < \theta(k) < \theta(j)$ implies that player k also belongs to coalition C_h, that is, $k \in C_h$. Thus, in the consistent order, players line up in a way that players in the same coalition are side-by-side. A set of all the orders on N consistent with $\mathcal{C}$ is denoted by $\Theta(N, \mathcal{C})$. Then, Owen's (1977) coalitional value CV is an average of player's marginal contributions when all the orders consistent with $\mathcal{C}$ occur with equal probability, being defined by,

$$\mathrm{CV}_i(N, v, \mathcal{C}) = \frac{1}{|\Theta(N, \mathcal{C})|} \sum_{\theta \in \Theta(N, \mathcal{C})} m_i^\theta(N, v), \text{ for each } i \in N.$$

Thus, according to the coalitional value, players in N appear in a way that the players in the same coalition appear successively. In other words, first coalitions enter subsequently in a random order and within each coalition the players enter subsequently in a random order.

An *external game* or a game played by the (representatives of the) coalitions $(M, v_{\mathcal{C}})$ is defined by $M = \{1, \ldots, m\}$ and $v_{\mathcal{C}}(H) = v(\bigcup_{k \in H} C_k)$ for each $H \subseteq M$.[1] For external game $(M, v_{\mathcal{C}})$, the Owen's coalitional value satisfies the following: for any $C_k \in \mathcal{C}$,

$$\sum_{i \in C_k} \mathrm{CV}_i(N, v, \mathcal{C}) = \mathrm{CV}_k(M, v_{\mathcal{C}}, \{M\}).$$

This property is called the intermediate game property. The coalitional value is characterized by the efficiency, the additivity, the null player property, the intermediate game property and the restricted equal treatment property which requires that if two players in $C_k \in \mathcal{C}$ are symmetric in (N, v), the two players should receive the equal payoff (see Owen 1977 and Peleg and Sudhölter 2003). Here, the first three axioms are the ones which are naturally extended to a game with a coalition structure. However, the null player property in this case may be a bit strong requirement because it implies that the null player gets nothing even though the coalition he belongs to is in very strong position. Thus, the coalitional value does not reflect a function of the formed coalition as system of mutual assistance. In Section 6, we provide a weaker version of the null player property in a game with a coalition structure to characterize our new solution, which is defined in the next section.

1.4.3 Non-symmetric generalization of the Shapley value and the coalitional value

Let $w = (w_i)_{i \in N}$ be a positive weight vector for N. We associate with a weight w a probability distribution $p(.; w)$ over $\Theta(N)$ as follows: for $\theta = (i_1, \ldots, i_n)$,

$$p(\theta; w) = \Pi_{j=1}^{n} \frac{w_{i_j}}{\sum_{h=1}^{j} w_{i_h}}. \tag{1.1}$$

This probability distribution can be constructed in such a way that one order is picked by choosing players one by one and placing each of them in turn at the *front* of the partially created line where the probability of choosing the player is the ratio between her/his weight and the total sum of the weights of the players who are not yet in the line. Thus, according to $p(.; w)$, some i becomes the last of the order in probability $\frac{w_i}{\sum_{j \in N} w_j}$.

The w-weighted Shapley value Sh^w for $(N, v) \in \mathbf{G}$ is

$$\mathrm{Sh}_i^w(N, v) = \sum_{\theta \in \Theta(N)} p(\theta; w)\, m_i^\theta(N, v)$$

[1]This game is referred to as an intermediate game in Peleg and Sudhölter (2003) and as a quotient game in Owen (1977).

for any $i \in N$.[2]

If $w_i = w_j$ for any $i \in N$ and for any $j \in N$, the w-weighted Shapley value clearly coincides with the Shapley value. Given coalition T of N, let (N, u_T) denote a T-unanimity game defined by $u_T(S) = 1$ if $S \supseteq T$ and $u_T(S) = 0$ otherwise. It is easily checked that $\mathrm{Sh}_i^w(N, u_T) = \frac{w_i}{\sum_{j \in T} w_j}$ if $i \in T$ and $\mathrm{Sh}_i^w(N, u_T) = 0$ otherwise.

For $\theta \in \Theta(N)$ and $S \subseteq N$, θ_S denotes an order of S such that for $i, j \in S$, $\theta_S(i) < \theta_S(j)$ if and only if $\theta(i) < \theta(j)$. Let $\Pi = \{S_1, \ldots, S_p\}$ be a partition of N, that is, $\cup_{k=1}^p S_k = N$ and $S_k \cap S_h = \emptyset$ for all $k \neq h$. Given a permutation $\theta \in \Theta(N)$ and a partition Π, θ^Π is an order of $\{1, \ldots, p\}$ such that $\theta^\Pi(k) < \theta^\Pi(h)$ if and only if there exists some $i \in S_k$ such that $\theta(i) < \theta(j)$ for all $j \in S_h$.[3]

A permutation $\theta \in \Theta(N)$ is said to be consistent with coalition structure $\mathcal{C}$ if according to the permutation, players in the same coalition in $\mathcal{C}$ appear successively, that is, for any $i \in C_k$, for any $j \in C_k$ and for any $h \in N$, $\theta(i) < \theta(h) < \theta(j)$ implies $h \in C_k$. A set of all the permutations that are consistent with $\mathcal{C}$ is denoted by $\Theta(N, \mathcal{C})$.

For $w = (w_i)_{i \in N}$ and for coalition S, we often use notations $w(S)$ and w_S to denote the summation of w_i over $i \in S$ and a restriction of w to S, respectively.

For a game with a coalition structure, it is possible to relax the symmetry treatment in two different directions: relaxing the intra-coalitional symmetry and relaxing the inter-coalitional symmetry. Given $(N, v, \mathcal{C})$, let $w = (w_i)_{i \in N}$ and $w^* = (w_k^*)_{k \in M}$ be positive weight vectors of the players in N and coalitions in $\mathcal{C}$. Thus, w and w^* represent the intra-coalitional asymmetry and inter-coalitional asymmetry, respectively. $\omega = (w, w^*)$ is a *weight structure* of $\mathcal{C}$. The *weighted value for a game with a coalition structure* or the *weighted coalitional value* (Levy and McLean 1989) is defined as follows:

$$\mathrm{CV}_i^\omega(N, v, \mathcal{C}) = \sum_{\theta \in \Theta(N, \mathcal{C})} p(\theta^{\mathcal{C}}; w^*) \left(\Pi_{k=1}^m p(\theta_{C_k}; w_{C_k})\right) m_i^\theta(N, v)$$

for each $i \in N$, where $w_{C_k} = (w_i)_{i \in C_k}$, and $p(.; w^*)$ and $p(.; w_{C_k})$ are probability distributions defined over $\Theta(M)$ and $\Theta(C_k)$ in a similar fashion to (1.1). Thus, the weighted coalitional value can be seen as an expected value of marginal contribution $m_i^\theta(N, v)$ where coalitions are arranged according to the probability distribution that an order $\theta^{\mathcal{C}} \in \Theta(M)$ occurs at probability $p(\theta^{\mathcal{C}}; w^*)$, and within the coalition, players in C_k are arranged according to the probability distribution that an order $\theta_{C_k} \in \Theta(C_k)$ occurs at probability $p(\theta_{C_k}; w_{C_k})$.

When either $\mathcal{C} = \{N\}$ and $w_i = w_j$ for all $i, j \in N$ or $\mathcal{C} = \{\{i\} : i \in N\}$ and $w_k^* = w_{k'}^*$ for all $k, k' \in M$, CV^ω is equal to the Shapley value. Moreover, it coincides with the Owen's (1977) coalitional value in the case that $w_i = w_j$ for all $i, j \in C_k \in \mathcal{C}$ and $w_k^* = w_{k'}^*$ for all $k, k' \in M$.

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[2] Kalai and Samet (1987) generalized positive weights to a weight system which is a pair of weights and an ordered partition on N in order to allow a weight of zero for some of the players.

[3] If we redefine $\theta^\Pi(k) < \theta^\Pi(h)$ by the condition that there exists some $j \in S_h$ such that $\theta(i) < \theta(j)$ for all $i \in S_k$, the discussion of this paper is unchanged. This is because we use this notation only for the order *consistent* with the partition (see the definition below).

Part I

Axiomatic approach

Chapter 2

Axiomatization of values of cooperative games by balanced cycle contributions property

2.1 Introduction

An important criterion in allocation problems is fairness. In the cooperative game theory, the widely used fairness criterion is the balanced contributions property introduced by Myerson (1980). The property requires that, for any pairs of players, the claim from one player against another is balanced with the counter claim from another against the player. Thus, if a solution satisfies this property, the outcome supported by the solution is fair in the sense that no one has a claim more against another. However, the property is rather strong since together with efficiency, which is also an important criterion in allocation problems, the only solution satisfying the property is the Shapley value.

In this chapter, we provide the fairness property, which is weaker than the balanced contributions property. In the weaker property, claims between two players cannot be balanced; however, claims among all players are balanced in a cyclical manner, i.e. for any order of players, the sum of the claims from each player against his predecessor is balanced with the sum of the claims from each player against his successor. This weaker balanced contributions property is satisfied by several solutions for cooperative games such as the Shapley value, the Egalitarian value, and the CIS value. Together with other basic axioms, the above-mentioned values are axiomatized, and these are our main results.

This chapter is organized as follows. In Section 2, the weaker fairness property is provided in Section 3. Sections 4 and 5 present axiomatizations of the Shapley and Egalitarian values, respectively. In Section 6, our results are generalized to the situations where players are asymmetric. Section 7 concludes the chapter.

2.2 Balanced contributions property

Let φ be a solution on **G**. The *balanced contributions property* (Myerson 1980) is the following. For any $(N, v) \in$ **G** and for any $\{i, j\} \subseteq N$,

$$\varphi_i(N, v) - \varphi_i(N \setminus j, v) = \varphi_j(N, v) - \varphi_j(N \setminus i, v),$$

where $(N \setminus j, v)$ and $(N \setminus i, v)$ are restrictions of (N, v) on $N \setminus i$ and $N \setminus j$, respectively.

An interpretation of the balanced contributions property is as follows. Assume that a value φ is commonly accepted as a distribution rule in the society, and that the claim by i against j is measured by i's contribution to j, i.e. $c(i,j) = \varphi_j(N,v) - \varphi_j(N \setminus i, v)$. The balanced contributions property is interpreted as a condition that claims between any two players are balanced with each other, i.e. $c(i,j) = c(j,i)$.

Myerson (1980) showed that the Shapley value is a unique efficient solution on **G**, satisfying the balanced contributions property.

2.3 Balanced cycle contributions property

Considering a weaker and minimal requirement of one described by the balanced contributions property makes sense from the following two reasons. One is that the balanced contributions property could be a too demanding property if we consider an application of the spirit of the condition to the real society because the population of the modern society is so large and it is hardly expected that the claims are balanced for *all* two individuals. The other is related to a solution theory in cooperative game. Because there is no efficient solution satisfying the balanced contributions property except for the Shapley value, exploring a weaker condition than the balanced contributions property enriches the solution theory.

Let $e(i,j)$ be the excess claim of i to j which is defined by $e(i,j) = c(i,j) - c(j,i)$. Then, the balanced contributions property is interpreted as a condition that for any two individuals i and j, the excess claim of i to j is zero, i.e., $e(i,j) = 0$. In this sense, the balanced contributions property requires that the excess claim should be zero for all two individuals. On the other hand, a weaker and minimal requirement is that the balancedness of the excess claims is attained as a whole of the member in a society. One of the possible expressions is that the sum of the excess claims among the society members is zero. Thus, given a coalition S with $|S| = s$ and an order $(i_1, i_2, \ldots, i_s)$ on S, the following holds:

$$e(i_1,i_2) + e(i_2,i_3) + \cdots + e(i_{n-1},i_s) + e(i_s,i_1) = \sum_{\ell=1}^{s} e(i_\ell, i_{\ell+1}) = 0, \tag{2.1}$$

where $i_{s+1} = i_1$. In fact, this is a weaker requirement than the balancedness of claims for each pair of individuals since if $e(i,j) = 0$ holds for each $\{i,j\} \subseteq N$, the above condition obviously holds. This can be seen as the condition that the sum of the excess claims of all the players is zero or the average of the excess claims among all the players is zero, irrespective of the order of players.

A similarity to the above condition is found in the general equilibrium theory of the standard micro economics. In this study, the excess demands of the individual economic agents are aggregated through the market and the total excess demand which is the sum of the individual demands of the economic agents becomes zero at the market equilibrium.

An order on S, $(i_1, i_2, \ldots, i_s)$, might be determined by some exogenous factor. Otherwise, both the group and the order might be endogenously determined, for example, by the following manner. A player j_1 will choose one of the excess claims, say $e(j_1, j_2)$, according to some judgment such as maximizing excess claims, and so on. Then, the second player, j_2, also chooses the one of his excess claims, say $e(j_2, j_3)$, and this process is continued until some player chooses the claim that come back to some earlier player. As a result, we obtain coalition $S = \{i_1, i_2, \ldots, i_s\}$ whose members can be indexed by the above procedure. The above conditions simply says that after the determination of the coalition and the order, the sum of the excess claims among the members of the coalition according to the order should be zero.

Since $e(i_\ell, i_{\ell+1}) = c(i_\ell, i_{\ell+1}) - c(i_{\ell+1}, i_\ell)$, Eq (2.1) can be reduced to

$$\sum_{\ell=1}^{s} c(i_\ell, i_{\ell+1}) = \sum_{\ell=1}^{s} c(i_{\ell+1}, i_\ell).$$

Thus, given an order $(i_1, i_2, \ldots, i_s)$, the sum of the claims from each player against his predecessor is balanced with the sum of the claims by each player against his successor. The LHS of the above equation is called the cycle contributions with respect to an order $(i_1, i_2, \ldots, i_s)$ and the RHS is the cycle contributions with respect to the inverse order. Thus, Eq. (2.1) can be seen as the balancedness of cycle contributions with respect to the order and its inverse order.

Together with the assumption that a claim by i against j is measured by $\varphi_j(N, v) - \varphi_j(N \setminus i, v)$, we obtain the following axiom:

Group balanced cycle contributions property (GBCC): For any $(N, v) \in$ **G**, any S with $s = |S| \geqq 2$, and any order $(i_1, i_2, \ldots, i_s)$ on S,

$$\sum_{\ell=1}^{s} \left(\varphi_{i_\ell}(N, v) - \varphi_{i_\ell}(N \setminus i_{\ell-1}, v)\right) = \sum_{\ell=1}^{s} \left(\varphi_{i_\ell}(N, v) - \varphi_{i_\ell}(N \setminus i_{\ell+1}, v)\right),$$

where $i_0 = i_s$ and $i_{s+1} = i_1$.

While GBCC requires the balancedness of cycle contributions for any group, the balancedness of cycle contributions for only the grand coalitions will make sense. Thus, we have the following:

Balanced cycle contributions property (BCC): For any $(N, v) \in$ **G** and any order $(i_1, i_2, \ldots, i_n)$ on N,

$$\sum_{\ell=1}^{n} \left(\varphi_{i_\ell}(N, v) - \varphi_{i_\ell}(N \setminus i_{\ell-1}, v)\right) = \sum_{\ell=1}^{n} \left(\varphi_{i_\ell}(N, v) - \varphi_{i_\ell}(N \setminus i_{\ell+1}, v)\right),$$

where $i_0 = i_n$ and $i_{n+1} = i_1$.

Since the term $\sum_{\ell=1}^{n} \varphi_{i_\ell}(N, v)$ is common to both sides, the condition described in the axiom is reduced to:

$$\sum_{\ell=1}^{n} \varphi_{i_\ell}(N \setminus i_{\ell-1}, v) = \sum_{\ell=1}^{n} \varphi_{i_\ell}(N \setminus i_{\ell+1}, v),$$

where $i_0 = i_n$ and $i_{n+1} = i_1$. This is more convenient representation of BCC.

Note that in a two person game $(\{i, j\}, v)$, the condition required by BCC is automatically satisfied because both the left- and right-hand sides of the equations of the condition are $\varphi_i(i, v) + \varphi_j(j, v)$.

It is obvius that BCC is weaker than GBCC. However, the following proposition shows the equivalence of the two axioms:

Proposition 2.1. *GBCC and BCC are equivalent.*

Proof. Is is enough to show that BCC implies GBCC. Suppose that ψ satisfies BCC. We will show that ψ satisfies the balancedness of the cycle contributions for any coalition S of size s.

Consider the case $s = 3$. When $n = 3$, it is obvious. Thus, we consider the case where $n \geqq 4$. For any $(N, v) \in$ **G** with $n \geq 4$, let $\{i, j, k\} \subseteq N$ and $a = (a_1, a_2, \ldots, a_m)$ be an order on the set $N \backslash \{i, j, k\}$. Hence, (i, j, k, a) is an order on N.

By BCC with respect to an order (i, j, k, a) on N,

$$\begin{aligned}\varphi_i(N\setminus a_m, v) + \varphi_j(N\setminus i, v) + \varphi_k(N\setminus j, v) + \varphi_{a_1}(N\setminus k, v) + \cdots + \varphi_{a_m}(N\setminus a_{m-1}, v)\\ = \varphi_i(N\setminus j, v) + \varphi_j(N\setminus k, v) + \varphi_k(N\setminus a_1, v) + \varphi_{a_1}(N\setminus a_2, v) + \cdots + \varphi_{a_m}(N\setminus i, v).\end{aligned} \tag{2.2}$$

By BCC with respect to an order (i, k, j, a),

$$\begin{aligned}\varphi_i(N\setminus a_m, v) + \varphi_k(N\setminus i, v) + \varphi_j(N\setminus k, v) + \varphi_{a_1}(N\setminus j, v) + \cdots + \varphi_{a_m}(N\setminus a_{m-1}, v)\\ = \varphi_i(N\setminus k, v) + \varphi_k(N\setminus j, v) + \varphi_j(N\setminus a_1, v) + \varphi_{a_1}(N\setminus a_2, v) + \cdots + \varphi_{a_m}(N\setminus i, v).\end{aligned} \tag{2.3}$$

$(1) - (2)$ equals

$$\begin{aligned}\varphi_i(N\setminus k, v) + \varphi_j(N\setminus i, v) + 2\varphi_k(N\setminus j, v) + \varphi_{a_1}(N\setminus k, v) - \varphi_{a_1}(N\setminus j, v)\\ = \varphi_i(N\setminus j, v) + 2\varphi_j(N\setminus k, v) + \varphi_k(N\setminus i, v) + \varphi_k(N\setminus a_1, v) - \varphi_j(N\setminus a_1, v).\end{aligned} \tag{2.4}$$

Similarly, by BCC with respect to two orders (j, k, i, a) and (j, i, k, a), we obtain

$$\begin{aligned}\varphi_j(N\setminus i, v) + \varphi_k(N\setminus j, v) + 2\varphi_i(N\setminus k, v) + \varphi_{a_1}(N\setminus i, v) - \varphi_{a_1}(N\setminus k, v)\\ = \varphi_j(N\setminus k, v) + 2\varphi_k(N\setminus i, v) + \varphi_i(N\setminus j, v) + \varphi_i(N\setminus a_1, v) - \varphi_k(N\setminus a_1, v),\end{aligned} \tag{2.5}$$

and by BCC with respect to two orders (k, i, j, a) and (k, j, i, a), we obtain

$$\begin{aligned}\varphi_k(N\setminus j, v) + \varphi_i(N\setminus k, v) + 2\varphi_j(N\setminus i, v) + \varphi_{a_1}(N\setminus j, v) - \varphi_{a_1}(N\setminus i, v)\\ = \varphi_k(N\setminus i, v) + 2\varphi_i(N\setminus j, v) + \varphi_j(N\setminus k, v) + \varphi_j(N\setminus a_1, v) - \varphi_i(N\setminus a_1, v).\end{aligned} \tag{2.6}$$

$\frac{1}{4}((3) + (4) + (5))$ equals

$$\varphi_j(N\setminus i, v) + \varphi_k(N\setminus j, v) + \varphi_i(N\setminus k, v) = \varphi_j(N\setminus k, v) + \varphi_k(N\setminus i, v) + \varphi_i(N\setminus j, v).$$

Thus, the balancedness of the cycle contributions holds for group S with $s = 3$.
The other cases $(s = 4, \ldots, n-1)$ can be proved by a similar manner. Thus we omit it. □

It is obvious that any value on **G** satisfying the balanced contributions property also satisfies BCC, thus satisfying GBCC.

It is clear that the *Egalitarian value* EV, which is defined as, for any $(N, v) \in$ **G** and any $i \in N$,

$$\mathrm{EV}_i(N, v) = \frac{v(N)}{n},$$

also satisfies BCC because

$$\sum_{\ell=1}^{n} \mathrm{EV}_{i_\ell}(N\setminus i_{\ell-1}, v) = \sum_{\ell=1}^{n} \mathrm{EV}_{i_\ell}(N\setminus i_{\ell+1}, v) = \sum_{i\in N} \frac{v(N\setminus i)}{n-1}.$$

Since the Egalitarian value is efficient, BCC is weaker than the balanced contributions property.

In addition to the Shapley value and the Egalitarian value, there are several famous values that satisfy BCC such as the CIS (center of gravity of the imputation set) value and the ENSC (egalitarian non-separable contribution) value (Driessen and Funaki 1991). Examples of solutions that do not satisfy BCC are the the nucleolus (Schmeidler 1969) and the τ-value (Tijs 1987).

BCC requires that cycle contributions among all players should be balanced between any order on the set of all players and its inverse order. Similarly, we can consider the property that cycle contributions among all groups of three (or more) players should be balanced between any order on the group and its inverse order as follows.

BCC for three players: For any $(N, v) \in \mathbf{G}$ and for any three player coalition $\{i, j, k\} \subseteq N$

$$\varphi_i(N \setminus k, v) + \varphi_j(N \setminus i, v) + \varphi_k(N \setminus j, v) = \varphi_i(N \setminus j, v) + \varphi_j(N \setminus k, v) + \varphi_k(N \setminus i, v).$$

It is straightforward that any value on **G** satisfying the balanced contributions property also satisfies BCC for three players. Thus, BCC for three players is a weaker property than the balanced contributions property as well as BCC is. The relationships between the BCC and the BCC for three players is as follows.

Proposition 2.2. *(i) If φ satisfies BCC, it also satisfies BCC for three players. (ii) If φ satisfies BCC for three players, it satisfies BCC.*

Proof. Let a value φ satisfy BCC. Note that both conditions are trivially satisfied for any game with two players or less. Moreover, for any three players game, BCC and BCC for three players are equivalent. Thus, we consider the case $n \geq 4$.

First, we show (i). This is proved by the same way of the proof of Proosition 2.1.

Next, we show (ii). Let a value φ satisfy BCC for three players. For any $(N, v) \in \mathbf{G}$ with $n \geq 4$, consider an order $(i_1, i_2, \ldots, i_n)$ on N. By BCC for three players with respect to (i_1, i_2, i_3),

$$\begin{aligned}\varphi_{i_1}(N \setminus i_3, v) &+ \varphi_{i_2}(N \setminus i_1, v) + \varphi_{i_3}(N \setminus i_2, v) \\ &= \varphi_{i_1}(N \setminus i_2, v) + \varphi_{i_2}(N \setminus i_3, v) + \varphi_{i_3}(N \setminus i_1, v).\end{aligned} \tag{2.7}$$

By BCC for three players with respect to (i_1, i_3, i_4),

$$\begin{aligned}\varphi_{i_1}(N \setminus i_4, v) &+ \varphi_{i_3}(N \setminus i_1, v) + \varphi_{i_4}(N \setminus i_3, v) \\ &= \varphi_{i_1}(N \setminus i_3, v) + \varphi_{i_3}(N \setminus i_4, v) + \varphi_{i_4}(N \setminus i_1, v).\end{aligned} \tag{2.8}$$

$(6) + (7)$ equals

$$\begin{aligned}\varphi_{i_1}(N \setminus i_4, v) &+ \varphi_{i_2}(N \setminus i_1, v) + \varphi_{i_3}(N \setminus i_2, v) + \varphi_{i_4}(N \setminus i_3, v) \\ &= \varphi_{i_1}(N \setminus i_2, v) + \varphi_{i_2}(N \setminus i_3, v) + \varphi_{i_3}(N \setminus i_4, v) + \varphi_{i_4}(N \setminus i_1, v).\end{aligned} \tag{2.9}$$

Next, by BCC for three players with respect to (i_1, i_4, i_5),

$$\begin{aligned}\varphi_{i_1}(N \setminus i_5, v) &+ \varphi_{i_4}(N \setminus i_1, v) + \varphi_{i_5}(N \setminus i_4, v) \\ &= \varphi_{i_1}(N \setminus i_4, v) + \varphi_{i_4}(N \setminus i_5, v) + \varphi_{i_5}(N \setminus i_1, v).\end{aligned} \tag{2.10}$$

$(8) + (9)$ equals

$$\begin{aligned}\varphi_{i_1}(N \setminus i_5, v) &+ \varphi_{i_2}(N \setminus i_1, v) + \varphi_{i_3}(N \setminus i_2, v) + \varphi_{i_4}(N \setminus i_3, v) + \varphi_{i_5}(N \setminus i_4, v) \\ &= \varphi_{i_1}(N \setminus i_2, v) + \varphi_{i_2}(N \setminus i_3, v) + \varphi_{i_3}(N \setminus i_4, v) + \varphi_{i_4}(N \setminus i_5, v) + \varphi_{i_5}(N \setminus i_1, v).\end{aligned}$$

Repeating a similar argument with respect to $(i_1, i_5, i_6), \ldots, (i_1, i_{n-1}, i_n)$, we obtain

$$\sum_{r=1}^{n} \varphi_{i_r}(N \setminus i_{r-1}, v) = \sum_{r=1}^{n} \varphi_{i_r}(N \setminus i_{r+1}, v).$$

□

Thus, this proposition implies the equivalence of BCC and BCC for three players.

In the above, we consider only groups of three players. In the following, we consider groups of r players, where $r \geq 4$.

BCC for r players: For any $(N, v) \in$ **G**, any $S \subseteq N$ such that $|S| = r$ and any order $(i_1, i_2, \ldots, i_r)$ on S,

$$\sum_{\ell=1}^{r} \left(\varphi_{i_\ell}(N, v) - \varphi_{i_\ell}(N \setminus i_{\ell-1}, v)\right) = \sum_{\ell=1}^{r} \left(\varphi_{i_\ell}(N, v) - \varphi_{i_\ell}(N \setminus i_{\ell+1}, v)\right),$$

where $i_0 = i_r$ and $i_{r+1} = i_1$.

The following proposition indicates that BCC and BCC for r players, where $r \geq 4$, are almost equivalent.

Proposition 2.3. *Let $r \geq 4$. (i) If φ satisfies BCC, it also satisfies BCC for r players. (ii) If φ satisfies BCC for r players, it satisfies BCC when there are r or more players.*

Proof. First, we show (i). By (i) of Proposition 2.2, if φ satisfies BCC, it also satisfies BCC for three players. By the proof of (ii) of Proposition 2.2, it is clear that "if φ satisfies BCC for three players, it satisfies BCC for r players with respect to $r \geq 4$ when there are r or more players." If there are less than r players, it is trivial that φ satisfies BCC for r players. Hence, (i) is obtained.

Next, we show (ii). Let $r \geq 4$. If φ satisfies BCC for r players, and there are more than r players in a game, let $i, j, k \subseteq N$ and $a = (a_1, a_2, \ldots, a_{r-3})$ be an order on $S \subseteq N \backslash \{i, j, k\}$ satisfying $|S| = r - 3$. Applying BCC for r players with respect to orders (i, j, k, a), (i, k, j, a), (j, k, i, a), (j, i, k, a), (k, i, j, a), (k, j, i, a) on $S \cup \{i, j, k\}$, we have that φ satisfies BCC for three players, i, j and k, as shown in the proof of (i) of Propositio 2.2. Thus, by (ii) of Proposition 2.2, it satisfies BCC. □

So, BCC for r players, where $r \geq 4$, is slightly weaker than BCC. The difference between Propositions 2.2 and 2.3 comes from the fact that while BCC for r players, where $r \geq 4$ is silent for a three person game (correctly speaking, a game with less than or equal to $r - 1$ players), BCC requires a non-trivial restriction to a game with three players.

Figure 2.1 shows the relationship between BC, GBCC, BCC and BCC for r players.

2.4 Axiomatization of the Shapley value

Together with two other basic axioms, BCC characterizes the Shapley value. The first axiom is a very fundamental one.

Efficiency: For any $(N, v) \in$ **G**,

$$\sum_{i \in N} \varphi_i(N, v) = v(N).$$

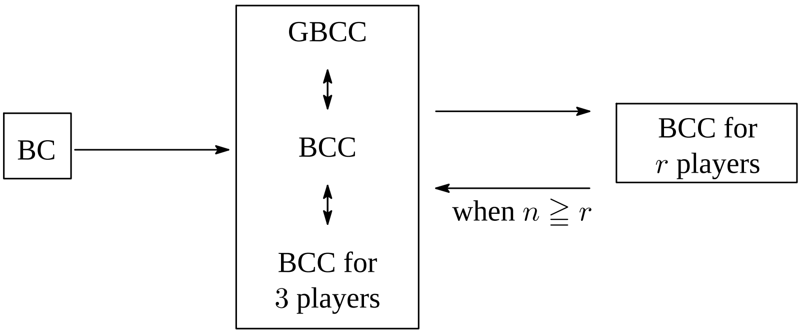

Figure 2.1: The relationship between BC, GBCC, BCC, BCC for 3 players, and BCC for r players ($r \geqq 4$)

The second property is related to null players, which was introduced by Derks and Haller (1999). A *null player* in (N, v) is a player $k \in N$, satisfying $v(S \cup k) = v(S)$ for any $S \subseteq N \setminus k$.

Null player out (NPO): For any $(N, v) \in \mathbf{G}$, if $k \in N$ is a null player in (N, v) for any $i \in N \setminus k$,

$$\varphi_i(N, v) = \varphi_i(N \setminus k, v).$$

NPO requires that a deletion of null players does not affect payoffs of the other players. Note that, in general, NPO has no relationship with the usual *null player property*.[1] However, together with Efficiency, NPO implies the null player property, since if $k \in N$ is a null player in a game (N, v),

$$\begin{aligned}\varphi_k(N, v) &= v(N) - \sum_{i \in N \setminus k} \varphi_i(N, v) \\ &= v(N) - \sum_{i \in N \setminus k} \varphi_i(N \setminus k, v) = v(N) - v(N \setminus k) = 0.\end{aligned}$$

It is easily checked that the Shapley value satisfies NPO and Efficiency. Thus, the Shapley value satisfies the above two properties and BCC. The next theorem states that the Shapley value is a unique value satisfying these three.

Theorem 2.1. *The Shapley value is the unique value on **G** that satisfies Efficiency, BCC, and NPO.*

Proof. We have already shown that the Shapley value satisfies Efficiency, BCC and NPO. Hence, it is sufficient to show the uniqueness.

Let φ be a value on **G** satisfying the three properties. We show the uniqueness of the value by the induction with respect to the number of players.

When $n = 1$, by Efficiency, $\varphi_i(N, v) = v(i)$ for $i \in N$.

Assume that, φ is uniquely determined, for any game with less than n players. We show that $\varphi(N, v)$ is uniquely determined when $N = \{1, 2, \ldots, n\}$.

Take any integer $k \in \mathbb{N} \setminus N$. Then, the null-extended game (N', w) of a game (N, v) with respect to k is defined as follows:

$$N' = N \cup k,$$

[1] The *null player property* requires that null players obtain nothing.

and for any $S \subseteq N'$,

$$w(S) = v(S \setminus k).$$

Clearly, k is a null player in (N', w) and $(N' \setminus j, w)$ for any $j \in N$. In addition, $(N' \setminus k, w) = (N, v)$ and $(N' \setminus \{j, k\}, w) = (N \setminus j, v)$ for any $j \in N$.

Consider an order $(1, k, 2, \dots, n)$ on N'. By BCC,

$$\begin{aligned}\varphi_1(N' \setminus n, w) + \varphi_k(N' \setminus 1, w) + \cdots + \varphi_n(N' \setminus (n-1), w) \\ = \varphi_1(N' \setminus k, w) + \varphi_k(N' \setminus 2, w) + \cdots + \varphi_n(N' \setminus 1, w).\end{aligned} \quad (2.11)$$

By Efficiency and NPO, $\varphi_k(N' \setminus 1, w) = \varphi_k(N' \setminus 2, w) = 0$. By NPO, $\varphi_i(N' \setminus j, w) = \varphi_i(N \setminus j, v)$ for any $\{i, j\} \subseteq N$. Therefore, (10) is equal to the following:

$$\begin{aligned}\varphi_1(N \setminus n, v) + \varphi_2(N, v) + \cdots + \varphi_n(N \setminus (n-1), v) \\ = \varphi_1(N, v) + \varphi_2(N \setminus 3, v) + \cdots + \varphi_n(N \setminus 1, v),\end{aligned}$$

or,

$$\begin{aligned}\varphi_1(N, v) - \varphi_2(N, v) = -\varphi_2(N \setminus 3, v) - \cdots - \varphi_n(N \setminus 1, v) \\ + \varphi_1(N \setminus n, v) + \varphi_3(N \setminus 2, v) + \cdots + \varphi_n(N \setminus (n-1), v).\end{aligned}$$

Let b_1 be the right-hand side of the above equation. By the induction hypothesis, b_1 is uniquely determined.

Applying the similar argument to the orders $(1, 2, k, 3, \dots, , n)$, $(1, 2, 3, k, 4, \dots, n)$, $\dots$, and $(1, 2, \dots, n-1, k, n)$, we obtain the following $(n-1)$ equations:

$$\begin{aligned}\varphi_1(N, v) - \varphi_2(N, v) &= b_1, \\ \varphi_2(N, v) - \varphi_3(N, v) &= b_2, \\ &\vdots \\ \varphi_{n-1}(N, v) - \varphi_n(N, v) &= b_{n-1},\end{aligned}$$

By Efficiency,

$$\varphi_1(N, v) + \varphi_2(N, v) + \cdots + \varphi_n(N, v) = v(N).$$

Since these n equations are linear independent, $\varphi(N, v)$ is uniquely determined. □

For the independence of Efficiency, BCC, and NPO, see Table 2.1.

Table 2.1: Independence of the axioms in Theorem 2.1

values / properties	Efficiency	BCC (for r)	NPO
The Banzhaf value (Banzhaf III 1965)	–	+	+
The τ-value (Tijs 1987)[2]	+	–	+
The Egalitarian value	+	+	–

+ : satisfy, $-$: not satisfy

[2]Note that the τ-value is defined on the class of the quasi-balanced games.

Since BCC and BCC for three players are equivalent, we obtain the fact that the Shapley value is a unique efficient value satisfying BCC for three players and NPO as a corollary of Theorem 2.1. On the other hand, since BCC for r players where $r \geq 4$ is weaker than BCC, we cannot obtain the axiomatization of the Shapley value through BCC for r players directly from Theorem 2.1 and Proposition 2.3. However, the following theorem shows that the Shapley value is axiomatized by Efficiency, BCC for r players and NPO.

Theorem 2.2. *Let $r \geq 3$. The Shapley value is the unique value on* **G** *that satisfies Efficiency, BCC for r players, and NPO.*

Proof. Let $(N, v) \in$ **G**. Since we know that the Shapley value satisfies Efficiency, NPO and BCC for r players, we show that the value φ satisfying the three axioms is uniquely determined. In case $n = 1$, Efficiency implies $v(i) = \mathrm{EV}_i(i, v)$. Consider $n \geq 2$. In what follows, we show that "if $n \geq 2$ and φ satisfies Efficiency, NPO and BCC for r players, then it must satisfy the balanced contributions property introduced by Myerson (1980)."

Take any integer $k_1, k_2, \ldots, k_{r-2} \in \mathbb{N} \setminus N$, and let $K = \{k_1, k_2, \ldots, k_{r-2}\}$. Then, the null-extended game (N', w) of a game (N, v) with respect to K is defined as follows:

$$N' = N \cup K,$$

and for any $S \subseteq N'$,

$$w(S) = v(S \setminus K).$$

Clearly, $|N'| \geq r$ and each $k \in K$ is a null player in (N', w) and its any restriction (N'', w) where $N'' \subseteq N'$ such that $k \in N'$. In addition, $(N' \setminus K, w) = (N, v)$.

Take any $i, j \in N \subseteq N'$ and consider an order $(i, j, k_1, k_2, \ldots, k_{r-2})$. By BCC for r players,

$$\begin{aligned} &\varphi_i(N' \setminus k_{r-2}, w) + \varphi_j(N' \setminus i, w) + \varphi_{k_1}(N' \setminus j, w) + \cdots + \varphi_{k_{r-2}}(N' \setminus k_{r-3}, w) \\ &\quad = \varphi_i(N' \setminus j, w) + \varphi_j(N' \setminus k_1, w) + \varphi_{k_1}(N' \setminus k_2, w) + \cdots + \varphi_{k_{r-2}}(N' \setminus i, w). \end{aligned} \tag{2.12}$$

By Efficiency ad NPO, (11) equals

$$\varphi_i(N, v) + \varphi_j(N \setminus i, v) = \varphi_i(N \setminus j, v) + \varphi_j(N, v)$$

$$\iff \varphi_i(N, v) - \varphi_i(N \setminus j, v) = \varphi_j(N, v) - \varphi_j(N \setminus i, v).$$

□

A remark on Theorems 2.1 and 2.2 is that it is easy to check that the proofs of Theorems 2.1 and 2.2 are applicable when we consider only restricted classes of games such as the zero-monotonic games, superadditive games, or convex games. Thus, Theorems 2.1 and 2.2 also hold, even though we replace **G** in the statement of the theorems by $\mathbf{G}^M$, $\mathbf{G}^S$, or $\mathbf{G}^C$, respectively.

2.5 Axiomatization of the Egalitarian value

Replacing NPO with the other property, the Egalitarian value is characterized in a similar manner.

A *proportional player* in (N, v) is a player $k \in N$, satisfying $v(S \cup k) - v(S) = \frac{1}{|S|} v(S)$ for all $S \subseteq N \setminus k$ with $S \neq \emptyset$. Thus, a proportional player is a player whose marginal contributions to any coalition (except the empty set) is directly proportional to the worth of the coalition, and it is inversely proportional to the size of the coalition. Similar to NPO, the following is considered.

Proportional player out (PPO): For any $(N, v) \in \mathbf{G}$, if $k \in N$ is a proportional player in (N, v), then for any $i \in N \setminus k$,

$$\varphi_i(N, v) = \varphi_i(N \setminus k, v).$$

Efficiency and PPO imply that any proportional player obtains an equal division of the worth of the grand coalition, since if $k \in N$ is a proportional player in (N, v),

$$\begin{aligned}\varphi_k(N, v) = v(N) - \sum_{i \in N \setminus k} \varphi_i(N, v) &= v(N) - \sum_{i \in N \setminus k} \varphi_i(N \setminus k, v) \\ &= v(N) - v(N \setminus k) = v(N) - \frac{n-1}{n} v(N) = \frac{v(N)}{n}.\end{aligned}$$

The Egalitarian value EV satisfies PPO, since if $k \in N$ is a proportional player in (N, v), then for any $i \in N \setminus k$,

$$\mathrm{EV}_i(N \setminus k, v) = \frac{v(N \setminus k)}{n-1} = \frac{(n-1)v(N)}{n} \cdot \frac{1}{n-1} = \frac{v(N)}{n}.$$

Following are the parallel results with Theorems 2.1 and 2.2.

Theorem 2.3. *The Egalitarian value is the unique value on* **G** *that satisfies Efficiency, BCC, and PPO.*

Proof. The proof is similar to that of Theorem 2.1. The difference between the two proofs is that in the proof of Theorem 2.3, we consider the proportional-extended game (N', w) of a game (N, v) with respect to $k \in \mathbb{N} \setminus N$ defined as follows:

$$N' = N \cup k,$$

and for any $S \subseteq N'$,

$$w(S) = \begin{cases} 0 & \text{if } S = \{k\} \\ \frac{|S|}{|S| - |S \cap k|} v(S \setminus k) & \text{otherwise.} \end{cases}$$

Clearly, k is a proportional player in (N', w) and $(N' \setminus j, w)$ for any $j \in N$. In addition, $(N' \setminus k, w) = (N, v)$ and $(N' \setminus \{j, k\}, w) = (N \setminus j, v)$ for any $j \in N$.

Assume that φ is the Egalitarian value, if there are less than or equal to $n - 1$ players. Consider an order $(1, k, 2, \ldots, n)$ on N'. By BCC, Efficiency, PPO, the induction hypothesis and the definition of the proportional-extended game, we have

$$\varphi_1(N, v) - \varphi_2(N, v) = 0.$$

Applying a similar argument to the orders $(1, 2, k, 3, \ldots, n), \ldots, (1, 2, \ldots, n-1, k, n)$, we have $\varphi_1(N, v) = \varphi_2(N, v) = \cdots = \varphi_n(N, v)$. By Efficiency, we conclude $\varphi_i(N, v) = v(N)/n$ for all $i \in N$. □

For the independence of Efficiency, BCC, and PPO, see Table 2.2.

Theorem 2.4. *Let $r \geq 3$. The Egalitarian value is the unique value on* **G** *that satisfies Efficiency, BCC for r players, and PPO.*

Table 2.2: Independence of the axioms in Theorem 2.2

values / properties	Efficiency	BCC (for r)	PPO
$\varphi = \begin{cases} \frac{\text{EV}}{2} & \text{if } n = 1 \text{ and } v(N) > 0 \\ \text{EV} & \text{otherwise} \end{cases}$	–	+	+
$\varphi_i(N, v) = \begin{cases} v(N) - v(N \setminus i) & \text{if } i \in P \\ \frac{(1-\lvert P\rvert)v(N)+\sum_{j \in P} v(N \setminus j)}{\lvert N \setminus P\rvert} & \text{otherwise} \end{cases}$	+	–	+
the Shapley value	+	+	–

P: a set of all proportional players, + : satisfy, $-$: not satisfy

Proof. We show the uniqueness of the value by the induction with respect to the number of players.

When $n = 1$, by Efficiency, $\varphi_i(N, v) = v(i)$ for $i \in N$. Let $(N, v) \in \mathbf{G}$, where $n = n$. Assume that, for any game with less than n players, φ is the Egalitarian value.

Take any integer $k_1, k_2, \ldots, k_{r-2} \in \mathbb{N} \setminus N$, and let $K = \{k_1, k_2, \ldots, k_{r-2}\}$. Then, the proportional-extended game (N', w) of a game (N, v) with respect to K is defined as follows:

$$N' = N \cup K,$$

and for any $S \subseteq N'$,

$$w(S) = \begin{cases} 0 & \text{if } S \subseteq K, \\ \frac{|S|}{|S|-|S \cap K|} v(S \setminus K) & \text{otherwise.} \end{cases}$$

Then, each $k \in K$ is a proportional player in (N', w) since for any $k \in K$ and any $S \subseteq N' \setminus k$, if $S \subseteq K$, then $w(S \cup k) = w(S) = 0$, and otherwise,

$$\begin{aligned} w(S \cup k) - w(S) &= \frac{|S| + 1}{(|S| + 1) - |(S \cup k) \cap K|} v((S \cup k) \setminus K) - \frac{|S|}{|S| - |S \cap K|} v(S \setminus K) \\ &= \frac{1}{|S| - |S \cap K|} v(S \setminus K) = \frac{1}{|S|} w(S). \end{aligned}$$

Similarly, each $k \in K$ is a proportional player in any restricted game (N'', w), where $N'' \subseteq N'$ and $k \in N''$. In addition, $(N' \setminus K, w) = (N, v)$.

Take any $i, j \in N \subseteq N'$, and consider an order $(i, j, k_1, k_2, \ldots, k_{r-2})$. By BCC for r players,

$$\begin{aligned} &\varphi_i(N' \setminus k_{r-2}, w) + \varphi_j(N' \setminus i, w) + \varphi_{k_1}(N' \setminus j, w) + \cdots + \varphi_{k_{r-2}}(N' \setminus k_{r-3}, w) \\ &\quad = \varphi_i(N' \setminus j, w) + \varphi_j(N' \setminus k_1, w) + \varphi_{k_1}(N' \setminus k_2, w) + \cdots + \varphi_{k_{r-2}}(N' \setminus i, w). \end{aligned}$$

Repeatedly applying PPO, $\varphi_i(N' \setminus k_{r-2}, w) = \varphi_i(N, v)$, $\varphi_j(N' \setminus k_1, w) = \varphi_j(N, v)$, $\varphi_i(N' \setminus j, w) = \varphi_i(N \setminus j, v)$, and $\varphi_j(N' \setminus i, w) = \varphi_j(N \setminus i, v)$. By Efficiency and PPO, $\varphi_{k_1}(N' \setminus j, w) = \frac{1}{|N'|-2} w(N' \setminus \{j, k_1\})$. Moreover, by repeatedly applying the definition of proportional players, $\frac{1}{|N'|-2} w(N' \setminus \{j, k_1\}) = \frac{v(N \setminus j)}{n-1}$. Similarly, $\varphi_{k_p}(N' \setminus k_{p-1}, w) = \frac{v(N)}{n}$ for each $p = 2, \ldots, r-2$, and $\varphi_{k_p}(N' \setminus k_{p+1}, w) = \frac{v(N)}{n}$ for each $p = 1, \ldots, r-3$.

Thus, the above equation can be reduced to

$$\varphi_i(N,v) + \varphi_j(N \setminus i, v) + \frac{v(N \setminus j)}{n-1} + (r-3)\frac{v(N)}{n} = \varphi_i(N \setminus j, v) + \varphi_j(N,v) + \frac{v(N \setminus i)}{n-1} + (r-3)\frac{v(N)}{n}$$

Moreover, by the induction hypothesis, the above equation is

$$\varphi_i(N,v) = \varphi_j(N,v).$$

Together with Efficiency, we have the desired result. □

Unlike the case of the Shapley value, the proofs of Theorems 2.3 and 2.4 are not applicable when we consider only restricted classes of games, such as the zero-monotonic games, superadditive games, or convex games. These differences come from the differences between the properties of null players and proportional players. Given a zero-monotonic, superadditive, or convex game, when we add a player who is a null player in the null-extended game,[3] the extended game is zero-monotonic, superadditive, or convex, respectively. However, when we add a player who is a proportional player in the proportional-extended game,[4] the extended game may not be zero-monotonic, superadditive, or convex. These are crucial in our proofs.

2.6 Weighted balanced cycle contributions property

In this section, we consider a non-symmetric generalization of the analysis in the previous sections. Let $w_i (> 0)$ denote a positive weight for a player i in the set of potential player $\mathbb{N}$. Let $w = (w_i)$, which is fixed throughout this section.

A non-symmetric generalization of BCC is as follows.

Weighted balanced cycle contributions property (WBCC): For any $(N,v) \in \mathbf{G}$ and any order $(i_1, i_2, \ldots, i_n)$ on N,

$$\sum_{\ell=1}^{n} w_{i_\ell} \varphi_{i_\ell}(N \setminus i_{\ell-1}, v) = \sum_{\ell=1}^{n} w_{i_\ell} \varphi_{i_\ell}(N \setminus i_{\ell+1}, v),$$

where $i_0 = i_n$ and $i_{n+1} = i_1$.

The weight for each player can be interpreted as, for instance, the different importance, the bargaining power, or the utility scale of players.

Since the w-weighted Shapley value satisfies the weighed balanced contributions property that requires, $w_i(\mathrm{Sh}_i^w(N,v) - \mathrm{Sh}_i^w(N \setminus j, v)) = w_j(\mathrm{Sh}_j^w(N,v) - \mathrm{Sh}_j^w(N \setminus i, v))$, for each pair of players $i, j \in N$, it also satisfies WBCC.

As the following theorem shows, WBCC with Efficiency and NPO axiomatizes the w-weighted Shapley value.

Theorem 2.5. *The w-weighted Shapley value is the unique value on **G** that satisfies Efficiency, WBCC, and NPO.*

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[3]For the definition of the null-extended game, see the proof of Theorem 2.1.

[4]For the definition of the proportional-extended game, see the proof of Theorem 2.3.

Proof. The proof of Theorem 2.5 are almost similar to the proof of Theorem 2.1. □

Similar to Theorem 2.2, the w-weighted Shapley value is also a unique value on **G**, satisfying Efficiency, WBCC for r players, and NPO. Moreover, these results also hold for restricted domains of games such as $\mathbf{G}^M$, $\mathbf{G}^S$ and $\mathbf{G}^C$.

The w-weighted Egalitarian value EV^w is defined by

$$\mathrm{EV}_i^w(N,v) = \frac{w_i}{\sum_{j\in N} w_j} v(N),$$

for each $i \in N$. When $w_i = w_j$ for all $i, j \in N$, the w-weighted Egalitarian value coincides with the Egalitarian value.

A *w-weighted proportional player* in (N,v) is a player $k \in N$ satisfying $v(S \cup k) - v(S) = \frac{w_k}{\sum_{j\in S} w_j} v(S)$ for all $S \subseteq N \setminus k$ with $S \neq \emptyset$. The following is a weighted version of PPO.

Weighted proportional player out (WPPO): For any $(N,v) \in \mathbf{G}$, if $k \in N$ is a w-weighted proportional player in (N,v), then for any $i \in N \setminus k$,

$$\varphi_i(N,v) = \varphi_i(N \setminus k, v).$$

The following theorem holds.

Theorem 2.6. *The w-weighted Egalitarian value is the unique value on* **G** *that satisfies Efficiency, WBCC, and WPPO.*

Proof. The proof of Theorem 2.6 are almost similar to the proof of Theorem 2.3. □

2.7 Concluding remarks

Except the Shapley and Egalitarian values, there are several famous values that satisfy BCC such as the CIS (center of gravity of the imputation set) value and the ENSC (egalitarian non-separable contribution) value (Driessen and Funaki 1991). The CIS value is characterized in a similar manner as we did in the note, while the ENSC value is not. For the case of the CIS value, we focus on the player $k \in N$ satisfying $v(S \cup k) - v(S) = \frac{1}{|S|}(v(S) - \sum_{i\in S} v(i))$ for all $S \subseteq N \setminus k$ with $S \neq \emptyset$. Efficiency, BCC (or BCC for r players) and the property that the elimination of the above-mentioned player does not affect the value of the other players, characterize the CIS value. Table 2.3 shows that what player is the one whose deletion does not affect the payoffs of the other players. With BCC and Efficiency, the invariance of the payoff from the deletion of such player characterizes the corresponding solutions. However, for the case of the ENSC value, we can show that there exists no player whose elimination does not affect the value of the other players.

Between Theorems 2.1 and 2.3, the player on which we pay attention is different. Hence, we cannot generalize our results to α-Egalitarian Shapley values (Joosten 1996, Brink, Funaki, and Ju 2007), which are convex combinations of the Egalitarian and Shapley values. If we pay attention to only null players and focus on the effect of the elimination of a null player in each value, all values we mention here (including the ENSC value) and all their convex combinations are characterized. This result will be included in another paper. Through this way that focuses on the deletion of null players, we may succeed in the class axiomatization of the solutions that satisfy BCC and Efficiency. This is one of our further researches on BCC.

Table 2.3: The players whose deletion does not affect the payoffs of other players

	k's marginal contribution for S
the Shapley value	0
the Egalitarian value	$\frac{v(S)}{\lvert S\rvert}$
the CIS value	$\frac{1}{\lvert S\rvert}(v(S) - \sum_{i \in S} v(i))$

Chapter 3

The Shapley-Egalitarian solution for games with coalition structures

3.1 Introduction

The purpose of chapter is to revisit a distributive analysis of a cooperative surplus among players when they already partition themselves into 'coalitions' before realizing cooperation. Two traditional works of Aumann and Dreze (1974) and Owen (1977) respectively propose distribution rules, solution concepts in a framework of a cooperative game with a coalition structure, which are different from each other but each of which is considered to be an extension of the Shapley value to the case. Because both studies implicitly or explicitly assume that players forms coalitions in order to affect their bargaining positions, they share a common presumption that a participant in a coalition who does not make an effective contribution to his fellows receive nothing. Thus, these studies lack a perspective that a (formed) coalition often has a tendency to a generous reallocation of the surplus within the members of the coalition as if it is a system of mutual assistance among the internal members, even though it is not established for such an end in the beginning.[1]

To reflect such a point of view, we introduce two new axioms in solution theory of a cooperative game with a coalition structure, both of which are slightly different from the ones considered in existing studies. The first axiom represents something like mutual aid of the formed coalitions. This is expressed by the statement on a *null player*: even a null player could obtain some portion of a bargaining surplus if a coalition that he belongs to generates it. The other is related to an *equity* criterion applied to members in the coalitions. This requires that two distinct players should be treated equally, *i.e.*, receive the same amount, if these two are judged to be in an equal position in their *internal* coalition. We show that these two axioms with the usual three axioms (Efficiency, Additivity and Coalitional Symmetry) lead to a unique solution concept which is also considered to be an extension of the Shapley value in cooperative games with coalition structures. This solution, named a Shapley-Egalitarian solution, is interpreted as an allocation of the cooperative surplus by using the Shapley value in two-step bargaining process: a bargaining

[1] Such tendency of groups is examined and explained in various contexts. Kropotkin (1972) explains this from a human evolution in the struggle for life. In a context of rent-seeking problem among two groups, Noh (1999) demonstrates that the group members can agree with egalitarian-like sharing rule among them to resolve a free rider problem in the group. Researchers in community psychology argue that the recent development of a number of mutual assistance organizations is due to stressful situations around ourselves (Levine 1988). Further, in the study of labor-management, reasons for and usefulness of profit sharing among employer and employees are examined (FitzRoy and Kraft 1986, 1987; Drago and Turnbull 1988; Kandel and Lazear 1992).

inter-coalitions and a bargaining intra-coalitions. Moreover, the bargaining surplus of the coalition is allocated among the intra-coalition members in egalitarian way. Thus, in the first step, each coalition obtains its Shapley value applied for a game among coalitions. The pure surplus of a coalition in the first step bargaining (its Shapley value obtained from the first step minus the worth of the coalition) is divided equally among players in the coalition. In the second step, players in the coalition receive their Shapley value applied for their own internal game. Thus, the Shapley-Egalitarian solution gives the sum of the payoffs in the first step and the second to each player. In addition to this definition, we give two expressions of this solution concept. One is an average of modified marginal contributions and the other is the weighted Shapley value of games with restricted communication derived from a coalition structure.

This chapter is organized as follows. In the next section, the correct statement about our new axioms is introduced and our main result on an axiomatic characterization of the solution concept is presented. Section 3 gives some remarks on the new solution concept.

3.2 Results

First, we define new axioms about null players and symmetric players. Let $(N, v, \mathcal{C}) \in$ **GC** and let ψ be a solution of cooperative games with coalition structures. Then,

Coalitional Null Player: If $i \in C_k$ is a null player in (N, v) and $k \in M$ is a dummy player in $(M, v_{\mathcal{C}})$ (that is, C_k is a dummy coalition), then $\psi_i(N, v, \mathcal{C}) = 0$.

Internal Equity: If $i \in C_k \in \mathcal{C}$ and $j \in C_k$ are symmetric in subgame $(C_k, v|_{C_k})$, then $\psi_i(N, v, \mathcal{C}) = \psi_j(N, v, \mathcal{C})$.

Thus, in the statement of Coalitional Null Player, the usual requirement on a null player (null player axiom in existing studies) that a null player obtains nothing in any situation is weakened so that he could obtain more than his own contributions because of the strong position of his coalition or mutual assistance between the internal members in the coalition. Internal Equity requires that two distinct players who are judged to be in an equal position in the internal situation (*i.e.*, subgame $(C_k, v|_{C_k})$) should be equally treated and thus receive the same amount of the surplus.

On the one hand, it is easily shown that the coalitional value satisfies Coalitional Null Player since it always gives nothing to the null player. On the other hand, it does not satisfy Internal Equity. In fact, consider a three-person game $(N, v, \mathcal{C})$ where $N = \{1, 2, 3\}$, $\mathcal{C} = \{\{1\}, \{2, 3\}\}$, and $v(\{1, 2\}) = v(\{1, 2, 3\}) = 1$ and $v(S) = 0$ otherwise. Then the coalitional value gives $(1/2, 1/2, 0)$ for the players. However players 2 and 3 are symmetric in $(\{2, 3\}, v|_{\{2,3\}})$.

Next theorem shows that there exists a unique solution on **GC** different from the coalitional value, satisfying these two axioms and usual three axioms (Efficiency, Additivity and Coalitional Symmetry).

Theorem 3.1. *There exists a unique solution of cooperative games with coalition structures on **GC** satisfying Coalitional Null Player, Internal Equity and the following three:*
(i) Efficiency: $\sum_{i \in N} \psi_i(N, v, \mathcal{C}) = v(N)$.
(ii) Additivity: $\psi(N, v, \mathcal{C}) + \psi(N, v', \mathcal{C}) = \psi(N, v + v', \mathcal{C})$, *where* $(v + v')(S) = v(S) + v'(S)$ *for any* $S \subseteq N$.
(iii) Coalitional Symmetry: If $k \in M$ *and* $h \in M$ *are symmetric in* $(M, v_{\mathcal{C}})$, *then* $\sum_{i \in C_k} \psi_i(N, v, \mathcal{C}) = \sum_{i \in C_h} \psi_i(N, v, \mathcal{C})$.

This solution is defined by the following formula:

$$\psi_i^\delta(N, v, \mathcal{C}) = \frac{\mathrm{Sh}_k(M, v_{\mathcal{C}}) - v(C_k)}{|C_k|} + \mathrm{Sh}_i(C_k, v|_{C_k}), \quad \textit{for } i \in C_k \in \mathcal{C}. \tag{3.1}$$

Proof. First, we show that ψ^δ satisfies all the five axioms. Efficiency and Additivity are obvious by the definition of ψ^δ since the Shapley value satisfies these two axioms. Next, ψ^δ satisfies Coalitional Symmetry because the summation of $\psi_i^\delta(N, v, \mathcal{C})$ over $C_k \in \mathcal{C}$ is player k's Shapley value of a game among coalitions $(M, v_{\mathcal{C}})$ and the Shapley value gives the equal payoffs to the symmetric players (symmetry axiom of the Shapley value, see Shapley 1953b). Since the first term of the definition of ψ^δ is the same for all the players in coalition $C_k \in \mathcal{C}$, the different payoffs among the players in C_k are caused by the subgame $(C_k, v|_{C_k})$. Thus ψ^δ fulfills Internal Equity because the Shapley value satisfies symmetry axiom. It also satisfies Coalitional Null Player axiom because the Shapley value gives dummy player i his stand-alone value $v(\{i\})$.

Next we will show the converse part. Let ψ be a solution on **GC** satisfying the five axioms. Given $T \subseteq N$, let (N, u_T) be a T unanimity game where $u_T(S) = 1$ if $S \supseteq T$ and $u_T(S) = 0$ otherwise. Given $c \in \mathbb{R}$, let cu_T be a T unanimity game u_T multiplied by a scalar c. Then, by Additivity axiom, it is enough to show that $\psi(N, cu_T, \mathcal{C})$ is uniquely determined by the five axioms.

For $T \subseteq N$, define $D \subseteq M$ by $\{\ k \in M\ :\ C_k \in \mathcal{C},\ C_k \cap T \neq \emptyset\}$. Then $(M, (cu_T)_{\mathcal{C}})$ is a D-unanimity game multiplied by c, *i.e.*, (M, cu_D). Then Coalitional Symmetry together with Coalitional Null Player axiom implies that for $k \in M \setminus D$ and for $i \in C_k$, $\psi_i(N, cu_T, \mathcal{C}) = 0$ and for $k \in D$, $\sum_{i \in C_k} \psi_i(N, cu_T, \mathcal{C}) = \frac{c}{|D|}$.

For an internal distribution of the members in $C_k, k \in D$, we consider two cases.

Case (a): $|D| \geqq 2$**.** Then, for any $k \in D$, $i \in C_k$ and $j \in C_k$ are symmetric in $(C_k, (cu_T)|_{C_k})$. Therefore $\psi_i(N, cu_T, \mathcal{C}) = \frac{c}{|C_k| \cdot |D|}$ for any $i \in C_k, k \in D$.

Case (b): $|D| = 1$**.** Then put $\mathcal{C} = \{C_1\}$. For $i \in C_1 \setminus T$, $\psi_i(N, cu_T, \mathcal{C}) = 0$ by Coalitional Null Player since player 1 is dummy in $(M, (cu_T)_{\mathcal{C}})$ and player i is null in (N, v). Moreover any $i, j \in T$ are symmetric in $(C_1, (cu_T)|_{C_1})$. Therefore by Internal Equity, $\psi_i(N, cu_T, \mathcal{C}) = \frac{c}{|T|}$ for any $i \in T$. □

We can interpret ψ^δ as a *two-step* Shapley value in the following sense. In the first step, each coalition $C_k \in \mathcal{C}$ acts like a single player and obtains player k's Shapley value of a game among coalitions, $(M, v_{\mathcal{C}})$. Thus, an allotment of first step for coalition C_k is $\mathrm{Sh}_k(M, v_{\mathcal{C}})$. In the second step, all the players in C_k agree with the following two things. First, they agree that $\mathrm{Sh}_k(M, v_{\mathcal{C}}) - v(C_k)$ is a pure surplus (it is non-negative if the game is superadditive) of the first step and therefore is split equally among the members in C_k. Second, they agree that remaining part $v(C_k)$ is distributed by the rule of the Shapley value for their subgame $(C_k, v|_{C_k})$. Thus, the pure bargaining surplus of the first stage is distributed within the members in the egalitarian way, which seems to reflect a generous reallocation or an aspect of mutual aid among the members embedded in our two axioms.

The solution ψ^δ is interpreted as the rights-egalitarian allocation rule for an allocation problem. An allocation problem, which is a triple (N, E, c), describes a situation that agents in N each of who has his monetary entitlement $c_i \in \mathbb{R}, c = (c_i)_{i \in N}$, discuss for the division of the budged $E \in \mathbb{R}$. The rights-egalitarian allocation rule F^{RE} for (N, E, c) is defined by

$$F_i^{RE}(N, E, c) = c_i + \frac{1}{n}\left(E - \sum_{i \in N} c_i\right)$$

for any $i \in N$ (Herrero, Maschler, and Villar 1999). To interpret ψ^δ in the context of an allocation problem, consider the following bargaining environment for the members in $C_k \in \mathcal{C}$. If they realize the cooperation for themselves, the worth $v(C_k)$ is distributed for them by the Shapley value, and thus each $i \in C_k$ obtains $\mathrm{Sh}_i(C_k, v)$ in this case. On the other side, if they form a coalition and bargain with the outside coalitions $C_h, h = 1, \ldots, k-1, k+1, \ldots, m$, the coalition obtains its Shapley value for $(M, v_{\mathcal{C}})$. Then, this situation can be described as an allocation problem (C_k, E, c) where $E = \mathrm{Sh}_k(M, v_{\mathcal{C}})$ and $c_i = \mathrm{Sh}_i(C_k, v)$ for all $i \in C_k$. It is easily confirmed that $F_i^{RE}(C_k, E, c) = \psi^\delta(N, v, \mathcal{C})$. Thus, ψ^δ is an allocation rule such that monetary entitlement of the players and the budget for the coalition is calculated by the Shapley value and then the rights-egalitarian solution is applied for an allocation problem derived from $(N, v, \mathcal{C})$ by such the manner. We name ψ^δ the *Shapley-Egalitarian solution* in a game with a coalition structure after this fact.

Before checking the independence of each axiom from the others in the theorem, the next remark is worth mentioning.

Remark 3.1. *It is worth mentioning that we can omit Efficiency from Theorem 1. In fact, the other axioms with non-emptiness of a solution which we implicitly assume, imply Efficiency. The reason why we add Efficiency in Theorem 1 is to easily compare our result with Owen's (1977) one in the next section.*

*The main logic is similar to Theorem 8.1.3 of Peleg and Sudhölter (2003). Let (N, v^0) be zero-game such that $v^0(S) = 0$ for any $S \subseteq N$ and $\mathcal{C}$ be a coalition structure on N. Then, $\psi(N, v^0, \mathcal{C})$ must be $0^N = (0, \ldots, 0) \in \mathbb{R}^N$ by Coalitional Null Player. Let $(N, v, \mathcal{C}) \in$ **GC**. By Additivity, $\psi(N, v, \mathcal{C}) + \psi(N, -v, \mathcal{C}) = \psi(N, v - v, \mathcal{C}) = \psi(N, v^0, \mathcal{C}) = 0^N$. So $\psi(N, v, \mathcal{C}) = -\psi(N, -v, \mathcal{C})$ holds. By the definition of a solution, $\sum_{i \in N} \psi_i(N, v, \mathcal{C}) \leqq v(N)$ and $\sum_{i \in N} \psi_i(N, v, \mathcal{C}) = -\sum_{i \in N} \psi_i(N, -v, \mathcal{C}) \geqq -(-v(N))$. Thus, $\sum_{i \in N} \psi_i(N, v, \mathcal{C}) = v(N)$ holds.*

Example 3.1. *We define the following solutions in order to check the independence of each axiom from the others. Let θ be an order on the set of all the integers $\mathbb{N}\{1, 2, 3, \ldots\}$. For any set $S \subset \mathbb{N}$, let $\theta[S]$ denote an order on S induced from θ such that for any $i, j \in S$, $\theta[S](i) < \theta[S](j)$ exactly if $\theta(i) < \theta(j)$.*

1. *For $(N, v) \in \Gamma$, let $\mathrm{Nu}(N, v)$ denote the nucleolus of (N, v) proposed by Schmeidler (1969).*[2] *For any $(N, v, \mathcal{C}) \in$ **GC**, we define a solution $\psi^{(i)}$ by $\psi_i^{(i)}(N, v, \mathcal{C}) = \frac{\mathrm{Nu}_k(M, v_{\mathcal{C}}) - v(C_k)}{|C_k|} + \mathrm{Nu}_i(C_k, v|_{C_k})$ for any $i \in C_k \in \mathcal{C}$. Then, $\psi^{(i)}$ satisfies Coalitional Symmetry, Internal Equity and Coalitional Null Player but not Additivity.*

2. *For any $i \in C_k \in \mathcal{C}$, define $\psi_i^{(ii)}(N, v, \mathcal{C}) = \frac{m_k^{\theta[M]}(M, v_{\mathcal{C}}) - v(C_k)}{|C_k|} + \mathrm{Sh}_i(C_k, v|_{C_k})$. Then, $\psi^{(ii)}$ satisfies Additivity, Internal Equity and Coalitional Null Player but not Coalitional Symmetry.*

3. *Then for any $i \in C_k \in \mathcal{C}$, define $\psi_i^{(iii)}(N, v, \mathcal{C}) = \frac{\mathrm{Sh}_k(M, v_{\mathcal{C}}) - v(C_k)}{|C_k|} + m_i^{\theta[C_k]}(C_k, v|_{C_k})$. Then, $\psi^{(iii)}$ satisfies Additivity, Coalitional Symmetry and Coalitional Null Player but not Internal Equity.*

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[2] The nucleolus of (N, v) is defined as follows. Give a payoff vector of N and coalition S, let $e(x, S)$ be the excess function of S at x defined by $e(x, S) = v(S) - \sum_{i \in S} x_i$. Let $e(x) = (e_k(x))$ be the $2^n - 1$ dimensional vector that is the decreasing order of $(e(x, S))_{S \subseteq N, S \neq \emptyset}$. For any two vector $x = (x_1, \ldots, x_K)$ and $y = (y_1, \ldots, y_K)$, the lexicographic order is defined by the following condition: $x >_L y$ if and only if there exist some $k = 1, \ldots, K$ such that $x_i = y_i$ for all $i < k$ and $x_k > y_k$, and $x =_L y$ if and only if $x = y$. Then the nucleolus is the payoff vector that minimizes $e(x)$ with respect to the lexicographic order.

4. *For any* $i \in C_k \in \mathcal{C}$, $\psi_i^{(iv)}(N, v, \mathcal{C}) = \psi_i^e(N, v, \mathcal{C}) = \frac{\mathrm{Sh}_k(M, v_{\mathcal{C}})}{|C_k|}$*. Then,* ψ^e *satisfies Additivity, Coalitional Symmetry and Internal Equity but not Coalitional Null Player.*

Among several solutions described in the above example, solution ψ^e, which uses the Shapley value for inter-coalitions and the Egalitarian value for intra-coalitions, also prepares the similar requirement of coalitions as generous reallocation system or mutual assistance considered in this chapter, but slightly give much weight to an egalitarian aspect within the internal members. In fact, this solution is axiomatized as follows.

Theorem 3.2. ψ^e *is a unique solution of cooperative games with coalition structures on* **GC** *satisfying Efficiency, Additivity, Coalitional Symmetry and the following two:*

Null Coalition: *If* $k \in M$ *is a null player in* $(M, v_{\mathcal{C}})$*, then* $\sum_{i \in C_k} \psi_i(N, v, \mathcal{C}) = 0$.

Internal Egalitarianism: *For any* $i \in C_k \in \mathcal{C}$ *and for any* $j \in C_k \in \mathcal{C}$, $\psi_i(N, v, \mathcal{C}) = \psi_j(N, v, \mathcal{C})$.

We omit the proof of Theorem 3.2 because this is similarly constructed to the proof of Theorem 3.1. For the independence of the axioms in the above theorem, consider solutions, for $i \in C_k \in \mathcal{C}$,

5. $\psi_i^{(v)}(N, v, \mathcal{C}) = \frac{\mathrm{Nu}_k(M, v_{\mathcal{C}})}{|C_k|}$,

6. $\psi_i^{(vi)}(N, v, \mathcal{C}) = \frac{m_k^{\theta[M]}(M, v_{\mathcal{C}})}{|C_k|}$, and

7. $\psi_i^{(vii)}(N, v, \mathcal{C}) = \frac{v(N)}{|N|}$.

Then, $\psi^{(v)}$, $\psi^{(vi)}$, ψ^{δ}, and $\psi^{(vii)}$ respectively show the independence of Additivity, Coalitional Symmetry, Internal Egalitarianism and Null Coalition from the other three.

3.3 Remarks

3.3.1 Comparison with the Owen's coalitional value

The coalitional value is characterized by Efficiency, Additivity, Coalitional Symmetry and the following two axioms. (See Owen 1977. However Peleg and Sudhölter 2003 show that by the same reason of Remark 3.1, we can conduct the axiomatization of the coalitional value without Efficiency.)

Null Player: If $i \in N$ is a null player in (N, v), then $\psi_i(N, v, \mathcal{C}) = 0$.

Restricted Equal Treatment: If $i, j \in C_k \in \mathcal{C}$ are symmetric in (N, v), then $\psi_i(N, v, \mathcal{C}) = \psi_j(N, v, \mathcal{C})$.

It is clear that Restricted Equal Treatment is weaker than Internal Equity and Null Player requires more than Coalitional Null Player. In fact, the coalitional value does not satisfy Internal Equity and the Shapley-Egalitarian solution does not satisfy Null Player. Thus, the coalitional value and the Shapley-Egalitarian solution are different in the judgment of application of the equity criterion. The coalitional value requires that two players in coalition C_k should be equally treated if these two are judged to be equal in the whole society. On the other hand, the Shapley-Egalitarian solution requires that two players in coalition C_k should be equally treated if these two are judged to be equal in the internal society. Moreover, these two solutions are different in the treatment of null players. While the coalitional value does not give any portion of surplus to a null player even if his coalition obtains large benefit, the Shapley-Egalitarian solution gives some portion of surplus to the null players if his coalition obtains the benefit. Thus, we understand that the essential difference between the two solutions lies in the treatment of null players and symmetric players.

3.3.2 Random arrival interpretation

It seems that the definition of ψ^δ is too much based on "two step bargaining process." We are, however, able to express ψ^δ as an average of the *modified* version of the marginal contributions as well as Sh and CV. Let $\theta \in \Theta(N, \mathcal{C})$. Let θ_M be an order on M derived from θ such that for $k, h \in M$, $\theta_M(k) < \theta_M(h)$ if and only if $\theta(i) < \theta(j)$ for all $i \in C_k$ and for all $j \in C_h$. θ_M is well-defined if θ is consistent with $\mathcal{C}$. For $i \in C_k \in \mathcal{C}$, we define modified marginal contribution of player i at order θ, $\bar{m}_i^\theta$ by

$$\bar{m}_i^\theta(N, v, \mathcal{C}) = \begin{cases} m_i^{\theta[C_k]}(C_k, v|_{C_k}) & \text{if } i \text{ is not last at order } \theta[C_k], \\ m_i^{\theta[C_k]}(C_k, v|_{C_k}) + m_k^{\theta_M}(M, v_{\mathcal{C}}) - v(C_k) & \text{if } i \text{ is last at order } \theta[C_k]. \end{cases}$$

The following theorem holds.

Theorem 3.3. *ψ^δ is expressed as follows: for $i \in N$,*

$$\psi_i^\delta(N, v, \mathcal{C}) = \frac{1}{|\Theta(N, \mathcal{C})|} \sum_{\theta \in \Theta(N, \mathcal{C})} \bar{m}_i^\theta(N, v, \mathcal{C}). \tag{3.2}$$

Proof. It is easily verified from this formula by Eq.(3.1) and the definition of the modified marginal contribution. □

3.3.3 Restricted communication

Myerson (1977) considers a situation that a communication between players is restricted on an undirected graph of N (see also Myerson 1980). Along this line of research, Aumann and Dreze's (1974) value, which is defined by $\mathrm{AD}_i(N, v, \mathcal{C}) = \mathrm{Sh}_i(C_k, v)$ for all $i \in C_k \in \mathcal{C}$, is considered to represent a situation that a coalition structure describes a communication restriction such that players in the same coalition communicate with each other, but each coalition is physically separated. This situation is also described as the graph such that each maximal component of the graph corresponds to a coalition in $\mathcal{C}$ and each subgraph on the component is a complete graph. Thus, Aumann and Dreze's value coincides with the Myerson value for such a graph situation. However, this interpretation of coalition structure does not fit the view that players form coalitions for the division of $v(N)$ since Aumann and Dreze's value does not satisfy the efficiency but the relative efficiency ($\sum_{i \in N} \psi_i(N, v, \mathcal{C}) = \sum_{C_k \in \mathcal{C}} v(C_k)$). This motivates another view of communication restriction by a coalition structure in the following sense:

(i) players in the same coalition $C_k \in \mathcal{C}$ can freely communicate with each other, and

(ii) players in C_k can communicate with players in the other coalitions if there is a permission of all the players in C_k.

Condition (i) means that players in any sub coalition $S \subseteq C_k \in \mathcal{C}$ can communicate with each other and thus obtain their worth of coalition, $v(S)$. In addition to (i), (ii) implies that there is a possibility of cooperation among players in the different coalitions. This is possible only if all the players in the relevant coalitions agree. Let $i \in C_k$ and $C_h \in \mathcal{C}, C_h \neq C_k$. While C_k and C_h obtain their worth $v(C_k \cup C_h)$, $C_k - i$ and C_h obtain the sum of $v(C_k - i)$ and $v(C_h)$ because there is no permission by player i or there is no permission of the party which the coalition represents and which requires the unanimous agreement.[3]

Definition 3.1. *Let $(N, v, \mathcal{C}) \in$ **GC**. $\mathcal{C}$-communication restricted game $(N, v^{\mathcal{C}})$ is defined as follows. For all $S \subseteq N$,*

$$v^{\mathcal{C}}(S) = v(\bigcup_{C_k \in \mathcal{C}(S)} C_k) + \sum_{T \in \mathcal{C}^0(S)} v(T)$$

where $\mathcal{C}(S) = \{C_k \in \mathcal{C} \ : \ C_k \subseteq S\}$ and $\mathcal{C}^0(S) = \{C_k \cap S \ : \ C_k \cap S \neq C_k, C_k \in \mathcal{C}\}$.

For the case of the bargaining for the profit division among employees and employers, $(N, v^{\mathcal{C}})$ can be seen as the situation where employees (employers) can negotiates with employers (employers) only in the case of unanimous agreement of all the members, and any sub group of employees (employers) can not contact with the employers (employers).

We obtain the following relationship between ψ^δ and the weighted Shapley value Sh^w. The following theorem holds.

Theorem 3.4. *Let $w = (w_i)_{i \in N}$ be a weight vector such that $w_i = \frac{1}{|C_k|}$ for all $i \in C_k \in \mathcal{C}$. Then,*

$$\psi^\delta(N, v, \mathcal{C}) = \mathrm{Sh}^w(N, v^{\mathcal{C}}). \tag{3.3}$$

Proof. Eq.(3.3) is obtained from Eq.(3.2) and the following fact. For $\theta \in \Theta(N)$, we define $\theta^* \in \Theta(M)$ by the condition that $\theta^*(k) > \theta^*(h)$ if and only if there exists $i \in C_k$ such that $\theta(i) > \theta(j)$ for all $j \in C_h$. Then, the probability that θ^* coincides with some order $\pi \in \Theta(M)$ when each θ appears according to the probability distribution $\mu^w(.)$ is just $\frac{1}{|M|!} = \frac{1}{|\Theta(M)|}$. □

Thus, Eq.(3.3) shows that ψ^δ is a weighted Shapley value for $\mathcal{C}$-communication restricted game. However, using reciprocal of the cardinality of a coalition as the weight of the member of the coalition does not seem to have much justification; rather usual non-weighted Shapley value appears to be more acceptable. The collective value ψ^γ, introduced in the next chapter, is such solution in a game with a coalition structure. Thus,

$$\psi^\gamma(N, v, \mathcal{C}) = \mathrm{Sh}(N, v^{\mathcal{C}}).$$

holds. The proof is provided in the next chapter.

[3]Carreras (1992) refers the similar restriction of coalition as “voting discipline” in the context of simple games.

Chapter 4

The collective value for games with coalition structures

4.1 Introduction

This chapter also studies a distribution rule of a cooperative surplus among players when they already partition themselves into 'coalitions' before realizing cooperation. A distribution rule, a solution concept in a framework of a cooperative game with a coalition structure, considered in this chapter departs from the existing solution concepts in two major directions. Similar to the Shapley-Egalitarian solution, the first point is to take into account the mutual-aid tendency of groups or generous allocation among members in the internal cooperation. The second is to treat the asymmetric sizes of coalitions as a factor affecting the bargaining outcome, which is not considered in Chapter 3. From the theoretical point of view, Kalai (1977) and Thomson (1986) show that in the context of bargaining problems, purely replications of players generate the size-dependent asymmetric weights of the Nash solution. On the other hand, from an empirical point of view, Metcalf, Wadsworth, and Ingram (1993) reported that in the observations of British manufacturing industry, strike incidence rose with the size of bargaining group, and it is known that the strike activity affects the bargaining outcome between employers and employees.[1]

The Aumann-Dreze value and the Owen's coalitional value, two traditional solution concepts in cooperative games with coalition structures and each of which is an extension of the Shapley value to a cooperative game with a coalition structure, do not satisfy both requirements mentioned above. On the one hand, both solutions give nothing to a null player whatever cooperation relationship he belongs to. Thus, according to these solution concepts, it does not happen that such player receives some portion of the cooperation surplus from his coalition due to its strong position, thus these solutions not having an essence of mutual assistance within the internal members. On the other hand, these solutions treat two distinct coalitions equally even if these are different in their sizes. As pointed out by Hart and Kurz (1983) and Winter (1992), a solution concept of a cooperative game with a coalition structure assumes the two levels interaction among players, *i.e.*, interactions inter- and intra- coalitions. In fact, the Owen's coalitional value satisfies the condition that the sum of the coalitional values of the players in a coalition coincides with the Shapley value of the coalition obtained from the game which is played by inter-coalitions. Thus, the coalitional value well describes the two levels interaction but not reflects an asymmetry in the interaction among coalitions pointed out by Kalai and Samet (1987)

[1]One reason is that most union power is partly derived from the threat of the strike (Ashenfelter and Johnson 1969).

and Levy and McLean (1989), caused by the different sizes of the coalitions.

The definition of our new solution concept, named the *collective value*, is established relying on a two-step bargaining process among players, a bargaining inter-coalitions in the first step and a bargaining intra-coalition in the second, and generous distribution tendency among the internal members. In the first step, each coalition obtains its *weighted* Shapley value applied for a game among coalitions. The pure surplus of a coalition in the first step bargaining (its weighted Shapley value obtained from the first step minus the worth of the coalition) is divided equally among players in the coalition. In the second step, players in the coalition receive their Shapley value applied for their own internal game. Thus, the collective value gives the sum of the payoffs in the first step and the second to each player. This definition indicates that the collective value is involved with the Egalitarian value as well as the Shapley value: the Egalitarian value is used for the bargaining surplus of a coalition and the Shapley value for the worth of the coalition. In fact, the collective value is interpreted as the rights-egalitarian solution, which is introduced by Herrero, Maschler, and Villar (1999), for an allocation problem, and we use the (weighted) Shapley value when we derive the allocation problem from the original game.

On the surface, our solution concept appears to lie in the very different line of research from existing studies. However, the collective value matches endogenous and exogenous interpretations of coalition structures. Aumann and Dreze (1974) discuss that the existing coalition structure arises from the *endogenous* formation of coalitions, given the game itself. They consider that lack of the superadditivity of the game leads to the formation of coalition structures. Here, we provide a different condition, a *quasi-partnership decomposition*, which is also considered as a reason of forming coalition structures, and show that the collective value is consistent with this this condition. Furthermore, in the line of Myerson (1977, 1980), a coalition structure can be considered as *exogenously* given communication restriction among players. We introduce a new interpretation of the coalition structure as restriction of communication among players, and show that the collective value coincides with the Shapley value applied for the game appropriately derived from the original game. Thus, the collective value is consistent with these interpretations of coalition structures.

Further, with the aid of research by Calvo and Santos (1997) and Bilbao (1998) on potential theory in cooperative games with communication restriction, we obtain a potential function for games with coalition structures, which is quite different from the one of Winter (1992). The collective value is expressed as the marginal contribution relative to this potential function. The potential function behind the solution concept inspires one of its properties similar to the balanced contributions of the Shapley value. We show that this property, called the *collective balanced contributions*, with some moderate additional conditions characterizes our solution. An axiomatization by the *additivity* axiom is also presented.

The rest of this chapter is organized as follows. The exact definition of a new solution concept is provided in Section 2. In Section 3, other expressions and interpretations of the solution are explained. In Section 4, we show that the collective value admits a potential function. Axiomatic characterizations of this solution are given in Section 5. Section 6 gives concluding remarks.

4.2 The collective value

As motivated by Hart and Kurz (1983) and Winter (1989), the coalition in $\mathcal{C}$ can be seen as a pressure group for the division of $v(N)$. So, van den Brink and van der Laan stated (van den Brink and van der Laan 2005, p195):

> to divide the worth of the grand coalition over all players, first this worth is dis-

> tributed over the coalitions in the a priori given coalition structure, and then the payoff assigned to a coalition is distributed over its players.

The Owen's coalitional value describes the above two level interactions, which are an interaction among coalitions and a one among players within a coalition, and has the consistent relation with the Shapley value's allocation. The coalitional value satisfies

$$\sum_{i \in C_k} \mathrm{CV}_i(N, v, \mathcal{C}) = \mathrm{CV}_k(M, v_{\mathcal{C}}, \{M\}) = \mathrm{Sh}_k(M, v_{\mathcal{C}})$$

for any $C_k \in \mathcal{C}$ because the coalitional value satisfies the intermediate game property and the coalitional value for a game with the grand coalition structure coincides with the Shapley value for the game.

There is an asymmetry of players in external game $(M, v_{\mathcal{C}})$ since players in the game represent the coalitions which may be different in size. In such a situation, the *weighted* Shapley value (Shapley 1953a) can be appropriate to deal with such asymmetries. Kalai and Samet stated in their paper (Kalai and Samet 1987, p221) as follows:

> It is important for applications in which the players themselves are, or are representing, groups of individuals. Such is the case for example when the players are parties, cities, or management boards. ... A natural candidate for a solution is the weighted Shapley value where the players are weighted by the size of the constituencies they stand for.

As the following definition will show, a new solution concept presented in this chapter is the very solution that reflects such a viewpoint in addition to the two level interactions.

A solution ψ^γ in a game with a coalition structure is defined in the following.

Definition 4.1. *For $(N, v, \mathcal{C}) \in$ **GC**, let w_k denote the weight for $k \in M$ such that $w_k = |C_k|$ and $w = (w_k)_{k \in M}$. Then, the collective value ψ^γ for $(N, v, \mathcal{C})$ is defined by*

$$\psi_i^\gamma(N, v, \mathcal{C}) = \frac{\mathrm{Sh}_k^w(M, v_{\mathcal{C}}) - v(C_k)}{|C_k|} + \mathrm{Sh}_i(C_k, v)$$

for any $i \in C_k \in \mathcal{C}$.

The definition of the collective value shows the close relation with the Shapley-Egalitarian solution defined in Chapter 3 and Kamijo (2007b). Only the difference is that in the definition of the collective value, the weighted Shapley value is applied to the external game. Moreover, the collective value can be interpreted as a two step approach: the first step is a negotiation among coalitions for the division of $v(N)$ and the second step is a negotiation among players for the division of the assignment of the coalition from the first step. The bargaining surplus of the coalition from the first step, $\mathrm{Sh}_k^w(M, v_{\mathcal{C}}) - v(C_k)$, is equally divided among its members. Moreover they obtain the Shapley value for their own game in the second step, $\mathrm{Sh}(C_k, v)$. Thus, this expression indicates that ψ^γ has a flavor of egalitarian rule in addition to the Shapley value: the Egalitarian value for the bargaining surplus of his coalition and the Shapley value for the worth of the coalition. As the result of this egalitarian part, ψ^γ does not satisfy the usual null player axiom but a weaker version. This point is considered in Section 6 to characterize the collective value by the additivity axiom.

The collective value is interpreted as the rights-egalitarian allocation rule for an allocation problem. An allocation problem, which is a triple (N, E, c), describes a situation that agents in N, each of who has his monetary entitlement $c_i \in \mathbb{R}, c = (c_i)_{i\in N}$, discuss for the division of the budged $E \in \mathbb{R}$. To interpret ψ^γ in the context of an allocation problem, consider the following bargaining environment for the members in $C_k \in \mathcal{C}$. If they realize the cooperation for themselves, the worth $v(C_k)$ is distributed for them by the Shapley value, and thus each $i \in C_k$ obtains $\mathrm{Sh}_i(C_k, v)$ in this case. On the other side, if they form a coalition and bargain with the outside coalitions $C_h, h = 1, \ldots, k-1, k+1, \ldots, m$, the coalition obtains its w-weighted Shapley value for $(M, v_{\mathcal{C}})$ with coalition-size weights. Then, this situation can be described as an allocation problem (C_k, E, c) where $E = \mathrm{Sh}^w_k(M, v_{\mathcal{C}})$ and $c_i = \mathrm{Sh}_i(C_k, v)$ for all $i \in C_k$. It is easily confirmed that $F^{RE}_i(C_k, E, c) = \psi^\gamma(N, v, \mathcal{C})$. Thus, ψ^γ is an allocation rule such that monetary entitlement of the players and the budget for the coalition is calculated by the Shapley value and the weighted Shapley value, and then the rights-egalitarian solution is applied for an allocation problem derived from $(N, v, \mathcal{C})$.

One difference between the collective value and the other values such as the coalitional value and the Shapley-Egalitarian solution is that in the definition above, each coalition, say C_k, receives $\mathrm{Sh}^w_k(M, v_{\mathcal{C}})$, *i.e.* the w-weighted Shapley value of the external game, instead of the usual Shapley value. Further, the weights are the sizes of each coalition, *i.e.*, $w_k = |C_k|$ for each $C_k \in \mathcal{C}$. From the definition, it is easily confirmed that ψ^γ satisfies

$$\sum_{i\in C_k} \psi^\gamma_i(N, v, \mathcal{C}) = \mathrm{Sh}^w_k(M, v_{\mathcal{C}}),$$

reflecting the asymmetries in the sizes of coalitions.

One may think that the definition of the collective value is a bid strange because it applies inconsistent treatment between a negotiation among coalitions and a negotiation within a coalition. However, this is not true; rather the collective value treats the two types of bargaining in consistent manner in terms of the players' sizes because in the subgame (C_k, v), the players in C_k are equal in their sizes, and the weighted Shapley value with equal weights among the players coincides with the Shapley value, that is, $\mathrm{Sh}^w(C_k, v) = \mathrm{Sh}(C_k, v)$, given $w_i = 1$ for all $i \in C_k$.

To obtain a better understanding on a two-step interpretation of ψ^γ, we introduce a "redistribution game" defined below. Let ϕ be a solution on **G** and $C_k \in \mathcal{C}$. Define a function $v^r(.|\phi) : 2^{C_k} \to \mathbb{R}$ by, for all $S \subseteq C_k$,

$$v^r(S|\phi) = \begin{cases} \phi_k(M, v_{\mathcal{C}}) & \text{if } S = C_k, \\ v(S) & \text{otherwise.} \end{cases}$$

A game $(C_k, v^r(.|\phi))$ is called a *redistribution game* for C_k over the coalitional bargaining surplus at distribution rule ϕ. Let $(N, v, \mathcal{C}) \in$ **GC**, $C_k \in \mathcal{C}$, and $M = \{k : C_k \in \mathcal{C}\}$. Let $w = (w_k)_{k\in M}$ with $w_k = |C_k|$ for any $k \in M$. The following theorem is easily derived from the definition of ψ^γ.

Theorem 4.1. *For $C_k \in \mathcal{C}$ and for $i \in C_k$,*

$$\psi^\gamma_i(N, v, \mathcal{C}) = \mathrm{Sh}_i(C_k, v^r(.|\mathrm{Sh}^w)).$$

Proof. Define (C_k, u) by $u(S) = \mathrm{Sh}^w_k(M, v_{\mathcal{C}}) - v(C_k)$ if $S = C_k$ and $u(S) = 0$ otherwise. Then, $v^r(.|\mathrm{Sh}^w) = v + u$. Since the Shapley value satisfies the additivity,

$$\mathrm{Sh}_i(C_k, v^r(.|\mathrm{Sh}^w)) = \mathrm{Sh}_i(C_k, v) + \mathrm{Sh}_i(C_k, u).$$

Furthermore, since the Shapley value satisfies the symmetry and the efficiency, $\mathrm{Sh}_i(C_k, u) = \frac{\mathrm{Sh}^w_k(M, v_{\mathcal{C}}) - v(C_k)}{|C_k|}$. □

Remark 4.1. *The Owen's coalitional value is also described as the Shapley value for the other type of redistribution game. For $C_k \in \mathcal{C}$, $(C_k, v^c(.|\phi))$ is defined by $v^c(S|\phi) = \phi_k(M, v^S_{\mathcal{C}})$ for all $S \subseteq C_k$ where $(M, v^S_{\mathcal{C}})$ is a game played by coalitions with C_k being replaced by $S \subseteq C_k$. That is, $v^S_{\mathcal{C}}(H) = \bigcup_{h\in H} C_h$ if $k \notin H$ and $v^S_{\mathcal{C}}(H) = \bigcup_{h\in H\setminus\{k\}} C_h \cup S$ if $k \in H$. Then, $\mathrm{CV}_i(N, v, \mathcal{C}) = \mathrm{Sh}_i(C_k, v^c(.|\mathrm{Sh}))$ holds (see, Owen 1977 and Winter 1992).*

Remark 4.2. *The Shapley-Egalitarian solution defined in Chapter 3 satisfies the following:*

$$\psi^\delta_i(N, v, \mathcal{C}) = \mathrm{Sh}_i(C_k, v^r(.|\mathrm{Sh}))$$

for any $C_k \in \mathcal{C}$ and for any $i \in C_k$.

4.3 Interpretations of the value and coalition structures

In this section, we consider an endogenous interpretation and an exogenous interpretation of coalition structures and show that the collective value fits these interpretations.

4.3.1 A value on $\mathcal{C}$-communication restricted situation

An exogenous interpretation of coalition structures is that they represent the some kinds of constraint on communication among players (see Aumann and Dreze 1974). Myerson (1977) considers a situation that a communication between players is restricted on an undirected graph of N (such game is called a graph-restricted game). Myerson (1980) considers more generalized situation that there is a sequence of conferences in which players communicate with each other and this communication restriction is expressed as the hyper-graph on N. Since Myerson's works, there are various kinds of research on games with restriction or constraint on communication among players (*e.g.*, a permission structure by Gilles, Owen, and van den Brink 1992; restricted coalitions by Derks and Peters 1993; a weighted hyper-graph by Amer and Carreras 1995, 1997; a probabilistic graph by Calvo, Lasaga, and van den Nouweland 1999; a partition system by Bilbao 1998).

Along this line of research, Aumann and Dreze's (1974) value, which is defined by $\mathrm{AD}_i(N, v, \mathcal{C}) = \mathrm{Sh}_i(C_k, v)$ for all $i \in C_k \in \mathcal{C}$, assumes a situation that a coalition structure describes a communication restriction such that players in the same coalition communicate with each other, but each coalition is physically separated. This situation is also described as the graph such that each maximal component of the graph corresponds to a coalition in the coalition structure and each subgraph on the component is a complete graph. Thus, Aumann and Dreze's value coincides with the Myerson value for such a graph situation. However, this interpretation of coalition structure does not fit the view that players form coalitions for the division of $v(N)$ since Aumann and Dreze's value does not satisfy the efficiency but the relative efficiency ($\sum_{i\in N} \mathrm{AD}_i(N, v, \mathcal{C}) = \sum_{k\in M} v(C_k)$). This motivates another view of communication restriction by a coalition structure. i.e., the $\mathcal{C}$-communication restricted situation defined in Chapter 3.

Let $(N, v^{\mathcal{C}})$ be a $\mathcal{C}$-communication restricted game of $(N, v, \mathcal{C})$. Then, ψ^γ is interpreted as a value on the $\mathcal{C}$-communication restricted game.

Theorem 4.2. *Let $(N, v, \mathcal{C}) \in$ **GC**. For $i \in N$,*

$$\psi^\gamma_i(N, v, \mathcal{C}) = \mathrm{Sh}_i(N, v^{\mathcal{C}}).$$

Proof. Take any order $\theta \in \Theta(N)$. Let $\theta[C_k]$ denote an order on C_k induced from θ such that for any $i, j \in C_k$, $\theta[C_k](i) < \theta[C_k](j)$ if and only if $\theta(i) < \theta(j)$, and let θ_M denote an order on M induced from θ such that for any $k, h \in M$, $\theta_M(k) < \theta_M(h)$ if and only if there is a player $i \in C_h$ such that $\theta(j) < \theta(i)$ for all $j \in C_k$. According to marginal contributions in $\mathcal{C}$-communication restricted game $v^{\mathcal{C}}$ at order θ, $i \in C_k$ obtains, when i is not the last in the order $\theta[C_k]$,

$$v(A_i^{\theta[C_k]} \cup i) - v(A_i^{\theta[C_k]}),$$

and when i is the last in the order, he obtains

$$[v\left(\bigcup_{h \in A_k^{\theta_M}} C_h \cup C_k\right) - v\left(\bigcup_{h \in A_k^{\theta_M}} C_h\right)] - v(C_k) + [v(C_k) - v(C_k - i)]$$

$$= m_k^{\theta_M}(M, v_{\mathcal{C}}) - v(C_k) + [v(C_k) - v(C_k - i)].$$

Because in the situation that each $\theta \in \Theta(N)$ occurs in equal probability, $\theta[C_k]$ coincides with one order on C_k in probability $1/|\Theta(C_k)|$, thus irrelevant to the selection of the order, and each $i \in C_k$ has a equal probability to be the last, we have

$$\mathrm{Sh}_i(N, v^{\mathcal{C}}) = \frac{1}{|C_k|} \frac{1}{|\theta(N)|} \sum_{\theta \in \Theta(N)} m_k^{\theta_M}(M, v_{\mathcal{C}}) - \frac{v(C_k)}{|C_k|} + \mathrm{Sh}_i(C_k, v).$$

Thus, it suffices to show that $\frac{1}{|\theta(N)|} \sum_{\theta \in \Theta(N)} m_k^{\theta_M}(M, v_{\mathcal{C}}) = \mathrm{Sh}_k^w(M, v_{\mathcal{C}})$.

We denote by $\mathrm{Prob}(\cdot)$ the probability that some phenomena happen in the situation that each $\theta \in \Theta(N)$ occurs in equal probability $1/|\Theta(N)|$. We will show that for any given order $\sigma \in \Theta(M)$, $\mathrm{Prob}(\theta_M = \sigma)$ is $\mu^w(\sigma)$ where $w_k = |C_k|$ for each $k \in M$. For simplifying explanation, let $\sigma = (\sigma_1, \ldots, \sigma_m)$ be $(1, \ldots, m)$. First, we consider the probability that $\theta_M(m)$ coincides with $\sigma_m = m$, that is, $\mathrm{Prob}(\theta_M(m) = m)$. Since this probability is equal to the probability that some player in C_m is the last position in order θ, we obtain

$$\mathrm{Prob}\left(\theta_M(m) = m\right) = \frac{|C_m|}{|N|} = \frac{w_m}{\sum_{h \in M} w_h}.$$

Further, assume that $\mathrm{Prob}\left(\theta_M(h) = h, h = k+1, \ldots, m\right) = \Pi_{h=k+1}^m \frac{w_h}{\sum_{h'=1}^h w_{h'}}$. Then, provided that $\theta_M(h) = h, h = k+1, \ldots, m$, the conditional probability that $\theta_M(k)$ coincides with k is

$$\frac{|C_k|}{\sum_{h'=1}^k |C_{h'}|} = \frac{w_k}{\sum_{h'=1}^k w_{h'}},$$

because this probability is equal to the probability that some $i \in C_k$ is the last player in the order that players in $C_h, h = k+1, \ldots, m$ are extracted from. Thus,

$$\begin{aligned}\mathrm{Prob}\left(\theta_M(h) = h, h = k, \ldots, m\right) &= \mathrm{Prob}\left(\theta_M(h) = h, h = k+1, \ldots, m\right) \\ &\quad \times \mathrm{Prob}\left(\theta_M(k) = k \mid \theta_M(h) = h, h = k+1, \ldots, m\right) \\ &= \Pi_{h=k}^m \frac{w_h}{\sum_{h'=1}^h w_{h'}}.\end{aligned}$$

Therefore, repeating this, we obtain $\mathrm{Prob}(\theta_M = \sigma) = \Pi_{h=1}^m \frac{w_h}{\sum_{h'=1}^h w_{h'}} = \mu^w(\sigma)$.

Thus, we have

$$\sum_{\theta\in\Theta(N)}\frac{1}{|\Theta(N)|}m_k^{\theta_M}(M,v_{\mathcal{C}})=\sum_{\sigma\in\Theta(M)}\mu^w(\sigma)m_k^{\sigma}(M,v_{\mathcal{C}})=\mathrm{Sh}_k^w(M,v_{\mathcal{C}}).$$

□

Remark 4.3. *The Shapley Egalitarian solution, ψ^δ, satisfies the following: Given $(N,v,\mathcal{C})\in$ **GC**, let $w=(w_i)_{i\in N}$ be such that $w_i=\frac{1}{|C_k|}$ for $i\in C_k\in\mathcal{C}$. Then,*

$$\psi^\delta(N,v,\mathcal{C})=\mathrm{Sh}^w(N,v^{\mathcal{C}}).$$

4.3.2 An endogenous interpretation of a coalition structure

Aumann and Dreze (1974) consider that one of the transparent explanations for the formation of coalition structures from games themselves is by the lack of the superadditivity (see the discussion of their paper). However, from the viewpoint that players form coalition structures for the bargaining of division of $v(N)$, we have to introduce another endogenous argument for the formation of coalition structures.

Let $(N,v)\in\mathbf{G}$. A coalition S is called a partnership in (N,v) if for any $T\subsetneq S$ and for any $R\subseteq N\setminus S$, $v(T\cup R)=v(R)$. Further, S is called a quasi-partnership in (N,v) if for any $T\subsetneq S$ and for any $R\subseteq N\setminus S$, $v(T\cup R)=v(T)+v(R)$. Thus, players in a quasi-partnership coalition T seem to have some rationale to act together because otherwise, they can not generate an additional surplus from cooperation with players outside T.

Let (N,v) be a game and $\mathcal{C}$ be a coalition structure on N. Then, $\mathcal{C}$ is called a *quasi-partnership decomposition* with respect to v if every $C_k\in C$ is a quasi-partnership in (N,v). The next theorem indicates that ψ^γ is consistent with this endogenous view of the coalition structure and the allocation of the Shapley value.

Theorem 4.3. *Let $(N,v,\mathcal{C})\in$ **GC**. If $\mathcal{C}$ is a quasi-partnership decomposition with respect to v, then*

$$\mathrm{Sh}(N,v)=\psi^\gamma(N,v,\mathcal{C}).$$

Proof. If $(N,v)=(N,v^{\mathcal{C}})$, Theorem 4.2 implies that $\mathrm{Sh}(N,v)=\mathrm{Sh}(N,v^{\mathcal{C}})=\psi^\gamma(N,v,\mathcal{C})$. Thus, it suffices to show $(N,v)=(N,v^{\mathcal{C}})$. For any $S\subseteq N$,

$$\begin{aligned}v^{\mathcal{C}}(S)&=v(\bigcup_{C_k\in\mathcal{C}(S)}C_k)+\sum_{T\in\mathcal{C}^0(S)}v(T)\\&=v(S)\end{aligned}$$

where the first equality is by the definition of $v^{\mathcal{C}}$ and the second is by the quasi-partnership of $C_k\in\mathcal{C}$. □

4.4 A potential function for games with coalition structures

Hart and Mas-Colell (1989) are the first to introduce a concept of a potential to cooperative game theory and show that a potential for a game exists (with an additional condition of the normalization, it is unique) and it derives the Shapley value. After Hart and Mas-Colell, the concept of potential was introduced to a non-cooperative game by Monderer and Shapley (1996) and has been considered for a cooperative game with several frameworks such as a game with a coalition structure by Winter (1992), a partition system by Bilbao (1998), a finite type continuum by Calvo and Santos (1997). Calvo and Santos (1997) also characterized the family of solutions which admitted a potential function.

Let P denote a real valued function on **G** which is normalized to $P(\emptyset, v) = 0$. Given $(N, v) \in$ **GC** and $i \in N$, define a marginal contribution of player i relative to P by

$$D_i P(N, v) = P(N, v) - P(N - i, v).$$

Thus, this marginal contribution is the difference of two situations measured by P where player i is there and he leaves. Function P is called a *potential* for games if it satisfies

$$v(N) = \sum_{i \in N} D_i P(N, v)$$

for any $(N, v) \in$ **GC**. Thus, a potential function is such that the allocation of marginal contributions (according to the potential function) always adds up exactly to the worth of the grand coalition. Hart and Mas-Colell (1989) show (in theorem A, p591) that (i) potential function P is uniquely determined, and (ii) the marginal contribution vector relative to the potential coincides with the Shapley value payoff vector, *i.e.*, $D_i P(N, v) = \mathrm{Sh}_i(N, v)$ for all $i \in N$.

They also consider a non-symmetric generalization of a potential approach. Let $w = (w_i)_i$ be a collection of the positive weights and P^w denote a real-valued function on **G** with $P^w(\emptyset, v) = 0$. Function P^w is called the *w-weighted potential* if it satisfies

$$v(N) = \sum_{i \in N} w_i D_i P(N, v)$$

for any $(N, v) \in$ **G**. They show (in theorem 5.2, p603) that (i) w-weighted potential function P^w is uniquely determined, and (ii) the marginal contribution relative to the potential multiplied by the corresponding weight coincides with the w-weighted Shapley value, *i.e.*, $w_i D_i P(N, v) = \mathrm{Sh}_i^w(N, v)$ for any $i \in N$.

According to Calvo and Santos (1997) and Bilbao (1998), a potential function for a game with a restricted communication is Hart and Mas-Colell's potential function (hereafter, the HM potential function. Similarly we use the term, the HM w-weighted potential function.) for the corresponding game which is appropriately defined to reflect the restriction on communication. Thus, the next theorem is an immediate consequence of Theorem 4.2.

Theorem 4.4. *Let P : **G** $\to \mathbb{R}$ denote the HM potential function. Then, given any $(N, v, \mathcal{C})$,*

$$\psi_i^\gamma(N, v, \mathcal{C}) = D_i P(N, v^{\mathcal{C}}) = P(N, v^{\mathcal{C}}) - P(N - i, v^{\mathcal{C}})$$

for any $i \in N$.

Bilbao (1998) also shows (in theorem 2, p135) that given a partition system $(N, \mathcal{F})$, for $S \subseteq N$ with $S \notin \mathcal{F}$,

$$P(S, v^{\mathcal{F}}) = \sum_{T \in \Pi_S} P(T, v^{\mathcal{F}}).$$

Hence, this result together with $(N, v^{\mathcal{C}}) = (N, v^{\mathcal{F}^{\mathcal{C}}})$ by Proposition 4.1 implies that for $i \in C_k \in \mathcal{C}$,

$$D_i P(N, v^{\mathcal{C}}) = P(N, v^{\mathcal{C}}) - P(N \setminus C_k, v^{\mathcal{C}}) - P(C_k - i, v^{\mathcal{C}}). \tag{4.1}$$

The next proposition gives another formula of $P(N, v^{\mathcal{C}})$ which seems to describe the restriction of communication by $\mathcal{C}$ and which is specific expression of the potential for the particular subclass of games with permission systems, which is different from the class Bilbao (1998) mainly considers.

Proposition 4.1. *Let* $(N, v, \mathcal{C}) \in$ ***GC*** *and* $M = \{k : C_k \in \mathcal{C}\}$*. Define a game* (M, u) *by*

$$u(L) = v(\bigcup_{k \in L} C_k) - \sum_{k \in L} v(C_k) + \sum_{k \in L} |C_k| P(C_k, v)$$

for each $L \subseteq M$*. Then,*

$$P(N, v^{\mathcal{C}}) = P^w(M, u),$$

where P^w *is the HM* w*-weighted potential function and* $w = (w_k)_{k \in M}$ *is such that* $w_k = |C_k|$ *for any* $k \in M$*.*

Proof. Let $(N, v, \mathcal{C}) \in$ **GC** be given. Put $M = \{k : C_k \in \mathcal{C}\}$. The proof proceeds by the way of mathematical induction of the number of $|M|$. For any $C_k \in \mathcal{C}$,

$$P(C_k, v^{\mathcal{C}}) = P(C_k, v) = \frac{1}{|C_k|} u(\{k\}) = P^w(\{k\}, u),$$

where the first equality is by $(C_k, v^{\mathcal{C}}) = (C_k, v)$, the second is by the definition of u, and the last is by the definition of the HM w-weighted potential function and $w_k = |C_k|$.

Assume that for any $L \subsetneq M$ and $N' = \bigcup_{k \in L} C_k$, $P(N', v^{\mathcal{C}}) = P^w(L, u)$ holds. We consider the case for $(N, v^{\mathcal{C}})$. By the definition of the potential function,

$$\begin{aligned}
v(\bigcup_{k \in M} C_k) &= v(N) \\
&= \sum_{i \in N} D_i P(N, v^{\mathcal{C}}) \\
&= \sum_{k \in M} \sum_{i \in C_k} \big(P(N, v^{\mathcal{C}}) - P(N \setminus C_k, v^{\mathcal{C}}) - P(C_k - i, v^{\mathcal{C}}) \big) \\
&= \sum_{k \in M} |C_k| \big(P(N, v^{\mathcal{C}}) - P(N \setminus C_k, v^{\mathcal{C}}) \big) \\
&\quad - \sum_{k \in M} |C_k| P(C_k, v^{\mathcal{C}}) + \sum_{k \in M} \sum_{i \in C_k} (P(C_k, v^{\mathcal{C}}) - P(C_k - i, v^{\mathcal{C}})) \\
&= \sum_{k \in M} |C_k| \big(P(N, v^{\mathcal{C}}) - P(N \setminus C_k, v^{\mathcal{C}}) \big) \\
&\quad - \sum_{k \in M} |C_k| P(C_k, v^{\mathcal{C}}) + \sum_{k \in M} \sum_{i \in C_k} \mathrm{Sh}_i(C_k, v^{\mathcal{C}}) \\
&= \sum_{k \in M} |C_k| \big(P(N, v^{\mathcal{C}}) - P(N \setminus C_k, v^{\mathcal{C}}) \big) - \sum_{k \in M} |C_k| P(C_k, v) + \sum_{k \in M} v(C_k),
\end{aligned}$$

where the third equality is by Equation (4.1), the second last equality is by one of the properties of the HM potential function, and the last is by the efficiency of the Shapley value and $(C_k, v^{\mathcal{C}}) = (C_k, v)$. Hence we obtain

$$\sum_{k \in M} |C_k| \big(P(N, v^{\mathcal{C}}) - P(N \setminus C_k, v^{\mathcal{C}}) \big) = v(\bigcup_{k \in M} C_k) + \sum_{k \in M} |C_k| P(C_k, v) - \sum_{k \in M} v(C_k),$$

By the assumption of the induction and the definition of u, this is equivalent to

$$\sum_{k\in M} w_k \left(P(N, v^{\mathcal{C}}) - P^w(M \setminus \{k\}, u)\right) = u(M),$$

where $w_k = |C_k|$ for any $k \in M$. Therefore the uniqueness of the weighted potential implies $P(N, v^{\mathcal{C}})$ must be $P^w(M, u)$. □

Thus, the above proposition shows that the potential for $(N, v, \mathcal{C})$ is expressed as the w-HM potential function for a game (M, u) which is a linear transformation of the external game $(M, v_{\mathcal{C}})$ on the utility space of the players. That is, for any $L \subseteq M$, $u(L) = \alpha v_{\mathcal{C}}(L) + \sum_{k\in L} \beta_k$ where $\alpha = 1$ and $\beta_k = |C_k|P(C_k, v) - v(C_k)$ for each $k \in L$. Because the HM w-weighted potential function is independent of the linear transformation in the sense that $P^w(N, \alpha v + \beta) = \alpha P^w(N, v) + \sum_{i\in N}(\beta_i/w_i)$,[2] we therefore conclude

$$P(N, v^{\mathcal{C}}) = P^w(M, u) = P^w(M, v_{\mathcal{C}}) + \sum_{k\in M} \left(P(C_k, v) - \frac{v(C_k)}{|C_k|}\right)$$

where $w_k = |C_k|$ for any $k \in M$.

4.5 Axiomatic characterizations

4.5.1 Collective balanced contributions

The balanced contributions property for the Shapley value was first considered by Myerson (1980). It means that any two players' marginal contributions to the other measured by the Shapley value balance. Formally, the Shapley value satisfies, given two players $i \in N$ and $j \in N$, $\mathrm{Sh}_i(N, v) - \mathrm{Sh}_i(N - j, v) = \mathrm{Sh}_j(N, v) - \mathrm{Sh}_j(N - i, v)$. Myerson (1980) shows that efficiency and this property characterize the Shapley value.

Extensions of the balanced contributions to a game with a coalition structure is considered by Calvo, Lasaga, and Winter (1996). They introduce two counterparts of the balanced contributions to that case and show that a unique efficient solution on **GC** satisfying these two properties is the Owen's coalitional value. These two are:

Individual Balanced Contributions: For $i \in C_k$ and $j \in C_k$, $C_k \in \mathcal{C}$,

$$\psi_i(N, v, \mathcal{C}) - \psi_i(N - j, v, \mathcal{C} - j) = \psi_j(N, v) - \psi_j(N - i, v, \mathcal{C} - i)$$

where for $i \in C_k \in \mathcal{C}$, $\mathcal{C} - i = \mathcal{C} \setminus \{C_k\} \cup \{C_k - i\}$.

Coalitional Balanced Contributions: For $C_k \in \mathcal{C}$ and for $C_h \in \mathcal{C}$,

$$\sum_{i\in C_k} \left(\psi_i(N, v, \mathcal{C}) - \psi_i(N \setminus C_h, v, \mathcal{C} \setminus \{C_h\})\right) = \sum_{i\in C_h} \left(\psi_i(N, v, \mathcal{C}) - \psi_i(N \setminus C_k, v, \mathcal{C} \setminus \{C_k\})\right).$$

We introduce different extensions of the balanced contributions for games with coalition structures. One is just the same requirement as the condition for the Shapley value, and the other is interpreted as an intermediate between Individual Balanced Contributions and Coalitional Balanced Contributions in the sense that "balance" is judged in the individual level, and contributions of a group instead of contributions of an individual are considered.

[2]Given $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^N$, a game $(N, \alpha v + \beta)$ is defined by $(\alpha v + \beta)(S) = \alpha v(S) + \sum_{i\in S} \beta_i$ for all $S \subseteq N$.

Balanced Contributions: For $i \in N$ and $j \in N$,

$$\psi_i(N, v, \{N\}) - \psi_i(N - j, v, \{N - j\}) = \psi_j(N, v, \{N\}) - \psi_j(N - i, v, \{N - i\}).$$

Collective Balanced Contributions: For every $i \in C_k \in \mathcal{C}$ and for every $j \in C_h \in \mathcal{C}$, $C_k \neq C_h$,

$$\psi_i(N, v, \mathcal{C}) - \psi_i(N \setminus C_h, v, \mathcal{C} \setminus \{C_h\}) = \psi_j(N, v, \mathcal{C}) - \psi_j(N \setminus C_k, v, \mathcal{C} \setminus \{C_k\}).$$

Since $(N, v, \{N\})$ is looked as the same situation as (N, v),[3] Balanced Contributions is the same requirement as the one which the Shapley value satisfies, and thus we use the same name. Collective Balanced Contributions requires that 'my group's contribution for your payoff measured by the solution balances with your group's contribution for my payoff measured by the solution.'

On the relationship between our axioms and ones of Calvo, Lasaga, and Winter (1996), Individual Balanced Contributions implies Balanced Contributions. Collective Balanced Contributions induces Coalitional Balanced Contributions only if $|C_k| = |C_h|$. However, in general, there is no general relationship between Collective Balanced Contributions and Coalitional Balanced Contributions. The next proposition shows that ψ^γ satisfies Balanced Contributions and Collective Balanced Contributions instead of Individual Balanced Contributions and Coalitional Balanced Contributions.

Proposition 4.2. *ψ^γ satisfies Balanced Contributions and Collective Balanced Contributions.*

Proof. First consider the case of $|\mathcal{C}| = 1$. By definition of ψ^γ, $\psi^\gamma(N, v, \{N\}) = \mathrm{Sh}(N, v)$ holds. We obtain the desired result because of the result of Myerson (1980).

Next we consider the case of $|\mathcal{C}| \geqq 2$. Note that by the definition of $\mathcal{C}$-communication restricted game, $(N \setminus C_k, v^{\mathcal{C}})$ which is a subgame of $(N, v^{\mathcal{C}})$ on $N \setminus C_k$, coincides with $(N \setminus C_k, v^{\mathcal{C} \setminus \{C_k\}})$ which is $\mathcal{C} \setminus \{C_k\}$-communication restricted game for game $(N \setminus \{C_k\}, v, \mathcal{C} \setminus \{C_k\})$. By Theorem 4.2, the property of the HM potential function and Equation (4.1), we have

$$\begin{aligned}
\psi_i^\gamma(N, v, \mathcal{C}) - {}&\psi_i^\gamma(N \setminus C_h, v, \mathcal{C} \setminus \{C_h\}) \\
&= \mathrm{Sh}_i(N, v^{\mathcal{C}}) - \mathrm{Sh}_i(N \setminus C_h, v^{\mathcal{C}}) \\
&= P(N, v^{\mathcal{C}}) - P(N \setminus C_k, v^{\mathcal{C}}) - P(C_k - i, v^{\mathcal{C}}) \\
&\quad - \big(P(N \setminus C_h, v^{\mathcal{C}}) - P(N \setminus (C_k \cup C_h), v^{\mathcal{C}}) - P(C_k - i, v^{\mathcal{C}})\big) \\
&= P(N, v^{\mathcal{C}}) - P(N \setminus C_k, v^{\mathcal{C}}) - \big(P(N \setminus C_h, v^{\mathcal{C}}) - P(N \setminus (C_k \cup C_h), v^{\mathcal{C}})\big) \\
&= P(N, v^{\mathcal{C}}) - P(N \setminus C_h, v^{\mathcal{C}}) - \big(P(N \setminus C_k, v^{\mathcal{C}}) - P(N \setminus (C_k \cup C_h), v^{\mathcal{C}})\big) \\
&= P(N, v^{\mathcal{C}}) - P(N \setminus C_h, v^{\mathcal{C}}) - P(C_h - j, v^{\mathcal{C}}) \\
&\quad - \big(P(N \setminus C_k, v^{\mathcal{C}}) - P(N \setminus (C_k \cup C_h), v^{\mathcal{C}}) - P(C_h - j, v^{\mathcal{C}})\big) \\
&= \mathrm{Sh}_j(N, v^{\mathcal{C}}) - \mathrm{Sh}_j(N \setminus C_k, v^{\mathcal{C}}) \\
&= \psi_j^\gamma(N, v, \mathcal{C}) - \psi_j^\gamma(N \setminus C_k, v, \mathcal{C} \setminus \{C_k\}).
\end{aligned}$$

□

This proposition means that, by the definition of ψ^γ, for every $C_k \in \mathcal{C}$ and for every $C_h \in \mathcal{C}$,

$$\frac{\mathrm{Sh}_k^w(M, v_{\mathcal{C}}) - \mathrm{Sh}_k^w(M \setminus \{h\}, v_{\mathcal{C}})}{|C_k|} = \frac{\mathrm{Sh}_h^w(M, v_{\mathcal{C}}) - \mathrm{Sh}_h^w(M \setminus \{k\}, v_{\mathcal{C}})}{|C_h|}.$$

[3]In fact, all the values for games with coalition structures considered in this chapter, AD, CV, ψ^δ, and ψ^γ, for $(N, v, \{N\})$ coincide with the Shapley value for (N, v).

This is the special case of the properties of the w-weighted Shapley value: For $(N, v) \in \mathbf{G}$, its weight $(w_i)_{i\in N}$, and for every $i, j \in N$,[4]

$$\frac{\mathrm{Sh}_i^w(N, v) - \mathrm{Sh}_i^w(N \setminus \{j\}, v)}{w_i} = \frac{\mathrm{Sh}_j^w(N, v) - \mathrm{Sh}_j^w(N \setminus \{i\}, v)}{w_j}.$$

Next theorem shows that Balanced Contributions and Collective Balanced Contributions are almost sufficient to characterize ψ^γ.

Theorem 4.5. *ψ^γ is a unique efficient solution satisfying the following two properties:*

(i) Balanced Contributions.
(ii) Collective Balanced Contributions.

Proof. We have known that ψ^γ satisfies efficiency, Balanced Contributions and Collective Balanced Contributions. Hence we will show the converse.

Let ψ be an efficient solution satisfying these two axioms. Fix $(N, v, \mathcal{C}) \in \mathbf{GC}$. We first show that ψ coincides with the Shapley value when $|\mathcal{C}| = 1$ or n. When $|\mathcal{C}| = n$, Collective Balanced Contributions coincides with balanced contributions. Because of the result of Myerson (1980), $\psi(N, v, [N]) = \mathrm{Sh}(N, v)$ where $[N] = \{\{i\} : i \in N\}$. Moreover, by Balanced Contributions, the same argument means that $\psi(N, v, \{N\}) = \psi(N, v, [N]) = \mathrm{Sh}(N, v)$.

Next we show the following claims.

Claim 1: For all $C_k \in \mathcal{C}$,

$$\sum_{i\in C_k} \psi_i(N, v, \mathcal{C}) = |C_k|\, D_k P^w(M, v_{\mathcal{C}}) \tag{4.2}$$

where P^w is the HM w-weighted potential function with weight vector $w = (w_k)_{k\in M}$ such that $w_k = |C_k|$ for each $k \in M$.

Let $(C_k, v, \{C_k\})$ be a subgame of $(N, v, \mathcal{C})$ to coalition C_k. Then the left hand side of (4.2) is

$$\sum_{i\in C_k} \psi_i(C_k, v, \{C_k\}) = v(C_k)$$

by the efficiency of ψ. The right hand side of (4.2) is

$$|C_k|\, D_k P^w(\{k\}, v_{\mathcal{C}}) = |C_k| \frac{v(C_k)}{|C_k|} = v(C_k).$$

Thus, condition (4.2) holds true for any subgame $(C_k, v, \{C_k\})$ of $(N, v, \mathcal{C})$.

We assume that (4.2) is satisfied for any $(N', v, \mathcal{C}')$ such that $L \subsetneq M$, $N' = \cup_{k\in L} C_k$ and $\mathcal{C}' = \{C_k :\ k \in L\}$. We now show that it holds true for $(N, v, \mathcal{C})$.

Condition (4.2) is equivalent to

$$\sum_{i\in C_k} \psi_i(N, v, \mathcal{C}) = |C_k|(P^w(M, v_{\mathcal{C}}) - P^w(M \setminus \{k\}, v_{\mathcal{C}})).$$

Equivalently,

$$P^w(M, v_{\mathcal{C}}) = \frac{\sum_{i\in C_k} \psi_i(N, v, \mathcal{C})}{|C_k|} + P^w(M \setminus \{k\}, v_{\mathcal{C}}).$$

[4] This property is pointed out in Hart and Mas-Colell (1989) and Amer and Carreras (1997).

We show that $\frac{\sum_{i\in C_k}\psi_i(N,v,\mathcal{C})}{|C_k|}+P^w(M\setminus\{k\},v_{\mathcal{C}})$ does not depend on $k\in M$. Take any $C_k\in\mathcal{C}$ and $C_h\in\mathcal{C}, C_k\neq C_h$. Then,

$$\frac{\sum_{i\in C_k}\psi_i(N,v,\mathcal{C})}{|C_k|}+P^w(M\setminus\{k\},v_{\mathcal{C}})-\left(\frac{\sum_{j\in C_h}\psi_j(N,v,\mathcal{C})}{|C_h|}+P^w(M\setminus\{h\},v_{\mathcal{C}})\right)$$

equal

$$\frac{\sum_{i\in C_k}\psi_i(N,v,\mathcal{C})}{|C_k|}-\frac{\sum_{j\in C_h}\psi_j(N,v,\mathcal{C})}{|C_h|}+\Big(P^w(M\setminus\{k\},v_{\mathcal{C}})-P^w(M\setminus\{h\},v_{\mathcal{C}})\Big). \quad (4.3)$$

The bracketed terms in (4.3) equals

$$P^w(M\setminus\{k\},v_{\mathcal{C}})-P^w(M\setminus\{k,h\},v_{\mathcal{C}})-\big(P^w(M\setminus\{h\},v_{\mathcal{C}})-P^w(M\setminus\{k,h\},v_{\mathcal{C}})\big).$$

By the definition of operator D and the induction hypothesis,

$$\begin{aligned}&=D_hP^w(M\setminus\{k\},v_{\mathcal{C}})-D_kP^w(M\setminus\{h\},v_{\mathcal{C}})\\&=\frac{\sum_{j\in C_h}\psi_j(N\setminus C_k,v,\mathcal{C}\setminus\{C_k\})}{|C_h|}-\frac{\sum_{i\in C_k}\psi_i(N\setminus\{C_h\},v,\mathcal{C}\setminus\{C_h\})}{|C_k|}.\end{aligned}$$

Substitute the above for the bracketed terms in (4.3), and we obtain

$$\begin{aligned}&\frac{\sum_{i\in C_k}\psi_i(N,v,\mathcal{C})}{|C_k|}-\frac{\sum_{j\in C_h}\psi_j(N,v,\mathcal{C})}{|C_h|}\\&\qquad+\frac{\sum_{j\in C_h}\psi_j(N\setminus C_k,v,\mathcal{C}\setminus\{C_k\})}{|C_h|}-\frac{\sum_{i\in C_k}\psi_i(N\setminus\{C_h\},v,\mathcal{C}\setminus\{C_h\})}{|C_k|}.\end{aligned}$$

Note that by Collective Balanced Contributions, $\psi_i(N,v,\mathcal{C})-\psi_i(N\setminus\{C_h\},v,\mathcal{C}\setminus\{C_h\})=\psi_j(N,v,\mathcal{C})-\psi_j(N\setminus\{C_k\},v,\mathcal{C}\setminus\{C_k\})$ is constant for every $i\in C_k$ and for every $j\in C_h$. Hence the above expression is zero and thus, (4.3) equals zero.

Therefore for some real number K which does not depend on $k\in M$,

$$\frac{\sum_{i\in C_k}\psi_i(N,v,\mathcal{C})}{|C_k|}+P^w(M\setminus\{k\},v_{\mathcal{C}})=K$$

holds true for any $k\in M$.

Then by efficiency of ψ, we obtain that

$$v_{\mathcal{C}}(M)=v(N)=\sum_{k\in M}\sum_{i\in C_k}\psi_i(N,v,\mathcal{C})=\sum_{k\in M}|C_k|(K-P^w(M\setminus\{k\},v_{\mathcal{C}}))$$

Therefore K is exactly the HM w-weighted potential function $P^w(M,v_{\mathcal{C}})$ because of its uniqueness.

Next we show the following claim.

Claim 2: $\psi_i(N,v,\mathcal{C})=\bar{C}+\psi_i(C_k,v,\{C_k\})$ for all $i\in C_k$ where $\bar{C}$ is a real number which does not depend on $i\in C_k$.

We prove Claim 2 by the induction on the cardinality of $\mathcal{C}$. When $|\mathcal{C}|=1$, this is obvious because we simply put $\bar{C}=0$.

Assume that the claim holds true when the number of elements in $\mathcal{C}$ is less than m $(m \geqq 2)$. For $(N, v, \mathcal{C})$ such that $|\mathcal{C}| = m$, by Collective Balanced Contributions, given $C_h \in \mathcal{C}$, we have

$$\psi_i(N, v, \mathcal{C}) - \psi_i(N \setminus \mathcal{C}_h, v, \mathcal{C} \setminus \{C_h\}) = \bar{C}_1 \quad \text{for every } i \in C_k, C_k \neq C_h$$

By the assumption of the induction, the left hand side of the above equation is

$$\psi_i(N, v, \mathcal{C}) - (\bar{C}_2 + \psi_i(C_k, v, \{C_k\})),$$

where $\bar{C}_2$ is constant for all $i \in C_k$. Therefore we obtain

$$\psi_i(N, v, \mathcal{C}) = \bar{C}_1 + \bar{C}_2 + \psi_i(C_k, v\{C_k\}) = \bar{C} + \psi_i(C_k, v, \{C_k\}).$$

This is the desired result.

By Claim 1, we know that the summation of $\psi_i(N, v, \mathcal{C})$ over $i \in C_k$ is exactly $|C_k|\, D_k P^w(M, v_{\mathcal{C}}) = \mathrm{Sh}_k^w(M, v_{\mathcal{C}})$. Then we conclude that

$$\bar{C} = \frac{\mathrm{Sh}_k^w(M, v_{\mathcal{C}}) - \sum_{i \in C_k} \psi_i(C_k, v, \{C_k\})}{|C_k|} = \frac{\mathrm{Sh}_k^w(M, v_{\mathcal{C}}) - v(C_k)}{|C_k|}$$

by efficiency of ψ. Therefore if $\psi_i(C_k, v, \{C_k\})$ is uniquely determined, $\psi(N, v, \mathcal{C})$ is also determined. However when $|\mathcal{C}| = 1$, we have shown that ψ equals the Shapley value Sh. Hence we obtain

$$\psi_i(N, v, \mathcal{C}) = \frac{\mathrm{Sh}_k^w(M, v_{\mathcal{C}}) - v(C_k)}{|C_k|} + \mathrm{Sh}_i(C_k, v).$$

□

As in the proof of Theorem 4.5, Balanced Contributions is necessary only to prove that if $\mathcal{C} = \{N\}$, the solution coincides with the Shapley value for (N, v). Thus the following theorems also hold.

Theorem 4.6. *ψ^γ is a unique efficient solution satisfying the following two properties:*

*(i) $\psi(N, v, \{N\}) = \mathrm{Sh}(N, v)$ for all $(N, v) \in$ **G**.*
(ii) Collective Balanced Contributions.

Theorem 4.7. *ψ^γ is a unique efficient solution satisfying the following two properties:*

*(i) Coincidence between the Grand and the Singleton Coalition Structure: For all $(N, v) \in$ **G**, $\psi(N, v, \{N\}) = \psi(N, v, [N])$,*
(ii) Collective Balanced Contributions.

4.5.2 Additivity

In this subsection, we provide an axiomatization of ψ^γ through the additivity axiom. Let ψ be a solution on **GC**. Let $(N, v, \mathcal{C}), (N, v', \mathcal{C}) \in$ **GC**.

Theorem 4.8. *ψ^γ is a unique efficient solution on **GC** satisfying the following four axioms.*

(i) Additivity: $\psi(N, v, \mathcal{C}) + \psi(N, v', \mathcal{C}) = \psi(N, v + v', \mathcal{C})$, where $(v + v')(S) = v(S) + v'(S)$ for all $S \subseteq N$.

(ii) Equal Power of Partnership Members: If $T \subseteq N$ is a partnership in (N, v) and $M' = \{k \in M : C_k \cap T \neq \emptyset\}$ is also a partnership in $(M, v_{\mathcal{C}})$, then $\psi_i(N, v, \mathcal{C}) = \psi_j(N, v, \mathcal{C})$ for any $i, j \in T$.

(iii) Internal Equity: If $i \in C_k$ and $j \in C_k$ are symmetric in (C_k, v), then $\psi_i(N, v, \mathcal{C}) = \psi_j(N, v, \mathcal{C})$.

(iv) Coalitional Null Player: If C_k is a dummy coalition (i.e., k is a dummy player in $(M, v_{\mathcal{C}})$) and $i \in C_k$ is a null player in (N, v), then $\psi_i(N, v, \mathcal{C}) = 0$.

Equal Power of Partnership Members says that all the members of partnership T obtain the equal payoff if its projection on M is also a partnership in $(M, v_{\mathcal{C}})$. The *extended* Shapley value defined by $\overline{\mathrm{Sh}}(N, v, \mathcal{C}) = \mathrm{Sh}(N, v)$ satisfies Equal Power of Partnership Members because all the players in T are symmetric in (N, v) and the Shapley value assigns equal payoff to symmetric players.

Internal Equity and Coalitional Null Player are the axioms introduced in Chapter 3 which togethering with efficiency and Additivity characterize ψ^{δ}. Internal Equity is stronger than the restricted equal treatment property which both the extended Shapley value and the Owen's coalitional value satisfy. Coalitional Null Player is weaker than the usual null player axiom. Furthermore the extended Shapley value satisfies the all the properties except for Internal Equity and the Owen's coalitional value does not satisfy Equal Power of Partnership Members nor Internal Equity.

Compared with Theorem 3.1 that characterizes the Shapley-Egalitarian solution, the differences between the collective value and the Shapley-Egalitarian solution lies in the axioms of Equal Power of Partnership Members and Coalitional Symmetry. More clearly, the collective value does not treat symmetric coalitions equally. Instead, the collective value requires that even the two players in the different coalitions should be treated equally if they are in the relation described by the partnership.

The next lemma is from Kalai and Samet (1987).

Lemma 4.1. *Let $w \in \mathbb{R}^N_{++}$ be a weight vector of N. If T is a partnership in (N, v), then $\mathrm{Sh}^w_i(N, v)/w_i = \mathrm{Sh}^w_j(N, v)/w_j$ for all $i, j \in T$.*

Proof. See the proof of Theorem 2 of Kalai and Samet (1987). □

Lemma 4.2. *Let ϕ be an efficient solution on **G** such that it satisfies the symmetry and the null player axioms. Let ψ be a two step solution on **G** defined by*

$$\psi_i(N, v, \mathcal{C}) = \frac{\mathrm{Sh}^w_k(M, v_{\mathcal{C}}) - v(C_k)}{|C_k|} + \phi_i(C_k, v).$$

*for all $(N, v, \mathcal{C}) \in$ **G** and for all $i \in C_k \in \mathcal{C}$, where $w = (w_k)_{k \in M}$ is such that $w_k = |C_k|$ for all $k \in M$. Then, ψ satisfies Equal Power of Partnership Members.*

Proof. Let $T \subseteq N$ be a partnership in (N, v) and $M' = \{k \in M : C_k \cap T \neq \emptyset\}$ be also a partnership in $(M, v_{\mathcal{C}})$. Let $k \in M'$.

Suppose $|M'| \geqq 2$. Since T is a partnership in (N, v), $v(S \cup C) = v(S)$ for any $S \subseteq C_k \setminus T$ and $C \subseteq T \cap C_k \subsetneq T$. Thus, for any $S \subseteq C_k$, $v(S) = v(S \cap (C_k \setminus T))$ and thus, any $i \in T \cap C_k$ is a null player in subgame (C_k, v). So $\phi_i(C_k, v) = 0$ for any $i \in T \cap C_k$ since ϕ satisfies the null player axiom. Because $w_k = |C_k|$ for any $k \in M$, $\frac{\mathrm{Sh}^w_k(M, v_{\mathcal{C}})}{|C_k|} = \frac{\mathrm{Sh}^w_h(M, v_{\mathcal{C}})}{|C_h|}$ for any $k, h \in M'$

by Lemma 4.1. By the partnership of M' in $(M, v_{\mathcal{C}})$, $v_{\mathcal{C}}(\{k\}) = v(C_k) = 0$ for each $k \in M'$. Thus $\psi_i(N, v, \mathcal{C}) = \psi_j(N, v, \mathcal{C})$ holds for any $i, j \in T$.

Suppose $|M'| = 1$. Since $T \subseteq C_k$ is a partnership in (N, v), all the players in T are symmetric in (N, v) and, of course, they are symmetric in (C_k, v). Thus $\phi_i(C_k, v)$ is constant over $i \in T$. Hence ψ satisfies Equal Power of Partnership Members. □

Proof of Theorem 4.8. From Lemma 4.2, we have shown that ψ^γ satisfies Equal Power of Partnership Members since the Shapley value satisfies the symmetry and the null player axioms. Furthermore, it is easy to check that it satisfies axioms (i), (iii) and (iv) by its definition.

Next we show the converse part. Let ψ be an efficient solution on **GC** which satisfies axioms (i) to (iv). Let $(N, v, \mathcal{C}) \in$ **GC**. Since ψ satisfies Additivity, it is sufficient to show that $\psi(N, cu_T, \mathcal{C})$ is uniquely determined for any $T \subseteq N$, where $c \in \mathbb{R}$ and cu_T is a scalar multiple of u_T by c. Let $D = \{k \in M : C_k \cap T \neq \emptyset\}$. Since $C_k \in \mathcal{C}$, $k \notin D$, is a dummy coalition and $i \in C_k$ is a null player, $\psi_i(N, cu_T, \mathcal{C}) = 0$ by Coalitional Null Player. Furthermore, efficiency means that $\sum_{k \in D} \sum_{i \in C_k} \psi_i(N, cu_T, \mathcal{C}) = c$.

Clearly T is a partnership in (N, cu_T) and D is also a partnership in $(M, (cu_T)_{\mathcal{C}})$. Therefore $\psi_i(N, cu_T, \mathcal{C}) = \psi_j(N, cu_T, \mathcal{C})$ for all $i, j \in T$ by Equal Power of Partnership Members.

Case a: $|D| = 1$. Let $k \in D$. Since C_k is a dummy coalition and $i \in C_k \setminus T$ is a null player, $\psi_i(N, cu_T, \mathcal{C}) = 0$ by Coalitional Null Player. Furthermore, $\psi_i(N, cu_T, \mathcal{C}) = \frac{c}{|T|}$ for all $i \in T$.

Case b: $|D| \geqq 2$. For each $C_k \in \mathcal{C}$, $k \in D$, $i \in C_k$ and $j \in C_k$ are symmetric in (C_k, v). Therefore $\psi_i(N, cu_T, \mathcal{C}) = \psi_j(N, cu_T, \mathcal{C})$ by Internal Equity. Moreover $\psi_i(N, cu_T, \mathcal{C}) = \psi_j(N, cu_T, \mathcal{C})$ for $i \in T \cap C_k$ and for $j \in T \cap C_h$. As a result, for any $i \in \cup_{k \in D} C_k$, $\psi_i(N, cu_T, \mathcal{C}) = \frac{c}{\sum_{h \in D} |C_h|}$. □

Remark 4.4. *The efficiency of a solution is derived from the four axioms in Theorem 4.8. In fact, consider a solution ψ satisfying these four. The main logic is similar to Theorem 8.1.3 of Peleg and Sudhölter (2003). Let (N, v^0) be zero-game such that $v^0(S) = 0$ for any $S \subseteq N$ and $\mathcal{C}$ be a coalition structure on N. Then, $\psi(N, v^0, \mathcal{C})$ must be $0^N \in \mathbb{R}^N$ by Coalitional Null Player. Let $(N, v, \mathcal{C}) \in$ **GC**. By Additivity, $\psi(N, v, \mathcal{C}) + \psi(N, -v, \mathcal{C}) = \psi(N, v - v, \mathcal{C}) = \psi(N, v^0, \mathcal{C}) = 0^N$ and thus, $\psi(N, v, \mathcal{C}) = -\psi(N, -v, \mathcal{C})$ holds. Since the payoff proposed by a solution must be feasible, $\sum_{i \in N} \psi_i(N, v, \mathcal{C}) \leqq v(N)$ and $\sum_{i \in N} \psi_i(N, v, \mathcal{C}) = -\sum_{i \in N} \psi_i(N, -v, \mathcal{C}) \geqq -(-v(N))$. Thus, $\sum_{i \in N} \psi_i(N, v, \mathcal{C}) = v(N)$ holds.*

Example 4.1. *The following solutions show the independence of each axiom from the others (except the efficiency) in Theorem 4.8. Let $(N, v, \mathcal{C}) \in$ **GC**.*

(i) Consider a solution ψ^n defined by

$$\frac{\mathrm{Sh}_k^w(M, v_{\mathcal{C}}) - v(C_k)}{|C_k|} + \mathrm{Nu}_i(C_k, v)$$

where $w \in \mathbb{R}_{++}^M$ is such that $w_k = |C_k|$ and Nu is the nucleolus introduced by Schmeidler (1969). Since Nu satisfies the symmetry and the null player axioms, ψ^n satisfies Equal Power of Partnership Member by Lemma 4.2. Moreover, ψ^n satisfies Internal Equity and Coalitional Null Player since Nu satisfies the symmetry and the null player axioms, but it does not satisfy the additivity because Nu does not satisfy the additivity.

(ii) ψ^δ is characterized by Additivity, Internal Equity, Coalitional Null Player and Coalitional Symmetry which is defined by, if $k \in M$ and $h \in M$ are symmetric in $(M, v_{\mathcal{C}})$, then $\sum_{i \in C_k} \psi_i(N, v, \mathcal{C}) = \sum_{i \in C_h} \psi_i(N, v, \mathcal{C})$. Since ψ^δ and ψ^γ are the different solutions, Equal Power of Partnership Members is independent of the other axioms.

(iii) *The extended Shapley value satisfies all the axioms except for Internal Equity.*

(iv) *The Egalitarian value defined by* $\psi_i^e(N, v, \mathcal{C}) = \frac{v(N)}{|N|}$ *for all* $i \in N$ *satisfies all the axioms except for Coalitional Null Player.*

4.6 Concluding remarks

Recently, Vidal-Puga (2005b) considered another value on games with coalition structures from a viewpoint of non-cooperative bargaining among the players. This solution also satisfies the condition that the sum of the payoffs of the players in C_k coincides with the weighted Shapley value of player k for the external game with coalition-size weights. Vidal-Puga (2005b) states that a generation of coalition size weights is due to "right to talk" of players. In contrast, in this chapter, we show that the generation of coalition size weights is due to communication restriction by coalitions.

Our solution can be extended to games with levels structures introduced by Winter (1989). Levels structure on N is a finite sequence of coalition structures, $\mathcal{C}^0, \dots, \mathcal{C}^l$ with $\mathcal{C}^0 = [N]$ and $\mathcal{C}^l = \{N\}$ such that if $k < h$, $\mathcal{C}^k$ is a finer coalition structure than $\mathcal{C}^h$. Consider the levels structure for six person game described by Table 4.1. Then, the payoff for player 1 is calculated as the following way.

level		coalition structure
3	$\mathcal{C}^3$	$\{\{1,2,3,4,5,6\}\}$
2	$\mathcal{C}^2$	$\{\{1,2,3\},\{4,5\},\{6\}\}$
1	$\mathcal{C}^1$	$\{\{1,2\},\{3\},\{4\},\{5\},\{6\}\}$
0	$\mathcal{C}^0$	$\{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}$

Table 4.1: Levels structure on $N = \{1,2,3,4,5,6\}$

First, in level $\mathcal{C}^2$, coalitions $\{1,2,3\}$, $\{4,5\}$ and $\{6\}$ bargain for the division of $v(N)$. As a result, coalition $\{1,2,3\}$ obtains $\mathrm{Sh}_1^w(M_1^2, v_{\mathcal{C}^2})$ where $M_1^2 = \{1,2,3\}$ and $w_1 = 3$, $w_2 = 2$ and $w_3 = 1$. Then, player 1 receives his dividend for this bargaining surplus, that is, $\frac{\mathrm{Sh}_1^w(M_1^2, v_{\mathcal{C}^2}) - v(\{1,2,3\})}{|\{1,2,3\}|}$. Next, in level $\mathcal{C}^1$, coalitions $\{1,2\}$ and $\{3\}$ bargain for the division of $v(\{1,2,3\})$ and $\{1,2\}$ obtains $\mathrm{Sh}_1^w(M_1^1, v_{\mathcal{C}^1})$ where $M_1^1 = \{1,2\}$ and $w_1 = 2$ and $w_2 = 1$. Player 1 receives $\frac{\mathrm{Sh}_1^w(M_1^1, v_{\mathcal{C}^1}) - v(\{1,2\})}{|\{1,2\}|}$. Finally, in level $\mathcal{C}^0$, players 1 and 2 bargain for the division of $v(\{1,2\})$ and player 1 obtains $\mathrm{Sh}_1(\{1,2\}, v)$. Therefore, the payoff for player 1 is

$$\frac{\mathrm{Sh}_1^w(M_1^2, v_{\mathcal{C}^2}) - v(\{1,2,3\})}{|\{1,2,3\}|} + \frac{\mathrm{Sh}_1^w(M_1^1, v_{\mathcal{C}^1}) - v(\{1,2\})}{|\{1,2\}|} + \mathrm{Sh}_1(\{1,2\}, v).$$

Generally, let $(N, v, \mathcal{L})$ be a game with levels structure where $(N, v) \in \mathbf{G}$ and $\mathcal{L} = \{\mathcal{C}^0, \dots, \mathcal{C}^l\}$ is a levels structure on N. For each $k = 0, \dots, l$, let $\mathcal{C}^k = \{C_1^k, \dots, C_{m_k}^k\}$ and $M^k = \{1, \dots, m_k\}$. For given $i \in N$, let $i(k)$ denote a coalitional index of coalition of level k which player i belongs to, *i.e.,* $C_{i(k)}^k \in \mathcal{C}^k$ and $i \in C_{i(k)}^k$. Further, put $M_i^k = \{h \in M^k : C_h^k \subseteq C_{i(k+1)}^{k+1}\}$ and $w_h^k = |C_h^k|$ for all $h \in M_i^k$. Of course, $i(k) \in M_i^k$. Let $(M_i^k, v_{\mathcal{C}^k})$ be a subgame of $(M^k, v_{\mathcal{C}^k})$ on M_i^k.

Definition 4.2. *A value ψ^γ for $(N, v, \mathcal{L})$ is defined by*

$$\begin{aligned}\psi_i^\gamma(N, v, \mathcal{L}) &= \sum_{k=0}^{l-1} \frac{\mathrm{Sh}_{i(k)}^{w^k}(M_i^k, v_{\mathcal{C}^k}) - v(C_{i(k)}^k)}{|C_{i(k)}^k|} + v(\{i\}) \\ &= \sum_{k=1}^{l-1} \frac{\mathrm{Sh}_{i(k)}^{w^k}(M_i^k, v_{\mathcal{C}^k}) - v(C_{i(k)}^k)}{|C_{i(k)}^k|} + \mathrm{Sh}_i(C_{i(1)}^1, v)\end{aligned}$$

for all $i \in N$.

Part II

Non-cooperative approach

Chapter 5

Bidding for social alternatives

5.1 Introduction

In the recent literature on the implementation of a solution concept established in cooperative game theory, the mechanisms based on individuals' bids have been intensively explored. In the pioneering work of Pérez-Castrillo and Wettstein (2001), they formulated the mechanism involving bidding stage where each individual is required to reveal payable bids to each of the other individuals in exchange for becoming a proposer in the subsequent stages, and they showed that in its subgame perfect equilibrium the Shapley value is realized as a final payoff to each individual. In another paper, Pérez-Castrillo and Wettstein (2005) also provided the mechanism which generates efficient network formation and proved that the payoff to the individuals coincide with the Shapley value in suitably defined cooperative game. Vidal-Puga and Bergantinos (2003) discussed the extension of this mechanism and succeeded in establishing the implementation of the Owen value. In the context of a local public goods economy, Mutuswami, Pérez-Castrillo, and Wettstein (2004) proposed the mechanism which ensures an efficient outcome and proved that their mechanism realizes the Shapley value in an appropriately defined cooperative game. Finally, in the context network allocation problem, Slikker (2007) applied the bidding mechanism to implement network allocation rules such as the Myerson value, the position value, and the component-wise egalitarian solution.

The purpose of this chapter is to examine possible applications of the bidding mechanism in the context of collective choice problem of social alternatives. Examples include a location of public facility such as a public school, a disposal center, and a nuclear-related equipment. We consider a standard quasi-linear environment with finite social alternatives that affects all of individuals in a society. The two companion papers of Pérez-Castrillo and Wettstein (2000, 2002) are the first to discuss such a collective choice problem. In their papers, they analyzed how to choose a single alternative among all the individuals' most favorite ones and provided a one-stage bidding mechanism that can always realize an efficient proposal. In the present paper, we consider a slightly generalized group decision problem in the sense that feasible alternatives are not restricted to individuals' most favorite ones. In a quasi-linear environment, as considered in the papers of Pérez-Castrillo and Wettstein, a concept of an efficiency is given as the maximization of sum of individuals' utilities. Assuming that the individuals' are self-regarding utility maximizers, there is no guarantee that individuals' proposals are socially efficient ones, and thus the realized social alternative is socially efficient. To realize a socially efficient alternative, we examine two alternative bidding mechanisms: one is a simple one-stage mechanism, and the other is a multi-stage mechanism.

In Section 2, we first examine the simple one-stage mechanism. In our one-stage mechanism, each individual is asked to make her/his bids paid to each of the other individuals to determine a proposer whose most favorite alternative will be realized as a social decision. For each individual, the net bid is calculated as the sum of the bids the individual pays to the other individuals minus the sum of the bids paid to the individual. Then, the individual with the highest net bid becomes the proposer. Our first result shows that this mechanism does work only in limited situations where the individuals' most favorite alternatives generate the same sums of individuals' utilities. Moreover, we show that, even in the cases where the mechanism can successfully work, it may fail to realize socially efficient alternative as an equilibrium outcome.

In Section 3, we examine an alternative bidding mechanism to realize socially efficient alternative. We provide a modified three-stage mechanism. The first stage of this modified mechanism is the same as the simple one-stage mechanism. In contrast to the one-stage mechanism, after a proposer is determined, the modified mechanism leads to the second stage where the proposer offers a social alternative that s/he wants to realize and monetary transfers to the other individuals. Then, in the last stage, the individuals other than the proposer sequentially replies "to accept the offer" or "to reject it." If the offer is unanimously accepted, the monetary transfers are carried out and the proposed social alternative is realized. In the case where the offer is rejected by at least one individual, the monetary transfers are not carried out and the proposer's most favorite alternative is realized as a social decision. We prove that this multi-stage bidding mechanism can work in any situation and always realizes a socially efficient alternative.

In Section 4, extensions and applications of the results obtained in the previous sections are considered. We show that if we replace the bidding game by the weighted bidding game, the necessity and sufficient condition on the existence of equilibrium in the bidding game is unchanged. Thus, the sum of the payoff of all the individuals obtained from subgame after the bidding game is irrelevant to who is the winner of the bidding game. This condition seems in a first look to be hard to be satisfied but if we think back on the suggestion of the Coase Theorem, it is not difficult for us to consider a situation that satisfies this condition because we only consider any type of bargaining after the determination of the winner of the bidding game. The application of the bidding mechanism to implement several solutions in cooperative game theory is considered.

5.2 Bidding mechanism

Let $N = \{1, \ldots, n\}$ be the set of n individuals. $X = \{x, y, z, \ldots\}$ is the set of social alternatives.[1] We assume that each individual's utility function U^i is linearly separable with respect to money, i.e. there exists $u^i : X \to \mathbb{R}$ such that, for all $(x, m) \in X \times \mathbb{R}$,

$$U^i(x, m) = u^i(x) + m. \tag{5.1}$$

We consider the social decision problem where each individual has a proposal, perhaps conflicting with each other, about the socially best alternative of X. For any $i \in N$, x^i denotes the individual i's proposal, i.e. the socially best alternative according to the i's evaluation. We assume that the socially best alternatives and i's most preferred alternative are uniquely determined, respectively.

In order to reconcile individuals' proposals and to determine the single socially best alternative, we consider the following one-stage *bidding mechanism*:

[1] We do not pose any assumption on the cardinality of X. Thus, X can be a finite or infinite set.

i). Each individual i makes a bid $b^i_j \in \mathbb{R}$ for each $j \neq i$, i.e. $(b^i_j)_{j\neq i} \in \mathbb{R}^{n-1}$.
ii). For each i, calculate a net bid $B^i := \sum_{j\neq i} b^i_j - \sum_{j\neq i} b^j_i$.
iii). An individual with the highest net bid becomes a proposer and her/his proposal is realized in the society in return for the actual payment of her/his bids to other individuals. If we have more than one individuals with the highest net bid then any one of them is randomly chosen.

We denote α the proposer, i.e. $\alpha \in \arg\max_{i\in N}\{B^i\}$. Consequently, the bidding mechanism defined above can be seen as a normal form game and we denote this normal form game by $B(N, X, (U^i)_{i\in N})$. We now examine a Nash equilibrium of this mechanism. Our first result shows that in any Nash equilibrium every individual's net bid is zero.

Lemma 5.1. *For any equilibrium strategy profile* $(b^i)_{i\in N}$, $B^i = 0$ *for any* $i \in N$.

proof. Define $\Omega = \{i : B^i \geq B^j\ \forall j \in N\}$. If $\Omega = N$, the fact that $\sum_{i\in N} B^i = 0$ trivially implies $B^i = 0$ for each $i \in N$. We now show that, for any equilibrium strategy profile $(b^i)_{i\in N}$, $\Omega = N$ follows. We prove this by contradiction. Suppose that $\Omega \subseteq N$ and $\Omega \neq N$. Then, we can find the two individuals $i \in \Omega$ and $k \in N\backslash\Omega$. Let $\delta > 0$, and consider the following profile $(\hat{b}^i)_{\in N}$ such that $\hat{b}^j = b^j\ \forall j \neq i$, and $\hat{b}^i_j = b^i_j + \delta/|\Omega|$ if $j \in \Omega\backslash\{i\}$; $\hat{b}^i_j = b^i_j - \delta$ if $j = k$; $\hat{b}^i_j = b^i_j$ otherwise. The new net bids are $\hat{B}^i = B^i - \delta/|\Omega|$; $\hat{B}^k = B^k + \delta$; $\hat{B}^j = B^j - \delta/|\Omega|\ \forall j \in \Omega\backslash\{j\}$; $\hat{B}^j = B^j\ \forall j \in N\backslash(\Omega \cup \{k\})$. Since $B^j > B^l$ holds for any $j \in \Omega$ and any $l \in N\backslash\Omega$, we still obtain $\hat{B}^j > \hat{B}^l$ for sufficiently small δ. Thus, $\hat{\Omega} := \{i : i = \hat{B}^i \geq \hat{B}^j\ \forall j \in N\}$ completely coincides with Ω. However, for the individual i, we have $\sum_{j\neq i} \hat{b}^i_j < \sum_{j\neq i} b^i_j$, and thus, her/his new strategy $\hat{b}^i$ increases her/his expected final payoff. □

From Lemma 5.1, every individual could be a proposer with the same probability. The next lemma tells that in any equilibrium each individual gain the same final payoff no matter who becomes a proposer.

Lemma 5.2. *For any equilibrium strategy profile* $(b^i)_{i\in N}$, *each player receives the same final payoff regardless of who becomes a proposer.*

proof. From Claim 1, the each player's net bid coincides each other in equilibrium. Thus, every player could become a proposer with the same probability. We prove the contrapositive of the claim. Suppose that some player i could get the highest payoff if s/he would become a proposer than in the case where some other player is a proposer. Then, sufficiently small increases in her/his bids to the other player improve her/his final payoff so that s/he will deviate from the equilibrium strategy. Similarly, if the individual i could obtain the biggest payoff when some other individual j is a proposer than in the other cases, s/he has an incentive to decrease her/his bid to the individual j. □

The two lemmas have an important and somewhat undesirable implication about the existence of an equilibrium of the mechanism. Let $(b^i)_{i\in N}$ be any equilibrium strategy profile. We now demonstrate that some severe restriction on the individuals' proposals is required to assure the profile $(b^i)_{i\in N}$ be an equilibrium of the mechanism. From Claim 1, B^i must be zero, or equivalently, we have

$$\sum_{j\neq i} b^i_j = \sum_{i\neq j} b^j_i, \tag{5.2}$$

and thus every player could become a proposer with a strictly positive probability. Moreover, from Claim 2, each player receives the same payment regardless of who becomes a proposer.

Therefore, for any two distinct individuals $j, k \in N$, the final payment of an individual $i \neq j, k$ must be the same:

$$u^i(x^k) + b^k_i = u^i(x^j) + b^j_i. \tag{5.3}$$

And also, each player i is indifferent between the two cases where, respectively, i her/himself becomes a proposer, or some other player k is a proposer. Let z (resp. x) be an alternative that is realized when i (resp. k) becomes a proposer. Thus, by (5.2) and (5.3), we have the following:

$$\begin{aligned} u^i(x^k) + b^k_i &= u^i(x^i) - \textstyle\sum_{j\neq i} b^i_j \\ &= u^i(x^i) - \textstyle\sum_{j\neq i} b^j_i = u^i(x^i) - \left(b^k_i + \textstyle\sum_{j\neq i,k}(b^k_i + u^i(x^k) - u^i(x^j))\right) \\ &= u^i(x^i) - (n-1)b^k_i - (n-2)u^i(x^k) + \textstyle\sum_{j\neq i,k} u^i(x^j). \end{aligned} \tag{5.4}$$

Hence, we obtain $nb^k_i = \sum_{j\in N} u^i(x^j) - nu^i(x^k)$, and then the equilibrium bid has to be uniquely determined as follows:

$$b^k_i = \tfrac{1}{n}\textstyle\sum_{j\in N} u^i(x^j) - u^i(x^k). \tag{5.5}$$

On the other hand, Lemma 5.1 shows that $\sum_{j\neq i} b^i_j - \sum_{j\neq i} b^j_i = 0 \ \forall i \in N$. Thus, the following two must be the same value:

$$\textstyle\sum_{j\neq i} b^i_j = \sum_{j\neq i}\left(\tfrac{1}{n}\sum_{k\in N} u^j(x^k) - u^j(x^i)\right); \tag{5.6}$$

$$\textstyle\sum_{j\neq i} b^j_i = \sum_{j\neq i}\left(\tfrac{1}{n}\sum_{k\in N} u^i(x^k) - u^i(x^j)\right). \tag{5.7}$$

It is obvious that (5.6) and (5.7) are not always compatible with each other. Consequently, we obtain the following proposition.

Proposition 5.1. *If a bidding game $B(N, X, (U^i)_{i\in N})$ has a Nash equilibrium, then the following condition must hold:*

$$\textstyle\sum_{k\in N} u^k(x^i) = \sum_{k\in N} u^k(x^j) \ \forall i, j \in N.$$

Proof. From (5.6) and (5.7),

$$\begin{aligned} \textstyle\sum_{j\neq i} b^i_j &= \textstyle\sum_{j\neq i}\left(\tfrac{1}{n}\sum_{k\in N} u^j(x^k) - u^j(x^i)\right) \\ &= \tfrac{1}{n}\textstyle\sum_{j\in N}\sum_{k\in N} u^j(x^k) - \tfrac{1}{n}\sum_{k\in N} u^i(x^k) - \sum_{j\neq i} u^j(x^i), \end{aligned} \tag{5.8}$$

and

$$\textstyle\sum_{j\neq i} b^j_i = \sum_{j\neq i}\left(\tfrac{1}{n}\sum_{k\in N} u^i(x^k) - u^i(x^j)\right) = \tfrac{n-1}{n}\sum_{k\in N} u^i(x^k) - \sum_{j\neq i} u^i(x^j). \tag{5.9}$$

Substituting (5.8) and (5.9) into (5.2), we obtain the following:

$$\begin{aligned} \textstyle\sum_{i\neq j} b^i_j = \sum_{j\neq i} b^j_i &\Leftrightarrow \tfrac{1}{n}\textstyle\sum_{j\in N}\sum_{k\in N} u^j(x^k) = \sum_{k\in N} u^i(x^k) + \sum_{j\neq i} u^j(x^i) - \sum_{j\neq i} u^i(x^j) \\ &\Leftrightarrow \tfrac{1}{n}\textstyle\sum_{j\in N}\sum_{k\in N} u^j(x^k) = \sum_{j\in N} u^j(x^i) \end{aligned} \tag{5.10}$$

The LHS of the last equation is the constant determined independently of the individual i. Thus, the proof is completed. □

Proposition 5.1 provides the necessary condition of the existence of an equilibrium of the bidding mechanism. There could exist an equilibrium of the bidding mechanism only if each individual's best alternative gives rise to the same sum of the individuals' utilities derived from the alternative, $\sum_{i\in N} u^i(x^j) = \sum_{i\in N} u^i(x^k)\ \forall j,k \in N$. The next proposition shows that the converse assertion is also true, i.e. if the bidding mechanism satisfies this condition then there exists an equilibrium in the mechanism.

Proposition 5.2. *For any bidding game $B(N, X, (U^i)_{i\in N})$ that satisfies the following condition:*

$$\textstyle\sum_{k\in N} u^k(x^i) = \sum_{k\in N} u^k(x^j)\ \forall i, j \in N,$$

there exists a Nash equilibrium. Moreover, under the above condition, the equilibrium bid is determined by equation (5.5) and the equilibrium payoff of $i \in N$ is $\frac{1}{n}\sum_{k\in N} u^i(x^k)$.

Proof. Let $(b^i)_{i\in N}$ be the profile of bids defined in (5.3), i.e. $b^k_i = \frac{1}{n}\sum_{j\in N} u^i(x^j) - u^i(x^k)$ $\forall i, k \in N$. We will show that $(b^i)_{i\in N}$ is a Nash equilibrium. It is easy to verify that $\sum_{j\neq i} b^i_j = \sum_{j\neq i} b^j_i$ follows for any $i, j \in N$, i.e. the net bid B^i is 0 for each $i \in N$, because $\sum_{k\in N} u^k(x^i) = \sum_{k\in N} u^k(x^j)$ holds for all $\forall i, j \in N$ now. Therefore, every individual could become a proposer with the same probability. We now show that the final payoff of each individual is the same regardless of who becomes a proposer. Fix an individual i arbitrarily. If some other individual j becomes a proposer, the final payoff of the individual i is

$$u^j_i + b^j_i = u^i(x^j) + \tfrac{1}{n}\textstyle\sum_{k\in N} u^i(x^k) - u^i(x^j) = \tfrac{1}{n}\sum_{k\in N} u^i(x^k). \tag{5.11}$$

On the other hand, if the individual i her/himself becomes a proposer, s/he will gain

$$\begin{aligned} u^i(x^i) - \textstyle\sum_{j\neq i} b^i_j &= u^i(x^i) - \textstyle\sum_{j\neq i}\left(\tfrac{1}{n}\sum_{k\in N} u^j(x^k) - u^j(x^i)\right) \\ &= \textstyle\sum_{j\in N} u^j(x^i) - \tfrac{1}{n}\sum_{k\in N}\left(\sum_{j\neq i} u^j(x^k) + u^i(x^k)\right) + \tfrac{1}{n}\sum_{k\in N} u^i(x^k) \\ &= \textstyle\sum_{j\in N} u^j(x^i) - \tfrac{1}{n}\left(n\sum_{j\in N} u^j(x^k)\right) + \tfrac{1}{n}\sum_{k\in N} u^i(x^k) \\ &= \tfrac{1}{n}\textstyle\sum_{k\in N} u^i(x^k). \end{aligned} \tag{5.12}$$

By the same argument as in the proof of Lemma 5.2, every individual has no incentive to deviate from the strategy profile $(b^i)_{i\in N}$. □

From Propositions 5.1 and 5.2, we obtain the following characterization of the existence of an equilibrium in the bidding mechanism.

Theorem 5.1. *A bidding game $B(N, X, (U^i)_{i\in N})$ has a Nash equilibrium if and only if the following condition holds:*

$$\textstyle\sum_{k\in N} u^k(x^i) = \sum_{k\in N} u^k(x^j)\ \forall i, j \in N.$$

Under the assumption of quasi-linearity of individuals' utility functions, a proposal of individual i, $x^i \in X$, can be said to be *efficient among all proposals* if, for any $j \in N$,

$$\textstyle\sum_{k\in N} u^k(x^i) \geq \sum_{k\in N} u^k(x^j).$$

From Theorem 5.1, it trivially follows the following result.

Corollary 5.1. *In an equilibrium of the bidding game, the adopted proposal is efficient among all proposals.*

More generalized welfare property is also considered as follows. An alternative $x \in X$ is *socially efficient* if, for any $y \in X$,

$$\sum_{k \in N} u^k(x) \geq \sum_{k \in N} u^k(y).$$

Assuming that each individual is a self-regarding utility maximizer, the adopted proposal in an equilibrium of the bidding game may fail to be socially efficient. To make sure of this point, we give the following example. Suppose that $N = \{1, 2\}$, $X = \{x, y, z\}$, and

$$u^1(x) = 4, \quad u^1(y) = 1, \quad u^1(z) = 3,$$
$$u^2(x) = 1, \quad u^2(y) = 4, \quad u^2(z) = 3.$$

In this case, $x^1 = x$ and $x^2 = y$, and $\sum_{i \in N} u^i(x^1) = \sum_{i \in N} u^i(x^2)$ holds. Thus, either x or y is realized as an equilibrium outcome of the mechanism. However, neither of them are efficient because z gives rise to $\sum_{i \in N} u^i(z) = 6 > 5$.

Hence, there are two problems concerning the simple one-stage bidding mechanism: one is that an equilibrium may fail to be exist in general, and the other is that an equilibrium outcome may not be socially efficient even when an equilibrium does exist. In the next section, we propose a modified bidding mechanism which overcomes these problems, i.e. the mechanism which always realizes an efficient social alternative as an equilibrium outcome.

5.3 Modified bidding mechanism

In the preceding section, we observed that our simple one-stage bidding mechanism can not always lead to an equilibrium. Moreover, the one-stage bidding mechanism may fail to realize socially efficient alternative. In their paper, Pérez-Castrillo and Wettstein (2000) have successfully avoided the former problem, i.e. the non-existence of an equilibrium, by the use of an extended strategy space[2]. However, since their one-stage mechanism is designed in the framework aimed at choosing a proposal from all the individuals' ones, not one social alternative from all feasible alternatives, the direct application of their one-stage mechanism in our model still fail to realize a socially efficient alternative in the case of the example we considered in the preceding section. In order to resolve each of the problems of the simple one-stage bidding mechanism, we now consider the multi-stage *modified bidding mechanism*. Our modified bidding mechanism proceeds in three stages:

> *Stage 1.* The first stage is similar to the simple one-stage mechanism. Each individual i makes a bid $b^i_j \in \mathbb{R}$ for each $j \neq i$, i.e. $(b^i_j)_{j \neq i} \in \mathbb{R}^{n-1}$. For each i, calculate a net bid $B^i := \sum_{j \neq i} b^i_j - \sum_{j \neq i} b^j_i$. A proposer is randomly chosen among those with the highest net bid. The proposer α pays her/his bids b^α_j for each other individual.
>
> *Stage 2.* The proposer makes an offer showing the social alternative x s/he wants to realize and a payment t_j, i.e. $(x, t_j) \in X \times \mathbb{R}$, to each $j \neq \alpha$. The payment y_j is

[2] In their one-stage bidding mechanism, each individual announces either of 0 or 1, interpreted as "really want" or "not," as well as her/his bids to the other individuals. Consequently, the strategy space of each individual becomes $\mathbb{R}^{n-1} \times \{0, 1\}$. A proposer is randomly chosen among the individuals who have the highest net bid and announces 1. If the set of such individuals is empty, a proposer is randomly chosen among all the individuals.

interpreted as a transfer (resp. claim) if it is positive (resp. negative) to an individual j.

Stage 3. Every individual other than α sequentially choose *to accept an offer* or *to reject an offer*. If we have a rejection by some individual, an offer is rejected. Otherwise, an offer is accepted. In the case of acceptance (resp. rejection), the alternative x is realized and t_j is paid to each $j \neq \alpha$ (resp. x^α is realized).

Our modified bidding mechanism is a three-stage mechanism, but never be a complicated one. After the first stage which is completely the same as in the simple one-stage bidding mechanism, our multi-stage mechanism asks a proposer to offer a monetary transfer for each of the other individuals, and in the last stage the individuals received the offers sequentially reply "accept it" or "reject it." In this mechanism, each individual receives the following final payment:

$$\text{an offer is accepted} \quad \Rightarrow \quad \begin{cases} \alpha & : \ u^\alpha(x) - \sum_{j \neq \alpha} b_j^\alpha - \sum_{j \neq \alpha} t_j \\ j \neq \alpha & : \ u^j(x) + b_j^\alpha + t_j, \end{cases} \tag{5.13}$$

$$\text{an offer is rejected} \quad \Rightarrow \quad \begin{cases} \alpha & : \ u^\alpha(x^\alpha) - \sum_{j \neq \alpha} b_j^\alpha \\ j \neq \alpha & : \ u^j(x^\alpha) + b_j^\alpha. \end{cases} \tag{5.14}$$

For the modified bidding mechanism, we obtain the following result.

Theorem 5.2. *There exists a subgame perfect equilibrium in the modified bidding mechanism. Moreover, in any subgame perfect equilibrium, socially efficient alternative is realized. Moreover, if x^* is a unique socially efficient alternative.*

Proof. It is obvious that, in any subgame that starts at $t = 3$, the following set of strategies is a unique SPE: every $i \neq \alpha$ accepts an offer (x, t_i) if $u^i(x) + t_i \geq u^i(x^\alpha)$, and rejects the offer otherwise.

Next, we consider an optimal offer of a proposer α at $t = 2$ who anticipates the actions of the other individuals which would follow at $t = 3$. We distinguish two cases: x^α is (i) socially efficient, or (ii) not. In the case of (ii), an optimal offer $(x^*, (t_i^*)_{i \neq \alpha}) \in X \times \mathbb{R}^{n-1}$ can be obtained as a solution of following maximization problem:

$$\max_{(x,(t_i)_{i \neq \alpha})} u^\alpha(x) - \textstyle\sum_{i \neq \alpha} t_i \quad s.t. \ \ u^i(x) + t_i \geq u^i(x^\alpha). \tag{5.15}$$

If we have more than one solutions, any of them are optimal offer. By definition, such an offer is accepted by every other individual $i \neq \alpha$. It is easy to verify that this offer is optimal for the proposer α. Since an optimal offer $(x, (t_i)_{i \neq \alpha})$ must satisfy the constraint with equality in (5.15), the optimal offer can be obtained as follows:

$$t_i^* = u^i(x^\alpha) - u^i(x^*), \ \forall i \neq \alpha, \tag{5.16}$$

$$x^* \in \arg\max_{x \in X} \textstyle\sum_{i \in N} u^i(x) - \sum_{i \neq \alpha} u^i(x^\alpha). \tag{5.17}$$

From (5.17), x^* is a socially efficient alternative. Because x^α is socially inefficient, we have

$$\textstyle\sum_{i \in N} u^i(x^*) - \sum_{i \in N} u^i(x^\alpha) > 0 \ \Rightarrow \ \sum_{i \in N} u^i(x^*) - \sum_{i \neq \alpha} u^i(x^\alpha) > u^\alpha(x^\alpha). \tag{5.18}$$

From (5.16), (5.18) can be rewritten as

$$u^i(x^*) - \textstyle\sum_{i \neq \alpha} t_i^* > u^\alpha(x^\alpha). \tag{5.19}$$

Therefore, from (5.19) and the fact that $(x^*, (t_i)_{i\neq\alpha})$ solves the problem (5.15), this offer is optimal for α. In the case of (ii), any of the optimal offers defined in (5.16) and (5.17) is still optimal for the proposer. Note that, in this case, the inequality in (5.18), thus also the one in (5.19), is replaced with equality. Thus, the payoff received by the proposer in the subgame that starts at $t = 2$ is equal to $u^\alpha(x^\alpha)$. In addition to these offers, it is easily verified that it is also optimal for the proposer to provide an offer that gives the payoff strictly less than $u^i(x^\alpha)$ to some individual i. Such an offer and the rejection by i at $t = 3$ together constitute SPE in the subgame that starts at $t = 2$.

As seen in the above argument, in any SPE of the game that starts at $t = 2$, a social alternative realized as an outcome of the game is always a socially efficient alternative. Given any SPE in the subgame that starts at $t = 2$, it follows from Theorem 5.1 that there exists a Nash equilibrium in the reduced game at $t = 1$. □

Our multi-stage mechanism can always realize a socially efficient alternative as an equilibrium. The keys are the monetary transfers offered and unanimously accepted, respectively, in the second and third stages, which lead a proposer to propose a socially efficient alternative which may not be the most favorable one for the proposer. As shown in the proof of Theorem 5.2, the surplus by such a conciliatory proposal is gained by the proposer through the monetary transfers.

5.4 Applications and extensions

5.4.1 Weighted bidding game

Let N, $|N| > 2$, be a set of individuals. For each $i \in N$, let Δ_i be an extensive form game related to player i. Let $\Delta = (\Delta_i)_{i\in N}$. A player set of Δ_i must include N. Let $w_i > 0$ be a positive weight for player $i \in N$ and $w = (w_i)_{i\in N}$. We define an extensive form game $\Gamma(N, w, \Delta)$ in which first a *weighted* bidding game is conducted among players in N and then a game Δ_α for the winner α of the bidding game follows after the appropriate transfer of the bids in the first bidding stage. Thus, $\Gamma(N, w, \Delta)$ is defined as follows:

i). Each individual i makes a bid $b^i_j \in \mathbb{R}$ for each $j \neq i$, i.e. $(b^i_j)_{j\neq i} \in \mathbb{R}^{n-1}$.
ii). For each i, calculate a weighted net bid $B^i(w) := \sum_{j\neq i} w_i b^i_j - \sum_{j\neq i} w_j b^j_i$.
iii). An individual with the highest weighted net bid is a winner of the bidding stage and only the winner actually pays her/his bids to other players.
iv). A game Δ_α for the winner α is played.

The final payoff of i in $\Gamma(N, w, \Delta)$ is the sum of her/his payoff received from the transfer of the bids in the weighted bidding stage and the payoff obtained from the subsequent game Δ_α. Let u^i_j, $i \in N$, $j \in N$, denote the subgame perfect equilibrium payoff for player j in Δ_i, the game related to player i. Then, the very similar arguments in the previous sections lead us to the following result.

Theorem 5.3. *Suppose that the sum of the equilibrium payoff u^i_k over k in Δ_i is irrelevant to the identity of i. That is,*

$$\sum_{k\in N} u^i_k = \sum_{k\in N} u^j_k$$

for any $i \in N$ and any $j \in N$. Then, the weighted bidding stage has a unique equilibrium such that each i announces

$$b^i_j = \sum_{j\in N} \frac{w_j}{\sum_{k\in N} w_k} u^k_j - u^i_j$$

for each $j \neq i$. Furthermore, the equilibrium payoff of i in $\Gamma(N, w, \Delta)$ is

$$\sum_{j \in N} \frac{w_j}{\sum_{k \in N} w_k} u_i^j.$$

The above theorem assures that under the condition described in the theorem, any random device which selects Δ_i proportional to their weight w can be replaced by the weighted bidding game. Moreover, the equilibrium payoff is achieved as an *realized* value instead of the *expected* one. Thus, the weighted bidding approach has an advantage to the random device approach in this meaning.

5.4.2 Applications to implementing cooperative solutions

A notable example of the application of our results is to implementing a solution established in cooperative game theory. A question mainly considered in cooperative game is how to allocate a certain amount of payoffs $v(N)$ derived from their cooperation and thus the condition described in Theorem 5.3 is likely satisfied ($\sum_{k \in N} u_k^i = v(N)$ for each $i \in N$). A cooperative game is a pair (N, v) where N is a set of individuals and $v : 2^N \to \mathbb{R}$ is a characteristic function with an interpretation that for each $S \subseteq N$, $v(S)$ is considered as a worth of S which members in S obtain for themselves without any cooperation of outside members. We usually assume $v(\emptyset) = 0$.

A solution ϕ for cooperative games is a function which associate any game (N, v) with a payoff vector $\phi(N, v) = (\phi_i(N, v))_{i \in N} \in \mathbb{R}^N$ with $\sum_{i \in N} \phi_i(N, v) \leqq v(N)$. If a solution ϕ always allocate $v(N)$ to the players, ϕ is called an efficient solution.

The study on the implementation of solutions in cooperative games considers a bargaining model or "mechanism" that is described by a non-cooperative game, and rational players obtain the payoff from this game which is the same as the one prescribed by the solution, i.e., in a subgame perfect equilibrium (SPE) of this game, each player obtain the payoff prescribed by the solution. To be a good mechanism, the mechanism must work in a large domain, lead to the Shapley value payoff in all SPE, and achieve the desired solution as the realized value.

In the following examples, we show that several kinds of solution in cooperative game theory are implemented as an equilibrium payoff of the suitably defined non-cooperative games. The proofs are obvious from our theorems and thus we omit them.

Example 5.1. *Consider the following steps of bargaining among players in N. This is defined by the recursive manner.*

(i) When there is only one player i, he obtains his value of stand-alone coalition, $v(\{i\})$.

(ii) Next consider the case of $|N| \geqq 2$. Suppose that the bargaining for the players with less than $|N|$ players are already defined. Then, the bargaining for N proceeds as follows:

Stage 1. The bidding game for N is played and the proposer α is determined after the transfer of hi/her bids to the other players.
Stage 2. The proposer α makes an offer $p_j \in \mathbb{R}$ to any other $j \neq \alpha$
Stage 3. Every individual other than α sequentially choose to accept an offer or to reject an offer.

If we have a rejection by some individual, an offer is rejected. Otherwise, an offer is accepted. In the case of acceptance, the proposer pays offer p_j to other $j \neq \alpha$ in return for obtaining the *value of their total cooperation* $v(N)$*. If there is some player rejecting the offer, the proposer leave the bargaining with obtaining* $v(\{\alpha\})$ *and the remaining players in* $N \setminus \{\alpha\}$ *continue the bargaining for* $N \setminus \{\alpha\}$*.*

This is the very bargaining of Pérez-Castrillo and Wettstein (2001) which implements the Shapley value when the underlying cooperative game is zero-monotonic.

The outline of the proof is as follows. This is conducted by the induction on the number of the players in the bargaining. When there is only one player i*, by the definition of the bargaining, he obtains* $v(\{i\}) = \mathrm{Sh}_i(N, v)$*. Suppose that in the bargaining for less than* $|N|$ *players, any subgame perfect equilibrium payoff coincides with their Shapley value. Consider the case with* $|N|$ *players. Let* α *be a proposer determined in step 1. Consider the responses of the other players for the offers by the proposer in step 3. Then, if one of them rejects the offer, players other than the proposer obtain the Shapley value for the game* $(N \setminus \{\alpha\}, v)$ *by the assumption of the induction. Then, by the very similar arguments of the proof of Theorem 5.3 and the weak condition on the characteristic function* v*, the subgame which starts from stage 2 with a proposer* α *generates a unique subgame perfect payoff such that* $i \neq \alpha$ *obtains*

$$\mathrm{Sh}_i(N \setminus \{\alpha\}, v)$$

and the proposer α *obtains*

$$v(N) - \sum_{i \neq \alpha} \mathrm{Sh}_i(N \setminus \{\alpha\}, v) = v(N) - v(N \setminus \{\alpha\}),$$

respectively. The zero-monotonicity of (N, v) *ensures that the proposer has the incentive to make such an offer. Therefore put* $u^i_j = \mathrm{Sh}_i(N \setminus \{i\}, v)$ *for* $j \neq i$ *and* $u^i_i = v(N) - v(N \setminus \{\alpha\})$ *and apply Theorem 5.3, the equilibrium payoff of player* i *of this bargaining is*

$$\frac{v(N) - v(N \setminus \{i\})}{n} + \frac{1}{n} \sum_{i \neq j} \mathrm{Sh}_i(N \setminus \{j\}, v).$$

This is the recursive formula of the Shapley value introduced by Maschler and Owen (1989).

Example 5.2. *If we replace the bidding stage in the bargaining of example 1 by the weighted bidding stage, this weighted bargaining implements the weighted Shapley value* Sh^w *if* (N, v) *is zero-monotonic.*

Example 5.3. *The weighted CIS (center of the imputation set) value is defined by*

$$\mathrm{CIS}^w_i(N, v) = \frac{w_i}{\sum_{k \in N} w_k} \left(v(N) - \sum_{j \in N} v(\{j\}) \right) + v(\{i\})$$

for each $i \in N$*.*

This solution is implemented by the three-stage bargaining similar to the one described in Example 1 when (N, v) *is zero-monotonic. The differences are that (i) the weighted bidding game is played instead of the bidding game in stage 1, and that (ii) in the case of rejection by some responder, each* $i \in N$ *obtains* $v(\{i\})$ *and the bargaining is over.*

By standard arguments for a proposal and sequential responses, under an appropriate condition, the subgame which starts from stage 2 with proposer α *generates a unique subgame perfect equilibrium payoff such that* α *obtains* $v(N) - \sum_{j \neq \alpha} v(\{j\})$ *and* $i \neq \alpha$ *obtains* $v(\{i\})$*. For each* $i \in N$*, put* $u^i_i = v(N) - \sum_{j \neq i} v(\{j\})$ *and* $u^i_j = v(\{j\})$ *for each* $j \neq i$ *and apply Theorem 5.3, we obtain the desired result. This bargaining works under the broad condition that* $v(N) \geq \sum_j v(\{j\})$ *for each* $i \in N$*.*

In the final of this subsection, let us consider the way to incorporate the bargaining and cooperative solutions in the above examples into the collective choice problem considered throughout this chapter. Given a set of alternatives X, we define a characteristic function v^X associated with X as follows:

$$v^X(S) = \max_{x \in X} \sum_{i \in S} u_i(x) \tag{5.20}$$

for all $S \subseteq N$.

To implement cooperative solutions with respect to a cooperative game (N, v^X) in the circumstance of the collective choice of social alternatives, all we have to do is to alter the bargaining games described in Examples 1 and 2 in three points: (i) the individual α selected as a proposer at the first stage proposes the social alternative x and a payment t_j to each $j \neq \alpha$, (ii) in the case of acceptance of all individuals other than α, the social alternative x is realized in return for the payment t_j from α to any other $j \neq \alpha$, and (iii) in the case of rejection of some individual, the proposer α is expelled from the society and s/he evaluates this situation as a negative infinity because s/he cannot live outside the society. After the modifications, the bargaining models described in Examples 1 and 2 implement $\mathrm{Sh}(N, v^X)$ and $\mathrm{Sh}^w(N, v^X)$ respectively. To implement the weighed CIS value, in addition to the modifications (i) and (ii) mentioned above, we need the slightly different change: (iii)' in the case of rejection of individual j, the individuals other than j are expelled from the society and individual j realizes his/her most favorite alternative x^j.

5.5 Conclusion

We examined two alternative bidding mechanisms in the framework where social alternatives are explicitly included in its description: one is the simple one-stage bidding mechanism, and the other is the multi-stage bidding mechanism. There are two serious defects in the simple one-stage mechanism. There does not always exist an equilibrium, and it may fail to generate a socially efficient alternative even if an equilibrium does exist. Our multi-stage bidding mechanism overcomes these two problems and always realizes a socially efficient alternative as an equilibrium.

Pérez-Castrillo and Wettstein (2000) showed that in an equilibrium of their one-stage bidding mechanism, every individual receives the final payoff more than or equal to the average of the payoffs, each obtained if each one proposal is realized. Thus, their bidding mechanism can be seen as an equitable social decision mechanism in both *ex ante* and *ex post* criteria: an equal probability to be a proposer, and the final payoffs more than or equal to their average payoffs, respectively. The same conclusion directly follows for our multi-stage bidding mechanism.

Chapter 6

Implementation of values in games with social structures

6.1 Introduction

This chapter studies the distribution of cooperative surplus among members of a society, who are subdivided into groups or coalitions, and explores a non-cooperative mechanism implementing a cooperative solution in such a situation. The cooperation relationship is expressed by the players' partition. The relationship among such coalitions can be classified into two major categories. One is the horizontal or equal–equal relationship between coalitions, where all of the coalitions have the same qualifications in the social economy. The other is the vertical or superior–inferior relationship between them, according to which some coalitions are judged, by some measure, to be in a position superior to the others.

One main stream of research on cooperative relationships is the study of individual incentives to the endogenous formation of cooperation structures (for the recent survey on various models of network formations, see the introduction of Bloch and Dutta 2008 for example). Another is to explore the solution concepts for a situation with exogenously given cooperation structure. The latter approach is mainly developed in cooperative solution theory, and various solutions for situations that admit some cooperation structure have been considered. Kalai and Samet (1987) extend the Shapley value to the situation in which players are partitioned into ordered coalitions, which can be seen as a *hierarchic structure* for them. Their extension also includes the non-symmetric generalization of the Shapley value in order to reflect the asymmetry among players caused by, for instance, the different importance of the players or the bargaining powers "determined by the strategic advantages conferred on players by the circumstances under which they bargain" (Binmore 1998, p. 80). In contrast, Owen (1977) proposes a generalization of the Shapley value, called the coalitional value, to a game with a *coalition structure* in which the players form coalitions that are in horizontal relationship with each other. A non-symmetric generalization of the coalitional value is studied by Levy and McLean (1989).

In this chapter, we first present a unified model, the *games with social structure*, in which there exist both the hierarchical and horizontal coalition structures, and define a weighted value for these games. This value is a generalization of the Shapley value to such a game, and thus, it coincides with the Shapley value, the weighted Shapley value with hierarchic structure, the coalitional value, and the weighted coalitional value, in some special cases. Then, in order to achieve this value through the decentralized decision-making process, we consider a non-cooperative bargaining model. Thus, our research follows the Nash Program, which fills the gap between the cooperative and non-cooperative approaches (for the detailed survey on the Nash Program, see Serrano 2005).

Our mechanism is established on the basis of the "one proposer and several responders" model a la Rubinstein and the *bidding game* for choosing the proposer by the participants' own decisions, where "mechanism" is a non-cooperative game and we use the term to avoid ambiguities with cooperative game. Pioneering work in this kind of bargaining has been undertaken by Pérez-Castrillo and Wettstein (2001). They formulate the mechanism involving the bidding stage in which each individual is required to reveal payable bids to each of the other individuals in exchange for becoming a proposer in the subsequent stage, and show that in its subgame perfect equilibrium (SPE), the Shapley value is realized as a final payoff to each individual. They also show that if the bidding stage is replaced by a weighted bidding stage, the weighted mechanism implements the weighted Shapley value without hierarchic structure. Vidal-Puga and Bergantinos (2003) discuss the extension of this mechanism and succeed in establishing the implementation of the coalitional value. Further, this approach is widely applied to several contexts, for instance, a local public goods economy (see Mutuswami, Pérez-Castrillo, and Wettstein 2004), a network formation (see Pérez-Castrillo and Wettstein 2005), a network allocation problem (see Slikker 2007), and the collective choice problem (see Pérez-Castrillo and Wettstein 2000, 2002 and Kamaga and Kamijo 2007).

We propose a bargaining model, called the *weighted social bidding mechanism*. In this bargaining, only the players in the highest ranked coalitions participate in the bidding stage to choose a proposer in the next stage. Then, the proposer determined in the previous stage offers a payoff distribution among all the players, and the players other than the proposer sequentially decide to either accept or reject the offer. In the case of acceptance by all players, the bargaining is over and their payoff is distributed according to the proposal. On the other hand, when there is a player who rejects the offer, the proposer leaves the bargaining and the remaining players continue the same bargaining process with the proposing coalition, i.e., the coalition that the proposer belongs to, retaining the right to choose a new proposer.

We demonstrate that our mechanism works in any *zero-monotonic* environment. That is, if the underlying cooperative game is zero-monotonic, in any SPE of the weighted social bidding mechanism, the equilibrium payoff vector coincides with the value defined in this chapter. This result has at least the following significance: (i) Our mechanism implements the Shapley value, the weighted Shapley value of Kalai and Samet (1987), the coalitional value, and the weighted coalitional value in some special cases. (ii) Since the weighted social bidding mechanism coincides with the bidding mechanism and the weighted bidding mechanism proposed by Pérez-Castrillo and Wettstein (2001) that implement the Shapley value and the weighted Shapley value without hierarchic structure, respectively, our result includes some of the results of their paper, and the weighted social bidding mechanism is an extension of these two mechanisms. (iii) Our result extends the domain of implementing the coalitional value compared with Vidal-Puga and Bergantinos (2003), who consider a two-step bidding mechanism to implement the coalitional value, because the mechanism proposed by Vidal-Puga and Bergantinos (2003) works in the strictly superadditive domain.

The rest of the chapter is organized as follows. In the next section, we provide the basic notations and definitions used in this chapter. In Section 3, we present a model of games with social structure and give a basic analysis for this class of games. In Section 4, we explain the bidding game and the relevant literature. In Section 5, we present the weighted social bidding mechanism and show that it implements the weighted value for a game with a social structure in any zero-monotonic environment. Section 6 is the conclusion.

6.2 Preliminary

In several situations, players partition themselves into some coalitions, each of which is in fifty-fifty relationship with the others. Such coalitions form a *coalition structure* $\mathcal{C} = \{C_1, \ldots, C_m\}$, which is a partition of N. Let $\mathcal{C} = \{C_1, \ldots, C_m\}$ be a coalition structure on N. Let $M = \{1, \ldots, m\}$ denote the set of coalitional indices of the coalition structure. A triple $(N, v, \mathcal{C})$ is called a *game with a coalition structure*. A solution for games with coalition structure is defined by the same manner as one for games without coalition structure. A well-known solution for games with coalition structure is the coalitional value by Owen (1977). A non-symmetric generalization of the coalitional value is provided by Levy and McLean (1989).

There also exist situations in which the players are organized into coalitions that are ranked by some measure. Let $\mathcal{H} = (H_1, \ldots, H_\ell)$ be an ordered partition of N and be called a *hierarchic structure* of N. Let $L = \{1, \ldots, \ell\}$ denote a set of indices of the elements in the hierarchic structure. A triple $(N, v, \mathcal{H})$ is a *game with a hierarchic structure*.

A permutation $\theta \in \Theta(N)$ is said to be consistent with hierarchic structure $\mathcal{H}$ if for any $i \in H_k$ and for any $j \in H_{k'}$, $k < k'$ implies $\theta(i) < \theta(j)$. Thus, according to the permutation consistent with hierarchic structure $\mathcal{H}$, players in the same coalition in $\mathcal{H}$ are successively ordered, and coalitions are arranged in the hierarchic order described by $\mathcal{H}$. A set of all the permutations that are consistent with $\mathcal{H}$ is denoted by $\Theta(N, \mathcal{H})$.

Let $w = (w_i)_{i \in N}$ be a positive weight vector of players in N. The *weighted value for a game with a hierarchic structure* or the *weighted Shapley value* (Kalai and Samet 1987) is defined as follows:[1]

$$\mathrm{HV}_i^w(N, v, \mathcal{H}) = \sum_{\theta \in \Theta(N, \mathcal{H})} \left(\Pi_{k=1}^{\ell} p(\theta_{H_k}; w_{H_k}) \right) m_i^\theta(N, v)$$

for each $i \in N$. Thus, HV^w is a weighted average of marginal contribution vectors where each order $\theta \in \Theta(N, \mathcal{H})$ has a weight proportional to $\Pi_{k=1}^{\ell} p(\theta_{H_k}; w_{H_k})$. In other words, HV^w for a player is an expected value of the marginal contributions in a situation in which players in H_k are arranged in order θ_{H_k} with probability $p(\theta_{H_k}; w_{H_k})$, and then coalitions in $\mathcal{H}$ are arranged in the hierarchic order. It is obvious that when $\mathcal{H} = (N)$ and w_i is constant over $i \in N$, HV^w coincides with the Shapley value, and when $\mathcal{H} = (N)$, it coincides with the weighted Shapley value originally considered by Shapley (1953a).

It is obvious from the definition of HV^w that deletion of players in higher ranked coalitions does not influence the payoff of the players in the lower ranked coalitions. So, for any $i \in H_1 \cup \cdots \cup H_k$,

$$\mathrm{HV}_i^w(N, v, \mathcal{H}) = \mathrm{HV}_i^w(N', v, \mathcal{H}'),$$

where $N' = H_1 \cup \cdots \cup H_k$ and $\mathcal{H}' = (H_1, \ldots, H_k)$.

For any coalition S, let (N, u_S) be an *S-unanimity game* defined by

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T, \\ 0 & \text{otherwise}, \end{cases}$$

for any $T \subseteq N$. It is well known that for any game (N, v), v is represented as a linear combina-

[1] In Kalai and Samet (1987), the pair of weight vector w and hierarchic structure $\mathcal{H}$ is referred to as a general weight system, and they treat the weighted Shapley value with a general weight system as a class of the solution for a game (N, v). In this chapter, we consider a hierarchic structure as a component of the game $(N, v, \mathcal{H})$ in order to keep the treatment of HV^w parallel to the treatment of CV^ω.

tion of unanimity games, that is,

$$v = \sum_{S \subseteq N; S \neq \emptyset} d_v(S) u_S,$$

where $d_v(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T)$ is a *dividend* of S.

For any coalition S, let $H(\mathcal{H}, S)$ denote the maximal ranked element H_k in $\mathcal{H}$ that has a non-empty intersection with S. For any coalition S, let $M(\mathcal{C}, S)$ denote the set of indices of coalitions that have non-empty intersections with S. The following propositions examine HV^w and CV^ω for unanimity game (N, u_S).

Proposition 6.1 (Kalai and Samet 1987). *For $S \subseteq N$ and for $i \in H_k \in \mathcal{H}$,*

$$\mathrm{HV}_i^w(N, u_S, \mathcal{H}) = \begin{cases} \frac{w_i}{w(S \cap H_k)} & \text{if } H(\mathcal{H}, S) = H_k, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6.2. *For $S \subseteq N$ and for $i \in C_k \in \mathcal{C}$,*

$$\mathrm{CV}_i^\omega(N, u_S, \mathcal{C}) = \begin{cases} \frac{w_k^*}{w^*(M(\mathcal{C}, S))} \frac{w_i}{w(S \cap C_k)} & \text{if } i \in S \cap C_k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $i \in N \setminus S$, player i's marginal contribution is 0 at any order $\theta \in \Theta(N, \mathcal{C})$. This implies that $\mathrm{CV}_i^\omega(N, u_S, \mathcal{C}) = 0$ for such i. When $i \in S$, by the fact that the weighted coalitional value is an expected value of marginal contributions, $\mathrm{CV}_i^\omega(N, u_S, \mathcal{C})$ coincides with the probability that player i is in the last player of S at order $\theta \in \Theta(N, \mathcal{C})$. Let $i \in C_k \in \mathcal{C}$. Because we focus only on the order consistent with $\mathcal{C}$, this probability is expressed by the product of the probability that k is in the last of $M(\mathcal{C}, S)$ at order $\theta^{\mathcal{C}}$ and the probability that i is the last of $C_k \cap S$ at order θ_{C_k}. By the definition of $p(.;.)$, the former probability and the latter one are $w_k^*/w^*(M(\mathcal{C}, S))$ and $w_i/w(S \cap C_k)$, respectively. Thus, we obtain the formula described in the proposition. □

Since the marginal contribution satisfies the linearity, that is, for $\alpha, \beta \in \mathbb{R}$ and for games (N, v) and (N, z), $m_i^\theta(N, \alpha v + \beta z) = \alpha m_i^\theta(N, v) + \beta m_i^\theta(N, z)$ holds, and in both formulas of the weighted Shapley value and the weighted coalitional value, the weight of the marginal contribution at any order θ is irrelevant to v, the two values also fulfill the linearity. Because, as mentioned above, any game v can be represented as linear combination of the unanimity games, Propositions 1 and 2 provide a useful method of calculating HV^w and CV^ω, respectively.

In the rest of this section, we explain the alternative representations of HV^w and CV^ω. These recursive representations are similar to the one for the Shapley value indicated by Maschler and Owen (1989) and Pérez-Castrillo and Wettstein (2001). When $\mathcal{H} = (N)$, Proposition 6.3 is the same statement as Lemma 1 of Pérez-Castrillo and Wettstein (2001), and when $\mathcal{C} = \{N\}$, Proposition 6.4 is the same statement as the lemma.

Proposition 6.3. *Let $(N, v, \mathcal{H})$ be a game with a hierarchic structure. For any i in the highest ranked coalition $H_\ell = H(\mathcal{H}, N)$,*

$$\mathrm{HV}_i^w(N, v, \mathcal{H}) = \frac{w_i}{w(H_\ell)} (v(N) - v(N - i)) + \sum_{j \in H_\ell - i} \frac{w_j}{w(H_\ell)} \mathrm{HV}_i^w(N - j, v, \mathcal{H} - j),$$

where $\mathcal{H} - j = (H_1, \ldots, H_{\ell-1}, H'_\ell)$ with $H'_\ell = H_\ell - j$.

Proof. According to the probability distribution on $\Theta(N, \mathcal{H})$ behind the definition of HV^w, some $i \in H_\ell$ becomes the last player in the order with probability $w_i/w(H_\ell)$. This means that $i \in H_\ell$ becomes the last and obtains her/his marginal contribution, $v(N) - v(N-i)$, with probability $w_i/w(H_\ell)$. On the other side, with probability $w_j/w(H_\ell)$, other $j \in H_\ell - i$ become the last player. Contingent on other j becoming the last at the order, the expected value of marginal contributions of player i is exactly $\mathrm{HV}_i^w(N-j, v, \mathcal{H}-j)$. Thus, we have the formula of the proposition. □

Proposition 6.4. *Let $(N, v, \mathcal{C})$ be a game with a coalition structure. For $i \in C_k \in \mathcal{C}$, put $\bar{w}_i = \frac{w_i}{w(C_k)} \frac{w_k^*}{w^*(M)}$. For any $i \in C_k \in \mathcal{C}$,*

$$\mathrm{CV}_i^\omega(N, v, \mathcal{C}) = \bar{w}_i \left(v(N) - v(N-i)\right) + \sum_{j \in C_k - i} \bar{w}_i \mathrm{HV}_i^w(N-j, v, \mathcal{H}') + \sum_{k' \in M-k} \frac{w_{k'}^*}{w^*(M)} \mathrm{CV}_i^\omega(N \setminus C_{k'}, v, \mathcal{C} - C_{k'}),$$

where $\mathcal{H}' = (N \setminus C_k, C_k - j)$ and $\mathcal{C} - C_{k'} = \mathcal{C} \setminus \{C_{k'}\}$.

Proof. The proof of this proposition proceeds in similar way to the proof of Proposition 6.3. Note that according to the probability distribution on $\Theta(N, \mathcal{C})$ behind the definition of CV^ω, some $i \in C_k$ becomes the last player in the order with probability $w_k^*/w^*(M) \times w_i/w(C_k) = \bar{w}_i$. Thus, s/he obtains $v(N) - v(N-i)$ with probability $\bar{w}_i$. On the other hand, we have to separate two cases when other $j \neq i$ becomes the last in the order. First, consider the case that $j \in C_k$. Because we only treat the order consistent with $\mathcal{C}$, this implies that C_k is the last coalition in the order. Thus, contingent on other $j \in C_k$ becoming the last player of the order, the expected value of i's marginal contributions is

$$\sum_{\theta \in \Theta(C_k - j)} p(\theta; w_{C_k - j}) \Big(v((N \setminus C_k) \cup P_i^\theta \cup i) - v((N \setminus C_k) \cup P_i^\theta) \Big) = \mathrm{HV}_i^w(N-j, v, \mathcal{H}').$$

Thus, for each $j \in C_k$, i obtains $\mathrm{HV}_i^w(N-j, v, \mathcal{H}')$ with probability $\bar{w}_j$. Second, consider the case that $j \in C_{k'}, C_k \neq C_{k'}$. This means that the position of coalition $C_{k'}$ is certainly determined, that is, the last position of the order. Thus, contingent on other $j \in C_{k'}$ becoming the last at the order, the expected value of i's marginal contributions is $\mathrm{CV}_i^\omega(N \setminus C_{k'}, v, \mathcal{C} - C_{k'})$, thus irrelevant to the identity of $j \in C_{k'}$. Because $\sum_{j \in C_{k'}} (w_{k'}^*/w^*(M))(w_j/w(C_{k'})) = w_{k'}^*/(w^*(M))$, we have the formula of the proposition. □

6.3 Games with social structure

In this section, we present a unified model of two kinds of games with cooperation structure. We consider a situation in which both horizontal and hierarchical structures exist simultaneously. An example of such a situation is the organization of the employees in a firm, where there are many employees in some levels and at the same time there also many employees in higher levels to the former. Let (N, v) be a game and $\mathcal{M} = (\mathcal{C}^1, \mathcal{C}^2, \ldots, \mathcal{C}^\ell)$ be a *social structure* in which for each $h = 1, \ldots, \ell$, $\mathcal{C}^h = \{C_1^h, C_2^h, \ldots, C_{m_h}^h\}$ is a coalition structure of subset N^h of N, and $(N^1, \ldots, N^\ell)$ is an ordered partition of N. Let $L = \{1, \ldots, \ell\}$ and $M^h = \{1, \ldots, m_h\}$ for each $h \in L$. A triple $(N, v, \mathcal{M})$ is a *game with a social structure*.

Next, we define a value for a game with a social structure in the way that it becomes a natural extension of the weighted Shapley value and the weighted coalitional value. For each order θ

of N, θ is consistent with social structure $\mathcal{M}$ if (i) $i \in N^h$ and $j \in N^{h'}$ with $h < h'$ imply $\theta(i) < \theta(j)$, and (ii) for any $i \in C_k^h$, for any $i' \in C_k^h$ and for any $j \in N$, $\theta(i) < \theta(j) < \theta(i')$ implies $j \in C_k^h$. The set of all the orders consistent with $\mathcal{M}$ is denoted by $\Theta(N, \mathcal{M})$.

Let $w = (w_i)_{i \in N}$ be a positive weight vector of players in N. For each $h \in L$, $w^{*h} = (w_k^{*h})_{k \in M^h}$ is a weight vector of coalitions in coalition structure $\mathcal{C}^h$. $\omega = (w, w^{*1}, \ldots, w^{*\ell})$ is a weight structure of social structure $\mathcal{M}$. Then, the weighted value for a game with a social structure is defined as

$$\Upsilon_i^\omega(N, v, \mathcal{M}) = \sum_{\theta \in \Theta(N,\mathcal{M})} \left(\Pi_{h=1}^{\ell} \left(p(\theta^{\mathcal{C}^h}; w^{*h}) \Pi_{k=1}^{m_h} p(\theta_{C_k^h}; w_{C_k^h}) \right) \right) m_i^\theta(N, v)$$

for each $i \in N$, where $\theta^{\mathcal{C}^h}$ is an order of M^h derived by the condition that $\theta^{\mathcal{C}^h}(k) < \theta^{\mathcal{C}^h}(k')$ if and only if for any $i \in C_k^h$ and for any $j \in C_{k'}^h$, $\theta(i) < \theta(j)$.

More simply, the weighted value for a game with a social structure can be rewritten as follows: for $i \in N^h$,

$$\Upsilon_i^\omega(N, v, \mathcal{M}) = \mathrm{CV}_i^{\omega^h}(N^h, v^h, \mathcal{C}^h) \tag{6.1}$$

where $\omega^h = (w_{N^h}, w^{*h})$ and v^h is defined by $v^h(S) = v(S \cup N^1 \cup \cdots \cup N^{h-1}) - v(N^1 \cup \cdots \cup N^{h-1})$ for each $S \subseteq N^h$. From this formula, it is easy to check that

$$\sum_{i \in N^h} \Upsilon_i^\omega(N, v, \mathcal{M}) = v(N^1 \cup \cdots \cup N^{h-1} \cup N^h) - v(N^1 \cup \cdots \cup N^{h-1}) \tag{6.2}$$

for each $h \in L$. This also implies that Υ^ω satisfies the efficiency ($\sum_{i \in N} \Upsilon_i^\omega(N, v, \mathcal{M}) = v(N)$).

For each $i \in C_k^\ell \in \mathcal{C}^\ell$, $\mathcal{M}_{-i} = (\mathcal{C}^1, \ldots, \mathcal{C}^{\ell-1}, \mathcal{C}^\ell \setminus \{C_k^\ell\}, \{C_k^\ell - i\})$ is a social structure derived from $\mathcal{M}$ by deleting player i such that in the new social structure, the coalition containing the deleted player now becomes the highest ranked coalition, with the other structure remaining unchanged. With understanding $(\mathcal{C}^1, \ldots, \mathcal{C}^{\ell-1}, \mathcal{C}^\ell \setminus \{C_k^\ell\}, \emptyset) = (\mathcal{C}^1, \ldots, \mathcal{C}^{\ell-1}, \mathcal{C}^\ell \setminus \{C_k^\ell\})$, $(\mathcal{C}^1, \ldots, \mathcal{C}^{\ell-1}, \emptyset, \{C_k^\ell - i\}) = (\mathcal{C}^1, \ldots, \mathcal{C}^{\ell-1}, \{C_k^\ell - i\})$ and $(\mathcal{C}^1, \ldots, \mathcal{C}^{\ell-1}, \emptyset, \emptyset) = (\mathcal{C}^1, \ldots, \mathcal{C}^{\ell-1})$, $\mathcal{M}_{-i}$ is well-defined.

The following propositions are parallel to Propositions 6.1 and 6.2 and Propositions 6.3 and 6.4.

Proposition 6.5. *For $S \subseteq N$, let $\hat{h}$ be the maximal integer h such that $S \cap N^h$ is not empty. Let $M(\mathcal{C}^{\hat{h}}, S) \subseteq M^{\hat{h}}$ denote the set of indices of coalitions in $\mathcal{C}^{\hat{h}}$ that have non-empty intersections with S. Then,*

$$\Upsilon_i^\omega(N, v, \mathcal{M}) = \begin{cases} \frac{w_k^{*\hat{h}}}{w^{*\hat{h}}(M(\mathcal{C}^{\hat{h}}, S))} \frac{w_i}{w(S \cap C_k^{\hat{h}})} & \text{if } i \in C_k^{\hat{h}} \cap S, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is directly followed from Proposition 6.2 and Equation (6.1). □

Proposition 6.6. *Let $(N, v, \mathcal{M})$ be a game with a social structure. For $i \in C_k^\ell \in \mathcal{C}^\ell$, put $\bar{w}_i = \frac{w_k^{*\ell}}{w^{*\ell}(M^\ell)} \frac{w_i}{w(C_k^\ell)}$*

(i) *For any $i \in C_k^\ell \in \mathcal{C}^\ell$,*

$$\begin{aligned} \Upsilon_i^\omega(N, v, \mathcal{M}) &= \bar{w}_i \left(v(N) - v(N - i)\right) \\ &+ \sum_{j \in C_k - i} \bar{w}_j \Upsilon_i^\omega(N - j, v, \mathcal{M}_{-j}) + \sum_{k' \in M^\ell - k} \frac{w_{k'}^{*\ell}}{w^{*\ell}(M^\ell)} \Upsilon_i^\omega(N \setminus C_{k'}^\ell, v, \mathcal{M} - C_{k'}^\ell), \end{aligned}$$

where $\mathcal{M} - C_{k'}^\ell = (\mathcal{C}^1, \ldots, \mathcal{C}^{\ell-1}, \mathcal{C}^\ell \setminus \{C_{k'}^\ell\})$.

(ii) *For any* $i \in N^\ell$,

$$\Upsilon_i^\omega(N, v, \mathcal{M}) = \bar{w}_i\left(v(N) - v(N-i)\right) + \sum_{j \in N^\ell - i} \bar{w}_j \Upsilon_i^\omega(N-j, v, \mathcal{M}_{-j}).$$

Proof. First, (i) of this proposition follows from Proposition 6.4 and Equation (6.1). Then, (ii) is obtained through the following:

$$\sum_{j \in N^\ell - i} \bar{w}_j \Upsilon_i^\omega(N-j, v, \mathcal{M}_{-j}) =$$

$$\begin{aligned}
&= \sum_{j \in C_k^\ell - i} \bar{w}_j \Upsilon_i^\omega(N-j, v, \mathcal{M}_{-j}) + \sum_{j \in N^\ell \setminus C_k^\ell} \bar{w}_j \Upsilon_i^\omega(N-j, v, \mathcal{M}_{-j}) \\
&= \sum_{j \in C_k^\ell - i} \bar{w}_j \Upsilon_i^\omega(N-j, v, \mathcal{M}_{-j}) + \sum_{k' \in M^\ell - k} \sum_{j \in C_{k'}^\ell} \bar{w}_j \Upsilon_i^\omega(N-j, v, \mathcal{M}_{-j}) \\
&= \sum_{j \in C_k^\ell - i} \bar{w}_j \Upsilon_i^\omega(N-j, v, \mathcal{M}_{-j}) + \sum_{k' \in M^\ell - k} \sum_{j \in C_{k'}^\ell} \bar{w}_j \Upsilon_i^\omega(N \setminus C_{k'}^\ell, v, \mathcal{M} - C_{k'}^\ell) \\
&= \sum_{j \in C_k^\ell - i} \bar{w}_j \Upsilon_i^\omega(N-j, v, \mathcal{M}_{-j}) + \sum_{k' \in M^\ell - k} \frac{w_{k'}^{*\ell}}{w^{*\ell}(M^\ell)} \Upsilon_i^\omega(N \setminus C_{k'}^\ell, v, \mathcal{M} - C_{k'}^\ell)
\end{aligned}$$

The third equality is by Equation (6.1). □

6.4 Bidding game

In this section, we explain a bargaining model in the first stage of which each individual is required to reveal payable bids to each of the other individuals in exchange for acquiring an advantageous position (or accepting an unfavorable position when bids are negative) in the subsequent stages.

For each $i \in N$, let Δ_i be an extensive form game related to player i. Let $\Delta = (\Delta_i)_{i \in N}$. A player set of Δ_i must include N. Let $w_i > 0$ be a positive weight for player $i \in N$ and $w = (w_i)_{i \in N}$. We define an extensive form game $\Gamma(N, w, \Delta)$ in which first a *weighted bidding game* is conducted among players in N, and then, a game Δ_α for the *winner* α of the bidding game follows after the appropriate transfer of the bids in the first bidding stage. Thus, $\Gamma(N, w, \Delta)$ is defined as follows:

i). Each player i simultaneously makes a bid $b_j^i \in \mathbb{R}$ for each $j \neq i$, i.e., s/he announces $b^i = (b_j^i)_{j \in N-i} \in \mathbb{R}^{N-i}$.
ii). For each i, calculate a weighted net bid

$$B^i(w) := \sum_{j \in N-i} w_i b_j^i - \sum_{j \in N-i} w_j b_i^j. \tag{6.3}$$

iii). A player with the highest weighted net bid is the winner of the bidding stage, and only the winner actually pays her/his bids to the other players. If there are two or more players with the highest net bid, then one of them is chosen randomly.
iv). A game Δ_α for the winner α is played.

The final payoff of i in $\Gamma(N, w, \Delta)$ is the sum of her/his payoff paid for, or received from, the transfer of the bids in the weighted bidding stage and the payoff obtained from the subsequent game Δ_α. Thus, given a payoff p_i^j which i obtains in Δ_j, the final payoff for the winner α is $-\sum_{j\in N-\alpha} b_j^\alpha + p_\alpha^\alpha$ and the final payoff for player $i \neq \alpha$ is $b_i^\alpha + p_i^\alpha$.

In the recent literature on the implementation of a solution concept established in cooperative game theory, the mechanisms based on the (weighted) bidding game have been explored intensively. A notable aspect of the mechanisms based on the bidding game is that they enable us to achieve some desirable payoff allocation as a *realized* value instead of an *expected* one, which is the reason why we employ it as a basis of our bargaining model in the next section. Pérez-Castrillo and Wettstein (2001) formulate the mechanism, called the bidding mechanism, according to which, in the first stage, the proposer in the next stages is chosen through the bidding game, and then the proposer makes an offer $x_j \in \mathbb{R}$ to any other j and the responders sequentially decide to accept the offer or reject it. In the case of acceptance by all the responders, the proposer actually pays her/his offer to the responders in return for obtaining the worth of total cooperation, $v(N)$. On the other hand, if any j rejects the offer, the proposer leaves the bargaining with the worth of her/his stand-alone coalition, and the other players continue the same bargaining process for $n-1$ players. They show that in any SPE of this recursively defined mechanism, the equilibrium payoff vector coincides with the Shapley value if the game is zero-monotonic.

Vidal-Puga and Bergantinos (2003) discuss the extension of this mechanism to a game with a coalition structure. They consider a two-step bidding mechanism: in the first step, the bidding mechanism is played within each coalition and the winner of the bidding stage whose offer is accepted by all the responders become a representative of her/his coalition by obtaining their *resource* of cooperation, instead of by obtaining their worth of coalition; in the second step, only the representatives of the coalitions play the bidding mechanism of Pérez-Castrillo and Wettstein (2001), taking into account their resources obtained from the first step. They demonstrate that this two-step bidding mechanism implements the coalitional value in some SPE of superadditive games, and to assure the uniqueness of this mechanism, the strict superadditivity of the games is needed. Finally, in the context of network allocation problem, Slikker (2007) applies the bidding mechanism to implement network allocation rules such as the Myerson value (Myerson 1977), the position value (Borm, Owen, and Tijs 1992), and the component-wise egalitarian solution.

6.5 Social bidding mechanism

In this section, we consider an extension of the bidding mechanism of Pérez-Castrillo and Wettstein (2001) to a game with a social structure. This is done by (i) restricting the participants of the bidding stage of the mechanism to the players who belong to the higher ranked coalitions and (ii) modifying a subsequent social structure after rejection of an offer made by a proposer in such a way that the proposing coalition, i.e., the coalition that the proposer belongs to, retains the right to choose a new proposer.

In the first stage of the mechanism, only the players in coalitions in the highest ranked coalition structures, $\mathcal{C}^\ell \in \mathcal{M}$, participate in the bidding game and thus, the proposer must be chosen from the players in N^ℓ. In the next stage, proposer $\alpha \in C_k^\ell$ makes an offer to *all* the players in $N-\alpha$, and the other players respond to the offer sequentially. Thus, while a proposer must be chosen from N^ℓ, her/his offers include the players in the lower ranked coalitions. In the case of acceptance by all players, the proposer pays her/his offer to any player $j \neq \alpha$ in return for obtaining their value of cooperation, $v(N)$, and the bargaining is over. On the other hand, when some player rejects the offer, the proposer α leaves the bargaining with her/his value of cooperation, $v(\{\alpha\})$, and the other players continue the same rule of bargaining for $n-1$ players with new social structure being $\mathcal{M}_{-\alpha} = (\mathcal{C}^1, \ldots, \mathcal{C}^{\ell-1}, \mathcal{C}^\ell \setminus \{C_k^\ell\}, \{C_k^\ell - \alpha\})$. Thus, after a rejection, proposing coalition $C_k^\ell - \alpha$ is in the higher ranked position and a proposer in the next round must be chosen from players in the coalition.

Let $(N, v, \mathcal{M})$ be a game with a social structure and ω be a weight structure of social structure $\mathcal{M}$. A bargaining model for a game with a social structure, referred to as the *ω-weighted social bidding mechanism* and denoted by $\mathrm{SBM}^\omega(N, v, \mathcal{M})$, is recursively defined as follows:

When $N = \{i\}$, player i obtains her/his value of stand-alone coalition, $v(\{i\})$, and the bargaining is over.

Suppose that $\mathrm{SBM}^\omega(N, v, \mathcal{M})$ is already defined for less than n players. The bargaining for the case with n players proceeds as follows:

Stage 1 When $N^\ell = \{i\}$, s/he is automatically selected as a proposer in the next stage. Otherwise, the weighted bidding game is played only by the members in the highest ranked coalition structure, i.e., members in N^ℓ. After the simultaneous choices of their bids, the weighted net bid $B_i(\bar{w})$ is calculated by formula (6.3) for each $i \in N^\ell$ where the weight is now measured by considering both intra- and inter-coalitional asymmetry. Thus, $\bar{w}_i = \frac{w_i}{w(C_k^\ell)} \frac{w_k^{*\ell}}{w^{*\ell}(M^\ell)}$ for each $i \in C_k^\ell \in \mathcal{C}^\ell$. Therefore, in this weighted bidding game, each player has a weight represented by her/his own weight $w_i/w(C_k^\ell)$ multiplied by her/his coalition's weight $w_k^{*\ell}/w^{*\ell}(M^\ell)$. A player with the highest weighted net bid becomes a proposer in the next stage in return for the actual payment of her/his bids to other players in N^ℓ. If we have two or more players with the highest net bid, then any one of them is randomly chosen.

Stage 2 Let α be a proposer chosen at the first stage. The proposer makes an offer $x_j \in \mathbb{R}$ to any $j \in N - \alpha$. In other words, proposer α chooses $n-1$ dimensional vector $x = (x_j)_{j \in N-\alpha}$.

Stage 3 Every player other than α sequentially chooses *to accept or reject the offer*. If we have a rejection by any responder, the offer is rejected; otherwise, the offer is accepted. In the case of acceptance, the proposer actually pays the offer x_j to any $j \neq \alpha$ and α obtains the worth of their total cooperation, $v(N)$. After that, the bargaining is over. Thus, the final payoff of proposer α is

$$v(N) - \sum_{j \in N^\ell - \alpha} b_j^\alpha - \sum_{j \in N - \alpha} x_j.$$

The final payoff of other $i \in N^\ell, i \neq \alpha$, is $b_i^\alpha + x_i$, and that of $i \in N \setminus N^\ell$ is x_i.

In the case of rejection, the proposer leaves the bargaining table with $v(\{\alpha\})$. The other players continue the bargaining for the division of $v(N - \alpha)$: they play $\mathrm{SBM}^\omega(N - \alpha, v, \mathcal{M}_{-\alpha})$. Thus, the final payoff of proposer α is $v(\{\alpha\}) - \sum_{j \in N^\ell - \alpha} b_j^\alpha$, that of i, $i \in N - \alpha$, is the sum of b_i^α and i's payoff obtained in $\mathrm{SBM}^\omega(N - \alpha, v, \mathcal{M}_{-\alpha})$, and that of i, $i \in N \setminus N^\ell$, is i's payoff obtained in $\mathrm{SBM}^\omega(N - \alpha, v, \mathcal{M}_{-\alpha})$.

Clearly, $\mathrm{SBM}^\omega(N, v, \mathcal{M})$ is identified as $\Gamma(N^\ell, (\bar{w}_i)_{i \in N^\ell}, (\Delta_i)_{i \in N^\ell})$, where for each $i \in N^\ell$, Δ_i is an extensive form game that starts from Stage 2 with i being chosen as the proposer.

When $\mathcal{M} = (\{N\})$ and $w_i = w_j$ for all $i, j \in N$, $\mathrm{SBM}^\omega(N, v, \mathcal{M})$ coincides with the bidding mechanism of Pérez-Castrillo and Wettstein (2001), and when $\mathcal{M} = (\{N\})$, it coincides with the weighted bidding mechanism also introduced by them. Thus, the following theorem includes theorems 1 and 2 of Pérez-Castrillo and Wettstein (2001) as special cases:

Theorem 6.1. *Let $(N, v, \mathcal{M})$ be a game with a social structure. Suppose that (N, v) is zero-monotonic. Then, in any SPE of the ω-weighted social bidding mechanism for $(N, v, \mathcal{M})$, the equilibrium payoff vector coincides with $\Upsilon^\omega(N, v, \mathcal{M})$.*

Proof. We prove this theorem by the induction on the number of the players. When $N = \{i\}$, according to the mechanism for one player case, player i obtains $v(\{i\}) = \Upsilon_i^\omega(N, v, \mathcal{M})$.

Now assume that the statement of the theorem holds for less than n players. We will show that the statement holds for n players. In the following, we use short-cut notations $\Upsilon^\omega(\mathcal{M})$ and $\Upsilon^\omega(\mathcal{M}_{-i})$ instead of $\Upsilon^\omega(N, v, \mathcal{M})$ and $\Upsilon^\omega(N - i, v, \mathcal{M}_{-i})$ for convenience.

Consider the following strategies:

At Stage 1, each player i, $i \in N^\ell$, makes the bid $b_j^i = \Upsilon_j^\omega(\mathcal{M}) - \Upsilon_j^\omega(\mathcal{M}_{-i})$ to any $j \in N^\ell - i$.

At Stage 2, player $\alpha \in N^\ell$, the proposer, offers $x_j = \Upsilon_j^\omega(\mathcal{M}_{-\alpha})$ to each $j \in N - \alpha$.

At Stage 3, player $i \in N - \alpha$ accepts any offer greater than or equal to $\Upsilon_i^\omega(\mathcal{M}_{-\alpha})$ and rejects any offer strictly smaller than $\Upsilon_i^\omega(\mathcal{M}_{-\alpha})$.

First we show that these strategies have $\Upsilon^\omega(N, v, \mathcal{M})$ as the final payoff. It is clear that these strategies yield $\Upsilon_i^\omega(\mathcal{M})$ for any $i \in N^\ell$ who is not the proposer, since $b_i^\alpha + x_i = \Upsilon_i^\omega(\mathcal{M})$, for $i \neq \alpha$. Additionally, any $i \in N \setminus N^\ell$ who is a member of lower ranked coalitions also obtains $\Upsilon_i^\omega(\mathcal{M})$ because $\Upsilon_i^\omega(\mathcal{M}_{-\alpha}) = \Upsilon_i^\omega(\mathcal{M})$ by the formula (6.1). Moreover, given that following the strategies the grand coalition is formed, the proposer α also obtains $\Upsilon_i^\omega(\mathcal{M})$ because Υ^ω satisfies efficeincy.

Note that the payoff determination is independent of the identity of the proposer α. Furthremore we note that, following the above mentioned strategies, weighted net bid $B_i(\bar{w})$ is zero for all $i \in N^\ell$ because

$$
\begin{aligned}
B^i(\bar{w}) &= \sum_{j \in N^\ell - i} \bar{w}_i b_j^i - \sum_{j \in N^\ell - i} \bar{w}_j b_i^j \\
&= \sum_{j \in N^\ell - i} \bar{w}_i \left(\Upsilon_j^\omega(\mathcal{M}) - \Upsilon_j^\omega(\mathcal{M}_{-i})\right) - \sum_{j \in N^\ell - i} \bar{w}_j \left(\Upsilon_i^\omega(\mathcal{M}) - \Upsilon_i^\omega(\mathcal{M}_{-j})\right) \\
&= \bar{w}_i(v(N) - v(N \setminus N^\ell) - \Upsilon_i^\omega(\mathcal{M})) - \bar{w}_i(v(N - i) - v(N \setminus N^\ell)) \\
&\quad - \sum_{j \in N^\ell - i} \bar{w}_j \Upsilon_i^\omega(\mathcal{M}) + \sum_{j \in N^\ell - i} \bar{w}_j \Upsilon_i^\omega(\mathcal{M}_{-j}) \\
&= \bar{w}_i(v(N) - v(N - i)) - \sum_{j \in N^\ell} \bar{w}_j \Upsilon_i^\omega(\mathcal{M}) + \sum_{j \in N^\ell - i} \bar{w}_j \Upsilon_i^\omega(\mathcal{M}_{-j}) \\
&= \bar{w}_i(v(N) - v(N - i)) - \Upsilon_i^\omega(\mathcal{M}) + \sum_{j \in N^\ell - i} \bar{w}_j \Upsilon_i^\omega(\mathcal{M}_{-j}) \\
&= 0,
\end{aligned}
$$

where the third equations follows from

$$\sum_{j \in N^\ell} \Upsilon_j^\omega(\mathcal{M}) = v(N) - v(N \setminus N^\ell)$$

and

$$\sum_{j \in N^\ell - i} \Upsilon_j^\omega(\mathcal{M}_{-i}) = v(N - i) - v(N \setminus N^\ell)$$

by (6.2), the fifth equation is by $\sum_{j\in N^\ell} \bar{w}_j = 1$, and the last equation follows from Proposition 6.6-(ii).

It remains to check the previous strategies constitute an SPE. Note first that the strategies at State 3 are best responses because if $j \neq \alpha$ rejects the offer, s/he plays the ω-weighted social bidding mechanism where the set of players is $N-\alpha$ and the social structure is $\mathcal{M}_{-\alpha}$; by the induction argument, the equilibrium payoff of this game is $\Upsilon^\omega(\mathcal{M}_{-\alpha})$. So, as long as $v(N) - v(\{\alpha\}) \geq \sum_{j\in N-\alpha} \Upsilon^\omega_j(\mathcal{M}_{-\alpha}) = v(N-\alpha)$, the strategy at State 2 is also best response.

Consider now the strategies at Stage 1. Consider a deviation of player i from above mentioned strategies at Stage 1. Let us denote any bid of player i by

$$c^i_j = b^i_j + a_j$$

for any $j \in N^\ell - i$ where b^i_j is the bid described in the above strategies. When player i changes her/his bid so that s/he should not become the proposer in any case, her/his payoff is not changed through the deviation. If player i deviates in a way that s/he becomes the winner of the bidding stage, her/his new net bid must satisfy

$$\hat{B}^i(\bar{w}) = \sum_{j\in N^\ell-i} \bar{w}_i b^i_j + \sum_{j\in N^\ell-i} \bar{w}_i a_j - \sum_{j\in N^\ell-i} \bar{w}_j b^j_i = \sum_{j\in N^\ell-i} \bar{w}_i a_j$$

$$\geqq \hat{B}^k(\bar{w}) = \sum_{j\in N^\ell-k} \bar{w}_k b^k_j - \sum_{j\in N^\ell-k} \bar{w}_j b^j_k - \bar{w}_i a_k = -\bar{w}_i a_k,$$

for all $k \in N^\ell - i$. So, $\sum_{j\in N^\ell-i} a_j \geqq -a_k$ for all $k \in N^\ell - i$. If $\sum_{j\in N^\ell-i} a_j < 0$, the condition is not satisfied for some k with $a_k < 0$. So, $\sum_{j\in N^\ell-i} a_j \geqq 0$. When $\sum_{j\in N^\ell-i} a_j = 0$, $c^i = b^i$ holds. On the other hand, if $\sum_{j\in N^\ell-i} a_j > 0$, her/his final payoff becomes $\Upsilon^\omega_i(\mathcal{M}) - \sum_{j\in N^\ell-i} a_j < \Upsilon^\omega_i(\mathcal{M})$. Thus, s/he can not be better off by changing her/his bid from the above mentioned strategies.

We now show that any SPE yields $\Upsilon^\omega(\mathcal{M})$. We proceeds by a series of claims:

Claim 1: In any SPE, at Stage 3, all players other than proposer α accept the offer if $x_i > \Upsilon^\omega_i(\mathcal{M}_{-\alpha})$ for every player $i \neq \alpha$. Moreover, if $x_i < \Upsilon^\omega_i(\mathcal{M}_{-\alpha})$ for at least some $i \neq \alpha$, then the offer is rejected.

Claim 2: In any SPE, at Stage 2, proposer α obtains $v(N) - v(N-\alpha)$ and every $i \neq \alpha$ obtains $\Upsilon^\omega_i(\mathcal{M}_{-\alpha})$, in addition to the transfer of the bids at Stage 1.

These two claims are almost equivalent to Claims (a) and (b) of Pérez-Castrillo and Wettstein (2001) and so we omit the proof. The only difference is that now player $i \neq \alpha$ obtains $\Upsilon^\omega_i(\mathcal{M}_{-\alpha})$ after a rejection by the induction hypothesis. A remark is that when $v(N)-v(\{\alpha\}) = v(N-\alpha)$, there are two types of SPEs: one is that the offer is accepted and the other is that the offer is rejected. However, their final payoff is the same in both cases.

The following two claims are on the behavior in the bidding stage. Suppose $|N^\ell| \geq 2$.

Claim 3: In any SPE, $B^i(\bar{w}) = 0$ for any $i \in N^\ell$.

Define $\Omega = \{i \in N^\ell : B^i(\bar{w}) \geq B^j(\bar{w})\ \forall j \in N^\ell\}$. If $\Omega = N^\ell$, the fact that $\sum_{i\in N^\ell} B^i(\bar{w}) = 0$ trivially implies $B^i(\bar{w}) = 0$ for each $i \in N^\ell$. We now show that, for any SPE, $\Omega = N^\ell$ follows. We prove this by contradiction. Let $(b^i)_{i\in N^\ell}$ be SPE strategies at Stage 1. Suppose that $\Omega \neq N^\ell$. Then, we can find two players $i \in \Omega$ and $k \in N^\ell \setminus \Omega$. Let $\delta > 0$, and consider player i's new strategy $\hat{b}^i$ such that $\hat{b}^i_j = b^i_j + \delta/|\Omega|$ if $j \in \Omega - i$; $\hat{b}^i_j = b^i_j - \delta$ if $j = k$; $\hat{b}^i_j = b^i_j$ otherwise. The new net bids are $\hat{B}^i(\bar{w}) = B^i(\bar{w}) - \bar{w}_i\delta/|\Omega|$; $\hat{B}^k(\bar{w}) = B^k(\bar{w}) + w_i\delta$; $\hat{B}^j(\bar{w}) = B^j(\bar{w}) - w_i\delta/|\Omega|$ for all $j \in \Omega - i$; $\hat{B}^j(\bar{w}) = B^j(\bar{w})$ for all $j \in N^\ell \setminus (\Omega \cup k)$. Since $B^j(\bar{w}) > B^l(\bar{w})$ holds for any $j \in \Omega$ and any $l \in N^\ell \setminus \Omega$, we still obtain $\hat{B}^j(\bar{w}) > \hat{B}^l(\bar{w})$ for sufficiently small δ. Thus, $\hat{\Omega} := \{i \in N^\ell : \hat{B}^i(\bar{w}) \geq \hat{B}^j(\bar{w})\ \forall j \in N^\ell\}$ completely coincides with Ω. However, for player i, we have $\sum_{j\in N^\ell - i} \hat{b}^i_j < \sum_{j\in N^\ell - i} b^i_j$, and thus, her/his new strategy $\hat{b}^i$ increases her/his expected final payoff: a contradiction.

Claim 4: For any SPE, each player's payoff is the same regardless of who is chosen as the proposer.

From Claim 3, each player's weighted net bid coincides each other in SPE. Thus, every player could become a proposer with the same probability. We prove the contrapositive of the claim. Suppose that some player i could get the highest payoff if s/he would become a proposer than in the case where some other player is a proposer. Then, sufficiently small increases in her/his bids to the other player improve her/his final payoff so that s/he will deviate from the SPE strategy. Similarly, if player i could obtain the biggest payoff when some other player j is a proposer than in the other cases, s/he has an incentive to decrease her/his bid to player j.

Claim 5: In any SPE, the final payment received by each of the players coincides with $\Upsilon^\omega_i(\mathcal{M})$.

For every $i \in N^\ell$, since by Claim 4 the payoff of player i is the same in the case where other $j \in N^\ell - i$ is chosen as the proposer and the case where other $j' \in N^\ell - i, j' \neq j$ is chosen as the proposer, we have

$$b^{j'}_i + \Upsilon^\omega_i(\mathcal{M}_{-j'}) = b^j_i + \Upsilon^\omega_i(\mathcal{M}_{-j}) \iff b^{j'}_i = b^j_i + \Upsilon^\omega_i(\mathcal{M}_{-j}) - \Upsilon^\omega_i(\mathcal{M}_{-j'}).$$

Moreover, Claim 4 also implies the payoff of i is the same in the case where i is chosen as the proposer and in the case that other j is chosen. Thus,

$$\begin{aligned} b^j_i + \Upsilon^\omega_i(\mathcal{M}_{-j}) &= v(N) - \sum_{j'\in N-i} \Upsilon^\omega_{j'}(\mathcal{M}_{-i}) - \sum_{j'\in N^\ell - i} b^i_{j'} \\ &= v(N) - v(N-i) - \sum_{j'\in N^\ell - i} (\frac{\bar{w}_{j'}}{\bar{w}_i}) b^{j'}_i \\ &= v(N) - v(N-i) - \sum_{j'\in N^\ell - i} (\frac{\bar{w}_{j'}}{\bar{w}_i}) \Big(b^j_i + \Upsilon^\omega_i(\mathcal{M}_{-j}) - \Upsilon^\omega_i(\mathcal{M}_{-j'})\Big) \end{aligned}$$

where the second equality is by the efficiency of Υ^ω and Claim 3. Since $\sum_{j'\in N^\ell} \bar{w}_{j'} = 1$, we have

$$\begin{aligned} (\frac{1}{\bar{w}_i}) b^j_i &= v(N) - v(N-i) + \sum_{j'\in N^\ell - i} (\frac{\bar{w}_{j'}}{\bar{w}_i}) \Upsilon^\omega_i(\mathcal{M}_{-j'}) - \sum_{j'\in N^\ell - i} (\frac{\bar{w}_{j'}}{\bar{w}_i}) \Upsilon^\omega_i(\mathcal{M}_{-j}) - \Upsilon^\omega_i(\mathcal{M}_{-j}) \\ &= v(N) - v(N-i) + \sum_{j'\in N^\ell - i} (\frac{\bar{w}_{j'}}{\bar{w}_i}) \Upsilon^\omega_i(\mathcal{M}_{-j'}) - (\frac{1}{\bar{w}_i}) \Upsilon^\omega_i(\mathcal{M}_{-j}) \end{aligned}$$

Thus, by Proposition 6.6-(ii), we have

$$b^j_i = \Upsilon^\omega_i(\mathcal{M}) - \Upsilon^\omega_i(\mathcal{M}_{-j}).$$

So, by the above result and Claim 2, for $i \in N^\ell$, her/his final payoff is $\Upsilon^\omega_i(\mathcal{M})$ and for $i \in N \setminus N^\ell$, i's final payoff is $\Upsilon^\omega_i(\mathcal{M}_{-\alpha}) = \Upsilon^\omega_i(\mathcal{M})$ where α is a player in N^ℓ. Moreover, when $|N^\ell| = 1$, $i \in N^\ell$ obtains $v(N) - v(N-i) = \Upsilon^\omega_i(\mathcal{M})$. □

With regard to the ω-weighted social bidding mechanism and Theorem 6.1, first, while the definition of Υ^ω is related to the order consistent with the social structure, the order of decision of the responders does not change the result. All we need is the information that the responders make their decision by turns and when one of them makes a decision, s/he knows not only the offers made by the proposer to the other responders but also the responses of her/his preceding responders.[2] In addition, as pointed out by Pérez-Castrillo and Wettstein (2001), other type of tie-breaking rule in the bidding stage is possible. For example, there is an order of priority of the players, and the player with highest priority is chosen when there are players that have equal highest weighted net bids. Further, the weighted bidding stage in SBM^ω can be replaced by the random selection of player $i \in N^\ell$ in a way that first one coalition C_k^ℓ in $\mathcal{C}^\ell$ is randomly chosen proportional to its normalized weight $w_k^{*\ell}/w^{*\ell}(M^\ell)$ and inside the chosen coalition, one player $i \in C_k^\ell$ is selected proportional to her/his normalized weight $w_i/w(C_k^\ell)$. However, this randomized mechanism achieves Υ^ω as an expected value.[3]

In addition to implementing Υ^ω as a realized value for any zero-monotonic game, Theorem 6.1 has the following significance related to the literature on implementing cooperative solutions.

- In the case of $\mathcal{M} = (\{N\})$, the ω-weighted social bidding mechanism coincides with the bidding mechanism and the weighted bidding mechanism of Pérez-Castrillo and Wettstein (2001). Therefore, the implementation of the Shapley value and the weighted Shapley value without hierarchic structure is obtained as the special cases of Theorem 6.1.

- If $\mathcal{C}^h = \{N^h\}$ for each $h \in L$, the ω-weighted social bidding mechanism implements HV^w for any zero-monotonic game.

- If $\mathcal{M} = (\mathcal{C})$ and $w_i = w_j$ for each $i, j \in C_k \in \mathcal{C}$ and $w_k^* = w_{k'}^*$ for each $k, k' \in M$, the ω-weighted social bidding mechanism implements the coalitional value for any zero-monotonic game. It should be emphasized that in the literature on the implementation of the coalitional value, Vidal-Puga and Bergantinos (2003) attain it for strictly superadditive environment and Vidal-Puga (2005a) attains it for strictly zero-monotonic environment.[4] Thus, our result widens the domain of the implementation of the coalitional value.

- If $\mathcal{M} = (\mathcal{C})$, the ω-weighted social bidding mechanism implements the family of all the weighted coalitional values introduced by Levy and McLean (1989).

One remark on the implementation of the coalitional value is that we are able to implement the coalitional value without the framework of a game with a social structure. To implement the coalitional value, it is suffice to consider a bargaining process such that after a rejection of offer by the first proposer, the proposing coalition retains the right to choose the second proposer and the second proposer is selected from the coalition by the bidding game. Therefore, the priority rights to select the proposer of the first proposing coalition is crucial to the implementation of the coalitional value. In fact, without a game with a social structure, Kamijo (2007a) implements the coalitional value by considering only the priority of choosing the proposer.

[2] As is pointed out by Pérez-Castrillo and Wettstein (2001), the sequential move of the responders is needed to avoid bad equilibria such that two or more responders choose to reject the offer even though the proposer makes offers that are beneficial for all of them.

[3] This point is similar to the relationship between the bidding mechanism of Pérez-Castrillo and Wettstein (2001) and the bargaining model of Hart and Mas-Colell (1996).

[4] The bargaining model of Vidal-Puga (2005a) is based on the one introduced by Hart and Mas-Colell (1996) and an extension of it to a (NTU) game with coalition structures. His model works in strictly zero-monotonic environment and attains the coalitional value in expected terms.

6.6 Concluding remarks

In this chapter, we propose games with social structure and a bargaining model for this class of games. The bargaining model is established on the basis of the bidding game and is an extension of the bidding mechanism of Pérez-Castrillo and Wettstein (2001). Similar to the result of Pérez-Castrillo and Wettstein (2001), the ω-weighted social bidding mechanism implements the weighted value for a game with a social structure in any SPE if the game is zero-monotonic. This result implies that we provide an implementation of the Shapley value, the weighted Shapley value with hierarchic structure, the coalitional value and the weighted coalitional value. Because the two-step bidding mechanism introduced by Vidal-Puga and Bergantinos (2003) works only in the strictly superadditive domain, our result also has implication on extending the domain of implementing the coalitional value.

Chapter 7

Implementation of two step values

7.1 Introduction

This chapter explores non-cooperative mechanisms that implement two solutions in game with coalition structure. In Chapter 3 and 4 of this thesis, we introduce two new solution concepts in game with coalition structures, ψ^{δ} and ψ^{γ}, and provide several axiomatic or non-axiomatic characterization of these solutions. To complete the analysis on the solutions, we provide two mechanisms implementing these solutions in this chapter.

The two mechanism considered here are also based on bidding mechanism of Pérez-Castrillo and Wettstein (2001). So, we refer the properties on the bidding game considered in Chapter 5 in order to prove the main results.

Both the two mechanisms proceed as follows. If the coalition structure is the grand coalition structure, then, it is the bidding mechanism of Pérez-Castrillo and Wettstein (2001). When there are more than one coalition in the coalition structure, the mechanism is as follows. In the first stage of the mechanisms, all the players participate in the bidding game. In the next stage, proposer $\alpha \in C_k$ makes an offer to *all* the players in $N - \alpha$, and the other players respond to the offer sequentially. In the case of acceptance by all players, the proposer pays her/his offer to any player $j \neq \alpha$ in return for obtaining their value of cooperation, $v(N)$, and the bargaining is over. On the other hand, when some player rejects the offer, (i) players in the proposing coalition C_k participate in the bidding mechanism of Pérez-Castrillo and Wettstein (2001) for themselves, and (ii) the remaining players in $N \setminus C_k$ continue the same bargaining. So, in contrast with the social bidding mechanism in Chapter 6, the proposing coalition does not retain the right to choose a new proposer. Rather, in exchange for a player in this coalition becoming a proposer, the proposing coalition has to take a risk to be separated from players in the other coalitions.

The difference between the two mechanism lies in the first stage. In one mechanism, the weighted bidding game where the weight of a player is reciprocal number of the cardinality of his coalition is played in the first stage. In another mechanism, the bidding game, i.e., the weighted bidding game with the weight of a player being 1, is played in the first stage. We show that the former mechanism implement the Shapley-Egalitarian solution and the latter implements the collective value for any superadditive game. Interestingly, while the definition of the collective value is related to the size relevant weight, the mechanism implementing the collective value is independent of such size relevant weight.

The rest of this chapter is as follows. In the next section, the mechanisms are explained and main results. In Section 3, the proofs of main theorems are provided.

7.2 Bargaining models for a game with a coalition structure

Let $(N, v, \mathcal{C})$ be a game with a coalition structure. We now define two bargaining models for a game with a coalition structure, denoted by $\mathrm{CBM}^{\delta}(N, v, \mathcal{C})$ and $\mathrm{CBM}^{\gamma}(N, v, \mathcal{C})$, respectively, in a recursive manner:

We first define $\mathrm{CBM}^{\delta}(N, v, \mathcal{C})$. When, $\mathcal{C} = \{N\}$, players in N play the bidding mechanism of Pérez-Castrillo and Wettstein (2001) for (N, v).

Suppose that $\mathrm{CBM}^{\delta}(N, v, \mathcal{C})$ is already defined for less than m coalitions in $\mathcal{C}$. The bargaining for the case with m coalitions proceeds as follows:

Stage 1 The weighted bidding game is played only by all players in N. After the simultaneous choices of their bids, the weighted net bid $B_i(w)$ is calculated by formula (6.3) for each $i \in N$ where $w_i = 1/|C_k|$ for any $i \in C_k \in \mathcal{C}$. A player with the highest weighted net bid becomes a proposer in the next stage in return for the actual payment of her/his bids to other players in N. If we have two or more players with the highest net bid, then any one of them is randomly chosen.

Stage 2 Let α be a proposer chosen at the first stage. The proposer makes an offer $x_j \in \mathbb{R}$ to any $j \in N - \alpha$. In other words, proposer α chooses $n-1$ dimensional vector $x = (x_j)_{j \in N-\alpha}$.

Stage 3 Every player other than α sequentially chooses *to accept or reject the offer*. If we have a rejection by any responder, the offer is rejected; otherwise, the offer is accepted. In the case of acceptance, the proposer actually pays the offer x_j to any $j \neq \alpha$ and α obtains the worth of their total cooperation, $v(N)$. After that, the bargaining is over. Thus, the final payoff of proposer α is

$$v(N) - \sum_{j \in N-\alpha} b_j^{\alpha} - \sum_{j \in N-\alpha} x_j.$$

The final payoff of other $i \in N - \alpha$ is $b_i^{\alpha} + x_i$.

In the case of rejection, the bargaining proceeds as follows: Let $C_k \in \mathcal{C}$ be a coalition such that $\alpha \in C_k$. Players in C_k discuss for the division of $v(C_k)$ and play $\mathrm{CBM}^{\delta}(C_k, v, \{C_k\})$. Players in $N \setminus C_k$ play $\mathrm{CBM}^{\delta}(N - C_k, v, \mathcal{C} \setminus \{C_k\})$. Thus, the final payoff of proposer α is the sum of $-\sum_{j \in N-\alpha} b_j^{\alpha}$ and α's payoff obtained in $\mathrm{CBM}^{\delta}(C_k, v, \{C_k\})$, that of i, $i \in C_k - \alpha$, is the sum of b_i^{α} and i's payoff obtained in $\mathrm{CBM}^{\delta}(C_k, v, \{C_k\})$, and that of i, $i \in N \setminus C_k$, is the sum of b_i^{α} and i's payoff obtained in $\mathrm{CBM}^{\delta}(N - C_k, v, \mathcal{C} \setminus \{C_k\})$.

Next, we define $\mathrm{CBM}^{\gamma}(N, v, \mathcal{C})$. The only difference between $\mathrm{CBM}^{\gamma}(N, v, \mathcal{C})$ and $\mathrm{CBM}^{\delta}(N, v, \mathcal{C})$ is Stage 1. In $\mathrm{CBM}^{\gamma}(N, v, \mathcal{C})$, Stage 1 is replaced by

Stage 1' The weighted bidding game is played only by all players in N. Here, each player's weight is identical, irrespective of the coalition that he belongs to. Thus, $w_i = 1$ for each $i \in N$.

Clearly, $\mathrm{CBM}^{\delta}(N, v, \mathcal{C})$ and $\mathrm{CBM}^{\gamma}(N, v, \mathcal{C})$ coincides with the bidding mechanism of Pérez-Castrillo and Wettstein (2001) when $\mathcal{C} = \{N\}$ or $\mathcal{C} = \{\{i\}\}_{i \in N}$. Moreover, $\mathrm{CBM}^{\delta}(N, v, \mathcal{C})$ (resp. $\mathrm{CBM}^{\delta}(N, v, \mathcal{C})$)is identified as $\Gamma(N, (w_i)_{i \in N}, (\Delta_i)_{i \in N})$, where for each $i \in N^{\ell}$, Δ_i is an extensive form game that starts from Stage 2 with i being chosen as the proposer and $w_i = 1/|C_k|$ for each $i \in C_k \in \mathcal{C}$ (resp, $w_i = 1$ for each $i \in N$).

Theorem 7.1. *Any subgame perfect equilibrium outcome of* $\mathrm{CBM}^{\delta}(N, v, \mathcal{C})$ *gives the same payoff as* $\psi^{\delta}(N, v, \mathcal{C})$ *if* (N, v) *is superadditive.*

Theorem 7.2. *Any subgame perfect equilibrium outcome of* $\mathrm{CBM}^{\gamma}(N,v,\mathcal{C})$ *gives the same payoff as* $\psi^{\gamma}(N,v,\mathcal{C})$ *if* (N,v) *is superadditive.*

The difference between the two mechanism and the social bidding mechanism considered in the previous chapter is in the bargaining after the rejection of the first proposer. In the two mechanism considered here, a proposing coalition is separated from the other coalitions. Thus, the proposing coalition in the two mechanisms has a risk to be separated from players in other coalitions exchange for a player in this coalition being a proposer. In contrast, in the the social bidding mechanism, a proposing coalition can still bargaining with the other coalitions and this coalition has the advantage in selecting the next proposer.

7.3 Proofs of the results

Proof of Theorem 7.1. We will prove this theorem by the induction on the number of elements in $\mathcal{C}$. If $|\mathcal{C}| = 1$, then the theorem holds because $\mathrm{CBM}^{\delta}(N,v,\mathcal{C})$ is the bidding mechanism of Pérez-Castrillo and Wettstein (2001) and $\psi^{\delta}(N,v,\mathcal{C})$ is $\mathrm{Sh}(N,v)$.

We assume that the theorem holds when $|\mathcal{C}| < m$. Then we will show that the theorem holds for $(N,v,\mathcal{C})$ with $|\mathcal{C}| = m$. First, consider the behaviors of the players in Stages 2 and 3. Let α be a proposer in Stage 2 and $C_k \in \mathcal{C}$ be a coalition such that $\alpha \in C_k$. Let y_i, $i \in N0\alpha$ be defined by, for $i \in C_h$,

$$y_i = \begin{cases} \psi_i^{\delta}(N \setminus C_k, v, \mathcal{C} \setminus \{C_k\}) & \text{if } C_h \neq C_k, \\ \mathrm{Sh}_j(C_k, v) & \text{if } C_h = C_k. \end{cases}$$

We have the following claims.

Claim 1: In any SPE, at Stage 3, all players other than proposer α accept the offer if $x_i > y_i$ for every player $i \neq \alpha$. Moreover, if $x_i < y_i$ for at least some $i \neq \alpha$, then the offer is rejected.

Claim 2: In any SPE, at Stage 2, proposer α obtains $v(N) - v(C_k) - v(N \setminus C_k) + \mathrm{Sh}_i(C_k, v)$ and every $i \neq \alpha$ obtains y_i, in addition to the transfer of the bids at Stage 1.

These two claims are almost equivalent to Claims (a) and (b) of Pérez-Castrillo and Wettstein (2001) and so we omit the proof. The only difference is that now player $i \neq \alpha$ obtains y_i after a rejection by the induction hypothesis. Note that the incentive of the proposer holds because

$$\begin{aligned} v(N) - \sum_{i \in N-\alpha} y_j &= v(N) - \left(\sum_{i \in C_k \setminus \alpha} \mathrm{Sh}_i(C_k, v) + \sum_{i \in N \setminus C_k} \psi_i^{\delta}(N \setminus C_k, v, \mathcal{C} \setminus \{C_k\}) \right) \\ &= v(N) - v(C_k) - v(N \setminus C_k) + \mathrm{Sh}_i(C_k, v) \geqq \mathrm{Sh}_i(C_k, v) \end{aligned}$$

A remark is that when $v(N) - v(C_k) - v(N - C_k) = 0$, there are two types of SPEs: one is that the offer is accepted and the other is that the offer is rejected. However, their final payoff is the same in both cases.

To consider the behavior in Stage 1, it should be emphasized that $\mathrm{CBM}^{\delta}(N,v,\mathcal{C})$ can be identified as $\Gamma(N, (w_i)_{i \in N}, (\Delta_i)_{i \in N})$. For each $i \in C_k \in \mathcal{C}$ and $j \in C_h \in CS$, let u_j^i be the payoff (without the transfer of the bids) of player j when i is a proposer. Thus,

$$u_j^i = \begin{cases} v(N) - v(C_k) - v(N \setminus C_k) + \mathrm{Sh}_i(C_k, v) & \text{if } i = j, \\ \mathrm{Sh}_j(C_k, v) & \text{if } i \neq j, C_h = C_k, \\ \psi_i^{\delta}(N \setminus C_k, v, \mathcal{C} \setminus \{C_k\}) & \text{if } i \neq j, C_h = C_k. \end{cases}$$

Then, it is easily confirmed that $\sum_{j\in N} u_j^i = v(N)$ and it is irrelevant to the identity of proposer i. Therefore, the necessity and sufficient condition for the existence of the bidding game holds and there exists unique bidding behavior in Stage 1. Moreover, Theorem 5.3 means that j's final payoff is the expected value of u_j^i when each i is selected proportional to his own weight $w_i = 1/|C_k|$. Since according to this probability distribution, some i is selected at probability $\frac{1}{m|C_k|}$, where $i \in C_k$, this expected value is

$$\begin{aligned}
&\frac{1}{m|C_k|}\left(v(N) - v(C_k) - v(N\setminus C_k) + \mathrm{Sh}_i(C_k, v)\right) \\
&+\frac{|C_k|-1}{m|C_k|}\mathrm{Sh}_i(C_k, v) + \frac{1}{m}\sum_{h\in M-k}\psi_i^\delta(N\setminus C_h, v, \mathcal{C}\setminus\{C_h\}) \\
&= \frac{1}{m|C_k|}\left(v(N) - v(C_k) - v(N\setminus C_k)\right) + \frac{1}{m}\mathrm{Sh}_i(C_k, v) \\
&+\frac{1}{m}\sum_{h\in M-k}\left(\frac{\mathrm{Sh}_k(M-h, v_{\mathcal{C}}) - v(C_k)}{|C_k|} + \mathrm{Sh}_i(C_k, v)\right) \\
&= -\frac{1}{m|C_k|}v(C_k) - \frac{m-1}{m|C_k|}v(C_k) + \frac{1}{m}\mathrm{Sh}_i(C_k, v) + \frac{m-1}{m}\mathrm{Sh}_i(C_k, v) \\
&+\frac{1}{|C_k|}\left(\frac{1}{m}\left(v(N) - v(N\setminus C_k)\right) + \frac{1}{m}\sum_{h\in M-k}\mathrm{Sh}_k(M-h, v_{\mathcal{C}})\right) \\
&= -\frac{v(C_k)}{|C_k|} + \mathrm{Sh}_i(C_k, v) + \frac{\mathrm{Sh}_k(M, v_{\mathcal{C}})}{|C_k|}.
\end{aligned}$$

We use Proposition 6.3 for the last equality. □

Proof of Theorem 7.2. Almost proof of this theorem is very similar to the proof of Theorem 7.1. One of the differences is that in $\mathrm{CBM}^\delta(N, v, \mathcal{C})$, for each $i \in C_k \in \mathcal{C}$ and $j \in C_h \in CS$, the payoff (without the transfer of the bids) of player j when i is a proposer, u_j^i, is

$$u_j^i = \begin{cases} v(N) - v(C_k) - v(N\setminus C_k) + \mathrm{Sh}_i(C_k, v) & \text{if } i = j, \\ \mathrm{Sh}_j(C_k, v) & \text{if } i \neq j, C_h = C_k, \\ \psi_i^\gamma(N\setminus C_k, v, \mathcal{C}\setminus\{C_k\}) & \text{if } i \neq j, C_h = C_k. \end{cases}$$

Since each i' weight is $w_i = 1$, the final payoff of $i \in C_k$ is

$$\begin{aligned}
&\frac{1}{n}\left(v(N) - v(C_k) - v(N\setminus C_k) + \mathrm{Sh}_i(C_k, v)\right) \\
&+\frac{|C_k|-1}{n}\mathrm{Sh}_i(C_k, v) + \sum_{h\in M-k}\frac{|C_h|}{n}\psi_i^\gamma(N\setminus C_h, v, \mathcal{C}\setminus\{C_h\}) \\
&= \frac{1}{n}\left(v(N) - v(C_k) - v(N\setminus C_k)\right) + \frac{|C_k|}{n}\mathrm{Sh}_i(C_k, v) \\
&+\sum_{h\in M-k}\frac{|C_h|}{n}\left(\frac{\mathrm{Sh}_k^\omega(M-h, v_{\mathcal{C}}) - v(C_k)}{|C_k|} + \mathrm{Sh}_i(C_k, v)\right) \\
&= -\frac{1}{n}v(C_k) - \frac{n-|C_k|}{n|C_k|}v(C_k) + \frac{|C_k|}{n}\mathrm{Sh}_i(C_k, v) + \frac{n-|C_k|}{n}\mathrm{Sh}_i(C_k, v)
\end{aligned}$$

$$+\frac{1}{|C_k|}\left(\frac{|C_k|}{n}\left(v(N)-v(N\setminus C_k)\right)+\sum_{h\in M-k}\frac{|C_h|}{n}\mathrm{Sh}_k^{\omega}(M-h,v_{\mathcal{C}})\right)$$

$$=-\frac{v(C_k)}{|C_k|}+\mathrm{Sh}_i(C_k,v)+\frac{\mathrm{Sh}_k^{\omega}(M,v_{\mathcal{C}})}{|C_k|},$$

where we use notation ω to describe a weight of the weighted Shapley value Sh^{ω} and here $\omega_k=|C_k|$ for all $k\in M$. The last equality follows from Proposition 6.3. □

Chapter 8

Conclusion and Further Topics

The final chapter summarizes the main contributions of this thesis from the literature on solution theory in cooperative game and then concludes by explaining the further topics.

In the thesis, I constructed the analysis on several solutions from axiomatic and non-cooperative approaches. Not only did I find new axiomatization results on solutions and new implementation results separately but also the implications from both approaches.

First, I provided new axiomatic foundation of the Shapley value, the most famous solution concept in cooperative game theory. I introduced a new axiom, called the balanced cycle contributions property (BCC), and axiomatized the Shapley value by BCC, efficiency and the axiom on the effect of the exclusion of a null player. One interesting point is that by using BCC, I also axiomatized both the Egalitarian value and the CIS value, and found the differences between the three solutions lies in what player' deletion does not affect the payoff of the other players. The result is summarized in Table 2.3 in Chapter 2.

Second, I introduced a new mechanism which implements the coalitional value of games with coalition structures. In the literature, Vidal-Puga and Bergantinos (2003) and Vidal-Puga (2005a) also presented mechanisms that implement the coalitional value. One merit of the mechanism introduced in this thesis, the social bidding mechanism, is that it implements the coalitional value in the larger domain than the one by the two mechanisms of Vidal-Puga and Bergantinos (2003) and Vidal-Puga (2005a). The crucial feature of the social bidding mechanism is that in the mechanism, a proposing coalition has the advantage in selecting the next proposer after the rejection of the previous proposer.

Third, I provided a new class of games, the game with social structure. This is a unified model of two kinds of games: a game with a horizontal structure and a game with a hierarchical structure. An example of such a situation is the organizational structure of the employees in a firm, where there are many employees in some level and at the same time there are also many employees in higher and lower levels. The weighted value defined in this class of games is an extension of the Shapley value to such a game, and thus, it coincides with the Shapley value, the weighted Shapley value with hierarchic structure, the coalitional value, and the weighted coalitional value, in some special cases.

Fourth, I provided two new solution concepts in games with coalition structures. These solutions, the Shapley-Egalitarian solution and the collective value, are two-step Shapley values in the following sense: an allocation of the cooperative surplus by using the Shapley value in two-step bargaining process, a bargaining inter-coalitions and a bargaining intra-coalitions. The bargaining surplus of the coalition is allocated among the intra-coalition members in egalitarian way. Thus, in the first step, each coalition obtains its Shapley value applied for a game among coalitions in the definition of the Shapley-Egalitarian solution. On the other hand, each coalition obtains its weighted Shapley value with size-relevant weight applied for a game among coalitions in the definition of the the collective value. In both solutions, the pure surplus of a coalition in the first step bargaining (its Shapley value obtained from the first step minus the worth of the coalition) is divided equally among players in the coalition. In the second step, players in the coalition receive their Shapley value applied for their own internal game.

Fifth, I gave axiomatic and non-cooperative foundations to the Shapley-Egalitarian solution and the collective value, and demonstrate the differences between the two solutions and the coalitional value. The coalitional value and the two-step Shapley values are different in the judgment of application of the equity criterion. The coalitional value requires that two players in coalition C_k should be equally treated if these two are judged to be equal in the whole society. On the other hand, the two-step Shapley values require that two players in coalition C_k should be equally treated if these two are judged to be equal in the internal society. I also found that the coalitional value and the two-step Shapley values are different in the treatment of null players. While the coalitional value does not give any portion of surplus to a null player even if his coalition obtains large benefits, the two-step Shapley values give some portion of the surplus to the null players if his coalition obtains some benefits. Moreover, from the analysis of non-cooperative foundation, I found that the difference between the two coalitional bidding mechanisms which implement the Shapley-Egalitarian solution and the collective value, respectively, and the social bidding mechanism which implements the coalitional value is in the bargaining after the rejection of the first proposer. In the coalitional bidding mechanisms, a proposing coalition is separated from the other coalitions. Thus, in exchange for a player in this coalition becoming a proposer, the proposing coalition has to take a risk to be separated from players in the other coalitions. Whereas, in the social bidding mechanism, a proposing coalition can still bargaining with the other coalitions and this coalition has the advantage in selecting the next proposer.

Finally, I explain the further topics related to this thesis. First is related to the new axiom BCC considered in Chapter 2. In Chapter 2, I provided axiomatization of the Shapley value and the Egalitarian value by using BCC. As mentioned in Chapter 2, this approach can be generalized when we pay attention to null players and focus on the effect of the exclusion of a null player in each value. By doing so, all values we mention here (including the ENSC value) and all their convex combinations are characterized. Through this way that focuses on the deletion of null players, it may be possible to succeed in the class axiomatization of the solutions that satisfy BCC and Efficiency.

Second is related to the extension of BCC to games with coalition structures. In this case, two types of BCC can be considered. One is BCC intra coalition and the other is BCC inter coalitions. One conjecture is that these two of BCC with efficiency and NPO axiomatize the coalitional value.

Third is related to an extension of the two new solutions, the Shapley-Egalitarian solution and the collective value, to NTU games. In the recent literature, the coalitional value is extended to NTU games with coalition structures (Bergantinos and Vidal-Puga 2005). Thus, it may also be possible to generalize our new solution concepts for game with coalition structure to an NTU case.

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