Essays on the Theory of R&D-based Economic Growth

R&Dに基づく経済成長の理論的研究

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Introduction

1.1 Purpose

This paper provides theoretical analyses of economic growth and business cycles by using the framework of the R&D-based growth model. The two phenomena, economic growth and business cycles, are examined separately in macroeconomic theoretical literature. Typical examples include the real business cycle (RBC) theory and the endogenous growth theory. The RBC theory focuses on the deviation from the trend caused by exogenous shocks to examine short-run business cycles, whereas the endogenous growth theory focuses on the property of steady-state growth to examine long-run economic growth.

Contrary to these theories, this study integrates the analyses of both short-run fluctuations and long-run sustained growth in a unified setup. It is not possible to investigate the interaction between growth and cycles as long as they are analyzed separately. Economic stabilization policies may inhibit long-run growth, and growth promotion policies may cause a cyclical economy. In addition, modifying the model to obtain valid long-run growth may affect the frequency and behavior of business cycles. Analyzing these phenomena is not feasible using the separated approach; however, it is possible using the integrated approach.

In addition, this study explains not only long-run growth but also economic fluctuations caused by endogenous factors. While business cycle models that rely on exogenous productivity shocks are useful to analyze the behavior of cycles, they cannot analyze their sources and conditions that arise in business cycles. This study is beneficial in this respect.

1.2 R&D-based endogenous growth theory

The endogenous growth theory was initially conceived and extended by Romer (1986, 1990), Lucas (1988), Barro (1990), Grossman and Helpman (1991b), and Aghion and Howitt (1992). The theory has been actively studied by macroeconomists over the past two decades; even now, it has impacts on many other areas of macroeconomics. Various models have been presented in the literature on endogenous growth theory to provide the appropriate microfoundation to avoide diminishing returns of capital or accumulative resources. This study applies the theoretical model that is known as the R&D-based growth model, which determines R&D and innovation as the main engine of long-run growth, as its name suggests. Moreover, we use the variety expanding model framework, which is one of the methods used to develop a model for explaining R&D driven technological progress.[1]

In the literature on growth theory, innovation is distinguished between process and product innovation. Product innovation can be further divided into horizontal and vertical innovation. While, the quality-ladder model determines that technology is improved through the quality of products using vertical innovation,[2] the variety expanding model shows that technological progress is improved through the expansion of the variety of goods using horizontal innovation. The monopolistic-competition model specified by Dixit and Stiglitz (1977) and Ethier (1982) is generally used to explain how such technological progress is driven by the private sector in tractable models. Judd (1985) constructed the first dynamic model in which the variety of goods was endogenously expanding, under the assumption of the monopolistic competition. It was then used to explain endogenous sustained growth by Romer (1987, 1990).

Two models of product innovation should be viewed as complements rather than as substitutes, as mentioned in Grossman and Helpman (1991a, chap. 4), Barro and Sala-i Martin (2004, chap. 7), and Gancia and Zilibotti (2005). One of the advantages of variety expanding models is their analytical

[1] The variety expanding model is also referred to as the \variety expansion model" or \product variety model."

[2] The most frequently referenced studies are Grossman and Helpman (1991b) and Aghion and Howitt (1992).

tractability. In fact, it helps to investigate complex theoretical issues in the growth cycle models and the monetary endogenous growth models. That's why we adopt variety expanding models throughout this thesis.

1.3 Deterministic cycles

This study applies two approaches to the R&D-based endogenous growth model in order to analyze cyclical growth. The first approach explains economic fluctuation using a deterministic periodic solution,[3] while the second explains economic volatility using indeterminate equilibria and sunspot.

Studies of the interaction between R&D and deterministic fluctuations were pioneered by Judd (1985) and Deneckere and Judd (1992). By applying the flip bifurcation theorem to the variety expanding model without capital accumulation, they found periodic or chaotic fluctuations between periods of innovation and periods of no innovation. However, in the above mentioned models, a sustained R&D effort did not contribute to long-run growth because it was canceled out by the obsolescence and dilution of knowledge.

Matsuyama (1999, 2001) modified the model in Deneckere and Judd (1992) by introducing capital accumulation and Romer's (1990) idea of endogenous growth, and investigated endogenous fluctuations with sustained long-run growth. Specifically, capital accumulation was derived from intertemporal optimization of the infinitely lived agents in Matsuyama (2001). Matsuyama (1999) did not present intertemporal optimization explicitly.[4] [5]

[3]The possibility of periodic cycles and chaos in the discrete-time optimal growth model were shown in Benhabib and Nishimura (1985) and Deneckere and Pelikan (1986). In a continuous-time optimal growth model with multisectors, Benhabib and Nishimura (1979) proved the existence of periodic equilibrium trajectories by applying the Hopf bifurcation theorem.

[4]As for the endogenous fluctuations model using the quality-ladder framework, Aghion and Howitt (1992) highlighted the possibility of endogenous cycles in their famous growth model. Francois and Lloyd-Ellis (2003, 2008, 2009) studied the multisectors' quality-ladder model with endogenous fluctuations on the basis of the theory of the implementation cycles of Shleifer (1986). Wälde (2005) also investigated sustained growth cycles in the one-sector quality-ladder model. However, it is noteworthy that cyclical behavior observed in his model depends on discontinuous jumps based on the uncertainty of R&D success.

[5]Helpman and Trajtenberg (1998) and Petsas (2003) illustrated long-run fluctuations based on the exogenous arrival of general purpose technologies (GPT). Comin and Gertler (2006) and Barlevy (2007) investigated the fluctuating R&D caused by exogenous productivity shocks.

More precise studies of the dynamics of Matsuyama's (1999) model were presented by Mitra (2001), Mukherji (2005), Gardini, Sushko, and Naimzada (2008), and Yano, Sato, and Furukawa (2011). Mitra (2001) analytically proved that Matsuyama's dynamics has a period-6 cycle and exhibites topological chaos under specific parameters. Mukherji (2005) showed the sufficient condition for stable period-2 cycles and examples of parameters for topological chaos that are more plausible than those explored by Mitra. Gardini, Sushko, and Naimzada (2008) showed that no stable cycle can exist except for period-2 cycles. Yano, Sato, and Furukawa (2011) demonstrated that Matsuyama's dynamics can exhibit ergodic chaos.

These models analyzed in Deneckere and Judd (1992) and Matsuyama (1999, 2001) assumed that inventors enjoy monopoly for only one period. Therefore, potential inventors engage in R&D activities only in the event of large productive resources relative to a technological level, because the existence of large productive resources guarantees the inventors' large monopoly profit and low cost of R&D. However, when the rate of technological improvement exceeds that of resource accumulation, resources become insufficient and eventually R&D will cease to continue. The ceasing of R&D will result in the economy concentrating on manufacturing and resource accumulation, which in turn will again result in sufficient resource accumulation, thereby restarting the R&D process. On the basis of this logic, the economy oscillates between two situations.[6]

The early chapters in this paper focus on periodic cycles due to the flip bifurcation, highlighted in Deneckere and Judd (1992) and Matsuyama's (1999, 2001) approach, and investigate the following issues: how R&D activities behave over business cycles and whether policies promoting long-run growth conflict or coincide with economic stabilization policies.

1.4 Indeterminacy in monetary endogenous growth models

When the equilibrium path cannot be uniquely determined, even if the initial condition of the predetermined variables is given, extrinsic uncertainty or

[6]Bental and Peled (1996) also provided the R&D-based growth model with finite-lived patent and showed that the economy continues to fluctuate between the search and no-search phases. Note that they considered process innovation and characterized R&D as taking random draws from a pool of technologies.

sunspots can lead to equilibrium cycles.[7]

Several studies, such as Benhabib, Perli, and Xie (1994), Evans, Honkapohja, and Romer (1998), Haruyama and Itaya (2006), Furukawa (2007a,b), Arnold and Kornprobst (2008), and Haruyama (2009), investigated economic volatility on the basis of this indeterminacy in R&D-based growth models. Benhabib, Perli, and Xie's (1994) most popular example modified the variety expanding model based on Romer (1990) by introducing the complementarity between intermediate inputs and endogenous accumulation of human capital; they showed that the balanced growth path might be locally indeterminate for large values of an elasticity of intertemporal substitution.

Evans, Honkapohja, and Romer (1998) also analyzed Romer's model with complementarity between intermediate inputs. They showed that the multiple balanced growth equilibria exist and global indeterminacy arises under convex adjustment costs to capital.[8] Furukawa (2007a) and Haruyama (2009) showed that local indeterminacy occurs easily in the discrete-time version of the variety expanding model without capital accumulation. Arnold and Kornprobst (2008) examined the occurrence of indeterminacy using the quality-ladder model.[9]

Chapters 5 and 6 investigate the issue of indeterminate equilibria in a monetary endogenous growth model. We propose a new long-run model by introducing exogenous money growth and nominal wage stickiness into R&D-based growth models. By analyzing the balanced growth path in these models, we examine how money growth influences economic growth and volatility. In addition, the consistency between policies to promote long-run growth and economic stability is examined in terms of monetary policy.

These studies follow the monetary growth theory first pioneered by Tobin (1965) and Sidrauski (1967).[10] It is known that indeterminate equilibria

[7] For details, see Azariadis (1981) and Benhabib and Farmer (1999). Benhabib and Farmer (1994) and Boldrin and Rustichini (1994) provided the first theoretical study on optimal growth model involving indeterminacy.

[8] Romer's original model has a unique balanced growth path with saddle-path stability, as shown in Arnold (2000a,b) and Garcia-Castrillo and Sanso (2002); therefore, no indeterminacy arises.

[9] As for other types of endogenous growth models, Palivos, Yip, and Zhang (2003) extended the one-sector endogenous growth model on the basis of Barro (1990) by introducing endogenous labor supply, and showed that dual balanced growth paths and indeterminacy arise. Benhabib and Perli (1994) and Xie (1994) examined determinacy property in the two-sector growth model based on Lucas (1988) (see also Mattana, 2004).

[10] For details on the monetary growth theory, see Zhang (2010).

are likely to arise in a dynamic model including money growth.[11] Many studies analyzed indeterminate equilibria in the context of the monetary endogenous growth theory, such as Itaya and Mino (2007, 2003), Mino and Itaya (2004), and Suen and Yip (2005); however, sustained long-run growth in these studies does not stem from R&D and does not include any rigidity. This study proposes a new channel attributed to nominal rigidities and endogenous R&D through which money growth influences the determinacy property of equilibria.

1.5 Structure

This thesis consists of five main chapters from Chapter 2 through Chapter 6. Chapters 2 through 4 are based on the endogenous growth model of Matsuyama (1999). On the other hand, Chapters 5 and 6 are based on the new Keynesian Dynamic General Equilibrium (DGE) model of Inoue and Tsuzuki (2011). These two types of models are included in this study; however both have the common structure as the R&D-based growth model.

The models used in this study can be divided in terms of the specification of R&D, too. One of the typical specifications of R&D is lab equipment specification, which assumes that R&D activity requires capital as inputs. Another specification is referred to as knowledge-driven specification, which assumes that labor and existing knowledge are invested into R&D. Chapters 2, 4, and 6 apply lab equipment specification, whereas Chapter 5 applies knowledge-driven specification. The model in chapter 3 has both specifications of R&D.

Further details of each chapter are as follows.

Chapter 2 is based on Shinagawa's (2007) study. This chapter considers an endogenous growth cycle model based on Matsuyama (1999), and examines the issue of an optimal patent policy using the concept of \patent breadth." Changes in patent breadth affect the economy through monopoly prices and patentees' market share. We find that while a broader patent leads to faster growth, it may make the balanced growth path unstable. Therefore, in terms of growth rates, economic stabilization policies are not desirable under certain conditions.

Chapter 3 is based on Shinagawa's (2009) study. It constructs an R&D-based growth model that has two specifications for R&D technology, lab

[11] For example, see Matsuyama (1990).

equipment and knowledge-driven specifications as stated above. There are three phases in this model, each distinguished by the resource allocation for R&D; the economy grows endogenously with fluctuations between more than one phase. We conclude that the fluctuating equilibrium path is possible with both technologies. When focusing on the period-2 cycles, two technologies are used alternately.

Chapter 4 is based on Shinagawa's (2013) studies.[12] This chapter investigates the endogenous fluctuations in a non-scale growth model. The literature on endogenous growth cycles predicts the countercyclical allocation of resources to R&D. However, this prediction is not supported by empirical studies. This chapter considers the R&D-based growth model with endogenous fluctuations introducing population growth and a negative externality that affects the productivity of R&D. We show that our simple modification makes R&D investment procyclical along sustained business cycles using both an overlapping generation framework and an infinitely-lived agent framework.

Chapter 5 is based on Shinagawa and Inoue's (2011) study.[13] This chapter extends the R&D-based growth model by introducing nominal wage stickiness and exogenous money growth, and examines how money growth affects long-run economic growth. We find that there exists a unique balanced growth path for sufficiently high rates of money growth, and that the economy exhibits sustained growth based on sustained R&D. Faster money growth results in greater employment and faster economic growth along such a balanced growth path. Furthermore, under some parameter restrictions, no balanced growth path exists for low rates of money growth; the economy is trapped in a steady state without long-run growth. These results suggest that money growth may be an important factor for long-run economic growth.

Chapter 6 is based on Shinagawa and Inoue's (2013) study. This chapter studies the R&D-based growth model with nominal wage stickiness, and examines how money growth affects not only long-run economic growth but also the determinacy property of the steady state. The model is extended by introducing capital accumulation and finite-lived patent. We find that

[12] Shinagawa (2013) reported at the 2010 Autumn Meeting of Japan Economic Association in Kwansei Gakuin University.

[13] Shinagawa and Inoue (2011) reported at the Glope II International Conference 2012 in Waseda University and the 2012 Spring Meeting of Japan Economic Association in Hokkaido University.

faster money growth results in faster balanced growth; however, this makes the balanced growth path more likely to be indeterminate. As a result, we establish that a policy trade-off exists between growth promotion and economic stabilization.

Chapter 2

Patent Policy and Endogenous Fluctuations

2.1 Introduction

This chapter examines the effect of a patent policy on economic growth and business cycles. Much of the literature on economic growth argued that R&D activities by private firms are important engines for economic growth. Private firms engage in R&D activities in search of monopoly profits, main source of which is the patent. Thus, patents play a central role in this line of studies.[1]

This chapter considers an endogenous growth cycle model based on Matsuyama (1999), and examines the issue of an optimal patent policy using the concept of \patent breadth."Patent breadth is an important factor that characterizes a patent, along with \patent length." Patent breadth generally means the scope of protection offered by a patent over its lifetime.[2] This study adopts the specification of patent breadth used in Klemperer (1990), who measured patent breadth by the distance in the space of certain characteristics between the patented product and the products that other firms can sell. Moreover, we assume that the transportation of patented goods requires resources, and the cost per kilometer increases as transportation

[1] Wälde (2005) and Boldrin and Levine (2002) proposed R&D-based endogenous growth models that have private R&D firms that do not require monopoly or a patent.

[2] For example, Gilbert and Shapiro (1990) explained patent breadth as "the flow rate of profit available to the patentee while the patent is in force."

distance increases.

Under an extremely narrow patent, inventors cannot obtain sufficient profit, and thus R&D never occurs. The government is required to maintain a certain level of broadness of a patent to achieve sustained economic growth based on R&D. This study investigates the issue of desired broadness of a patent from two standpoints, growth enhancement and economic stabilization. That is, we assume that the government aims to stabilize the economy and promote economic growth.

The main results are summarized as follows. A broader patent induces firms to conduct higher levels of R&D activity through higher monopoly prices and higher market share of patentees. Therefore, a broader patent promotes economic growth, regardless of whether the equilibrium path is fluctuating. With respect to economic stabilization, when patent breadth is adjusted into the appropriate interval, the stable balanced growth path is allowed to exist. However, when the market share of patentees is at a maximum, a broader patent increases the volatility of the economy. As a result, with respect to growth rate, economic stabilization policies may not be desirable. That is, a trade-off exists between growth-enhancing policy and economic stabilization policy. Matsuyama (1999) did not present any policy implications, and thus these results are original ones obtained by this study.

Studies on economic stabilization using the variety expanding framework include Deneckere and Judd (1992), Aloi and Lasselle (2007), and Haruyama (2009). Aloi and Lasselle (2007) considered the endogenous growth cycle model based on Matsuyama (1999) and examined economic stabilization policies by subsidizing R&D. They concluded that economic stabilization policies promote long-run growth and increase welfare. In contrast, Deneckere and Judd (1992) investigated economic stabilization policies through lump-sum taxation on households. They argued that economic stabilization decreases welfare when the discount factor is sufficiently large. In more recent work, Haruyama (2009) examined R&D subsidies using a model that has the possibility of revealing indeterminacy and found a trade-off between economic stabilization and growth enhancements or welfare improvements. Our result is similar to Deneckere and Judd (1992) and Haruyama (2009) with respect to the undesirability of economic stabilization.

The rest of this chapter is organized as follows. The next section provides the details of the assumption of patent breadth used in our model. Section 2.3 sets up the model used in our theoretical investigation. Section 2.4 examines the dynamic properties of the model and illustrates that the equilibrium path fluctuates endogenously. Section 2.5 examines the effects of patent policies on economic growth and business cycles. Section 2.6 concludes the chapter.

2.2 Patent breadth

2.2.1 The economic implications of patent breadth

Nordhaus (1969, 1972) and Scherer (1972) pioneered theoretical studies on optimal patent design. From their studies to the present, this line of research focused on analyses on patent length. However, patents are characterized not only by patent length but also by patent breadth. In many cases, patent breadth captures the profit that the patentees earn per unit time.[3] It is determined by the strength of protection and the width of the coverage of an individual patent. The patent with the stronger protection can seek larger amounts of damage claims when it is infringed. The patent that has the wider coverage can exclude the products with the lower degree of similarity in addition to the highly similar products.[4] Both patents can earn larger profits per unit time.[5]

For example, Tandon (1982) considered that all patented goods are subject to compulsory licensing and patentees receive a royalty fee that is paid by firms producing the patented goods over the lifetime of the patents. Under such a patent system, Tandon interpreted that increasing the royalty fee is equivalent to increasing patent breadth. However, the specifications of patent breadth differ in each model.[6]

O'Donoghue, Scotchmer, and Thisse (1998) and O'Donoghue and Zweimuller (2004) distinguished between leading breadth and lagging breadth. Lagging breadth of patents is identified as the scope of restrictions on the imitation

[3]Patent breadth assumed in Gallini (1992) cannot be captured by this way.

[4]Scotchmer (2004) explained the width of the coverage of a patent from legal perspective by the \claims" in the patent document and the \doctrine of equivalents."

[5]In addition, Motohashi (2003) provided the extension of protection to new spheres as another example of broadening patent protection.

[6]Nordhaus (1972) considered that patent breadth measures the degree of leakage of an invented new technology in the context of a process innovation. The narrower patent means that the followers can produce goods by the nearer level of the technology with the inventor. Gallini (1992) considered that patent breadth corresponds to the entry cost to produce imitations. For other literature, useful survey has been presented by Denicolò (1996, Sec. 2).

of patented goods. In contrast, leading breadth of patents is identified as the scope of restrictions on future inventions that is similar to patented technology. This chapter focuses on lagging breadth.[7]

2.2.2 Patent breadth in Klemperer

This study applies the specification of patent breadth used in Klemperer (1990),[8] who considered patent breadth as the distance from the patentee within which imitations are prohibited. That is, patent breadth ω allows competing firms with a distance ω from the patentee to produce the patented product. We refer to the boundary that demarcates the farthest limit of a patent's protection as the patent boundary.

We assume that the monopoly price of originals produced by the patentee is p_m and the price of imitations is p_i. No difference exists in the quality of originals and imitations. Consumers of the patented products are required to pay transport cost $T > 0$ per unit distance per unit purchase of imitations. Then, consumers prefer to purchase the originals at price p_m rather than purchase imitations from their imitators on the patent boundary at price p_i if and only if $T\omega + p_i \geq p_m$.[9] That is, the limit price at which the patentee can rule out other competing firms from the market is equal to $T\omega + p_i$. The broader patent (the larger value of ω) makes it possible for the patentees to set the higher monopoly prices and earn larger monopoly profits.

This study assumes that transport cost positively depends on the number of consumers that transport goods.[10] This assumption allows an equilibrium in which the patentee and imitators coexist in the same market. Patent breadth affects the economy through the markup of the monopoly price and the market share of patentees.[11]

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[7] Li (2001) presented a specification in which no distinction between them is required.

[8] Klemperer (1990) used the term "patent width" instead of "patent breadth."

[9] All imitations are supplied by the firm just on the patent boundary. The consumers do not demand the products by the firms located farther from the patent boundary, because of the excessive transport cost.

[10] Klemperer (1990) assumed that the transport costs are distributed with a certain density function, and differ among consumers.

[11] Futagami and Iwaisako (2003, 2007) and Kwan and Lai (2003) also examined the issue of an optimal patent policy using the variety expanding framework and the lab equipment specification. However, in their models, the coexistence of the patentee and the imitators cannot occur, or the market share of the patentees has no influence on the economy. The studies on patent breadth using dynamic general equilibrium model include Li (2001),

2.3 Model

We consider the dynamic general equilibrium model based on Matsuyama (1999). Time is discrete and indexed by $t = 0, 1, 2, \ldots$. We assume two-period-lived overlapping generation (OLG) households, that inelastically supply labor when young. A single final good represents a numeraire produced using intermediate goods and labor inputs, which can be consumed or invested. A new variety of intermediate goods is invented by allocating capital for R&D activities. Inventors enjoy a one-period monopoly through patent protection. The available intermediate goods are produced by multiple intermediate firms using capital.

As previously mentioned, our specification of patent breadth draws from Klemperer (1990) and is denoted by ω. A patent is an exclusive right granted to inventors of new intermediate goods. In this model, a patent guarantees that patentee is a unique producer of each patented good within the patent boundary for one period.

We assume that transportation requires T units of capital per unit distance per unit purchase of imitations. Then, the unit transport cost is Tr_t, where r_t denotes the price of capital. No difference exists in the quality of originals and imitations. Furthermore, imitation is costless. We rule out the distance among consumers, final goods firms, and intermediate goods firms within the patent boundary. Firms within the patent boundary implemented all R&D activities, and diffusion of new varieties from outside the boundary never occurs. The time lag in the diffusion of invention is also ruled out.

2.3.1 Final goods

We assume that perfect competition prevails in the final goods market. Final goods firms are continuously distributed, and their total number is normalized at unity. All final goods firms have identical production technology and produce identical goods. The production function of final goods firm indexed by σ 2 [0, 1] is

$$y_t(\sigma) = A[L(\sigma)]^{1-\alpha} \int_0^{N_t} [x_t(z,\sigma)]^{\alpha} dz, \quad 0 < \alpha < 1, \quad A > 0, \tag{2.1}$$

Goh and Olivier (2002), Futagami and Iwaisako (2003), and O'Donoghue and Zweimuller (2004).

where $y_t(\sigma)$ represents the final output, $L(\sigma)$ represents labor input, $x_t(z,\sigma)$ represents the amount of the intermediate goods indexed by z, and $1/(1-\alpha)$ denotes the elasticity of substitution between all pairs of intermediate goods. It also follows that $(1-\alpha)$ represents the labor share of the economy and N_t represents the number of available intermediate goods in period t which represents the technology level of the economy. We assume that labor is equally divided among firms; that is, $L(\sigma) = L$, $\forall\sigma \in [0,1]$. Because $w_t = (1-\alpha)y_t(\sigma)/L(\sigma)$ holds, this assumption means $y_t(\sigma) = Y_t \equiv \int_0^1 y(\sigma)d\sigma$, $\forall\sigma \in [0,1]$.

Given limited patent protection, only the "new" intermediate goods, $(N_{t-1}, N_t]$, are patented. We assume that final goods firm $\sigma \in [0, S_t]$ purchases all patented products from their patentees at a price p_{mt}, whereas firm $\sigma \in (S_t, 1]$ purchases from imitators at a price p_{it}.[12] $S_t \in [0,1]$ represents the market share of patentees.[13] Under these assumptions, profit maximization of firm $\sigma \in [0, S_t]$ yields the demand for each patented intermediate good $z \in (N_{t-1}, N_t]$ as

$$x_t(z,\sigma) = x_{mt} \equiv A^{\frac{1}{1-\alpha}} L \alpha^{\frac{1}{1-\alpha}} p_{mt}^{-\frac{1}{1-\alpha}}, \quad \text{for } z \in (N_{t-1}, N_t] \text{ and } \sigma \in [0, S_t]. \tag{2.2}$$

The aggregate demand function of the products of the patentee is given by

$$X_{mt}(z) = \int_0^{S_t} x_t(z,\sigma)d\sigma = S_t x_{mt}, \quad \text{for } z \in (N_{t-1}, N_t].$$

Similarly, profit maximization of firm $\sigma \in (S_t, 1]$ yields

$$\begin{aligned} x_t(z,\sigma) = x_{it} \equiv A^{\frac{1}{1-\alpha}} L \alpha^{\frac{1}{1-\alpha}} [p_{it} + \omega T(S_t) r_t]^{-\frac{1}{1-\alpha}}, \\ \text{for } z \in (N_{t-1}, N_t] \text{ and } \sigma \in (S_t, 1], \end{aligned} \tag{2.3}$$

where $T(S_t) \geq 0$ is the unit transport cost which depends on S_t. We assume that $T(\cdot)$ is a C^2 function and $T'(S_t) < 0$. That is, the unit transport cost is increasing in the number of final goods firms that transport imitations, which is denoted by $1 - S_t$. This assumption captures the congestion effect for the transportation of imitations.

[12] All intermediate goods are produced by the same technology; therefore, the prices of the intermediate goods need not be distinguished by z in equilibrium.

[13] To simplify the explanation, we exclude the circumstance that one firm simultaneously purchases from both the patentees and the imitators.

Finally, non-patented intermediate goods $z \in [0, N_{t-1}]$ are supplied by intermediate firms within the patent boundary at a competitive price p_{ct}. Therefore, all final goods firms have an identical demand function for each intermediate good $z \in [0, N_{t-1}]$.

$$x_t(z, \sigma) = x_{ct} \equiv A^{\frac{1}{1-\alpha}} L \alpha^{\frac{1}{1-\alpha}} p_{ct}^{-\frac{1}{1-\alpha}}, \quad \text{for } z \in [0, N_{t-1}] \text{ and } \sigma \in [0, 1]. \quad (2.4)$$

2.3.2 Intermediate goods

Each of the intermediate goods is produced using one unit of capital. Regardless of whether patented or non-patented and whether imitations or originals, all intermediate goods are produced using identical technology. The "old" intermediate goods, $[0, N_{t-1}]$, are competitively supplied. Hence, their price equals marginal cost, $p_{ct} = r_t$, for $z \in [0, N_{t-1}]$. The same equation holds for the price of the imitations, $p_{it}(z) = r_t$, for $z \in (N_{t-1}, N_t]$.

2.3.2.1 Market share of patentees

If transport costs are sufficiently high and $[\omega T(S_t) + 1]r_t$ is larger than p_{mt}, the purchase of imitations is more costly than the purchase of originals; thus, S_t is increasing. In contrast, if $[\omega T(S_t) + 1]r_t$ is smaller than p_{mt}, S_t is decreasing. According to the previously described adjustment process, S_t converges to the value that satisfies the following equation:[14]

$$[\omega T(S_t) + 1]r_t = p_{mt}. \quad (2.5)$$

However, when $[\omega T(1) + 1]r_t > p_{mt}$ holds, the corner solution, $S_t = 1$, is realized and (2.5) does not hold.[15] Similarly, when $[\omega T(0) + 1]r_t < p_{mt}$ holds, $S_t = 0$ is realized.

Summarizing the above, the patentees' market share, S_t, is given by the function of μ_t and ω as follows:

$$S_t = \hat{S}(\mu_t, \omega) \equiv \begin{cases} 1, & \text{if } \mu_t \leq \omega T(1), \\ T^{-1}(\mu_t/\omega), & \text{if } \mu_t \in (\omega T(1), \omega T(0)), \\ 0, & \text{if } \mu_t \geq \omega T(0), \end{cases} \quad (2.6)$$

[14] This result depends on the assumption, $T'(S_t) < 0$. If $T'(S_t)$ is positive, S_t will converge to either $S_t = 0$ or $S_t = 1$.

[15] In this case, because p_{mt} is extremely small, even if the transport cost is minimum, purchasing imitations is more costly. Therefore, there does not exist a final goods firm that deals with imitators.

where $\mu_t \equiv (p_{mt}/r_t) - 1$ denotes the markup (the rate of profit) of the patentees. Because $T(\cdot)$ is monotonically decreasing, the inverse function of $T(\cdot)$ denoted by $T^{-1}(\cdot)$ is well defined. Note that T^{-1} is a decreasing function.

2.3.2.2 Optimization of patentees

The monopoly profit of the patentees is given by

$$\Pi_t = X_{mt}\mu_t r_t = \hat{S}(\mu_t, \omega) x_{mt}\mu_t r_t. \tag{2.7}$$

Maximization of the monopoly profit (2.7) subject to the demand function (2.2) and the patentees' market share (2.6) yields[16]

$$\mu_t = \mu(\omega) \equiv \begin{cases} \hat{\mu}(\omega), & \text{if } \hat{\mu}(\omega) > \omega T(1), \\ \mu_1(\omega) \equiv \min\left\{\frac{1}{\alpha} - 1, \omega T(1)\right\}, & \text{if } \hat{\mu}(\omega) \leq \omega T(1). \end{cases} \tag{2.8}$$

$\hat{\mu}(\omega)$ is defined as the root of the following implicit function:

$$F(\mu) \equiv \frac{1}{(1-\alpha)(1+\mu)} - \frac{\alpha}{1-\alpha} - \varepsilon\left(\frac{\mu}{\omega}\right) = 0. \tag{2.9}$$

$\hat{\mu}(\omega)$ is smaller than $\omega T(0)$ for any positive values of ω; therefore, $S_t = T^{-1}(\mu_t/\omega) > 0$ holds. $\varepsilon(\mu/\omega)$ denotes the absolute value of the elasticity of $T^{-1}(\mu/\omega)$ with respect to μ/ω as follows:

$$\varepsilon\left(\frac{\mu}{\omega}\right) \equiv -\frac{(T^{-1})'(\mu/\omega)\cdot(\mu/\omega)}{T^{-1}(\mu/\omega)} > 0,$$

which is equal to the inverse of the absolute value of the elasticity of $T(S)$ with respect to S. We will investigate the model under the following assumption.[17]

Assumption 2.1

$$\varepsilon'\left(\frac{\mu}{\omega}\right) \geq 0, \quad \text{for all } \frac{\mu}{\omega} > 0, \quad \text{and} \quad \varepsilon(0) < 1.$$

[16] Appendix 2.A provides detailed derivations.

[17] When $(T^{-1})''(\mu/\omega) \leq -\frac{1+\varepsilon(\mu/\omega)}{\mu/\omega}(T^{-1})'(\mu/\omega)$ is satisfied, Assumption 2.1 holds. The right-hand side of the above equation is positive since $(T^{-1})'(\mu/\omega) < 0$. Therefore, if $(T^{-1})''(\mu/\omega) \leq 0$ or $T''(S) \leq 0$, $\varepsilon'(\mu/\omega)$ is positive and Assumption 2.1 holds.

When $\varepsilon^{q}(\mu/\omega) \geq 0$, $F(\mu)$ is decreasing in μ. Thus, the root of the implicit function, $F(\mu) = 0$, is unique, if it exists.

Substituting (2.8) into (2.6) gives $S_t = S(\omega) \equiv \hat{S}(\mu(\omega), \omega)$. The maximized monopoly profit is

$$\Pi_t(\omega, r_t) = \mu(\omega)S(\omega)x_{mt}r_t.$$

The transport cost is given by

$$s\omega T(S(\omega))r_t = \mu(\omega)r_t, \quad \text{if } S(\omega) < 1. \tag{2.10}$$

When $S(\omega) = 1$, the transport cost need not be considered. Finally, by using (2.2), (2.3), (2.4), and (2.5), we obtain

$$[1 + \mu(\omega)]^{-\frac{1}{1-\alpha}}x_{ct} = x_{mt} = x_{it}. \tag{2.11}$$

2.3.3 R&D

The number of intermediate goods expands according to the following equation:

$$N_t - N_{t-1} = \eta R_t, \quad \eta > 0,$$

where R_t represents the amount of capital allocated to R&D and η is the parameter that reflects the productivity of R&D.

Each inventor enjoys one-period monopoly and earns profit $\Pi_t(\omega, r_t)$. Therefore, in equilibrium, the following free-entry condition must be satisfied:

$$\Pi_t(\omega, r_t) \leq \eta^{-1}r_t, \quad \text{with an equality whenever } N_t > N_{t-1}.$$

Because $S(\omega) \neq 0$, this inequality is written as

$$x_{mt} \leq \bar{x}_{mt} \equiv \frac{1}{\eta\mu(\omega)S(\omega)}, \quad \text{with an equality whenever } N_t > N_{t-1}, \tag{2.12}$$

where $\bar{x}_{mt}$ is the breakeven point of x_{mt}. $\bar{x}_{mt}$ becomes larger for a small value of $S(\omega)$ because the inventors need to earn sufficient profits from fewer final goods firms.

Finally, clearing the capital market requires

$$\begin{aligned} K_{t-1} = {} & N_{t-1}x_{ct} + (N_t - N_{t-1})S(\omega)x_{mt} + R_t \\ & + (N_t - N_{t-1})[1 - S(\omega)]x_{it}[1 + \omega T(S(\omega))]. \end{aligned} \tag{2.13}$$

where K_{t-1} represents the amount of capital accumulated in period $t-1$ and available in period t. Available capital is utilized by (1) producing competitive intermediate goods, (2) patentees that produce the new intermediate goods, (3) R&D, (4) imitators that produce the new intermediate goods, and (5) transportation of imitation goods, as shown on the right-hand side of (2.13). In the case of $S(\omega) = 1$, the terms of (4) and (5) are equal to zero. In the period during which no R&D occurs, only the term of (1) is positive.

2.3.4 Consumers

Each consumer lives for two periods. When young, he/she supplies one unit of labor and earns wage w_t, which is divided into savings and consumption. When old, he/she only consumes using his/her savings. Let c_{1t} and c_{2t+1} denote the consumption in periods t and $t+1$, respectively, of consumers born in period t. Each consumer chooses c_{1t} and c_{2t+1} that maximize his/her utility, $U_t = (1-s)\log c_{1t} + s \log c_{2t+1}$, subject to the budget constraint $c_{2t+1} = (w_t - c_{1t})r_{t+1}$.

The solution to this simple maximization problem is characterized by the following linear saving function:

$$K_t = sw_tL = s(1-\alpha)Y_t, \tag{2.14}$$

where L represents the number of consumers born in each period.

2.3.5 Equilibrium

Substituting (2.10), (2.11), and (2.12) into (2.13) yields

$$\frac{N_t - N_{t-1}}{N_{t-1}} = \max\{0, [1 + \mu(\omega)]^{\frac{\alpha}{1-\alpha}}(k_{t-1} - 1)\}, \tag{2.15}$$

$$x_{ct} = \max\left\{\frac{K_{t-1}}{N_{t-1}}, \frac{1}{\Omega(\omega)}\right\}, \tag{2.16}$$

where we define $k_{t-1} \equiv \Omega(\omega)(K_{t-1}/N_{t-1})$, and

$$\Omega(\omega) \equiv \frac{1}{[1+\mu(\omega)]^{\frac{1}{1-\alpha}}\bar{x}_{mt}} = \frac{\eta S(\omega)\mu(\omega)}{[1+\mu(\omega)]^{\frac{1}{1-\alpha}}} > 0.$$

If $k_{t-1} > 1$ holds, i.e., the economy has a sufficient stock of capital relative to its technological level, a positive amount of capital is allocated for R&D and $N_t > N_{t-1}$ holds. In contrast, if $k_{t-1} \leq 1$, neither R&D occurs nor technological progress arises.

Patent breadth ω influences the occurrence of R&D through the markup μ and the patentees' market share S. The larger values of μ and S increase inventors' monopoly profits, and make R&D easier to occur for a given N_{t-1} and K_{t-1}.

Substituting (2.11), (2.15), and (2.16) into (2.1) illustrates that the total output is given by

$$Y_t \equiv \int_0^1 y(q)dq = AL^{1-\alpha}\Omega(\omega)^{-\alpha}N_{t-1}\psi(k_{t-1}), \tag{2.17}$$

where

$$\psi(k_{t-1}) \equiv \begin{cases} k_{t-1}^{\alpha}, & \text{if } k_{t-1} \leq 1, \\ k_{t-1}, & \text{if } k_{t-1} > 1. \end{cases}$$

Summarizing (2.15), (2.17), and (2.14) yields the following one-dimensional dynamical system:

$$k_t = \phi_\omega(k_{t-1}) \equiv \Phi(k_{t-1}, \omega) \equiv \begin{cases} G(\omega)k_{t-1}^{\alpha}, & \text{if } k_{t-1} \leq 1, \\ \dfrac{G(\omega)k_{t-1}}{1+[1+\Theta(\omega)](k_{t-1}-1)}, & \text{if } k_{t-1} > 1, \end{cases} \tag{2.18}$$

where we define $G(\omega) \equiv s(1-\alpha)AL^{1-\alpha}\Omega(\omega)^{1-\alpha}$ and $\Theta(\omega) \equiv [1+\mu(\omega)]^{\frac{\alpha}{1-\alpha}} - 1$. $\phi_\omega(k_{t-1})$ is a unimodal form with a kink at $k_{t-1} = 1$ on the (k_{t-1}, k_t) plane. If the initial values of k are given, the law of motion (2.18) characterizes the equilibrium path $\{k_t\}_0^1$, whose properties depend on patent breadth, ω.

2.4 Dynamics

When $G(\omega) \leq 1$, $\phi_\omega(1) \leq 1$ holds and the fixed point of the law of motion (2.18) belongs to $(0, 1]$ as shown in Figure 2.1(a). We obtain this fixed point as $k^*(\omega) = G(\omega)^{\frac{1}{1-\alpha}}$, which is always stable. We refer to this steady state as the no-growth steady state. In this case, no R&D occurs except for the initial period, and the economy cannot sustain growth; that is, $g_Y^* = g_K^* = g_N^* = g^* = 0$, where g_X^* denotes the growth rate of a variable X at the fixed point k^*.

If $G(\omega) > 1$, the law of motion (2.18) has a unique fixed point that is larger than 1, as shown in Figure 2.1(b). This fixed point is derived as

$$k^{**}(\omega) \equiv 1 + \frac{G(\omega) - 1}{\Theta(\omega) + 1} > 1.$$

At this fixed point, K, Y, and N continue to grow at constant rates, i.e., the economy achieves balanced growth. We define this steady state as the balanced growth path (BGP). The balanced growth rates of K, Y, and N are derived as $g_K^{**} = g_Y^{**} = g_N^{**} = g^{**}(\omega) \equiv G(\omega) - 1$, where g_X^{**} is defined as the growth rate of a variable X at the fixed point k^{**}.

Some algebra shows that the slope of the map, ϕ_ω, at the fixed point k^{**} is

$$\lambda(\omega) \equiv \frac{\partial \phi_\omega}{\partial k_{t-1}}(k^{**}(\omega)) = -\frac{\Theta(\omega)}{G(\omega)}.$$

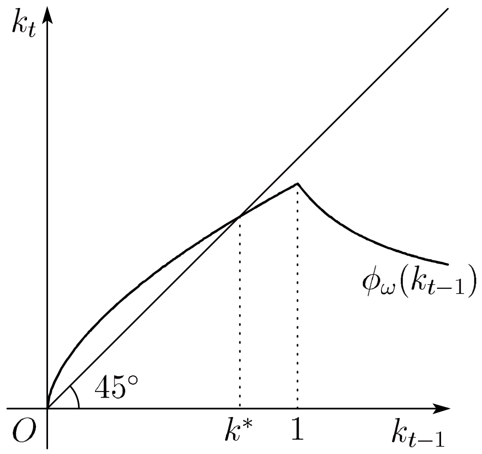
When $G(\omega) > \Theta(\omega)$, $|\lambda(\omega)|$ is smaller than 1, and the fixed point k^{**} is globally stable. Because λ is negative, the equilibrium path exhibits damped oscillations and eventually converges to k^{**}.

In contrast, when $1 < G(\omega) < \Theta(\omega)$, $|\lambda(\omega)|$ exceeds 1 and the fixed point k^{**} is locally unstable. In this case, we prove the existence of period-2 cycles,[18] and the trajectory continues to fluctuate in the trapping region, $[\phi_\omega^2(1), \phi_\omega(1)]$.[19] Note that $\phi_\omega(1) = G(\omega) > 1$ and $\phi_\omega^2(1) = \phi_\omega(G(\omega)) < 1$ always hold as long as $\Theta(\omega) > G(\omega) > 1$.

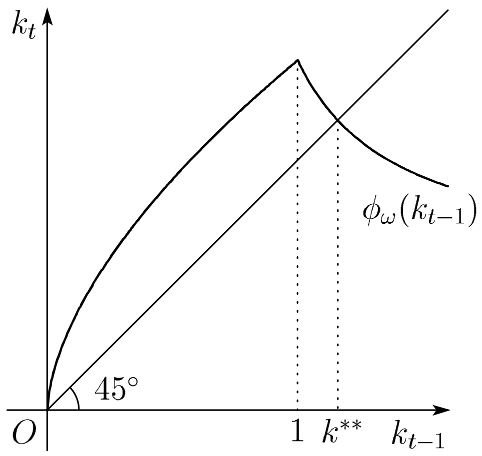
The average growth rate of K, Y, and N over period-2 cycles are derived as $g_K^c = g_Y^c = g_N^c = g^c(\omega) \equiv G(\omega)(k^L(\omega))^{-\frac{1-\alpha}{2}} - 1$, where k^L is one of the fixed points of period-2 that is smaller than 1. We can easily show that $g^c(\omega) > g^{**}(\omega)$.

[18] By slightly changing a parameter that has an influence on $G(\omega)$ or $\Theta(\omega)$, the fixed point loses it stability, and the flip bifurcation (period-doubling bifurcation) occurs.

[19] We define $\phi_\omega^2(k) = \phi_\omega(\phi_\omega(k))$, and $\phi_\omega^n(k) = \phi_\omega(\phi_\omega^{n-1}(k))$, for $n = \{3, 4, \ldots\}$.



(a) $G(\omega) < 1$.



(b) $G(\omega) > 1$.

Figure 2.1: Two types of fixed points.

Summarizing, we obtain a result that is very similar to Matsuyama (1999), as follows.[20]

Proposition 2.1

(a) When $G(\omega) \leq 1$, the law of motion (2.18) has a unique fixed point $k^* \leq 1$, which is globally stable, and k monotonically converges to k^{**} for any initial condition k_0.

(b) When $G(\omega) \in (1, \Theta(\omega))$, the law of motion (2.18) has a unique fixed point $k^{**} > 1$, which is locally unstable and k continues to fluctuate in $[\phi_\omega^2(1), \phi_\omega(1)]$ for almost all initial conditions.[21]

(c) When $G(\omega) > \Theta(\omega)$, the law of motion (2.18) has a unique fixed point $k^{**} > 1$, which is globally stable and k oscillatory converges to k^{**} for any initial condition k_0.

If $G(\omega) = \Theta(\omega)$ and $\lambda(\omega)$ are just equal to -1, $\phi_\omega^2(k) = k$ holds for any $k \in [1, \phi_\omega(1)]$. That is, fixed points of period-2 continuously exist in $[1, \phi_\omega(1)]$. The average growth rate over these period-2 cycles is equal to g^{**}. In this case, the equilibrium path converges to either these cycles or the fixed point k^{**}.

2.5 Patent policy

This section investigates the effects of patent policies. First, we provide the following lemmas.

Lemma 2.1 Under Assumption 2.1, both $\hat{\mu}(\omega)$ and $\mu_1(\omega)$ are non-decreasing functions of ω. $\hat{\mu}(\omega)$ does not depend on ω if and only if $\varepsilon'(\mu/\omega) = 0$.

proof. From (2.8), $\mu_1(\omega)$ is clearly a non-decreasing function. As for $\hat{\mu}(\omega)$, applying an implicit function theorem to (2.9) yields

$$\hat{\mu}'(\omega) = -\frac{\partial F/\partial \omega}{\partial F/\partial \hat{\mu}} = \frac{\varepsilon'(\hat{\mu}/\omega)(\hat{\mu}/\omega)}{\frac{\omega}{(1-\alpha)(1+\hat{\mu})^2} + \varepsilon'(\hat{\mu}/\omega)} \geq 0. \tag{2.19}$$

Furthermore, if $\varepsilon'(\hat{\mu}/\omega)$ is equal to 0, $\hat{\mu}'(\omega) = 0$ holds. □

[20] We can provide a more formal proof of proposition in a similar way as Matsuyama (1999).

[21] The initial condition, k_0, that satisfies $\phi_\omega^{n'}(k_0) = k^{**}$ for a finite n' is excluded.

When $\varepsilon'(\hat{\mu}/\omega) = 0$ and, thus, ε is constant, the root of (2.9) can be derived the closed-form $\hat{\mu} = \frac{1}{\alpha+(1-\alpha)\varepsilon} - 1$.

Lemma 2.2 Under Assumption 2.1, $T^{-1}(\hat{\mu}(\omega)/\omega)$ is increasing in ω.

proof. From (2.19), if $\varepsilon'(\hat{\mu}/\omega) \geq 0$, $\hat{\mu}'(\omega) < \hat{\mu}(\omega)/\omega$ holds. Therefore, we obtain

$$\frac{\partial T^{-1}(\hat{\mu}(\omega)/\omega)}{\partial \omega} = \frac{1}{\omega}\left[\hat{\mu}'(\omega) - \frac{\hat{\mu}(\omega)}{\omega}\right](T^{-1})'\left(\frac{\hat{\mu}(\omega)}{\omega}\right) > 0. \tag{2.20}$$

□

When $S_t = T^{-1}(\hat{\mu}(\omega)/\omega) < 1$, a rise in patent breadth, ω, has three effects that affect the patentees' market share, S_t, as follows. The first and second effects raise S_t through an increase in the distance to transport and an increase in unit transport cost. The third effect reduces S_t through an increase in the monopoly price of the originals supplied by the patentees. However, the first and second effects always dominate the third one.

Lemma 2.3 Let Assumption 2.1 hold. $S(\omega) < 1$ is satisfied for sufficiently small ω if and only if $\varepsilon(T(1)) < 1$ holds.[22]

proof. A necessary and sufficient condition for $S(\omega) < 1$ is $\hat{\mu}(\omega) > \omega T(1)$. Under Assumption 2.1, $F(\mu)$ is decreasing in μ and $\lim_{\mu\to\infty} F(\mu) < 0$. Therefore, if $F(\omega T(1)) > 0$ is satisfied, a unique root of the implicit function, $F(\hat{\mu}) = 0$, exists such that $\hat{\mu} > \omega T(1)$.

When $\varepsilon(T(1)) < 1$ holds, $\lim_{\omega\to 0} F(\omega T(1)) = 1 - \varepsilon(T(1)) > 0$; thus, there exists $\hat{\mu}(\omega)$ such that $\hat{\mu}(\omega) > \omega T(1)$ for sufficiently small values of ω. In contrast, if $\varepsilon(T(1)) \geq 1$ holds, $\lim_{\omega\to 0} F(\omega T(1)) \leq 0$. Because $F(\omega T(1))$ is decreasing in ω, there does not exist $\hat{\mu}(\omega)$ that satisfies $\hat{\mu}(\omega) > \omega T(1)$ for any $\omega > 0$. □

If $\varepsilon(T(1)) < 1$ holds, there exists positive threshold values of ω, $\bar{\omega}$, and $\bar{\bar{\omega}}$, as follows:

$$\bar{\omega} \equiv \frac{(1-\alpha)[1-\varepsilon(T(1))]}{[\alpha + (1-\alpha)\varepsilon(T(1))]T(1)}, \quad \bar{\bar{\omega}} \equiv (1-\alpha)/[\alpha T(1)].$$

[22] The condition, $\varepsilon(T(1)) < 1$, can be rewritten as $T(1) < |T'(1)|$.

Note that $\bar{\omega} < \bar{\bar{\omega}}$ holds. When ω is smaller than $\bar{\omega}$, we have $\mu_t = \hat{\mu}(\omega)$ and $S(\omega) = T^{-1}(\hat{\mu}(\omega)/\omega) < 1$; that is, final goods firms that deal with imitators exist. In this situation, an increase in patent breadth, ω, increases patentees' market share, S_t, and the markup of the patented intermediate goods, μ_t, except for the case of $\varepsilon'(\mu/\omega) = 0$.

When $\omega \in [\bar{\omega}, \bar{\bar{\omega}})$, no imitation occurs; therefore, the patentees occupy all demand for the intermediate goods by limit pricing. That is, $S(\omega) = 1$. The markup is $\mu_t = \omega T(1)$, which increases with increasing patent breadth. When $\omega > \bar{\bar{\omega}}$, $\mu_t = \frac{1-\alpha}{\alpha}$ holds and a change in ω has no influence on the economy.

Summarizing the above, we obtain following equations:

$$\mu(\omega) = \begin{cases} \hat{\mu}(\omega), & \text{if } \omega \in (0, \bar{\omega}), \\ \omega T(1), & \text{if } \omega \in [\bar{\omega}, \bar{\bar{\omega}}), \\ \frac{1-\alpha}{\alpha}, & \text{if } \omega \geq \bar{\bar{\omega}}, \end{cases}$$

$$S(\omega) = \begin{cases} T^{-1}(\hat{\mu}(\omega)/\omega), & \text{if } \omega \in (0, \bar{\omega}), \\ 1, & \text{if } \omega \geq \bar{\omega}. \end{cases}$$

2.5.1 Economic stabilization

We investigate how a patent policy can achieve long-term stable and balanced growth. As shown in Proposition 2.1, the economy has a stable BGP by adjusting patent breadth, ω, to satisfy $1 < G(\omega) < \Theta(\omega)$. To examine the effect of patent breadth on the equilibrium path, we show the following lemmas.

Lemma 2.4 Under Assumption 2.1, $G(\omega)$ is increasing in ω for $\omega \in (0, \bar{\bar{\omega}})$.

proof. To prove the lemma, we just have to show that $\Omega(\omega)$ is an increasing function. The derivative of $\Omega(\omega)$, which is well-defined for $\omega \in (0, \bar{\bar{\omega}}) \setminus \{\bar{\omega}\}$, is derived as follows:

$$\begin{aligned} \Omega'(\omega) &= \frac{\eta\mu(\omega)}{[1+\mu(\omega)]^{\frac{1}{1-\alpha}}} S'(\omega) + \frac{\eta[1-\alpha-\alpha\mu(\omega)]}{(1-\alpha)[1+\mu(\omega)]^{\frac{\alpha}{1-\alpha}}} S(\omega)\mu'(\omega), \\ &= \Omega(\omega)\left\{\frac{S'(\omega)}{S(\omega)} + \frac{\mu'(\omega)}{\mu(\omega)[1+\mu(\omega)]}\left(1 - \frac{\alpha}{1-\alpha}\mu\right)\right\}. \end{aligned}$$

From Lemmas 2.1 and 2.2, which are valid under Assumption 2.1, we show that $\Omega'(\omega)$ is positive for $\omega \in (0, \bar{\omega})$. As for $\omega \in (\bar{\omega}, \bar{\bar{\omega}})$, $S'(\omega) = 0$ and $\mu'(\omega) = T(1) > 0$ holds; therefore, $\Omega'(\omega)$ is positive. □

Lemma 2.5 Under Assumption 2.1, $|\lambda|$ is increasing in ω for $\omega \in (\bar{\omega}, \bar{\bar{\omega}})$. If $\varepsilon(\mu/\omega)$ is constant, $|\lambda|$ is decreasing in ω for $\omega \in (0, \bar{\omega})$.

proof. The derivative of $|\lambda|$ with respect to ω, which is well-defined for $\omega \in (0, \bar{\bar{\omega}}) \setminus \{\bar{\omega}\}$, is derived as follows:

$$\begin{aligned}\frac{\partial|\lambda|}{\partial\omega} =& \frac{\mu'(\omega)}{G(\omega)}\left\{\frac{\alpha}{1-\alpha}[1+\mu(\omega)]^{\frac{\alpha}{1-\alpha}-1} - \frac{[1-\alpha-\alpha\mu(\omega)]}{[1+\mu(\omega)]\mu(\omega)}\Theta(\omega)\right\}\\ &-(1-\alpha)\frac{\Theta(\omega)}{G(\omega)}\frac{S'(\omega)}{S(\omega)},\end{aligned}$$

$\mu'(\omega) = T(1) > 0$ and $S'(\omega) = 0$ hold for $\omega \in (\bar{\omega}, \bar{\bar{\omega}})$, therefore $\partial|\lambda|/\partial\omega$ is positive.

When $\omega < \bar{\omega}$, $S(\omega) = T^{-1}(\mu/\omega)$ and $S'(\omega) = \partial T^{-1}/\partial\omega$ hold and substituting (2.20) into the previous equation yields[23]

$$\frac{\partial|\lambda|}{\partial\omega} = \frac{\alpha}{1-\alpha}\frac{\hat{\mu}'(\omega)}{G(\omega)}[1+\hat{\mu}(\omega)]^{\frac{\alpha}{1-\alpha}-1} - (1-\alpha)\frac{\Theta(\omega)\varepsilon}{G(\omega)\omega},$$

which is negative for the sufficiently small value of $\hat{\mu}'(\omega)$.[24] When $\varepsilon(\mu/\omega)$ is constant, $\hat{\mu}'(\omega) = 0$ holds as shown in Lemma 2.1, and $\partial|\lambda|/\partial\omega$ is negative. □

Summarizing these lemmas, we obtain the following proposition.

Proposition 2.2 Let Assumption 2.1 and $G(\bar{\bar{\omega}}) > 1$ hold.

(a) There exists a unique positive threshold value of ω, ω_0, such that the economy exhibits sustained growth for $\omega > \omega_0$.

(b) Let $G(\bar{\omega}) > \Theta(\bar{\omega})$ and $G(\bar{\bar{\omega}}) < \Theta(\bar{\bar{\omega}})$ hold. There exists a unique threshold value of ω, $\omega_2 \in (\bar{\omega}, \bar{\bar{\omega}})$, such that the BGP is stable for $\omega \in (\bar{\omega}, \omega_2)$, whereas the BGP is unstable for $\omega > \omega_2$.

[23] Using (2.9), we get $-\frac{1-\alpha-\alpha\hat{\mu}}{1+\hat{\mu}} = -\frac{1}{1+\hat{\mu}} + \alpha = (1-\alpha)\varepsilon$.

[24] For $\omega \in (0, \bar{\omega})$, ω affects $|\lambda|$ through $\mu(\omega)$ and $S(\omega)$. These effects work are opposite in nature, and thus the sign of $|\lambda'(\omega)|$ is not generally specified.

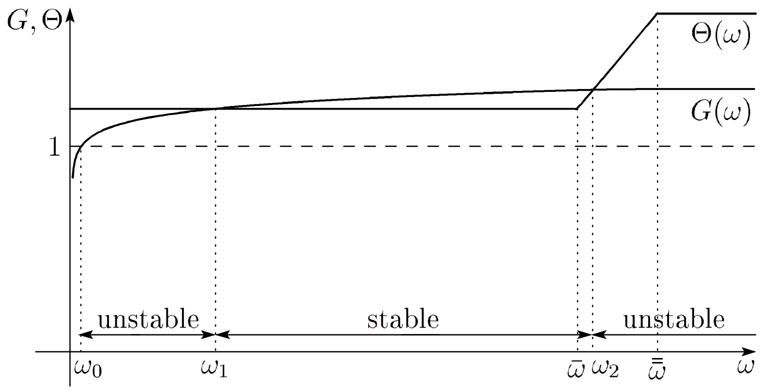


Figure 2.2: Stability of the balanced growth path.

(c) Let $G(\bar{\omega}) > \Theta(\bar{\omega}) > 1$ and $\varepsilon'(\mu/\omega) = 0$ hold. There exists a unique threshold value of ω, ω_1 2 $(\omega_0, \bar{\omega})$, such that the BGP is unstable for ω 2 (ω_0, ω_1), whereas the BGP is stable for ω 2 $(\omega_1, \bar{\omega})$.

Let $G(\bar{\bar{\omega}}) > 1$ hold. Then, from Lemma 2.4, there exists a unique threshold value of ω, ω_0, such that $G(\omega_0) = 1$. In this case, the economy is trapped in a long-term, no-growth steady state for $\omega < \omega_0$. However, the steady state of the economy may change to the BGP, and long-run growth occurs through a policy of extending patent breadth up to $\omega > \omega_0$.

Let $G(\bar{\omega}) > \Theta(\bar{\omega}) > 1$ hold. In such a case, Lemma 2.5 implies that if the BGP is unstable for $\omega > \bar{\omega}$, stabilizing it by narrowing patent breadth up to close enough to $\bar{\omega}$ is possible. If the BGP is unstable for $\omega < \bar{\omega}$, extending patent breadth makes it stable when $\varepsilon(\mu/\omega)$ is constant.

An examples of the graphs of $G(\omega)$ and $\Theta(\omega)$ appears in Figure 2.2, which assumes that $\varepsilon(\mu/\omega)$ is constant and, thus, $\Theta(\omega)$ is drawn as a horizontal line for $\omega < \bar{\omega}$. As is shown, $G(\bar{\bar{\omega}}) > 1$ holds and ω_0 exists. Therefore, the economy can get out of the no-growth steady state by extending patent breadth. However, in this figure, $G(\bar{\bar{\omega}}) < \Theta(\bar{\bar{\omega}})$ holds and, thus, the BGP is unstable for sufficiently large values of patent breadth. If the policy of extending patent breadth is excessive, the economy becomes volatile.

2.5.2 Growth enhancement and economic stabilization

This section investigates the consistency between growth-enhancing policies and economic stabilization policies. Let $\Theta(\bar{\omega}) < G(\omega)$ hold; that is, the BGP is stable for ω close enough to $\bar{\omega}$, and economic stabilization policies are effective.

2.5.2.1 The case of $\Theta(\bar{\bar{\omega}}) < G(\bar{\bar{\omega}})$

From Lemma 2.5, when $\Theta(\bar{\bar{\omega}}) < G(\bar{\bar{\omega}})$ is satisfied, the BGP is stable for all $\omega \geq \bar{\omega}$. The balanced growth rate, $g^{**} = G(\omega) - 1$, is maximized for $\omega \geq \bar{\bar{\omega}}$, as shown in Lemma 2.4. Therefore, if no $\omega < \bar{\omega}$ exists such that the BGP is unstable, the long-run growth rate is maximized at $\omega \geq \bar{\bar{\omega}}$. In contrast, if there exists $\omega < \bar{\omega}$ such that the BGP is unstable, additional discussion is needed to compare the growth rates between the BGP and cycles.

Lemma 2.6 Let Assumption 2.1 hold and ω^0 and ω^{00} exist such that $\omega^0 < \omega^{00}$ and the BGP is stable for ω^{00} and unstable for ω^0. When $\phi^2_{\omega^0}(1) > \phi^{-1}_{\omega^0}(k^{**}(\omega^0))$ is satisfied, the average growth rate along any equilibrium path fluctuates in the trapping region, $[\phi^2_{\omega^0}(1), \phi_{\omega^0}(1)]$, for ω^0 is lower than the balanced growth rate for ω^{00}.

proof. The trapping region for $\omega = \omega^0$ except for the fixed point $k^{**}(\omega^0)$ is divided three intervals as follows:

$$P_1 = [\phi^2_{\omega^0}(1), 1], \quad P_2 = (1, k^{**}(\omega^0)), \quad P_3 = (k^{**}(\omega^0), \phi_{\omega^0}(1)].$$

When $\phi^2_{\omega^0}(1) > \phi^{-1}_{\omega^0}(k^{**}(\omega^0))$, the orbit cannot stay in P_1 for two periods in a row once trapped by the trapping region.

If k_t belongs to P_3 in period t, k_{t-1} belongs to $P_1 \cup P_2$. When $k_{t-1} \in P_2$, the average growth rate over two periods, $t-1$ and t, is given by $(Y_{t+1}/Y_{t-1})^{\frac{1}{2}} - 1 = G(\omega) - 1$. In contrast, when $k_{t-1} \in P_1$, we obtain $(Y_{t+1}/Y_{t-1})^{\frac{1}{2}} - 1 = G(\omega)(k_{t-1})^{-\frac{1-\alpha}{2}} - 1$. Let us define $\zeta(k_{-1}) \equiv G(\omega)(k_{t-1})^{-\frac{1-\alpha}{2}} - 1$, which is a decreasing function of k_{t-1}. The minimum value of k_{t-1} is $\phi^2_{\omega^0}(1)$; therefore, the following inequality holds for arbitrary t:

$$\left(\frac{Y_t}{Y_{t-1}}\right)^{\frac{1}{2}} - 1 \leq \zeta(\phi^2_{\omega^0}(1)) = G(\omega^0)\left[\frac{G(\omega^0)^2}{1 + [1 + \Theta(\omega^0)][G(\omega^0) - 1]}\right]^{-\frac{1-\alpha}{2}} - 1.$$

From the assumption, there exists $\varpi \in [\omega', \omega'')$ such that $G(\varpi) = \Theta(\tilde{\varepsilon})$ holds. Note that $\phi^2_{\varpi}(1)$ is equal to 1. $\zeta(\phi^2_{\omega}(1))$ is increasing in ω under Assumption 2.1; therefore, we obtain $\zeta(\phi^2_{\omega'}(1)) \leq \zeta(\phi^2_{\varpi}(1)) = G(\varpi) < G(\omega'')$. $\square$

Let the condition of Lemma 2.6 be satisfied for all $\omega < \bar{\omega}$ such that the BGP is unstable or neutrally stable.[25]

Assumption 2.2 $\phi^2_{\omega}(1) > \phi^{-1}_{\omega}(k^{**}(\omega))$ holds for any ω that satisfies $\omega < \bar{\omega}$ and $1 < G(\omega) \leq \Theta(\omega)$.

Lemma 2.6 implies that even if there exists $\omega < \bar{\omega}$ that makes the BGP unstable, the long-run growth rate is maximized by applying the patent policy such that $\omega \geq \bar{\bar{\omega}}$. Because the BGP is stable for such ω, growth-maximizing policies are consistent with economic stabilization policies.

2.5.2.2 The case of $\Theta(\bar{\bar{\omega}}) > G(\bar{\bar{\omega}})$

In contrast, for $\Theta(\bar{\bar{\omega}}) > G(\bar{\bar{\omega}})$, the BGP is unstable for $\omega \geq \bar{\bar{\omega}}$. Therefore, the economy becomes volatile by excessively extending patent breadth, as shown in Figure 2.2.

In this case, there exists $\hat{\omega} > \bar{\omega}$ such that $\Theta(\hat{\omega}) > G(\hat{\omega})$ is satisfied and a period-2 cycle is stable. To achieve balanced growth in the long-run, patent breadth must be at least lower than such $\hat{\omega}$. The growth rates along the BGP and the period-2 cycles are given by $g^{**}(\omega)$ and $g^c(\omega)$, respectively, and the inequalities $g^{**}(\omega) < g^{**}(\hat{\omega}) < g^c(\hat{\omega})$ hold for any ω that satisfies $\Theta(\omega) < G(\omega)$. The economy experiences the faster long-run growth in exchange for accepting the fluctuating equilibrium path. Therefore, growth-maximizing policies are inconsistent with economic stabilization policies.

The case of $\Theta(\bar{\bar{\omega}}) = G(\bar{\bar{\omega}})$ can be applied to a similar analysis. In this case, the long-run growth rate is maximized by patent policies with $\omega \geq \bar{\bar{\omega}}$. However, under such policies, the BGP is neutrally stable and the economy converges to period-2 cycles that exist continuously for almost all initial conditions.

Summarizing the above yields the following proposition:

[25] Let $\varepsilon(\mu/\omega)$ be constant. If $\varepsilon \geq \bar{\varepsilon} \equiv [(3+\sqrt{5})/2]^{-\frac{1-\alpha}{\alpha}} - \frac{\alpha}{1-\alpha}$ holds, $\phi^2_{\omega}(1) > \phi^{-1}_{\omega}(k^{**}(\omega))$ is satisfied for $\omega \geq \bar{\omega}$. When $\alpha < \bar{\alpha} \simeq 0.9267$, $\bar{\varepsilon}$ is negative, and thus $\varepsilon \geq \bar{\varepsilon}$ is satisfied for all $\varepsilon > 0$.

Proposition 2.3 Let Assumptions 2.1 and 2.2 and $\Theta(\bar{\omega}) < G(\bar{\omega})$ hold.

(a) When $\Theta(\bar{\bar{\omega}}) < G(\bar{\bar{\omega}})$ holds, the patent policies with $\omega \geq \bar{\bar{\omega}}$ make the BGP stable and maximize the long-run growth rate.

(b) When $\Theta(\bar{\bar{\omega}}) \geq G(\bar{\bar{\omega}})$ holds, the patent policies for which the BGP is stable cannot maximize the long-run growth rate.

2.6 Conclusions

This study examined the issue of an optimal patent policy using an endogenous fluctuation model based on Matsuyama (1999). We focused on patent breadth, an important factor that characterizes patents. Changes in patent breadth influence the economy through monopoly prices of originals and patentees' market share.

When patentees' market share equals to 1, extending patent breadth has an effect only on the monopoly price. In this situation, the broader patent may trigger the instability of the BGP and create volatility in the economy. In contrast, a broader patent induces private firms to conduct higher levels of R&D activity through higher monopoly prices of originals and higher patentees' market share. Therefore, extending patent breadth always has a positive effect on economic growth. As a result, the possibility exists that economic stabilization policies might be undesirable with respect to growth rate.

Appendix

2.A Derivation of Equation (2.8)

When $S = T^{-1}(\mu_t/\omega)$, the monopoly profit (2.7) is written as

$$\Pi_{T^{-1}t} = T^{-1}\left(\frac{\mu_t}{\omega}\right) A^{\frac{1}{1-\alpha}} L\alpha^{\frac{1}{1-\alpha}} [r_t(1+\mu_t)]^{-\frac{1}{1-\alpha}} \mu_t r_t.$$

The first-order condition of maximization $\Pi_{T^{-1}t}$ is given by

$$(T^{-1})^0\left(\frac{\mu_t}{\omega}\right) + T^{-1}\left(\frac{\mu_t}{\omega}\right)\frac{1}{1+\mu_t}\left(\frac{1}{\mu_t} - \frac{\alpha}{1-\alpha}\right) = 0. \tag{2.21}$$

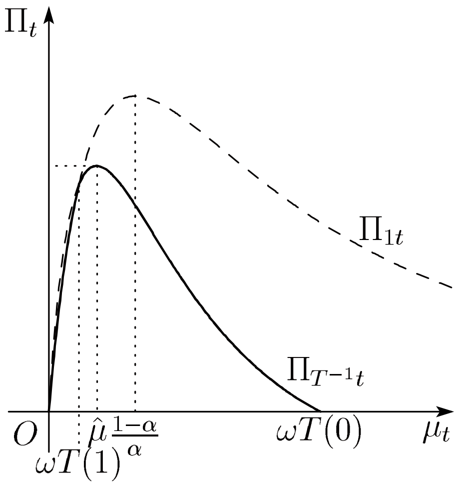
Because $(T^{-1})'(\mu/\omega)$ is negative, if $T^{-1}(\mu/\omega) \leq 0$, the previous equation has no root, thus ruling out the case of $T^{-1}(\hat{\mu}/\omega) \leq 0$ and satisfying $\hat{\mu} < \omega T(0)$. Dividing both sides of (2.21) by $T^{-1}(\mu_t/\omega)$ yields the implicit function (2.9) and we derive $\hat{\mu}(\omega)$. The second-order condition is written as $-\frac{\varepsilon(\hat{\mu}_t/\omega)}{\omega} - \frac{1}{1-\alpha} < 0$, which is satisfied under Assumption 2.1.

When $S = 1$, the monopoly profit (2.7) is

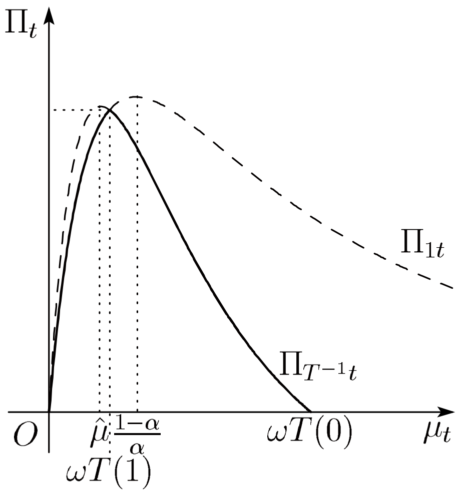
$$\Pi_{1t} = A^{\frac{1}{1-\alpha}} L \alpha^{\frac{1}{1-\alpha}} [r_t(1+\mu_t)]^{-\frac{1}{1-\alpha}} \mu_t r_t,$$

which is maximized at $\mu_t = (1-\alpha)/\alpha$. As long as $\varepsilon(\mu/\omega) > 0$, $\hat{\mu}(\omega)$ is always smaller than $(1-\alpha)/\alpha$.

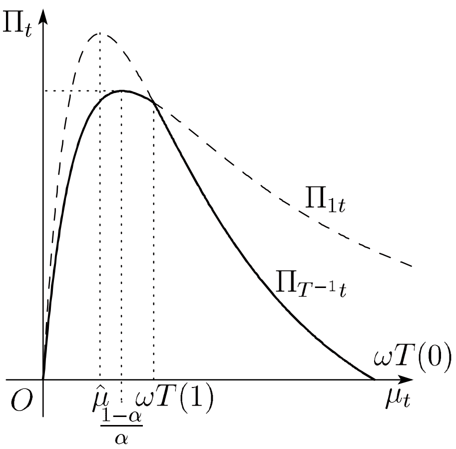
The realized monopoly profit is $\Pi_t = \min\{\Pi_{T^{-1}t}, \Pi_{1t}\}$. If and only if $\hat{\mu}(\omega) > \omega T(1)$, $S = T^{-1}(\mu_t/\omega) < 1$ holds and Π_t is maximized at $\mu_t = \hat{\mu}(\omega)$, as shown in Figure 2.3(a). In contrast, when $\hat{\mu}(\omega) \leq \omega T(1)$, S equals 1. If $\hat{\mu}(\omega) < \omega T(1) < (1-\alpha)/\alpha$, Π_t is maximized at $\mu_t = \omega T(1)$, whereas if $(1-\alpha)/\alpha < \omega T(1)$, Π_t is maximized at $\mu_t = (1-\alpha)/\alpha$ [see Figures 2.3(b) and 2.3(c)]. To summarize, we obtain the markup of patentees as (2.8).



(a) $\mu = \hat{\mu}(\omega)$.



(b) $\mu = \omega T(1)$.



(c) $\mu = (1-\alpha)/\alpha$.

Figure 2.3: Maximization of monopoly profit.

Chapter 3

Factor-Intensive R&D and Endogenous Fluctuations

3.1 Introduction

Rivera-Batiz and Romer (1991) proposed two types of specification for R&D technology, which are heavily used in endogenous growth studies on the basis of a variety-expanding framework. The first specification assumes that labor and existing knowledge are invested as inputs into R&D activity, as assumed in Romer (1990) and Grossman and Helpman (1991a, Chap. 3). They referred to this type of specification as *knowledge-driven specification* of R&D. Another specification is referred to as *lab equipment specification* of R&D,which assumes that R&D activity requires capital or output, as assumed in Romer (1987).[1]

This chapter considers an endogenous fluctuation model based on Matsuyama (1999), and introduces two specifications for R&D technology. As one of the results, we obtain three types of balanced growth paths distinguished by the allocation of resources for R&D. Furthermore, we obtain the fluctuating equilibrium path along which both R&D technologies are used alternately or periodically. The fluctuating equilibrium paths that are analyzed in Matsuyama (1999) have periods in which no R&D occurs. In contrast, along our fluctuating equilibrium paths, R&D occurs every period.

The structure of the chapter is as follows. The next section sets up the

[1]Matsuyama (1995) and Gancia and Zilibotti (2005) have provided comparative analysis between both specifications.

model. Section 3.3 shows the resource allocation in equilibrium and describes the law of motion. Section 3.4 provides steady states and examines their stabilities. Section 3.5 focuses on the endogenous fluctuating equilibrium path. Section 3.6 concludes.

3.2 Model

We consider the dynamic model based on Matsuyama (1999). Time is discrete and indexed by $t = 0, 1, 2, \ldots$. We assume two-period-lived overlapping generation (OLG) households. The young work and receive an income; when older, individuals consume all of their savings. A single final good is taken as a numeraire that is produced using intermediate goods and labor inputs, and it can be consumed or invested. The two types of R&D are based on the knowledge-driven and the lab equipment specification. We refer to the types of R&D as knowledge-driven R&D and lab equipment R&D, respectively. Inventors enjoy a one-period monopoly because of patent protection. The available intermediate goods are produced using capital by intermediate firms.

3.2.1 Final goods

Production technology for the final goods is given by

$$Y_t = AL_{Yt}^{1-\alpha}\int_0^{N_t} x_t(z)^\alpha dz, \quad 0 < \alpha < 1, \quad A > 0, \tag{3.1}$$

where Y_t denotes the output level of the final goods, L_{Yt} represents the amount of labor employed for the production of the final goods, $x_t(z)$ represents the amount of intermediate goods indexed by z, and $1/(1-\alpha)$ denotes the elasticity of substitution between all pairs of intermediate goods. N_t represents the number of available intermediate goods in period t for the technology level of the economy.

Profit maximization yields

$$\begin{aligned} w_t &= (1-\alpha)AL_{Yt}^{-\alpha}\int_0^{N_t} x_t(z)^\alpha dz = (1-\alpha)\frac{Y_t}{L_{Yt}}, \\ p_t(z) &= \alpha AL_{Yt}^{1-\alpha}x_t(z)^{-(1-\alpha)}, \quad \text{for } z \ 2 \ [0, N_t], \end{aligned} \tag{3.2}$$

where w_t represents the real wage rate and $p_t(z)$ represents the price of intermediate goods z.

3.2.2 Intermediate goods

Each intermediate good is produced using one unit of capital. Because of limited patent protection, the "old" intermediate goods, $[0, N_{t-1}]$, are competitively supplied. Hence, the price is equal to the marginal cost, $p_t(z) = r_t$ for $z \in [0, N_{t-1}]$, where r_t is the rental price of capital. However, the "new" intermediate goods invented during period $t-1$, $(N_{t-1}, N_t]$, are monopolistically supplied and sold at the monopoly price, $p_t(z) = r_t/\alpha$, for $z \in (N_{t-1}, N_t]$. All intermediate goods symmetrically enter into the production of the final goods, i.e., $x_t(z) = x_{ct}$ for $z \in [0, N_{t-1}]$ and $x_t(z) = x_{mt}$ for $z \in (N_{t-1}, N_t]$. From (3.2), we can easily illustrate that $x_{mt} = \alpha^{\frac{1}{1-\alpha}} x_{ct}$ holds and the maximized monopoly profits are denoted by the following:

$$\Pi_t(z) = \Pi_t \equiv \frac{1-\alpha}{\alpha} x_{mt} r_t, \quad \text{for } z \in (N_{t-1}, N_t].$$

Considering these results for the profit maximization of intermediate goods firms, we rewrite the production function (3.1) as follows:

$$Y_t = A L_{Yt}^{1-\alpha} (\alpha^{\frac{1}{1-\alpha}} x_{ct})^{\alpha} N_{t-1} \left[\frac{N_t}{N_{t-1}} - 1 + \alpha^{-\frac{\alpha}{1-\alpha}} \right]. \tag{3.3}$$

3.2.3 R&D

The number of intermediate goods, N, expands according to the following equation:

$$N_t - N_{t-1} = \eta_{LE} K_{Nt} + \eta_{KD} N_{t-1} L_{Nt}, \quad N_0 > 0. \tag{3.4}$$

The first term on the right-hand side represents new intermediate goods inventions using lab equipment R&D. K_{Nt} is the amount of capital allocated to R&D. The second term on the right-hand side represents knowledge-driven R&D, and L_{Nt} represents the amount of labor employed for R&D. Following the formalism adopted in much of the literature, we assume that the stock of existing knowledge has a positive effect on the productivity of present R&D.[2] $\eta_{LE} > 0$ and $\eta_{KD} > 0$ are parameters.

[2] Here, we retain the linear relationship between increase in knowledge and stock of knowledge on the basis of the first generation R&D-based endogenous growth model (Romer, 1990). However, Jones (1995a,b) argued that assuming this linearity is problematic. Surveys of this issue are presented by Jones (2005, 1999) and Li (2002, 2000).

Finally, clearing the capital market and the labor market requires the following:

$$L = L_{Yt} + L_{Nt}, \quad L > 0, \tag{3.5}$$

$$K_{t-1} = K_{Nt} + (N_t - N_{t-1})x_{mt} + N_{t-1}x_{ct}, \tag{3.6}$$

where L represents the total amount of labor (a constant) and K_{t-1} represents the amount of capital accumulated in period $t-1$ and available in period t. Substituting $x_{mt} = \alpha^{\frac{1}{1-\alpha}} x_{ct}$ into (3.6), we obtain the following:

$$x_{ct} = \frac{\alpha^{-\frac{1}{1-\alpha}}(K_{t-1} - K_{Nt})}{\left[\eta_{KD}L_{Nt} + \frac{\eta_{LE}K_{Nt}}{N_{t-1}} + \alpha^{-\frac{1}{1-\alpha}}\right] N_{t-1}}. \tag{3.7}$$

3.2.4 Consumers

Each consumer lives for two periods. When young, he/she supplies one unit of labor and earns wage w_t, which is divided into savings and consumption. When old, he/she only consumes his/her savings. Let c_{1t} and c_{2t+1} denote the consumption in periods t and $t+1$, respectively, for consumers born in period t. Each consumer chooses c_{1t} and c_{2t+1} to maximize their utility, $U_t = (1-s)\log c_{1t} + s \log c_{2t+1}$, where s 2 $(0, 1]$, subject to the budget constraint $c_{2t+1} = (w_t - c_{1t})r_{t+1}$.

The solution to this simple maximization problem is characterized by the following linear saving function:

$$K_t = s w_t L = s(1-\alpha)Y_t = s(Y_t - r_t K_{t-1}), \tag{3.8}$$

Substituting (3.3) into (3.8) yields

$$K_t = s\left[1 - \frac{\alpha^{1-\frac{\alpha}{1-\alpha}} K_{t-1}}{\left[\frac{N_t}{N_{t-1}} - 1 + \alpha^{-\frac{\alpha}{1-\alpha}}\right] x_{ct}N_{t-1}}\right] Y_t. \tag{3.9}$$

3.3 Equilibrium

This section derives the allocation of resources in equilibrium. In this model, each productive factor may be, respectively allocated to two sectors. Labor is needed for R&D and the production of final goods. Capital is needed for R&D and the production of intermediate goods.

In equilibrium, the following free-entry conditions must be satisfied:

$$\begin{aligned} \frac{r_t}{\eta_{LE}} &\geq \Pi_t, \quad \text{with an equality whenever } K_{Nt} > 0, \\ \frac{w_t}{\eta_{KD} N_{t-1}} &\geq \Pi_t, \quad \text{with an equality whenever } L_{Nt} > 0. \end{aligned} \tag{3.10}$$

By using (3.10), (3.5), and (3.6), we obtain the following results:

- If $\kappa^+ > 1$ holds,

$$\begin{aligned} L_{Nt} > 0, K_{Nt} = 0, &\quad \text{for } k_{t-1} \in [0, \kappa^-], \\ L_{Nt} > 0, K_{Nt} > 0, &\quad \text{for } k_{t-1} \in (\kappa^+, \kappa^-), \\ L_{Nt} = 0, K_{Nt} > 0, &\quad \text{for } k_{t-1} \in [\kappa^+, \infty). \end{aligned}$$

- If $\kappa^+ \leq 1$ holds,

$$\begin{aligned} L_{Nt} = K_{Nt} = 0, &\quad \text{for } k_{t-1} \in [0, 1], \\ L_{Nt} = 0, K_{Nt} > 0, &\quad \text{for } k_{t-1} \in (1, \infty), \end{aligned}$$

where

$$k_{t-1} \equiv \alpha^{\frac{\alpha}{1-\alpha}} (1-\alpha) \eta_{LE} \frac{K_{t-1}}{N_{t-1}}, \quad \kappa^+ \equiv \alpha^{\frac{1}{1-\alpha}} L \eta_{KD}, \quad \kappa^- \equiv \frac{1 + \alpha \kappa^+}{1 + \alpha}.$$

κ^- depends positively on κ^+, and has a value between 1 and κ^+. Note that knowledge-driven R&D never occurs when $\kappa^+ \leq 1$ holds. In this case, this model behaves identically to the model from Matsuyama (1999).

3.3.1 No-R&D regime

When $\max\{\kappa^+, k_{t-1}\} \leq 1$ holds, no resource is allocated to R&D and $N_t/N_{t-1} - 1 = 0$. In this case, the economy is in the no-R&D regime. The total output and savings in this regime are given by the following equations:

$$\begin{aligned} Y_t &= \frac{A L^{1-\alpha} N_{t-1}}{[\alpha^{\frac{\alpha}{1-\alpha}} (1-\alpha) \eta_{LE}]^\alpha} k_{t-1}^\alpha, \\ K_t &= s(1-\alpha) Y_t. \end{aligned} \tag{3.11}$$

3.3.2 Lab equipment R&D regime

When $k_{t-1} > \max\{1, \kappa^+\}$ holds, positive capital, $K_{Nt} = \eta_{LE}^{-1}\alpha^{-\frac{\alpha}{1-\alpha}}(k_{t-1} - 1)N_{t-1}$, is allocated to R&D, and the new intermediate goods are invented by lab equipment R&D. In this case, the economy is in the lab equipment R&D regime. The growth rate of N is given by the following equations:

$$\frac{N_t}{N_{t-1}} - 1 = \alpha^{-\frac{\alpha}{1-\alpha}}\left(k_{t-1} - 1\right). \tag{3.12}$$

Substituting (3.12) and (3.7) into (3.3) and (3.9) yields

$$\begin{aligned} Y_t &= \frac{AL^{1-\alpha}N_{t-1}}{[\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\eta_{LE}]^{\alpha}}k_{t-1}, \\ K_t &= s(1-\alpha)Y_t. \end{aligned} \tag{3.13}$$

3.3.3 Knowledge-driven R&D regime

When $\kappa^+ > 1$ and $k_{t-1} \leq \kappa^-$ hold, the new intermediate goods are invented by knowledge-driven R&D. In this case, the economy is in the knowledge-driven R&D regime. The amount of labor employed for R&D is given by

$$L_{Nt} = \alpha L\frac{\kappa^+ - \kappa^-}{\kappa^+} = \eta_{KD}^{-1}\alpha^{-\frac{\alpha}{1-\alpha}}\left[\alpha\kappa^- + (1-\alpha)\kappa^+ - 1\right].$$

The growth rate of N is given by

$$\frac{N_t}{N_{t-1}} - 1 = \alpha^{-\frac{\alpha}{1-\alpha}}\left[\alpha\kappa^- + (1-\alpha)\kappa^+ - 1\right], \tag{3.14}$$

which does not depend on N_{t-1} and K_{t-1}. The total output and saving in this regime are given by

$$\begin{aligned} Y_t &= AL^{1-\alpha}\frac{[\alpha\kappa^- + (1-\alpha)\kappa^+]^{2-\alpha}}{[\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\eta_{LE}]^{\alpha}(\kappa^+)^{1-\alpha}(\kappa^-)^{\alpha}}k_{t-1}^{\alpha}N_{t-1}, \\ K_t &= s\left[1 - \frac{\alpha\kappa^-}{\alpha\kappa^- + (1-\alpha)\kappa^+}\right]Y_t. \end{aligned} \tag{3.15}$$

3.3.4 Mixed R&D regime

When $\kappa^+ > 1$ and $k_{t-1} \in (\kappa^-, \kappa^+)$ hold, both lab equipment R&D and knowledge-driven R&D are carried out. In this case, the economy is in the mixed R&D regime. The resources are allocated such that the unit costs of both R&D types are equivalent.

$$\begin{aligned} K_{Nt} &= \eta_{LE}^{-1}\alpha^{-\frac{\alpha}{1-\alpha}}\left[(1+\alpha)k_{t-1} - \alpha\kappa^+ - 1\right] N_{t-1}, \\ L_{Nt} &= \eta_{KD}^{-1}\alpha^{-\frac{\alpha}{1-\alpha}}(\kappa^+ - k_{t-1}). \end{aligned}$$

Substituting these equations into (3.4), we obtain the growth rate of N as follows:

$$\frac{N_t}{N_{t-1}} - 1 = \alpha^{-\frac{\alpha}{1-\alpha}}\left[\alpha k_{t-1} + (1-\alpha)\kappa^+ - 1\right]. \tag{3.16}$$

R&D is promoted by the larger value of k_{t-1}. The total output and capital accumulation are

$$\begin{aligned} Y_t &= AL^{1-\alpha}\frac{[\alpha k_{t-1} + (1-\alpha)\kappa^+]^{2-\alpha}}{[\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\eta_{LE}]^{\alpha}(\kappa^+)^{1-\alpha}}N_{t-1}, \\ K_t &= s\left[1 - \frac{\alpha k_{t-1}}{\alpha k_{t-1} + (1-\alpha)\kappa^+}\right] Y_t. \end{aligned} \tag{3.17}$$

3.3.5 Law of motion

Using (3.11) through (3.17) shows that the dynamics of the market equilibrium can be summarized by the first-order difference equation, $k_t = \phi(k_{t-1})$, $\phi : \mathbb{R}_+ \to \mathbb{R}_+$. We must consider the following two cases that are distinguished by the value of κ^+.

- When $\kappa^+ > 1$ holds,

$$\begin{aligned} k_t &= \phi(k_{t-1}) \\ &= \begin{cases} \phi_{KD}(k_{t-1}) \equiv \dfrac{G(\kappa^+/\kappa^-)^{\alpha}[\alpha\kappa^- + (1-\alpha)\kappa^+]^{1-\alpha}k_{t-1}^{\alpha}}{1 + \alpha^{-\frac{\alpha}{1-\alpha}}[\alpha\kappa^- + (1-\alpha)\kappa^+ - 1]}, & \text{if } k_{t-1} \in [0, \kappa^-], \\ \phi_{M}(k_{t-1}) \equiv \dfrac{G(\kappa^+)^{\alpha}[\alpha k_{t-1} + (1-\alpha)\kappa^+]^{1-\alpha}}{1 + \alpha^{-\frac{\alpha}{1-\alpha}}[\alpha k_{t-1} + (1-\alpha)\kappa^+ - 1]}, & \text{if } k_{t-1} \in (\kappa^-, \kappa^+), \\ \phi_{LE}(k_{t-1}) \equiv \dfrac{Gk_{t-1}}{1 + \alpha^{-\frac{\alpha}{1-\alpha}}(k_{t-1} - 1)}, & \text{if } k_{t-1} \in [\kappa^+, \infty). \end{cases} \end{aligned} \tag{3.18}$$

- When $\kappa^+ \leq 1$ holds,

$$k_t = \phi(k_{t-1}) = \begin{cases} \phi_N(k_{t-1}) \equiv G k_{t-1}^{\alpha}, & \text{if } k_{t-1} \in [0, 1], \\ \phi_{LE}(k_{t-1}), & \text{if } k_{t-1} \in (1, \infty), \end{cases} \tag{3.19}$$

where $G \equiv [\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\eta_{LE}]^{1-\alpha} s(1-\alpha)AL^{1-\alpha}$.

(3.18) and (3.19) can be summarized as

$$\phi(k_{t-1}) = \mathrm{F}(k_{t-1}; \max\{1, \kappa^-, k_{t-1}\}, \max\{1, \kappa^+, k_{t-1}\}),$$

where $\mathrm{F}(k_{t-1}; X, Z) \equiv \frac{GX^{-\alpha}Z^{\alpha}[\alpha X + (1-\alpha)Z]^{1-\alpha} k_{t-1}^{\alpha}}{1 + \alpha^{-\frac{\alpha}{1-\alpha}}[\alpha X + (1-\alpha)Z - 1]}$.

(3.18) and (3.19) appear in Figure 3.1(a) and Figure 3.1(b) with k_{t-1} on the horizontal axis and k_t on the vertical axis. In the case of $\kappa^+ > 1$, the graph of $\phi(k_{t-1})$ is a unimodal form with two kinks. Note that $\phi'_{KD}(k) > 0$, $\phi''_{KD}(k) < 0$, $\phi'_{M}(k) < 0$, $\phi''_{M}(k) > 0$, $\phi'_{LE}(k) < 0$, and $\phi''_{LE}(k) > 0$ hold. In contrast, when $\kappa^+ \leq 1$ holds, the graph of $\phi(k_{t-1})$ takes a unimodal form that has a unique kink, and this case is identical to the model analyzed in Matsuyama (1999).

3.4 Steady state

$\phi(k)$ has a unique fixed-point except for the origin. This section shows the regime in which fixed point of $\phi(k)$ belongs. When κ^+ is larger than 1, the following relationship holds:

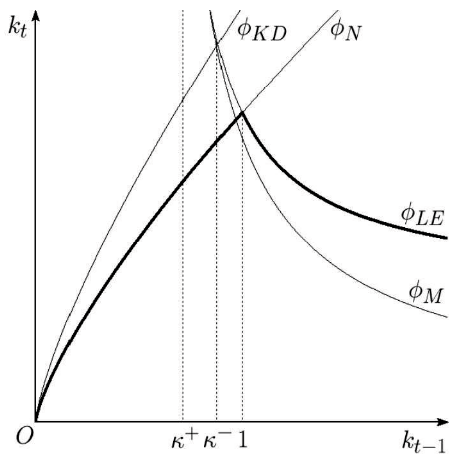
$$\begin{aligned} G \gtreqless G^- \quad &\Leftrightarrow \quad \phi(\kappa^-) \gtreqless \kappa^-, \\ G \gtreqless G^+ \quad &\Leftrightarrow \quad \phi(\kappa^+) \gtreqless \kappa^+, \end{aligned} \tag{3.20}$$

where

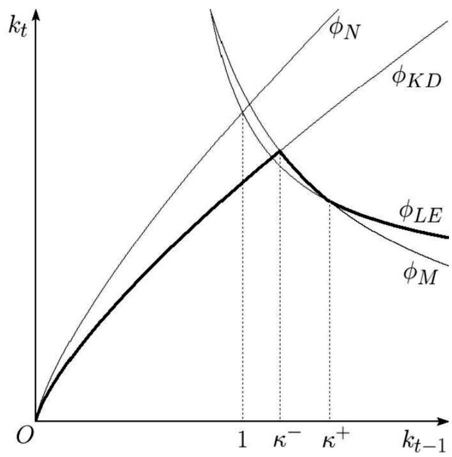
$$\begin{aligned} G^- &= G^-(\kappa^+) \equiv \frac{1 + \alpha^{-\frac{\alpha}{1-\alpha}}[\alpha\kappa^- + (1-\alpha)\kappa^+ - 1]}{(\kappa^+)^{\alpha}[\alpha\kappa^- + (1-\alpha)\kappa^+]^{1-\alpha}} \kappa^-, \\ G^+ &= G^+(\kappa^+) \equiv 1 + \alpha^{-\frac{\alpha}{1-\alpha}}[\kappa^+ - 1]. \end{aligned}$$

$G^-(\kappa^+)$ and $G^+(\kappa^+)$ are increasing in κ^+, and $G^+(\kappa^+) > G^-(\kappa^+) > 1$ holds as long as $\kappa^+ > 1$.

The argument in (3.20) can be rewritten in the following way.



(a) Case of $\kappa^+ \leq 1$.



(b) Case of $\kappa^+ > 1$.

Figure 3.1: Law of motion.

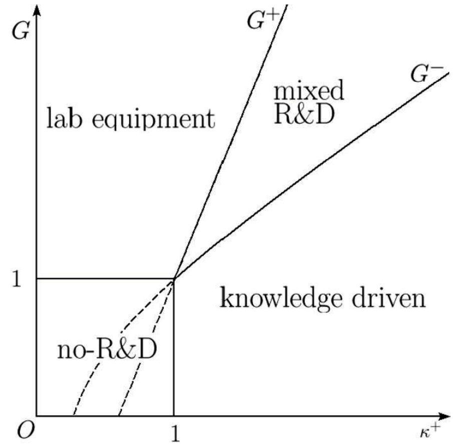


Figure 3.2: Regime in which the fixed point of ϕ is located.

Proposition 3.1 Let $\kappa^+ > 1$ hold, then:

- The fixed point of $\phi(k)$ belongs to the knowledge-driven R&D regime for G 2 $(0, G^-(\kappa^+)]$.
- The fixed point of $\phi(k)$ belongs to the mixed R&D regime for G 2 $(G^-(\kappa^+), G^+(\kappa^+))$.
- The fixed point of $\phi(k)$ belongs to the lab equipment R&D regime for G 2 $[G^+(\kappa^+), 1\ )$.

In contrast, when $\kappa^+ \leq 1$, the fixed point belongs to the no-R&D regime for the parameters that satisfy $G \leq 1$, and in the lab equipment R&D regime for $G > 1$, as shown in Matsuyama (1999).

Figure 3.2 summarizes these results. The region that corresponds to $\kappa^+ > 1$ is separated into three regions using the graphs of G^+ and G^-, which have upward slopes. When $\kappa^+ > 1$ holds, the economy achieves balanced growth based on sustained R&D regardless of the regime that involves the fixed point. That is, our model has three types of the balanced growth paths (BGPs). Along each BGP, Y, K, and N continue to grow at the same rate.

In contrast, the region that corresponds to $\kappa^+ \leq 1$ is separated into two regions using the horizontal line for $G = 1$.

3.4.1 Steady state in the knowledge-driven R&D regime

The fixed point that belongs to the knowledge-driven R&D regime is given by

$$k^*_{KD} = [\alpha\kappa^- + (1-\alpha)\kappa^+]\left[\frac{G(\kappa^+/\kappa^-)^\alpha}{1+\alpha^{-\frac{\alpha}{1-\alpha}}[\alpha\kappa^- + (1-\alpha)\kappa^+ - 1]}\right]^{\frac{1}{1-\alpha}}.$$

In this steady state, Y, K, and N grow at the constant rate $g^*_{KD} = \alpha^{-\frac{\alpha}{1-\alpha}}[\alpha\kappa^- + (1-\alpha)\kappa^+ - 1]$. This balanced growth rate is increasing in κ^+; however, it does not depend on the value of G and k^*_{KD}.

For the stability of the steady state, the following proposition is easy to verify.

Proposition 3.2 The fixed point k^*_{KD} is globally stable, and k monotonically converges to k^*_{KD}.

3.4.2 Steady state in the lab equipment R&D regime

The fixed point located in the lab equipment R&D regime is given by $k^*_{LE} \equiv 1+\alpha^{\frac{\alpha}{1-\alpha}}(G-1)$. The balanced growth rate in this steady state is $g^*_{LE} = G-1$, which is independent of κ^+.

By using $\phi'_{LE}(k^*_{LE}) = -(\alpha^{-\frac{\alpha}{1-\alpha}} - 1)/G$, the stability of the steady state, k^*_{LE}, is summarized as follows:

Proposition 3.3 (a) When $G > \max\{1, G^+(\kappa^+), (\alpha^{-\frac{\alpha}{1-\alpha}} - 1)\}$, the fixed point, k^*_{LE}, is stable and k oscillatory converges to k^*_{LE} for any initial condition k_0.

(b) When $\max\{1, G^+(\kappa^+)\} < G < \alpha^{-\frac{\alpha}{1-\alpha}} - 1$, k^*_{LE} is unstable, and k continues to fluctuate for almost all initial conditions.[3]

As shown in Proposition 3.3, when $\max\{1, G^+(\kappa^+)\} < G < \alpha^{-\frac{\alpha}{1-\alpha}} - 1$ holds, $|\phi'_{LE}(k^*_{LE})|$ exceeds 1 and the steady state is locally unstable. In this case, the fluctuating equilibrium paths exist, and the trajectory continues to fluctuate in the trapping region, $[\phi^2(\max\{1, \kappa^-\}), \phi(\max\{1, \kappa^-\})]$.

[3] $\alpha^{-\frac{\alpha}{1-\alpha}} - 1$ is larger than 1 when $\alpha > 1/2$ holds, and it is larger than $G^+(\kappa^+)$ when $\kappa^+ > 2(\alpha^{-\frac{\alpha}{1-\alpha}} - 1)/\alpha^{-\frac{\alpha}{1-\alpha}}$ holds. Note that $\alpha \gtreqless 1/2$, $2(\alpha^{-\frac{\alpha}{1-\alpha}} - 1)/\alpha^{-\frac{\alpha}{1-\alpha}} \gtreqless 1$ holds.

3.4.3 Steady state in the mixed R&D regime

We obtain the fixed point in the mixed-R&D regime, k_M^*, as the root of the implicit function, $k_M^* - \phi_M(k_M^*) = 0$. The growth rate of this steady state is given by $g_M^* \equiv \alpha^{-\frac{\alpha}{1-\alpha}}[\alpha k_M^* + (1-\alpha)\kappa^+ - 1]$, which is increasing in both G and κ^+.

The stability of this fixed point can be shown in the following way.

Proposition 3.4 Let $\kappa^+ > 1$ and $G \in (G^-(\kappa^+), G^+(\kappa^+))$ hold.

(a) If $\kappa^+ \leq \kappa_1$ holds, the fixed point, k_M^*, is unstable for any $G \in (G^-(\kappa^+), G^+(\kappa^+))$.

(b) If $\kappa^+ \geq \kappa_2$ holds, the fixed point, k_M^*, is stable for any $G \in (G^-(\kappa^+), G^+(\kappa^+))$.

(c) If $\kappa^+ \in (\kappa_1, \kappa_2)$ holds, there exists the threshold value of G, $G_c(\kappa^+) \in (G^-(\kappa^+), G^+(\kappa^+))$, such that k_M^* is stable for $G < G_c(\kappa^+)$ and is unstable for $G > G_c(\kappa^+)$.

where[4]

$$\kappa_1 \equiv 1 + \alpha^{\frac{\alpha}{1-\alpha}}\left[\frac{\alpha(\alpha^{-\frac{\alpha}{1-\alpha}} - 1)}{1-\alpha^2} - 1\right],$$

$$\kappa_2 \equiv \frac{1+\alpha}{\alpha}\frac{\sqrt{\gamma_b^2 - 4\gamma_a\gamma_c} - \gamma_b}{2\gamma_a} - \frac{1}{\alpha},$$

and

$$\gamma_a \equiv (1-\alpha^3) > 0, \quad \gamma_b \equiv \alpha - 2 + (1 + \alpha^2 - \alpha^3)\alpha^{\frac{1}{1-\alpha}} > 0,$$
$$\gamma_c \equiv (1-\alpha)(1-\alpha^{\frac{1}{1-\alpha}}) > 0.$$

$G_c(\kappa^+)$ is defined for $\kappa^+ \in (\kappa_1, \kappa_2)$ as follows:

$$G_c(\kappa^+) \equiv \frac{1 + \alpha^{-\frac{\alpha}{1-\alpha}}[\alpha X^*(\kappa^+) + (1-\alpha)\kappa^+ - 1]}{(\kappa^+)^{\alpha}[\alpha X^*(\kappa^+) + (1-\alpha)\kappa^+]^{1-\alpha}} X^*(\kappa^+),$$

$$X^*(\kappa^+) \equiv \frac{\sqrt{\gamma_e(\kappa^+)^2 - 4\gamma_d\gamma_f(\kappa^+)} - \gamma_e(\kappa^+)}{2\gamma_d}, \tag{3.21}$$

[4] The necessary and sufficient condition for $\kappa_2 > \kappa_1 > 1$ is given by $\alpha > \hat{\alpha} \simeq 0.5841$.

$$\gamma_d \equiv \alpha^2(1-\alpha) > 0, \quad \gamma_e(\kappa^+) \equiv (2\alpha-\alpha^2)[(1-\alpha)\kappa^+ + \alpha^{\frac{\alpha}{1-\alpha}} - 1] > 0,$$
$$\gamma_f(\kappa^+) \equiv (1-\alpha)\kappa^+[(1-\alpha)\kappa^+ + \alpha^{\frac{\alpha}{1-\alpha}} - 1] > 0.$$

proof. See Appendix 3.A. □

3.4.4 Steady state in the no-R&D regime

When $\kappa^+ \leq 1$ and $k_{t-1} < 1$ hold, the fixed point is located in the no-R&D regime. The fixed point is given by $k_N^* \equiv G^{\frac{1}{1-\alpha}}$, and is globally stable. In this case, no R&D occurs except for during the initial period, and the economy cannot sustainably grow.

Figure 3.3 summarizes the results of this section.[5] The dotted region corresponds to the parameters for which the steady state is unstable. The parameter set that has the steady state in the lab equipment R&D regime is separated into two regions by the horizontal line $G = \alpha^{-\frac{\alpha}{1-\alpha}} - 1$, as shown in Proposition 3.3. Similarly, the parameter set with the steady state in the mixed R&D regime is separated by the graph of $G_c(\kappa^+)$, as shown in Proposition 3.4.

Figure 3.4 rewrites Figure 3.3 in the (η_{KD}, η_{LE})-plane. When knowledge-driven R&D has a sufficiently high productivity relative to lab equipment R&D, technological progress done by knowledge-driven R&D in the steady state. In contrast, lab equipment R&D has a high productivity relative to knowledge-driven R&D, resources are allocated only to lab equipment R&D in the steady state. Moreover, if both η_{KD} and η_{LE} are sufficiently large in a balance, both R&D continue to be carried out in the steady state. If η_{KD} and η_{LE} are not extensive, steady state become unstable and equilibrium path may continue to fluctuate.

3.5 Fluctuating equilibrium path

When the steady state belonging to the lab equipment R&D regime or the mixed-R&D regime loses its stability, the endogenous fluctuations arise. The fluctuating equilibrium paths that are analyzed in Matsuyama (1999, 2001)

[5] If $\alpha > (\sqrt{5}-1)/2 \simeq 0.618$, $G^+(\kappa^+) = \alpha(\alpha^{-\frac{\alpha}{1-\alpha}} - 1)/(1-\alpha^2)$ is larger than $\alpha^{-\frac{\alpha}{1-\alpha}} - 1$, as shown in Figure 3.3.

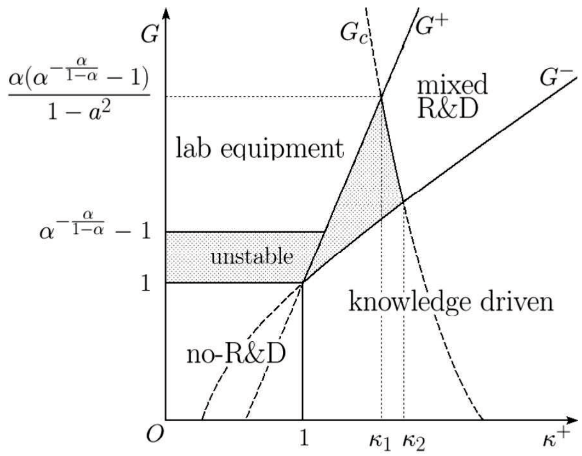


Figure 3.3: Stability of the steady state ((κ^+, G)-plane).

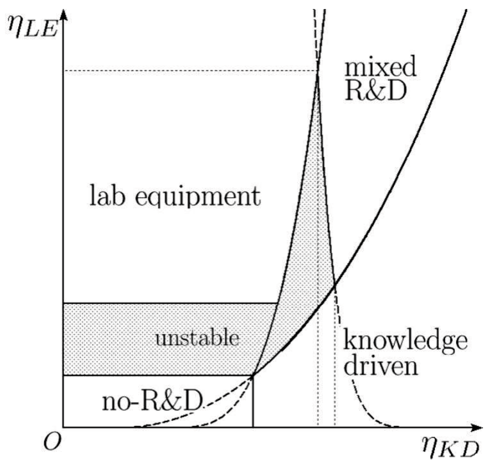


Figure 3.4: Stability of the steady state ((η_{KD}, η_{LE})-plane).

and Deneckere and Judd (1992) have periods in which no R&D occurs. Conversely, along our fluctuating equilibrium path, R&D occurs in every period without stopping. This section analyzes the growth rate of such a fluctuating equilibrium path. In particular, we consider the case in which the unstable steady state is included in the lab equipment R&D regime.

For the fluctuation equilibrium path, we can verify the following proposition.

Proposition 3.5 Let $\kappa^+ > 1$ and $G\ 2\ (G^+(\kappa^+), \alpha^{-\frac{\alpha}{1-\alpha}} - 1)$ hold.

(a) There exists a finite integer s such that $\phi^s(k_{t-1}) < \kappa^+$ is satisfied for any $k_{t-1}\ 2\ [\kappa^+, 1\ )\ \mathrm{n}\, k^*_{LE}$.

(b) When $\phi_M(k) \geq \phi_{LE}(k)$ is satisfied for $k\ 2\ (\kappa^-, \kappa^+)$, there exists a finite integer s such that $\phi^s(k_{t-1}) \leq \kappa^-$ holds for any $k_{t-1}\ 2\ (\kappa^-, 1\ )\ \mathrm{n}\, k^*_{LE}$.

proof. See Appendix 3.B. □

Proposition 3.5(a) establishes that the economy cannot continue to stay only in the lab equipment R&D regime except for when in the steady state. Furthermore, it cannot continue to stay in the region that is lower than k^*_{LE}. Therefore, the fluctuating equilibrium path must go through more than one regime including the lab equipment R&D regime.

Moreover, Proposition 3.5(b) establishes that if the graph of ϕ_M is located above the graph of ϕ_{LE} in the mixed-R&D regime, as shown in Figure 3.1(a), the economy cannot continue to stay in the union of the lab equipment R&D regime and the mixed R&D regime except in the steady state.[6] That is, the fluctuating equilibrium path must go through the knowledge-driven R&D regime. We can confirm that the trapping region, $[\phi^2(\kappa^-), \phi(\kappa^-)]$, which includes the knowledge-driven R&D regime if the condition of Proposition 3.5(b) is satisfied. In particular, over the period-2 cycles, the economy alternately experiences the lab equipment R&D regime and the knowledge-driven R&D regime.[7]

[6]When $(1-\alpha)\alpha^{-2}(\alpha^{-\frac{\alpha}{1-\alpha}} - 1) > 1$, $\alpha > \underline{\alpha}\ ;\ 0.6532$ holds, $\phi_M(\kappa^-) > \phi_{LE}(\kappa^-)$ is satisfied for all $\kappa^+ > 1$. Even if $\alpha < \underline{\alpha}$ holds, $\phi_M(\kappa^-) > \phi_{LE}(\kappa^-)$ holds for sufficiently large values of κ^+.

[7]In the case of $\kappa^+ \leq 1$, Matsuyama (1999) has shown that the economy moves back and forth between the no-R&D regime and the lab equipment R&D regime.

When the steady state is unstable and the economy is trapped by a period-n cycle, its average growth rate is derived as follows:

$$g_n \equiv \left(\frac{Y_{t+n}}{Y_t}\right)^{\frac{1}{n}} - 1 = G\left[\prod_{k_i \in (0,\kappa^-]} \Psi_{KD}(k_i) \prod_{k_i \in (\kappa^-,\kappa^+)} \Psi_M(k_i)\right]^{\frac{1}{n}} - 1, \tag{3.22}$$
$$i \in \{t, t+1, \ldots, t+n\}$$

where Ψ_{KD} and Ψ_M are defined as [8]

$$\Psi_{KD}(k) \equiv \left(\frac{\kappa^+}{\kappa^-}\right)^{\alpha}\left[\frac{\alpha\kappa^- + (1-\alpha)\kappa^+}{k}\right]^{1-\alpha},$$
$$\Psi_M(k) \equiv \left(\frac{\kappa^+}{k}\right)^{\alpha}\left[\frac{\alpha k + (1-\alpha)\kappa^+}{k}\right]^{1-\alpha}.$$

These are decreasing functions of k and satisfy the following relationship:

$$\Psi_{KD}(k_1) > \Psi_M(k_2) > 1, \quad \forall k_1 \in (0,\kappa^-], \quad k_2 \in (\kappa^-,\kappa^+). \tag{3.23}$$

(3.22) implies that the average growth rate over the cycles depends on the number of the fixed points of period n located in each regime. When the economy is in the lab equipment R&D regime, the gross growth rate of K is equal to G.[9] Similarly, when the economy is in the mixed R&D regime (the knowledge-driven R&D regime), the gross growth rate of K is $G\Psi_M(k_{t-1})$ ($G\Psi_{KD}(k_{t-1})$). Because Y, N, and K grow at the same rate over the entire cycle, the average gross growth rate is derived by raising their product to the power of $1/n$.

Recall that except for the case in which the system is conservative, the fluctuating equilibrium path cannot continue to stay only in the lab equipment R&D regime; therefore, the average gross growth rate over the cycle, (3.22), is guaranteed to be larger that the gross growth rate of the BGP, $1 + g^*_{LE} = G$, from (3.23).

3.5.1 Period-2 cycles

We consider the period-2 cycles over which the economy moves back and forth between the knowledge-driven R&D regime and the lab equipment R&D

[8] Note that $\Psi_{KD}(k) \gtreqless \Psi_M(k) \Leftrightarrow k \gtreqless \kappa^-$ holds.

[9] The gross growth rate equals one plus the (net) growth rate.

regime. In the case in which the BGP belongs to the lab equipment R&D regime, this type of period-2 cycle may exist when the BGP loses its stability, as shown in Proposition 3.5.

Let us assume that the fixed points of period-2, k_L and k_H, exist, and $k_{t-1} = k_{t+2} = k_L$, $k_t = k_{t+1} = k_H$, and $k_L < k_H = \phi_{KD}(k_L)$ hold. From (3.22), the average growth rate over the cycle is $G[\Psi_{KD}(k_L)]^{\frac{1}{2}} - 1 > g^*_{LE}$. Furthermore, let g^{LE}_X denote the growth rate of the variable X in the lab equipment R&D regime. Similarly, g^{KD}_X denotes the growth rate of X in the knowledge-driven R&D regime. By using (3.12) through (3.15), we obtain the growth rates of K, Y, and N in each period as follows:

$$g^{KD}_N = \alpha^{-\frac{\alpha}{1-\alpha}}[\alpha\kappa^- + (1-\alpha)\kappa^+ - 1] < g^{LE}_N = \alpha^{-\frac{\alpha}{1-\alpha}}(k_H - 1),$$

$$g^{KD}_K = G\Psi_{KD}(k_L) - 1 > g^{LE}_K = G - 1,$$

$$g^{KD}_Y = \frac{\kappa^+ G\Psi_{KD}(k_L)}{\alpha\kappa^- + (1-\alpha)\kappa^+} - 1 > g^{LE}_Y = \frac{[\alpha\kappa^- + (1-\alpha)\kappa^+]G}{\kappa^+} - 1.$$

By using these equations, we argue the following results. First, the growth rate of N is negatively correlated with the growth rate of Y; that is, the productivity improvement is countercyclical. This countercyclical behavior of N is inherited from Matsuyama's model. In this respect, Chapter 4 provides a detailed analysis.

Second, the amplitude of the growth rate of Y is larger than the amplitude of the growth rate of K. If the aggregate savings rate, K_t/Y_t, was constant as in Chapter 2 and Matsuyama (1999), both amplitudes would be equal. However, in our model, the aggregate savings rate is higher in the knowledge-driven R&D regime than in the lab equipment R&D regime, which is caused by the higher labor share in the knowledge-driven R&D regime.[10]

Finally, we confirm that $g^{KD}_Y > g^*_{LE} > g^{LE}_Y$ holds; that is, growth in total output is slower in the lab equipment R&D regime than in the steady state despite the fact that faster average growth is achieved over the cycle.[11] This situation is also caused by the fluctuating behavior of the aggregate savings rate. In Matsuyama's model, the cycles enhance the entire growth, with no degradation in growth during each period. Conversely, in our model, faster growth requires sacrifice from the specific generations.

[10]As long as $\kappa^+ > 1$, $G^-(\kappa^+) > \kappa^+/[\alpha\kappa^- + (1-\alpha)\kappa^+]$ always holds. Because the cycles require $G > G^-(\kappa^+)$ to exist; $[\alpha\kappa^- + (1-\alpha)\kappa^+]G/\kappa^+$ is larger than 1.

[11]In other words, knowledge-driven R&D is carried out procyclically, whereas lab-equipment R&D is carried out coutercyclically.

3.6 Conclusions

This chapter constructed the R&D-based growth model and introduced two specifications for R&D technology, that are heavily used in endogenous growth studies on the basis of the variety-expanding framework. We examined endogenous growth and endogenous fluctuations using such a dynamic model.

When knowledge-driven R&D has sufficiently high productivity, three regimes are distinguished by the allocation of resources for R&D: (1) the knowledge-driven R&D regime, (2) the lab equipment R&D regime, and (3) the mixed R&D regime. The technology level and the amount of capital stock determine the regime to which the economy belongs.

In the long run, when the steady state is located in the lab equipment R&D regime or the mixed R&D regime, the possibility exists that the steady state loses its stability and endogenous fluctuations arise. Along such a fluctuating equilibrium path, both R&D technologies are alternately or periodically used.[12]

Appendix

3.A Proof of Proposition 3.4

We define the following function:

$$\lambda(k,\kappa^+) \equiv \{|\phi_M'(k)| \,|\, k = \phi(k)\},$$

where it is easy to show that

$$\lambda(k,\kappa^+) = \frac{\alpha k[(\alpha^{-\frac{\alpha}{1-\alpha}} - 1) + \alpha\chi(k,\kappa^+)]}{[\alpha k + (1-\alpha)\kappa^+]\chi(k,\kappa^+)},$$

and

$$\chi(X,Z) \equiv 1 + \alpha^{-\frac{\alpha}{1-\alpha}}[\alpha X + (1-\alpha)Z - 1].$$

The stability of the fixed point depends on whether $\lambda(k_M^*,\kappa^+)$ exceeds 1. Using the following relationship is useful:

$$\lambda(k_M^*,\kappa^+) \gtreqless 1 \quad , \qquad \theta(k_M^*,\kappa^+) \lesseqgtr 0,$$

[12]However, these results crucially depend on the linearity of the knowledge spillover in the technologies of R&D. We will examine this issue in the following chapter.

where

$$\theta(X,Z) \equiv \chi(X,Z) - \frac{\alpha X(\alpha^{-\frac{\alpha}{1-\alpha}} - 1)}{\alpha X + (1-\alpha)Z - \alpha^2 X}.$$

Note that $\partial\theta(X,\kappa^+)/\partial X > 0$ holds for any $X > 0$.

When $\kappa^+ \geq (1-\alpha^{\frac{\alpha}{1-\alpha}})/(1-\alpha)$ holds, we have $\theta(0,\kappa^+) = \chi(0,\kappa^+) \geq 0$; therefore, $\theta(X,\kappa^+) = 0$ has no positive root. In contrast, when $\kappa^+ < (1-\alpha^{\frac{\alpha}{1-\alpha}})/(1-\alpha)$ holds, $\theta(X^*,\kappa^+) = 0$ has a unique positive root, X^*, which is given by (3.21).

(a) We define κ_1 as the positive value of κ^+ such that $X^*(\kappa^+) = \kappa^+$. Because $\theta(X,X)$ is increasing in X, and $\theta(0,0) < 0$ holds, κ_1 uniquely exists.

If $\kappa^+ \leq \kappa_1$ holds, the following relationship is satisfied for any $X \in (\kappa^-,\kappa^+)$:

$$\theta(X,\kappa^+) < \theta(\kappa^+,\kappa^+) \leq \theta(\kappa_1,\kappa_1) = 0.$$

Recall that k_M^* must belong to (κ^-,κ^+), and $\theta(k_M^*,\kappa^+) < 0$ holds; that is, we have $|\phi_M(k_M^*)| > 1$ as long as $\kappa^+ \leq \kappa_1$.

(b) We define κ_2 as the positive value of κ^+ such that $X^*(\kappa^+) = \kappa^-$. Note that $\kappa^- = \kappa^-(\kappa^+)$ also depends on κ^+. Some algebra shows that

$$\frac{d\theta(\kappa^-(X),X)}{dX} = \frac{\alpha^{-\frac{\alpha}{1-\alpha}}}{1+\alpha} + \frac{\alpha(1-\alpha^2)(\alpha^{-\frac{\alpha}{1-\alpha}} - 1)}{[X(1-a^3) + \alpha(1-\alpha)]^2} > 0,$$

Because $\theta(\kappa^-(0),0)$ is negative, κ_2 uniquely exists.

If $\kappa^+ \geq \kappa_2$, the following relationship holds for $X \in (\kappa^-(\kappa^+),\kappa^+)$:

$$\theta(X,\kappa^+) > \theta(\kappa^-(\kappa^+),\kappa^+) \geq \theta\left(\kappa^-(\kappa_2),\kappa_2\right) = 0,$$

which implies that $\theta(k_M^*,\kappa^+) > 0$ and $|\phi_M(k_M^*)| < 1$.

(c) When $\kappa^+ \in (\kappa_1,\kappa_2)$, $\theta(\kappa^+,\kappa^+) > 0$ and $\theta(\kappa^-,\kappa^+) < 0$ hold, and $X^*(\kappa^+)$ belongs to (κ^-,κ^+). If $k_M^* < X^*(\kappa^+)$ holds, then we obtain $|\phi_M(k_M^*)| > 1$, whereas if $k_M^* > X^*(\kappa^+)$ holds, we obtain $|\phi_M(k_M^*)| < 1$. A one-to-one correspondence exists between k_M^* and G, and k_M^* is monotonically increasing in G. Therefore, there uniquely exists the threshold value of G, $G_c(\kappa^+) \in (G^-(\kappa^+), G^+(\kappa^+))$, such that $k_M^* = X^*(\kappa^+)$. □

3.B Proof of Proposition 3.5

At first, we will prove Proposition 3.5(b). Let $\kappa^+ > 1$ and $G > G^+(\kappa^+)$ hold; thus, the fixed point is $k^*_{LE} \in (\kappa^+, \infty)$. The following equation holds for any k_{t-1}:

$$\frac{1}{\phi_{LE}(k_{t-1})} - \frac{1}{k^*_{LE}} = -\frac{\alpha^{-\frac{\alpha}{1-\alpha}} - 1}{G}\left(\frac{1}{k_{t-1}} - \frac{1}{k^*_{LE}}\right). \tag{3.24}$$

Therefore, for $k_{t-1} \in [\kappa^+, k^*_{LE})$

$$\frac{1}{\phi^2(k_{t-1})} - \frac{1}{k^*_{LE}} = \left(\frac{\alpha^{-\frac{\alpha}{1-\alpha}} - 1}{G}\right)^2 \left(\frac{1}{k_{t-1}} - \frac{1}{k^*_{LE}}\right) > 0. \tag{3.25}$$

That is, $\phi^2(k_{t-1}) < k_{t-1}$ holds because $G < \alpha^{-\frac{\alpha}{1-\alpha}} - 1$.

As for $k_{t-1} \in (\kappa^-, \kappa^+)$, using $k^*_{LE} < \phi_M(k_{t-1}) \leq \phi_{LE}(k_{t-1})$ and (3.24), we obtain

$$\begin{aligned}\frac{1}{\phi^2(k_{t-1})} - \frac{1}{k^*_{LE}} &= -\frac{\alpha^{-\frac{\alpha}{1-\alpha}} - 1}{G} \times \left(\frac{1}{\phi_M(k_{t-1})} - \frac{1}{k^*_{LE}}\right) \\ &\geq \left(\frac{\alpha^{-\frac{\alpha}{1-\alpha}} - 1}{G}\right)^2 \left(\frac{1}{k_{t-1}} - \frac{1}{k^*_{LE}}\right) > 0.\end{aligned} \tag{3.26}$$

Therefore, $\phi^2(k_{t-1}) < k_{t-1}$ holds. Using (3.25) and (3.26), $\phi^2(k_{t-1}) < k_{t-1}$ holds for any $k_{t-1} \in (\kappa^-, k^*_{LE})$. By iterating this process, we obtain a finite integer n such that $\phi^{2n}(k_{t-1}) \leq \kappa^-$ holds. From (3.24), the equilibrium path cannot continue to stay in (k^*_{LE}, ∞); therefore, Proposition 3.5(b) is justified. By using (3.25), Proposition 3.5(a) is justified. $\square$

Chapter 4

Endogenous Fluctuations with Procyclical R&D

4.1 Introduction

This chapter examines how R&D activity varies over the business cycle using the framework of the R&D-based growth models. In business cycles, when any economic quantity is positively correlated with the business condition of the economy, it is said to be procyclical. Countercyclical is the opposite of procyclical.[1] Most of the preceding literature on endogenous growth cycles have predicted the countercyclical allocation of resources to R&D. See for example Matsuyama (1999, 2001), Wälde (2002), Bental and Peled (1996), and Francois and Lloyd-Ellis (2003, 2008).[2]

The prediction that R&D expenditures are countercyclical is difficult to justify from empirical studies. Wälde and Woitek (2004) have studied the cyclical properties of R&D in G7 countries using annual data from 1973 to 2000. They found that aggregate R&D expenditures tended to be procyclical and argued that the prediction of Matsuyama (1999, 2001) was counterfactual. Fatás (2000) and Comin and Gertler (2006) also have found a highly procyclical tendency of R&D expenditures using U.S. data. In particular, Comin and Gertler (2006) focused on longer-term oscillations than conven-

[1] In this chapter, the growth rate of real GDP is used as a procyclical economic indicator.

[2] Francois and Lloyd-Ellis (2008) did not interpret the activity that was a source of productivity improvements as R&D, but as an "entrepreneurial search." However, its process was formally identical to the R&D process in the earlier models of Grossman and Helpman (1991b) and Francois and Lloyd-Ellis (2003).

tional business cycles. They termed these oscillations the "medium-term cycle" that includes frequencies between 6 months and 50 years. In this respect, there is a close relationship between their empirical study and our theoretical analysis. Geroski and Walters (1995) argued that their analysis of the U.K. data revealed that productivity improvements were also procyclical. Barlevy (2007), using data from both the National Science Foundation (NSF) and Standard & Poor's Compustat database of publicly traded companies, found a positive correlation between the growth rates of output and R&D expenditures at the industry level as well as the aggregate level.

The main purpose of this chapter is to include the procyclical behavior of R&D into the endogenous fluctuation model. We modify the variety-expanding model in Matsuyama (1999, 2001), introducing population growth and a negative externality that affects the productivity of R&D. We assume that finding new knowledge becomes more difficult as economies become technologically more advanced, as in the semi-endogenous growth model in Jones (1995a) and Segerstrom (1998).[3] This assumption has been first proposed to eliminate the scale effect, which is a serious counterfactual prediction in the first-generation R&D-based endogenous growth models, such that an economy with a large population grows faster.[4]

Relevant related literature includes Wälde (2005), Francois and Lloyd-Ellis (2009), Comin and Gertler (2006), and Barlevy (2007). Francois and Lloyd-Ellis (2009) have studied the endogenous business cycle model based on their previous work (Francois and Lloyd-Ellis, 2003). They decomposed the innovation process into three distinct stages: R&D, commercialization, and innovation. Their model illustrated the procyclical movement of R&D, and they determined that the countercyclical movement of commercialization played a central role in this new result. Furthermore, they showed that the total expenditure for innovation, defined as the sum of expenditures for R&D and commercialization, moved procyclically. Wälde (2005) also illustrated procyclical R&D behavior by using a quality-ladder framework with capital accumulation. The Francois and Lloyd-Ellis (2009) and Wälde (2005) models are similar to ours in that they assumed a negative externality of knowledge

[3] Jones (1995a) called such an externality the fishing-out effect.

[4] Jones (1995a) was also the study based on the variety-expanding model in Romer (1990). However, its balanced growth path (BGP) has a saddle property and no endogenous fluctuation occurs as proven by Arnold (2006). Note that in order to examine the dynamics analytically, Arnold (2006) assumed non-diminishing returns to labor in R&D, which was not assumed in Jones' original model.

accumulation and derive non-scale growth with endogenous fluctuations. On the other hand, Comin and Gertler (2006) and Barlevy (2007) have discussed the cyclicality of R&D over the business cycles that were caused by exogenous shocks. The former was based on a variety-expanding framework and used similar approach to Francois and Lloyd-Ellis (2009), i.e., decomposing the innovation process.[5] The latter, using a quality-ladder framework, showed that the equilibrium R&D was procyclical in a decentralized market. However, optimal R&D was found to be countercyclical by a central planner's problem.[6]

As the aforementioned studies illustrate, the theoretical explanation of the procyclicality of R&D is one of the most controversial topics in the studies of R&D and business cycles. This study achieves the procyclical R&D behavior under an assumption that is simpler than those of Francois and Lloyd-Ellis (2009) and Wälde (2005). In addition, it does not require exogenous shocks, unlike the Comin and Gertler (2006) and Barlevy (2007) models.

The rest of this chapter is organized as follows. Section 4.2 sets up the model used in our theoretical investigation and derives the law of motion that characterizes the equilibrium path of the economy. Section 4.3 examines the dynamic properties of the model and illustrates that the equilibrium path fluctuates endogenously. Section 4.4 focuses on period-2 cycles and studies the cyclicality of R&D investment. Section 5 studies the model with infinitely-lived agents to show the robustness of our results. Section 6 provides conclusions.

4.2 Model

Our model considers the dynamic model based on Matsuyama (1999). Time is discrete and indexed by $t = 0, 1, 2, \ldots$. There is a single final good taken as a numeraire that is produced using intermediate goods and labor. It can be consumed or invested. A new variety of intermediate goods is invented by allocating capital for R&D activities. Inventors enjoy a one-period monopoly by patent protection. The available intermediate goods are produced by multiple intermediate firms using capital. Finally, we assume two-period-lived overlapping generation (OLG) households, who inelastically supply labor when young.

[5] They introduced the stage of \adoption" instead of commercialization.

[6] For other recent work on procyclicality of R&D, see Nuno (2011).

4.2.1 Final goods

We assume that perfect competition prevails in the final goods market. The production function is given by

$$Y_t = AL_t^{1-\alpha} \int_0^{N_t} x_t(z)^\alpha dz, \quad 0 < \alpha < 1, \quad A > 0, \tag{4.1}$$

where Y_t is the final output, L_t is inelastically supplied labor, $x_t(z)$ is the amount of the intermediate good indexed by z, and $1/(1-\alpha)$ denotes the elasticity of substitution between all pairs of intermediate goods. N_t is the number of available intermediate goods in period t that represents the technology level of the economy.

Profit maximization yields $w_t = (1-\alpha)Y_t/L_t$ and the inverse demand function for each intermediate good z as

$$p_t(z) = \alpha AL_t^{1-\alpha} x_t(z)^{-(1-\alpha)}, \quad \text{for } z \in [0, N_t], \tag{4.2}$$

where w_t is the real wage rate and $p_t(z)$ is the price of the intermediate good z.

4.2.2 Intermediate goods

Each intermediate good is produced by using one unit of capital. Because of limited patent protection, the \old" intermediate goods, $[0, N_{t-1}]$, are supplied competitively. Hence, the price is equal to the marginal cost, $p_t(z) = r_t$, for $z \in [0, N_{t-1}]$, where r_t is the rental price of capital. However, the \new" intermediate goods invented in period $t-1$, $(N_{t-1}, N_t]$, are supplied monopolistically and sold at the monopoly price, $p_t(z) = r_t/\alpha$, for $z \in (N_{t-1}, N_t]$. All intermediate goods enter symmetrically into the production of the final good, i.e., $x_t(z) = x_{ct}$ for $z \in [0, N_{t-1}]$ and $x_t(z) = x_{mt}$ for $z \in (N_{t-1}, N_t]$. From (4.2), we can easily illustrate that $x_{mt} = \alpha^{\frac{1}{1-\alpha}} x_{ct}$ holds and the maximized monopoly profits are

$$\Pi_t(z) = \Pi_t \equiv \frac{1-\alpha}{\alpha} x_{mt} r_t, \quad \text{for } z \in (N_{t-1}, N_t]. \tag{4.3}$$

Considering these results of the profit maximization of the intermediate goods firms, we can rewrite the production function (4.1) as

$$Y_t = AL_t^{1-\alpha}(\alpha^{\frac{1}{1-\alpha}} x_{ct})^\alpha N_{t-1} \left[\frac{N_t}{N_{t-1}} - 1 + \alpha^{-\frac{\alpha}{1-\alpha}} \right]. \tag{4.4}$$

4.2.3 R&D

The number of intermediate goods N expands according to the following equation: [7]

$$N_t - N_{t-1} = \eta \frac{R_t}{N_{t-1}^{\phi}}, \quad N_0 > 0, \quad \phi > 0, \quad \eta > 0,$$

where R_t is the amount of the capital allocated to R&D. Following the formation adopted in Jones (1995a), we assume that the past discoveries make inventing a new machine more difficult. This external effect is captured by ϕ.

Each inventor enjoys a one-period monopoly and earns profits Π_t. Therefore, in equilibrium, the following free-entry condition must be satisfied:

$$\Pi_t \leq \eta^{-1} N_{t-1}^{\phi} r_t, \quad \text{with an equality whenever } N_t > N_{t-1}. \tag{4.5}$$

The breakeven point of x_{mt} is given by $\bar{x}_{mt} \equiv \frac{\alpha}{1-\alpha}\eta^{-1}N_{t-1}^{\phi}$. It becomes larger for a large value of ϕ, since R&D becomes costlier for any given N_{t-1} and L_t.

Finally, clearing the capital market requires

$$K_t = R_t + (N_t - N_{t-1})x_{mt} + N_{t-1}x_{ct}, \tag{4.6}$$

where K_t is the amount of capital accumulated in period $t-1$ and available in period t. The available capital is utilized by R&D, producing monopolistic intermediate goods, and producing competitive intermediate goods, as shown on the right-hand side of (4.6).

4.2.4 Consumers

Each consumer lives for two periods. When young, he/she supplies one unit of labor and earns wage w_t, which is divided into savings and consumption. When old, he/she only consumes using his/her savings. Let c_{1t} and c_{2t+1} denote the consumption in periods t and $t+1$, respectively, of the consumers born in period t. Each consumer chooses c_{1t} and c_{2t+1} that maximizes their utility, $U_t = (1-s)\log c_{1t} + s \log c_{2t+1}$, where s 2 $(0, 1]$, subject to the budget constraint, $c_{2t+1} = (w_t - c_{1t})r_{t+1}$.

[7] This specification is based on Rivera-Batiz and Romer's (1991) "lab equipment model."

The solution to this simple maximization problem is characterized by the following linear saving function:

$$K_{t+1} = sw_tL_t = s(1-\alpha)Y_t, \tag{4.7}$$

where L_t represents the number of consumers born in period t, which increases at the exogenous rate n, i.e., $L_t = (1+n)L_{t-1}$.

4.2.5 Equilibrium

Substituting (4.3), (4.5), and $x_{ct}/x_{mt} = \alpha^{-\frac{1}{1-\alpha}}$ into (4.6) yields:

$$\frac{N_t - N_{t-1}}{N_{t-1}} = \mu(k_{t-1}) \equiv \max\{0, \alpha^{-\frac{\alpha}{1-\alpha}}(k_{t-1} - 1)\}, \tag{4.8}$$

$$x_{ct} = \max\left\{\frac{K_t}{N_{t-1}}, \alpha^{-\frac{1}{1-\alpha}}\bar{x}_{mt}\right\}, \tag{4.9}$$

where we define $k_{t-1} \equiv \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\eta K_t/N_{t-1}^{1+\phi}$. If $k_{t-1} > 1$ holds, i.e., the economy has a sufficient stock of capital relative to its technological level, the positive amount of capital is allocated for R&D and $N_t > N_{t-1}$ holds. In contrast, if $k_{t-1} \leq 1$, neither R&D occurs nor technological progress arises.

Substituting (4.8) and (4.9) into (4.4) illustrates that the total output is equal to

$$Y_t = \frac{AL_t^{1-\alpha}N_{t-1}^{1+\alpha\phi}}{[\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\eta]^{\alpha}}\psi(k_{t-1}), \tag{4.10}$$

where

$$\psi(k_{t-1}) \equiv \begin{cases} k_{t-1}^{\alpha}, & \text{if } k_{t-1} \leq 1, \\ k_{t-1}, & \text{if } k_{t-1} > 1. \end{cases}$$

To describe the equilibrium path of this economy, we define the new variable

$$\ell_{t-1} \equiv [s(1-\alpha)A]^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\eta\frac{L_t}{N_{t-1}^{\phi}}.$$

Summarizing (4.7), (4.8), and (4.10) yields the following two-dimensional dynamical system:

$$
\begin{aligned}
\ell_t = f^{\ell}(k_{t-1}, \ell_{t-1}) &= \begin{cases} (1+n)\ell_{t-1}, & \text{if } k_{t-1} \leq 1, \\ \dfrac{(1+n)\ell_{t-1}}{[1+\alpha^{-\frac{\alpha}{1-\alpha}}(k_{t-1}-1)]^{\phi}}, & \text{if } k_{t-1} > 1, \end{cases} \\
k_t = f^{k}(k_{t-1}, \ell_{t-1}) &= \begin{cases} \ell_{t-1}^{1-\alpha} k_{t-1}^{\alpha}, & \text{if } k_{t-1} \leq 1, \\ \dfrac{\ell_{t-1}^{1-\alpha} k_{t-1}}{[1+\alpha^{-\frac{\alpha}{1-\alpha}}(k_{t-1}-1)]^{1+\phi}}, & \text{if } k_{t-1} > 1. \end{cases}
\end{aligned} \tag{4.11}
$$

If the initial values of k_0 and ℓ_0 are given, the law of motion, (4.11), characterizes the equilibrium path $\{k_t, \ell_t\}_0^1$, whose properties depend on parameter values, α, n, and ϕ.

4.3 Dynamics

The law of motion, (4.11), has a unique positive fixed point (k^*, ℓ^*), where

$$
k^* = 1 + \alpha^{\frac{\alpha}{1-\alpha}}[(1+n)^{\frac{1}{\phi}} - 1] > 1, \quad \ell^* = (1+n)^{\frac{1+\phi}{\phi(1-\alpha)}}. \tag{4.12}
$$

In the long run, $k \leq 1$ is unsustainable by an exogenous population growth. Therefore, (k^*, ℓ^*) is a unique non-trivial fixed point of the dynamical system (4.11).[8] At this fixed point, K, Y (or per capita output $y \equiv Y_t/(L_{t-1} + L_t)$), and N grow at constant rates. That is, the economy achieves balanced growth. The balanced growth rate of per capita output is derived as $g_y^* = (1+n)^{\frac{1}{\phi}} - 1$, which is independent of population L.

4.3.1 Stability

The two-dimensional system, (4.11), has two predetermined variables, k and ℓ. If the fixed point is a sink, it is locally stable.

Proposition 4.1 There is a unique bifurcation point of ϕ, ϕ_b, that satisfies $B(\phi_b) - \Lambda(\phi_b) = 0$. The fixed point (k^*, ℓ^*) is a sink for $\phi < \phi_b$, whereas it is a saddle point for $\phi > \phi_b$, where $B(\phi)$ and $\Lambda(\phi)$ are defined as follows:

$$
B(\phi) \equiv \frac{2 - \phi(1+\alpha)}{2 + \phi(1+\alpha)}, \quad \Lambda(\phi) \equiv \frac{\alpha^{-\frac{\alpha}{1-\alpha}} - 1}{(1+n)^{\frac{1}{\phi}}}.
$$

[8] Substituting $\ell_t = \ell_{t-1} > 0$ into $\ell_t = f^{\ell}(k_{t-1}, \ell_{t-1})$, and solving for k_{t-1}, we obtain k^* uniquely. ℓ^* uniquely exists by corresponding with a unique k^*.

Furthermore, in the sufficiently small neighborhood of (k^*, ℓ^*), the system (4.11) has a periodic orbit of period-2 on one side of the bifurcation point ϕ_b.

proof. See Appendix 4.A. □

Proposition 4.1 argues that the unique fixed point loses its stability for sufficiently large ϕ. Moreover, the flip bifurcation (period-doubling bifurcation) occurs by slightly changing a bifurcation parameter ϕ. If this bifurcation is supercritical, there are stable period-2 cycles for $\phi > \phi_b$ in the neighborhood of ϕ_b. Conversely, if the bifurcation is subcritical, period-2 cycles with a saddle property exist for $\phi < \phi_b$.

4.4 Period-2 cycles

The existence of the period-2 cycles is verified by the following method.

Proposition 4.2 If and only if $\phi \geq \bar{\phi}_1$, the system, (4.11), has a pair of the fixed points of period-2, (k^H, ℓ^H) and (k^L, ℓ^L), such that $k^L \leq 1 < k^H$, where $\bar{\phi}_1$ is defined as follows:

$$\bar{\phi}_1 \equiv \frac{\log(1+n)}{\log \chi},$$

where

$$\chi \equiv \frac{(1+n)^{\frac{1+\alpha}{2}} + \sqrt{(1+n)^{1+\alpha} - 4\alpha^{\frac{\alpha}{1-\alpha}}\left(1-\alpha^{\frac{\alpha}{1-\alpha}}\right)}}{2\alpha^{\frac{\alpha}{1-\alpha}}}.$$

proof. See Appendix 4.B. □

When the parameters satisfy the conditions of Proposition 4.2, the system, (4.11), has the period-2 cycles moving back and forth between the two phases, as shown in Deneckere and Judd (1992) and Matsuyama (1999). In one phase, capital is allocated to R&D and new intermediate goods are invented. In the other phase, all capital is allocated to the intermediate goods sector and no invention occurs. We shall refer to each phase as the R&D phase and the no R&D phase, respectively. The average growth rates of per capita output over the cycles are given by $g_y^{cycle} = (1+n)^{\frac{1}{\phi}} - 1$, which is equal to the growth rate along its balanced growth path (BGP), g^*.

Our main purpose is to clarify whether R&D investment is procyclical or countercyclical over business cycles. Let g_X^1 denote the growth rate of the variable X in the R&D phase. Similarly, g_X^0 denotes the growth rate of X in the no R&D phase.

Proposition 4.3 (a) If $\alpha > 1/2$, a threshold $\bar{\phi}_2$ exists such that $g_y^1 > g_y^0$ holds for $\phi \in [\bar{\phi}_1, \bar{\phi}_2)$, while $g_y^1 < g_y^0$ holds for $\phi > \bar{\phi}_2$ where

$$\bar{\phi}_2 \equiv \frac{\log(1+n)}{\log(1-\alpha^{\frac{\alpha}{1-\alpha}}) - \log \alpha^{\frac{\alpha}{1-\alpha}}} > \bar{\phi}_1.$$

(b) If $\alpha \leq 1/2$, $g_y^1 > g_y^0$ holds for any $\phi \geq \bar{\phi}_1$.

proof. See Appendix 4.C. □

$g_y^1 > g_y^0$ means that the R&D phase achieves faster growth than the no R&D phase; that is, R&D investment is procyclical. In contrast, when $g_y^1 < g_y^0$, R&D investment is high with low growth.

The results of Propositions 4.2 and 4.3 appear in Figure 4.1 with α on the horizontal axis, ϕ on the vertical axis, and two downward curves. The region above the graph of $\bar{\phi}_1$ corresponds to the set of α and ϕ for which period-2 cycle, described in Proposition 4.2, exist. Furthermore, that region is separated into two regions by the graph of $\bar{\phi}_2$. The lower region corresponds to the set of parameters associated with procyclical R&D investment, while the upper region corresponds to the set of parameters with the countercyclical R&D investment. We can see that when $\alpha \leq 1/2$, in which case the elasticity of substitution between each intermediate good is low or the markup of the monopoly price is high, there does not exist a value of ϕ that causes the countercyclical R&D investment. Whereas, when $\alpha > 1/2$, the sign of the inequality between g_y^1 and g_y^0 may change depending on the values of ϕ.

4.4.1 Examples

We assume that the parameter values are $\alpha = \{1/3, 0.9\}$ and $n = (1.012)^{10} - 1 \simeq 0.1267$. The rate of population growth chosen means a population growth rate of $1.2\%/year$ and patent length of 10 years.

Example 4.1 If $\alpha = 0.9$, $n = 0.1267$, and $\phi = 0.27$, the fixed point (k^*, ℓ^*) is a saddle point, and the fixed points of period-2 exist. The growth rates

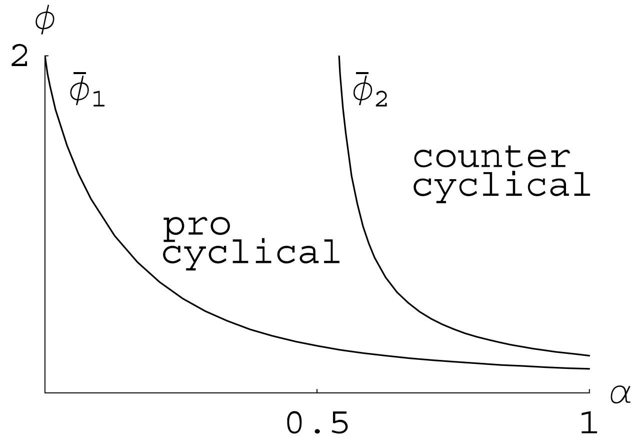


Figure 4.1: The cyclicality of R&D.

of each phase are $(g_y^1, g_y^0) = (0.5552, 0.5558)$. Therefore, R&D investment is countercyclical.

Example 4.2 If $\alpha = 0.9$, $n = 0.1267$, and $\phi = 0.25$, the fixed point (k^*, ℓ^*) is a saddle point and the fixed points of period-2 exist. The growth rates of each phase are $(g_y^1, g_y^0) = (0.6118, 0.6111)$. Therefore, R&D investment is procyclical.

The value of the parameter, $\alpha = 0.9$, corresponds to the markup of the monopoly price, $1/\alpha \simeq 1.1111$. For example, Rotemberg and Woodford (1995) estimated as 1.115. $\alpha = 0.9$ is consistent with their estimation. The threshold values of ϕ are $\phi_b = 0.1572$, $\bar{\phi}_1 = 0.1007$, and $\bar{\phi}_2 = 0.2604$.

Example 4.3 If $\alpha = 1/3$, $n = 0.1267$, and $\phi = 0.61$, the fixed point (k^*, ℓ^*) is a saddle point and the fixed points of period-2 exist. The growth rates of each phase are $(g_y^1, g_y^0) = (0.2458, 0.1869)$. Therefore, R&D investment is procyclical.

In our model, α equals the share of capital, conventionally considered to be around $1/3$. For $\alpha = 1/3$, $\phi_b = 0.4298$ and $\bar{\phi}_1 = 0.4280$, R&D moves procyclically regardless of the value of ϕ.

4.5 Model with infinitely lived agents

In this section, we consider an infinitely-lived agent economy instead of an OLG framework, and show the robustness of our results. Using an OLG framework is unsuitable in the temporally patent model as in Matsuyama (1999) or Aloi and Lasselle (2007), because the one period of the discrete time has two distinct interpretations: patent length and half of a lifetime.[9] There is no reason for these two interpretations to be identical.

4.5.1 Model

We assume the same structure as in Section 4.2, except for households. Therefore, (4.1) through (4.6) and (4.8) through (4.10) hold, where L_t is the number of infinitely-lived households who each supply one unit of labor inelastically and grow at n. The other parameters and variables are defined in Section 4.2.

As for consumers or households, assuming infinitely-lived agents implies that the optimal consumption path is characterized by an Euler equation, instead of the savings function (4.7). Each household chooses a consumption path that maximizes their discounted utility, $\sum_{t=0}^{\infty} \beta^t \log \tilde{c}_t$, subject to the budget constraint, $\tilde{k}_{t+1} = w_t + r_t \tilde{k}_t - \tilde{c}_t - n\tilde{k}_{t+1}$, where $\beta \in (0, 1)$ is the discount factor. $\tilde{c}_t = C_t/L_t$ and $\tilde{k}_t = K_t/L_t$ are per capita consumption and stock of capital, respectively. The final goods market clears when

$$Y_t = K_{t+1} + C_t. \tag{4.13}$$

The solution to this simple maximization problem is characterized by an Euler equation and a transversality condition as follows:

$$\frac{\tilde{c}_t}{\tilde{c}_{t-1}} = \frac{\beta r_t}{1+n}, \tag{4.14}$$

$$\lim_{T \to \infty} \beta^T \frac{\tilde{k}_{T+1}}{\tilde{c}_T} = 0. \tag{4.15}$$

[9] Matsuyama (1999) did not present utility maximization explicitly; however, his savings function can be derived from the conventional OLG assumptions. Matsuyama (2001) studied an infinitely-lived agent economy.

4.5.2 Equilibrium

We define the following new variables:

$$\hat{\ell}_{t-1} \equiv A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} (1-\alpha)\eta \frac{L_t}{N_{t-1}^{\phi}}, \quad c_{t-1} \equiv \alpha^{\frac{\alpha}{1-\alpha}} (1-\alpha)\eta \frac{C_{t-1}}{N_{t-1}^{1+\phi}}.$$

In equilibrium, $Y_t = r_t K_t + w_t L_t$ holds. Therefore the rate of return on capital is

$$r_t = \alpha \frac{Y_t}{K_t} = \alpha \hat{\ell}_{t-1}^{1-\alpha} \psi(k_{t-1}) k_{t-1}^{-1}. \tag{4.16}$$

Summarizing (4.8), (4.10), (4.13), (4.14), and (4.16) provides the three-dimensional dynamical system as follows:

$$\begin{aligned}
k_t &= f^k(k_{t-1}, \hat{\ell}_{t-1}, c_{t-1}) \equiv \begin{cases} \hat{\ell}_{t-1}^{1-\alpha} k_{t-1}^{\alpha} \left[1 - \alpha\beta \frac{c_{t-1}}{k_{t-1}}\right], & \text{for } k_{t-1} \leq 1, \\ \frac{\hat{\ell}_{t-1}^{1-\alpha} [k_{t-1} - \alpha\beta c_{t-1}]}{[1+\alpha^{-\frac{\alpha}{1-\alpha}} (k_{t-1}-1)]^{1+\phi}}, & \text{for } k_{t-1} > 1, \end{cases} \\
\hat{\ell}_t &= f^{\ell}(k_{t-1}, \hat{\ell}_{t-1}), \\
c_t &= f^c(k_{t-1}, \hat{\ell}_{t-1}, c_{t-1}) \equiv \begin{cases} \alpha\beta \hat{\ell}_{t-1}^{1-\alpha} k_{t-1}^{-(1-\alpha)} c_{t-1}, & \text{for } k_{t-1} \leq 1, \\ \frac{\alpha\beta \hat{\ell}_{t-1}^{1-\alpha} c_{t-1}}{[1+\alpha^{-\frac{\alpha}{1-\alpha}} (k_{t-1}-1)]^{1+\phi}}, & \text{for } k_{t-1} > 1, \end{cases}
\end{aligned} \tag{4.17}$$

where $f^{\ell}(\cdot,\cdot)$ was defined in (4.11). If the initial value of $(k_0, \hat{\ell}_0)$ is given, the equilibrium path $\{k_t, \hat{\ell}_t, c_t\}_0^{\infty}$ is characterized by the law of motion (4.17) and the transversality condition (4.15).

4.5.3 Dynamics

The law of motion (4.17) has a unique positive fixed point, $(k^*, \hat{\ell}^*, c^*)$, where

$$\hat{\ell}^* = \left[\frac{(1+n)^{\frac{1+\phi}{\phi}}}{\alpha\beta}\right]^{\frac{1}{1-\alpha}}, \quad c^* = k^* \left(\frac{1-\alpha\beta}{\alpha\beta}\right).$$

At this fixed point, the economy achieves balanced growth. Moreover, since $\tilde{k}$ and $\tilde{c}$ grow at the same rate along the BGP, the transversality condition (4.15) is satisfied as long as $\beta < 1$.

The three-dimensional system (4.17) has two predetermined variables, k and $\hat{\ell}$, and one non-predetermined variable, c. The local saddle path stability requires a two-dimensional locally stable manifold. We can verify the following proposition through a local stability analysis.

Proposition 4.4 There is a unique threshold of ϕ, ϕ_b, which satisfies $B(\phi_b) - \Lambda(\phi_b) = 0$. If $\phi > \phi_b$, the fixed point $(k^*, \hat{\ell}^*, c^*)$ is locally unstable.

proof. See Appendix 4.D. □

According to the proposition, for a sufficiently large value of ϕ, only the one-dimensional locally stable manifold exists. Therefore, the economy that begins close to the fixed point will move away from it. Since a trajectory cannot approach the unique fixed point asymptotically, the equilibrium dynamics of the economy exhibit endogenous fluctuations for almost all initial conditions.

4.5.4 Period-2 cycles

With respect to the existence of the period-2 cycles, we can show a similar result with Proposition 4.2, i.e., the three-dimensional dynamical system (4.17) has the period-2 cycles fluctuating between the R&D regime and the no R&D regime; this exists for $\phi > \bar{\phi}_1$. Such periodic orbits satisfy the transversality condition. As for the cyclical properties of R&D, it is possible to illustrate an identical result, as shown in Proposition 4.3 and Figure 4.1. The discount factor β does not affect these results.

4.6 Conclusions

This chapter has examined the cyclicality of the R&D investment over the business cycles by using the variety-expanding model with limited patent protection. We have illustrated that the unique fixed point loses its stability. In addition, period-2 cycles moving back and forth exist between the R&D and the no R&D phases. Moreover, we examined the possibility and conditions that R&D investment is procyclical over the period-2 cycles. The preceding literature on endogenous growth cycles, such as Matsuyama (1999, 2001), predicted the countercyclical allocation of resources to R&D. However, empirical evidence does not support these predictions. We have proven the existence of the parameter set that achieves procyclical R&D, shown in many empirical studies. In our model, countercyclical R&D requires a large capital share and a sufficiently strong external effect. In other cases, R&D investment is procyclical.

We assume exogenous population growth and the negative externality of the stock of knowledge that works in R&D, following the formation of the semi-endogenous growth model. [10] In our model, the parameter of this externality plays the central role in the decision of the cyclicality of R&D.

Appendix

4.A Proof of Proposition 4.1

In order to examine local stability, we linearize the system (4.11) around the fixed point (k^*, ℓ^*).

$$\begin{bmatrix} k_t - k^* \\ \ell_t - \ell^* \end{bmatrix} = \mathbf{J} \begin{bmatrix} k_{t-1} - k^* \\ \ell_{t-1} - \ell^* \end{bmatrix}, \quad \text{where} \quad \mathbf{J} \equiv \begin{bmatrix} f_1^{k*} & f_2^{k*} \\ f_1^{\ell *} & f_2^{\ell *} \end{bmatrix}.$$

A stability type of the fixed point depends on the trace ($\mathrm{tr}\,\mathbf{J}$) and the determinant ($\det \mathbf{J}$) of the Jacobian matrix. We define seven regions separated by three lines, $\det \mathbf{J} = \mathrm{tr}\,\mathbf{J} - 1$, $\det = -\mathrm{tr}\,\mathbf{J} - 1$, and $\det \mathbf{J} = 1$, as shown in Figure 4.2.[11] We also know that if the Jacobian was somehow to move from inside the triangle with sink stability to outside, a bifurcation would occur.

$\det \mathbf{J}$ and $\mathrm{tr}\,\mathbf{J}$ are derived as

$$\begin{aligned} \det \mathbf{J} &= -\alpha\phi - (1+\alpha\phi)\frac{\alpha^{-\frac{\alpha}{1-\alpha}} - 1}{(1+n)^{\frac{1}{\phi}}}, \\ \mathrm{tr}\,\mathbf{J} &= -\phi + 1 - (1+\phi)\frac{\alpha^{-\frac{\alpha}{1-\alpha}} - 1}{(1+n)^{\frac{1}{\phi}}}. \end{aligned} \tag{4.18}$$

It is clear that $\det \mathbf{J} < 1$ and $\det \mathbf{J} > \mathrm{tr}\,\mathbf{J} - 1$, that is, the pair of $\det \mathbf{J}$ and $\mathrm{tr}\,\mathbf{J}$ does not belong to the shaded region in Figure 4.2.[12] In addition, from

[10] As a result, even if the economy grows along the fluctuating equilibrium path, the long-run growth is not endogenous and requires positive population growth, as shown in the literature using a similar assumption such as Jones (1995a) and Segerstrom (1998). Young (1998), Peretto (1998), and Howitt (1999), indicated this problem and proposed models that have non-scale endogenous growth. A survey of this issue is presented in Jones (1999, 2005). Li (2000, 2002) argued that the predictions of these models depend on the knife-edge assumption and that the semi-endogenous growth prediction is more general.

[11] See Azariadis (1993, Ch.6) for further details.

[12] Therefore, the possibilities of the saddle-node bifurcation and the Hopf bifurcation can be ruled out.

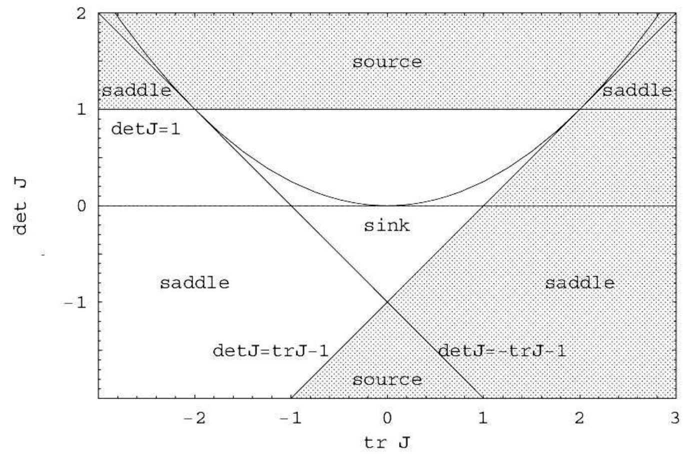


Figure 4.2: Local stability on the plane.

(4.18), we obtain the following relationship:

$$\Lambda(\phi) \text{ Q } B(\phi) \quad , \qquad \det \mathrm{J} \text{ R } -\mathrm{tr}\mathrm{J} - 1.$$

□

4.B Proof of Proposition 4.2

Solving $\ell^H = f^\ell(f^k(k^H, \ell^H), f^\ell(k^H, \ell^H))$ for k^H, we obtain

$$k^H \equiv 1 + \alpha^{\frac{\alpha}{1-\alpha}}[(1+n)^{\frac{2}{\phi}} - 1] > 1. \tag{4.19}$$

Furthermore, k^L, ℓ^L, and ℓ^H are

$$k^L \equiv [k^H(1+n)^{\frac{1-\alpha}{2} - \frac{1+\phi}{\phi}}]^{\frac{2}{1+\alpha}}, \tag{4.20}$$
$$\ell^L \equiv (k^H)^{\frac{1}{1+\alpha}}(1+n)^{\frac{2\alpha}{1-\alpha^2}\left(\frac{1+\phi}{\phi} - \frac{1-\alpha}{2}\right)},$$

and $\ell^H = (1+n)\ell^L$. Certain algebra shows that $k^L \leq 1$ is satisfied if and only if $(1+n)^{\frac{1}{\phi}} \leq \chi$ or $\phi \geq \bar{\phi}_1$.[13] □

[13]Since $\alpha^{\frac{\alpha}{1-\alpha}}(1 - \alpha^{\frac{\alpha}{1-\alpha}}) \leq 0.25$, χ and $\bar{\phi}_1$ are real numbers.

4.C Proof of Proposition 4.3

The growth rate of per capita output is given by $g_{yt} = \frac{k_t[1+\mu(k_{t-1})]^{1+\phi}}{k_{t-1}(1+n)} - 1$. Using (4.19) and (4.20), we obtain $g_y^1 = (k^H)^{\frac{1-\alpha}{1+\alpha}}(1+n)^{\frac{2\alpha}{(1+\alpha)\phi}} - 1$ and $g_y^0 = (k^H)^{-\frac{1-\alpha}{1+\alpha}}(1+n)^{\frac{2}{(1+\alpha)\phi}} - 1$. It is possible to show that $g_y^1 > g_y^0$ holds if and only if $(1+n)^{\frac{1}{\phi}} < \alpha^{-\frac{\alpha}{1-\alpha}} - 1$. For $\alpha < 0.5$, $\alpha^{-\frac{\alpha}{1-\alpha}} - 1 < 1$ holds(Case(b)). Therefore, no positive pair of ϕ and n exists such that $(1+n)^{\frac{1}{\phi}} < \alpha^{-\frac{\alpha}{1-\alpha}} - 1$ is satisfied. On the other hand, when $\alpha > 1/2$, $\alpha^{-\frac{\alpha}{1-\alpha}} - 1$ is larger than 1 (Case(a)). In this case, $(1+n)^{\frac{1}{\phi}} < \alpha^{-\frac{\alpha}{1-\alpha}} - 1$ holds for $\phi > \bar{\phi}_2$. Since $\chi \geq \alpha^{-\frac{\alpha}{1-\alpha}} - 1$, $\bar{\phi}_2 \geq \bar{\phi}_1$ holds. □

4.D Proof of Proposition 4.4

We linearize the system (4.17) around the fixed point $(k^*, \hat{\ell}^*, c^*)$.

$$\begin{bmatrix} k_t - k^* \\ \hat{\ell}_t - \hat{\ell}^* \\ c_t - c^* \end{bmatrix} = \hat{\mathbf{J}} \begin{bmatrix} k_{t-1} - k^* \\ \hat{\ell}_{t-1} - \hat{\ell}^* \\ c_{t-1} - c^* \end{bmatrix}, \quad \text{where} \quad \hat{\mathbf{J}} = \begin{bmatrix} \hat{f}_1^{k*} & \hat{f}_2^{k*} & \hat{f}_3^{k*} \\ f_1^{\ell*} & f_2^{\ell*} & 0 \\ f_1^{c*} & f_2^{c*} & f_3^{c*} \end{bmatrix}.$$

It is easily shown that $f_2^{\ell*} = f_3^{c*} = 1$ and $\hat{f}_3^{k*} = -1$. Therefore, the eigenvalues of the Jacobian matrix, $\hat{\mathbf{J}}$, denoted as λ, are obtained by solving the following characteristic equation:

$$\begin{aligned} P(\lambda) \equiv |\hat{\mathbf{J}} - \lambda \mathbf{I}| = &-\lambda^3 + (\hat{f}_1^{k*} + 2)\lambda^2 + (-f_1^{c*} + f_1^{\ell*}\hat{f}_2^{k*} - 2\hat{f}_1^{k*} - 1)\lambda \\ &+ (\hat{f}_1^{k*} - f_1^{\ell*} f_2^{c*} + f_1^{c*} - f_1^{\ell*}\hat{f}_2^{k*}) = 0. \end{aligned} \tag{4.21}$$

Here, $\hat{f}_1^{k*}$, $\hat{f}_3^{k*} f_1^{c*}$, $f_1^{\ell*}\hat{f}_2^{k*}$, and $f_1^{\ell*} f_2^{c*}$ are

$$\begin{aligned} \hat{f}_1^{k*} &= \frac{1}{\alpha\beta} - (1+\phi)(\Lambda(\phi) + 1), \\ f_1^{c*} &= -(1+\phi)\left(\frac{1}{\alpha\beta} - 1\right)(\Lambda(\phi) + 1), \\ f_1^{\ell*}\hat{f}_2^{k*} &= -\phi(1-\alpha)(\Lambda(\phi) + 1), \\ f_1^{\ell*} f_2^{c*} &= -\phi(1-\alpha)\left(\frac{1}{\alpha\beta} - 1\right)(\Lambda(\phi) + 1). \end{aligned}$$

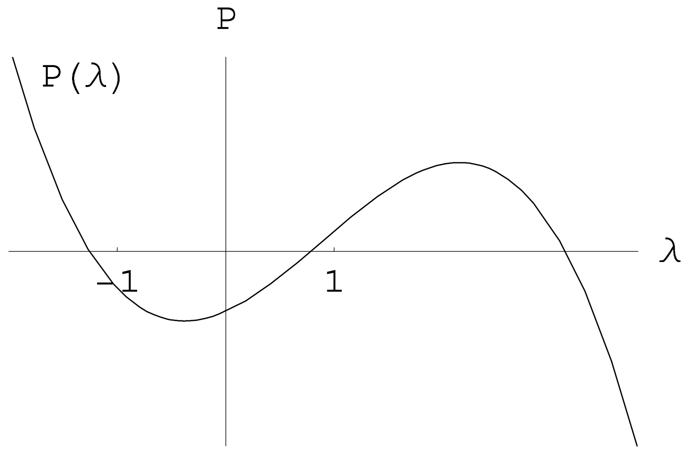


Figure 4.3: Characteristic equation for $\phi > \phi_b$.

From $\lim_{\lambda \to 1} P(\lambda) = -\infty$ and $P(1) = -f_1^{\ell *} f_2^{c*} > 0$, there is at least one real root that is larger than 1. On the other hand, $P(-1)$ is given by

$$P(-1) = \frac{4(\alpha\beta + 1)}{\alpha\beta} - \frac{2(\alpha\beta + 1) + [\alpha^2\beta + \alpha(1+\beta) + 1]\phi}{\alpha\beta}(\Lambda(\phi) + 1),$$

then, $P(-1) = 0$ requires that the parameters satisfy

$$B(\phi) - \Lambda(\phi) = 0. \tag{4.22}$$

$P(-1)$ is monotonically decreasing in ϕ, and $\lim_{\phi \to 0} P(-1) = \frac{2(1+\alpha\beta)}{\alpha\beta} > 0$ and $\lim_{\phi \to 1} P(-1) = -\infty$. Therefore, there exists a unique value of ϕ, ϕ_b, that satisfies (4.22). When $\phi > \phi_b$, $P(-1) < 0$ and $\lim_{\lambda\to-\infty} P(\lambda) = 1$ hold. As such, (4.21) has at least one root that belongs to $(-1, -\infty)$, as shown in Figure 4.3. Similarly, from $P(1) > 1$ and $P(-1) < 0$, (4.21) has a root in $(1, -1)$.

Summarizing these results, we show that the Jacobian matrix, $\hat{\mathbf{J}}$, has the three real eigenvalues, $\lambda_0 > 1$, $\lambda_1 \in (1, -1)$, and $\lambda_2 < -1$ for $\phi > \phi_b$. There is only one eigenvalue in a unit circle. Therefore, the fixed point $(k^*, \hat{\ell}^*, c^*)$ is unstable.

□

Chapter 5

R&D-based Growth Model with Nominal Wage Stickiness

5.1 Introduction

Macroeconomists discuss the long-run and short-run theories separately. The foundation of the former is the optimal growth theory[1] or the endogenous growth theory, which analyzes the supply side of the economy. The central underpinning of the latter is the new Keynesian theory, in which prices or nominal wages are supposedly sticky and the price adjustment process is analyzed.[2]

It is possible that such a divided framework is justified by the natural rate hypothesis.[3] The conventional wisdom among macroeconomists is that the natural rate hypothesis is valid. However, if price stickiness remains during the steady state of the short-run model, money is not superneutral in the long run and the natural rate hypothesis loses its validity.[4] In this situation,

[1] See Ramsey (1928), Cass (1965), and Koopmans (1965).

[2] For details on the new Keynesian theory, see Woodford (2003) and Gali (2008).

[3] For the natural rate hypothesis, see Friedman (1968) and Lucas (1972).

[4] Akerlof, Dickens, and Perry (2000, 1996) and Inoue, Shinagawa, and Tsuzuki (2011) proposed a long-run Phillips curve that is vertical for comparatively high inflation rates and downward sloping for lesser inflation rates. That is, their long-run Phillips curve is downward sloping in the low inflationary and deflationary economy as Japan in the 1990s-2000s. This study focuses on such a situation. For other empirical evidence that justifies the downward slope of the long-run Phillips curve, see Graham and Snower (2008, Sec. 1).

price stickiness must be considered in the long-run model.

In view of the above, this chapter proposes a new long-run model involving a price adjustment process by introducing nominal wage stickiness into an R&D-based growth model. Note that we derive the new Keynesian Phillips curve (NKPC), under which the natural rate hypothesis does not hold.[5]

Inoue and Tsuzuki (2011) and Tsuzuki and Inoue (2010) proposed the Dynamic General Equilibrium (DGE) model with the NKPC and technological change. In their model, the natural rate hypothesis did not hold, and the long-run output gap existed when the money growth rate was lower than that of technological change.[6] However, their analyses assumed exogenous technological change, as did the Solow model.[7]

This study provides the new Keynesian DGE model on the basis of Inoue and Tsuzuki (2011) with endogenous technological change, rather than exogenous growth, by introducing explicit R&D activities.[8] That is, in this study, the new Keynesian theory that represents the short-run theory is integrated with the endogenous growth theory that represents the long-run theory. Using such a model, we examine how money growth affects long-run output, employment, and economic growth along the balanced growth path.

We focus on the steady-state economic growth and employment. For sufficiently high money growth rates, there is a unique balanced growth path, and the economy exhibits sustained growth based on sustained R&D. Faster money growth causes greater employment and faster economic growth along the balanced growth path. Furthermore, under some parameter restrictions, there is no balanced growth path for low money growth rates, and the economy is trapped in a steady state without long-run growth. These results

[5]That is, the long-run Phillips curve derived from our NKPC is downward sloping as the traditional Keynesian's Phillips curve. On the contrary, the other type of NKPC, which inherits the property of Friedman's expectations-augmented Phillips curve, is conceivable. Under such a NKPC, the long-run Phillips curve is vertical at the natural rate of unemployment, i.e., the natural rate hypothesis holds. Also see footnote 15.

[6]Some studies such as Christiano, Motto, and Rostagno (2003) also proposed new Keynesian models that introduced an exogenous technological trend. However, they did not analyze the long-run output gap.

[7]See Solow (1956).

[8]Annicchiarico, Pelloni, and Rossi (2011), Kühn (2010), Rannenberg (2009), and Vaona (2012) have proposed a new Keynesian model in which sustained growth becomes endogenous through learning-by-doing or simple externality. Tsuzuki and Inoue (2011) have also proposed a new Keynesian endogenous growth model introducing human capital accumulations, as in Lucas (1988).

suggest that money growth may be an important factor for long-run economic growth. That is, financial authorities are required to maintain high money growth rates to achieve sustained and faster economic growth. In Inoue and Tsuzuki's (2011) model, the long-run growth rate was determined by the exogenous rate of technological change and was not affected by a monetary policy. Therefore, these results with respect to long-run growth are newly obtained from our study.

Most of the preceding theoretical studies on money and endogenous growth have concluded that a higher money growth is associated with a lower rate of long-run growth, which is contrary to the conclusion of this study. See for example Marquis and Reffett (1995, 1991), Jones and Manuelli (1995), Pecorino (1995), and Mino (1997).[9] Some authors argued that inflation has a negative impact on economic growth (Fischer, 1993, Barro, 1995, 1996). However, Levine and Zervos (1993) and Ericsson, Irons, and Tryon (2001) pointed out that the negative correlation between inflation and growth is not robust. Bruno and Easterly (1998) concluded that growth and inflation are negatively related only in the extremely high inflationary economy.

A number of empirical studies showed positive relationships between inflation (or money growth) and economic growth for advanced countries. McCandless and Weber (1995) reported a positive correlation between real growth and money growth (M0, M1, and M2) for a subsample of OECD countries using the data for the period 1960 { 1990. Aleskerov and Alper (2000) showed a positive and statistically significant correlation between money (M1 and M2) and real GDP growth rates as well as between CPI and real GDP growth rates for the OECD countries using the data for the more recent period 1960–1996. They also found a positive and significant correlation between the growth rates of M1 and real GDP for countries with inflation rates no greater than 15%.[10]

Pollin and Zhu (2006) reported that the effects of inflation on economic growth are positive and significant when the inflation rate is below its threshold level of 15-18% using the panel data for 80 countries over the period 1961-2000. Kremer, Bick, and Nautz (2013) found that inflation is positively

[9] Mino and Shibata (2000, 1995) have demonstrated the positive relationship between a monetary expansion and long-run growth using *the infinitely lived overlapping-generation* models.

[10] Furthermore, applying cluster analysis, they found positive correlations of money growth and inflation with growth rates of real GDP for some clusters including G7 countries (except for Italy).

and significantly correlated with economic growth in industrial countries if the inflation rate is less than 2.53%.[11] Lee and Wong (2005) used a threshold regression model to investigate a relationship between inflation and economic growth for Japan using the data for the period 1970-2001. They showed that inflation promotes economic growth as long as inflation rate falls within the range between 2.52% and 9.66%. Our study provides a theoretical explanation for these empirical results.

The remainder of this chapter is organized as follows. Section 2 sets up the model used in our theoretical investigation. Section 5.3 derives the law of motion and the steady state, which characterize the equilibrium path of the economy. It also investigates the existence and the uniqueness of the steady state. Section 5.4 examines the local determinacy of the steady state. Section 5.5 concludes the chapter.

5.2 Model

We consider the continuous-time version of the dynamic model based on Inoue and Tsuzuki (2011) and Grossman and Helpman (1991a, Chap. 3). Let us assume an economy populated by many infinitely-lived households under monopolistic competition in the labor market, and there are rigidities of nominal wage. There is a single final good, which is produced using intermediate goods and supplied competitively. A new variety of intermediate goods is invented by allocating labor for R&D activities, and inventors enjoy infinitely-lived monopoly power. The available intermediate goods are produced by multiple intermediate firms using labor. Finally, as a monetary policy rule, we use the k-percent rule under which financial authorities expand money supply at a constant rate.[12]

5.2.1 Employment agency

The manufacturing and R&D sectors regard each household's labor as an imperfect substitute for any other household's labor. To simplify the analysis,

[11] A Similar result was shown by Hwang and Wu (2011) for the Chinese economy. Villavicencio and Mignon (2011) and Khan and Senhadji (2001) also found a positive growth effect of moderate inflation for advanced countries.

[12] For the k-percent rule, see Friedman (1969). Fujisaki and Mino (2007) and Mino and Itaya (2004) have discussed the monetary endogenous growth model with the Taylor rule.

we assume that an employment agency combines differentiated labor forces into a composite labor force according to the Dixit-Stiglitz function[13]

$$\ell = \left[\int_0^1 \ell_j^{\zeta} dj \right]^{\frac{1}{\zeta}}, \quad \zeta \in (0, 1),$$

and supplies composite labor to the intermediate goods and the R&D sectors. ℓ_j denotes differentiated labor supplied by household $j \in [0, 1]$, and ℓ is the composite labor force. The number of households is normalized to 1. $\upsilon = 1/(1-\zeta)(> 1)$ is the elasticity of substitution between each pair of differentiated labor inputs.

Cost minimization of the employment agency yields the following demand functions for differentiated labor j:

$$\ell_j = \left(\frac{W_j}{W} \right)^{-\frac{1}{1-\zeta}} \ell,$$

where W_j denotes the nominal wage rate of labor force j, and W denotes the nominal wage rate of the composite labor force, which is given by

$$W = \left[\int_0^1 W_j^{-\frac{\zeta}{1-\zeta}} dj \right]^{-\frac{1-\zeta}{\zeta}}.$$

5.2.2 Final goods sector

We assume that perfect competition prevails in the final goods market. The final goods firm produces the quantity y according to the Dixit-Stiglitz function as follows:

$$y = \left[\int_0^N x_i^{\alpha} di \right]^{\frac{1}{\alpha}}, \quad \alpha \in (0, 1),$$

where x_i is the quantity of intermediate goods indexed by $i \in [0, N]$, and $\phi = 1/(1-\alpha)$ (> 1) represents the elasticity of substitution between every pair of intermediate goods. N is the number of available intermediate goods and represents the technology level of the economy. The final goods firm faces

[13] See Dixit and Stiglitz (1977) and Blanchard and Kiyotaki (1987).

diminishing returns with each intermediate good; therefore, greater values of N imply higher productivity.[14]

Cost minimization by the final-goods producing firm yields the following demand functions for intermediate goods $i \in [0, N]$:

$$x_i = \left(\frac{p_i}{p}\right)^{-\frac{1}{1-\alpha}} y, \tag{5.1}$$

where p_i is the price of intermediate goods i, and p is the price of the final good or the price level, which is given by

$$p = \left[\int_0^N p_i^{-\frac{\alpha}{1-\alpha}} di\right]^{-\frac{1-\alpha}{\alpha}}.$$

5.2.3 Intermediate goods sector

Each intermediate good is produced using one unit of composite labor; thus, marginal cost is equal to the nominal wage level, W. Because patents have an infinite life, all intermediate goods are supplied monopolistically. Maximization of the monopoly profit, $\Pi_i = (p_i - W)x_i$, subject to the demand function (5.1) yields

$$p_i = p_x \equiv \frac{1}{\alpha}W, \quad x_i = x \equiv \frac{\ell_x}{N}, \quad \forall i \in [0, N]. \tag{5.2}$$

where ℓ_x represents the amount of composite labor allocated to the production of the intermediate goods. All intermediate goods enter symmetrically into production of the final good. Moreover, the maximized monopoly profit is

$$\Pi_i = \Pi = \frac{1-\alpha}{\alpha}Wx_i = \frac{1-\alpha}{\alpha}W\frac{\ell_x}{N}, \quad \forall i \in [0, N]. \tag{5.3}$$

From (5.2), the market equilibrium levels of output, y, and the price of the final good, p, are obtained as

[14] BilBiie, Ghironi, and Melitz (2008) and Fujiwara (2007) have provided dynamic new Keynesian models with product-variety framework and endogenous entry based on Melitz (2003). However, no endogenous long-run growth occurs in their models.

$$y = N^{\frac{1}{\alpha}} x = N^{\frac{1-\alpha}{\alpha}} \ell_x, \tag{5.4}$$

$$p = N^{-\frac{1-\alpha}{\alpha}} p_x = N^{-\frac{1-\alpha}{\alpha}} \frac{1}{\alpha} W. \tag{5.5}$$

We can rewrite (5.5) as

$$w \equiv \frac{W}{p} = \alpha N^{\frac{1-\alpha}{\alpha}}. \tag{5.6}$$

5.2.4 R&D sector

The number of intermediate goods, N, expands according to the following equation:

$$\frac{\dot{N}}{N} = \eta \ell_n, \quad N(0) > 0, \tag{5.7}$$

where $\eta(> 0)$ is the parameter that reflects the productivity of R&D. ℓ_n represents the amount of composite labor allocated to R&D, and clearing the labor market requires $\ell = \ell_x + \ell_n$.

In equilibrium, the following free-entry condition must be satisfied:

$$V \leq \frac{W}{\eta N}, \quad \text{with an equality whenever } \dot{N} > 0. \tag{5.8}$$

The right-hand side is the nominal unit cost of R&D. V represents the value of the patent, which is given by the discounted stream of the monopoly profit:

$$V(t) = \int_t^{\infty} \Pi(\tau) e^{-\int_t^{\tau} R(\iota) d\iota} d\tau, \tag{5.9}$$

where R is the nominal interest rate. Differentiating (5.9) with respect to time, t, yields the following no-arbitrage condition:

$$R = \frac{\Pi + \dot{V}}{V}. \tag{5.10}$$

5.2.5 Households

Household j possesses nominal money balances, M_j, and share of the monopoly firms, S_j. The share S_j yields returns at rate R. Thus, the budget constraint of household j is given by

$$\dot{A}_j = \dot{M}_j + \dot{S}_j = W_j \ell_j + RS_j - pc_j + p\tau, \quad \forall j \in [0, 1],$$

where A_j is the nominal assets of household j, ℓ_j is labor supplied elastically by household j, and c_j is consumption of household j. $p\tau$ is nominal transfer income from the financial authorities in a lump-sum fashion. The final goods market clears when $y = c \equiv \int_0^1 c_j dj$. We can rewrite the budget constraint in real terms as

$$\dot{a}_j = \frac{W_j}{p}\ell_j + ra_j - Rm_j - c_j + \tau,$$

where $r \equiv R - \pi$ is the real interest rate, $\pi \equiv \dot{p}/p$ is the inflation rate, $m_j \equiv M_j/p$ is real money balances, and $a_j \equiv A_j/p$ is the stock of assets in real terms.

Household j obtains utility from consumption, c_j, and real money balances, m_j, and it encounters disutility from the labor supply, ℓ_j, and wage negotiations. Thus, the instantaneous utility function of household j is as follows:

$$u(c_j, m_j, \ell_j, \omega_j) = \log c_j + \delta_m \log m_j - \delta_\ell \frac{\ell_j^{1+\psi}}{1+\psi} - \frac{\gamma}{2}\omega_j^2,$$

where $\psi (> 0)$ is the elasticity of the marginal disutility of the labor supply. $\gamma (\geq 0)$ denotes the scale of the nominal wage adjustment cost from wage negotiations and $\omega_j \equiv \dot{W}_j/W_j$.[15] If $\gamma = 0$, the nominal wage is flexible; however, if $\gamma > 0$, the nominal wage is sticky. $\delta_m (> 0)$ and $\delta_\ell (> 0)$ denote the utility weights on real money balances and labor supply, respectively.

[15] We specify the adjustment cost function as a quadratic expression following Rotemberg (1982). The adjustment cost can be defined as $\gamma\frac{(\omega_j - \omega^*)^2}{2}$ instead of $\gamma\frac{\omega_j^2}{2}$, where ω^* is the steady-state value of ω_j. If we choose such an expression, wage stickiness will vanish in the long run and the natural rate hypothesis will be valid.

Summarizing the above, household j faces the following dynamical optimization problem:

$$\max_{c_j, m_j, \omega_j} \int_0^{\infty} \left[\log c_j + \delta_m \log m_j - \delta_\ell \frac{\ell_j^{1+\psi}}{1+\psi} - \frac{\gamma}{2}\omega_j^2 \right] e^{-\rho t} dt,$$

$$\begin{aligned} \text{subject to} \quad & \dot{a}_j = r a_j + \frac{W_j}{p}\ell_j - c_j - R m_j + \tau, \\ & \dot{W}_j = \omega_j W_j, \\ & \ell_j = \left(\frac{W_j}{W}\right)^{-\frac{1}{1-\zeta}} \ell, \end{aligned} \tag{5.11}$$

where $\rho(> 0)$ is the subjective discount rate. Since all households behave symmetrically according to the same equations, $W_j = W$, $c_j = c$, $w_j = w$, $\ell_j = \ell$, and $m_j = m$ hold. When $\gamma > 0$, the solution to the optimization problem above is characterized by the Euler equation and the wage version of the NKPC, as follows:[16]

$$\frac{\dot{c}}{c} + \rho + \pi = R = \delta_m \frac{c}{m}, \tag{5.12}$$

$$\dot{\omega} = \omega\rho + \frac{\zeta}{1-\zeta}\frac{\ell w}{c\gamma} - \frac{\delta_\ell}{1-\zeta}\frac{\ell^{1+\psi}}{\gamma}, \tag{5.13}$$

where $m \equiv \int_0^1 m_j dj$ is real money balances for the entire economy. The transversality condition for the households is given by

$$\lim_{t \to \infty} \frac{a(t)}{c(t)} e^{-\rho t} = 0. \tag{5.14}$$

On the other hand, when $\gamma = 0$ the following equation holds instead of the NKPC (5.13):

$$\zeta \frac{w}{c} = \delta_\ell \ell^{\psi}. \tag{5.15}$$

5.2.6 Money growth

Financial authorities are assumed to change money supply, M, at a constant rate θ. That is, the financial policy rule is given by $\dot{M}/M = \theta$. Therefore,

[16]Appendix 5.A provides detailed derivations.

the following equation holds:

$$\frac{\dot{m}}{m} = \theta - \pi.$$

All seignorage is transfered to households; that is, $p\tau = \dot{M}$.

5.3 Steady state

When the nominal wage is sticky ($\gamma > 0$), and the positive composite labor is allocated to R&D at any time ($\ell_n > 0$) the equilibrium path is characterized by the transversality condition (5.14) and the following differential equations:[17]

$$\dot{R} = R^2 - (\theta + \rho)R, \tag{5.16}$$

$$\dot{\chi} = R\chi - (\rho + \omega)\chi, \tag{5.17}$$

$$\dot{\omega} = \rho\omega + \left(\frac{\ell}{\chi} - \delta_\ell \ell^{1+\psi}\right)\frac{\upsilon}{\gamma}, \tag{5.18}$$

where $\chi \equiv \ell_x/(\alpha\zeta)$ and

$$\ell = \ell(R, \chi, \omega) = \frac{\omega - R}{\eta} + \zeta\chi. \tag{5.19}$$

When R, χ, and ω are given, we obtain the ℓ_x, ℓ_n, and π as follows:

$$\ell_x = \alpha\zeta\chi, \tag{5.20}$$

$$\ell_n = \frac{\omega - R}{\eta} + \frac{\zeta}{\phi}\chi, \tag{5.21}$$

$$\pi = \pi(R, \chi, \omega) = \omega - \frac{1}{\alpha\phi}\eta\ell_n. \tag{5.22}$$

ϕ and υ were defined as $\phi \equiv 1/(1-\alpha)$ and $\upsilon \equiv 1/(1-\zeta)$.

[17] Full derivations are given in Appendix 5.B. We can show that a similar differential equations system is derived from the lab equipment model based on Rivera-Batiz and Romer (1991).

5.3.1 Balanced growth path

If the law of motion (5.16) through (5.18) has fixed points, they are derived as follows:

$$R^* = \theta + \rho, \quad \omega^* = \theta, \quad \chi^* \equiv \chi^*(\ell^*), \quad \ell^* > \underline{\ell} \equiv \alpha\phi\frac{\rho}{\eta},$$

where $\chi^*(\ell^*)$ is the increasing function of ℓ^* defined as

$$\chi^*(\ell^*) = \frac{\ell^*}{\zeta} + \frac{\rho}{\zeta\eta}. \tag{5.23}$$

When ℓ^* is given, the steady-state value of χ is derived according to (5.23). ℓ^* is determined by the following wage version of the long-run Phillips curve:

$$\omega^* = \Omega(\ell^*) \equiv \frac{\upsilon}{\gamma\rho}\left[\delta_\ell(\ell^*)^{1+\psi} - \frac{\ell^*}{\chi^*(\ell^*)}\right]. \tag{5.24}$$

The steady-state values of ℓ_x and ℓ_n are

$$\ell_x^*(\ell^*) = \alpha\ell^* + \alpha\frac{\rho}{\eta}, \quad \ell_n^*(\ell^*) = \frac{\ell^*}{\phi} - \alpha\frac{\rho}{\eta}. \tag{5.25}$$

However, to guarantee that ℓ_n^* is positive, ℓ^* must be greater than $\underline{\ell}$.

If it is the case that $\ell^* > \underline{\ell}$, at this fixed point y and N grow at constant rates. That is, the economy achieves balanced growth. We shall define this steady state as the balanced growth path (BGP). From (5.4) and (5.7), the balanced-growth rate of output is derived as

$$g_y^*(\ell^*) = \frac{1}{\alpha\phi}\eta\ell_n^*(\ell^*).$$

From (5.22), the inflation rate along the BGP is given by the difference between the money growth rate and the long-run growth rate, as shown by Siegel (1983); that is,

$$\pi^* = \theta - g_y^*(\ell^*). \tag{5.26}$$

However, the long-run growth rate is exogenous and constant in Siegel (1983).[18]

[18] Siegel's equation includes the population growth rate, which is assumed to be zero in our model.

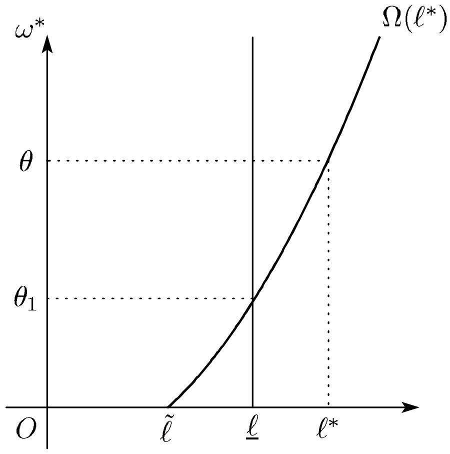


Figure 5.1: The long-run Phillips curve at the BGP (for $\theta > 0$).

5.3.2 Natural employment level

We refer to the output and employment level in the flexible-price economy (i.e., when $\gamma = 0$) as the natural output level and the natural employment level. The employment gap is the difference between the actual and natural employment levels. In the flexible-price economy, the employment level, ℓ, is characterized by (5.15) instead of NKPC (5.13). Then, substituting (5.4), (5.6), (5.25), and $y = c$ into (5.15), we obtain the natural employment level along the BGP, $\tilde{\ell}$, as the root of the following implicit function:

$$\frac{\tilde{\ell}}{\chi^*(\tilde{\ell})} - \delta_\ell(\tilde{\ell})^{1+\psi} = 0.$$

5.3.3 Existence and uniqueness of the balanced growth path

5.3.3.1 Case of non-negative money growth

When θ $(= \omega^*)$ belongs to $[0, 1\ )$, the long-run Phillips curve (5.24) is upward sloping on a (ℓ^*, ω^*)-plane as shown in Figure 5.1. Note that the horizontal axis measures the employment level instead of the unemployment rate or the employment gap. Therefore, the usual Phillips curve is flipped backward in Figure 5.1.

When $\theta \geq 0$ is given, the BGP level of employment, ℓ^*, is uniquely determined according to the long-run Phillips curve. However, for a small value of θ, the root of the equation, $\theta = \Omega(\ell)$, is smaller than $\underline{\ell}$; it is inappropriate for the BGP value. This threshold is given by

$$\theta_1 \equiv \frac{v}{\gamma\rho}\left[\delta_\ell\left(\alpha\phi\frac{\rho}{\eta}\right)^{1+\psi} - \alpha\zeta\right].$$

These results may be summarized as follows:

Proposition 5.1 Let $\theta \geq 0$. If and only if $\theta > \theta_1$, a unique BGP, $(R^*, \chi^*(\ell^*), \omega^*)$, exists. On the other hand, if $\theta \leq \theta_1$, there is no BGP.

When the R&D sector is sufficiently productive and the parameters satisfy

$$\eta > \eta_0 \equiv \alpha\phi\rho\left(\frac{\alpha\zeta}{\delta_\ell}\right)^{-\frac{1}{1+\psi}},$$

$\theta_1 < 0$ holds; thus, $\theta \geq 0 > \theta_1$ always holds. In this case, when the financial authorities apply a monetary policy with $\theta = 0$, $\ell^* = \bar{\ell}$ holds and the employment gap caused by nominal wage stickiness is eliminated.

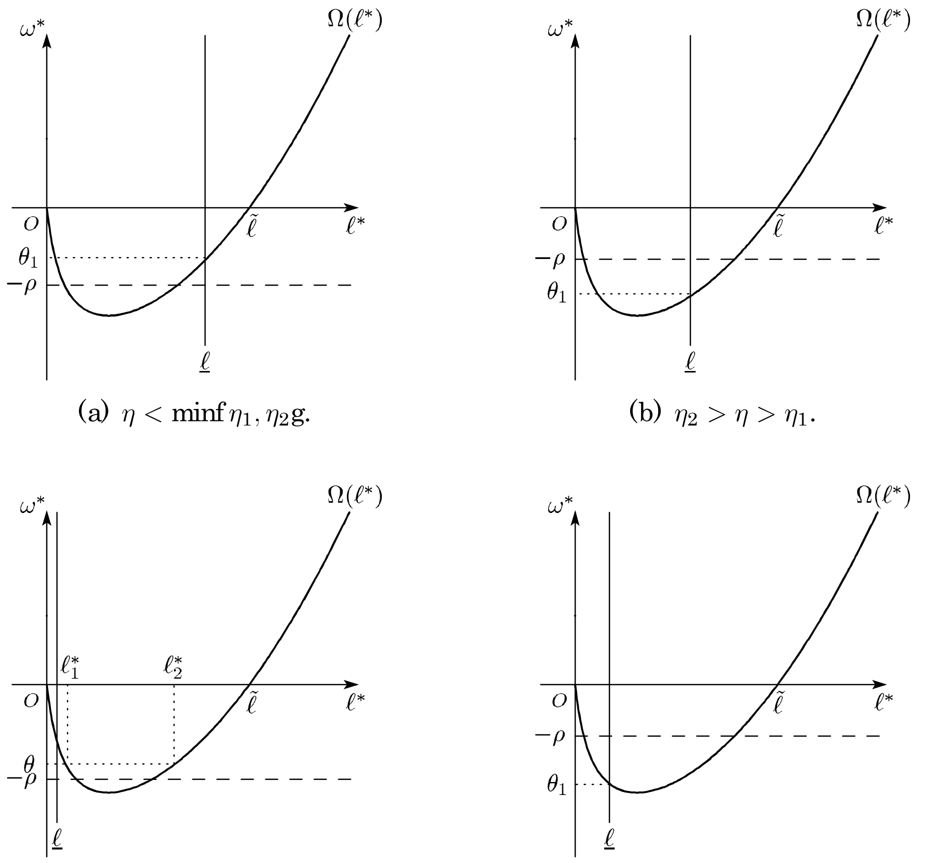
If $\eta \leq \eta_0$, the existence of the BGP requires that the money growth rate, θ, is sufficiently high. When θ is small and the BGP does not exist, there is only the no-growth steady state mentioned below.

5.3.3.2 Case allowing money contraction

Some algebra shows that $\Omega(0) = 0$, $\Omega'(0) < 0$ and $\Omega''(\ell) > 0, \forall\ell > 0$. Therefore, when we allow a negative value of θ, $\Omega(\ell)$ is convex and a unimodal form through the origin as shown in Figure 5.2. However, θ is bounded by $-\rho$ to guarantee that the BGP value of the nominal interest rate, $R^* = \theta + \rho$, is positive.

When the parameters satisfy

$$\eta < \eta_2 \equiv \alpha\phi\rho\left[\frac{\alpha\zeta}{\delta_\ell\phi(1+\psi)}\right]^{-\frac{1}{1+\psi}},$$



(a) $\eta < \min\{\eta_1, \eta_2\}$. (b) $\eta_2 > \eta > \eta_1$.

(c) $\eta_1 > \eta > \eta_2$. (d) $\eta > \max\{\eta_1, \eta_2\}$.

Figure 5.2: The long-run Phillips curve at the BGP (for $\theta > -\rho$).

$\Omega'(\underline{\ell}) > 0$ [see Figures 5.2(a) and 5.2(b)]. In contrast, for $\eta > \eta_2$, $\Omega'(\underline{\ell}) < 0$ holds [see Figures 5.2(c) and 5.2(d)]. Moreover, if the parameters satisfy

$$\eta < \eta_1 \equiv \begin{cases} \alpha\phi\rho\left[\frac{1}{\delta_\ell}\left(\alpha\zeta - \frac{\gamma\rho^2}{\upsilon}\right)\right]^{-\frac{1}{1+\psi}}, & \text{if } \alpha\zeta\upsilon > \gamma\rho^2, \\ 1\ , & \text{otherwise,} \end{cases}$$

θ_1 is greater than $-\rho$ [see Figures 5.2(a) and 5.2(c)]. For $\eta > \eta_1$, $\theta_1 < \rho$ holds [see Figures 5.2(b) and 5.2(d)].[19] To sum up these findings, we can see four

[19] η_0 is smaller than both of η_1 and η_2.

cases as shown in Figures 5.2(a) through 5.2(d).[20]

At first, in the cases of Figures 5.2(b) and 5.2(d), $\theta = \Omega(\ell)$ has a unique root such that $\ell = \ell^* > \underline{\ell}$ for all $\theta > -\rho$. That is, a unique BGP exists for all possible money growth rates.

In the case of Figure 5.2(a), $\theta > \theta_1$ is again a necessary and sufficient condition for the existence of a unique BGP. That is, sufficiently high rates of money growth are required to achieve sustained economic growth.

The following proposition summarizes the above properties.

Proposition 5.2 (a) If the parameters satisfy $\eta > \eta_1$, a unique BGP, $(R^*, \chi^*(\ell^*), \omega^*)$, exists for all $\theta > -\rho$.

(b) Let the parameters satisfy $\eta < \min\{\eta_1, \eta_2\}$. If and only if $\theta > \theta_1(> -\rho)$, a unique BGP, $(R^*, \chi^*(\ell^*), \omega^*)$, exists. In contrast, if $\theta \in (-\rho, \theta_1]$, there is no BGP.

On the other hand, in the case of Figure 5.2(c), it is possible that $\theta = \Omega(\ell)$ has dual roots, ℓ_1^* and ℓ_2^*, which belong to $(\underline{\ell}, \infty)$ under a contractionary monetary policy.[21] To put it more precisely, we can state the following proposition.

Proposition 5.3 Let $\eta \in (\eta_2, \eta_1)$ hold. For $\theta \geq \theta_1$, a unique BGP, $(R^*, \chi^*(\ell^*), \omega^*)$, exists. For $\theta < \theta_1$ close enough to θ_1, dual BGPs, $(R^*, \chi^*(\ell_1^*), \omega^*)$ and $(R^*, \chi^*(\ell_2^*), \omega^*)$, exist.

Letting $\ell_1^* < \ell_2^*$, we obtain $g_y^*(\ell_1^*) < g_y^*(\ell_2^*)$. Therefore, when the money growth rate, θ, is smaller than $\theta_1(< 0)$ and belongs to the neighborhood of θ_1, BGPs with a high and low growth rate coexist. Our model has no mechanism to choose between them. That is, global indeterminacy arises.[22] The behavior of the economy is determined by agents' expectations. If the minimum value of $\Omega(\ell)$ is greater than $-\rho$, by decreasing θ toward $-\rho$, a saddle-node bifurcation will occur and the BGPs will vanish.[23]

The arguments of Propositions 5.1 through 5.3 are summarized in Table 5.1 for the case of $\eta_2 < \eta_1 < \infty$.

[20] If $\alpha\zeta\upsilon \geq \gamma\rho^2$ holds, $\eta_1 = \infty$ and the cases of Figures 5.2(b) and 5.2(c) cannot arise.

[21] When γ is sufficiently large and the parameters satisfy $\gamma > \alpha\zeta\upsilon(\psi + \alpha)/[(1 + \psi)\rho^2]$, $\eta_1 > \eta_2$ holds.

[22] Regarding local indeterminacy, Section 5.4 provides detailed analyses.

[23] For the saddle-node bifurcation of multi-dimensional systems, see Theorem 3.4.1 in Guckenheimer and Holmes (1983).

	$\theta < \theta_1$	$\theta = \theta_1$	$\theta > \theta_1$
$\eta \leq \eta_2$	no BGP	no BGP	a unique BGP
$\eta \in (\eta_2, \eta_1)$	dual BGPs or no BGP	a unique BGP	a unique BGP
$\eta > \eta_1$	—	—	a unique BGP

Table 5.1: The existence and uniqueness of BGP ($\eta_2 < \eta_1 < 1$).
"—" shows that no such combinations of parameters exist because $\theta_1 < -\rho$.

5.3.4 Money growth, inflation, and economic growth

Let a unique BGP exist. Then, we obtain the following proposition.

Proposition 5.4 Let $\theta > \max\{-\rho, \theta_1\}$ hold and a unique BGP exists. In response to a permanent increase in the money growth rate, θ, the economy experiences greater employment and faster economic growth along the unique BGP.

This proposition can be proved as follows. As shown in Figures 5.1 and 5.2, when a unique BGP exists, ℓ^* lies on the upward-slope of the long-run Phillips curve. Therefore, an increase in θ raises the BGP level of employment, ℓ^*. Since $(\ell_x^*)'(\ell^*) > 0$ and $(\ell_n^*)'(\ell^*) > 0$, an increase in ℓ^* raises labor allocated to each sector.[24] As a result, since $(g_y^*)'(\ell^*) > 0$, the greater value of θ raises g_y^*. That is, economic growth accelerates with money growth.[25]

Furthermore, consider the following two facts. First, the growth acceleration effect of money growth is attributed purely to nominal wage stickiness. A small value of γ diminishes the impact of money growth on employment and economic growth. In a flexible-price economy, a change in the money growth rate has no effect on employment and economic growth. That is, money is superneutral.

This result depends on the assumption of the money-in-utility-function. If we adopt a cash-in-advance approach instead of the money-in-utility-function approach, the superneutrality of money does not hold even in a flexible-price economy. A rise in the rate of money growth has a growth deceleration effect attributable to the cash-in-advance constraint. Therefore, in the sticky-price

[24] In addition, ℓ_n/ℓ_x increases.

[25] It is more realistic to assume the upper limit of labor supply, as in Inoue, Shinagawa, and Tsuzuki (2011). This study focuses on the situation in which employment does not reach the upper limit of labor supply.

economy, the growth acceleration effect, which is argued in Proposition 5.4, is weakened by the opposite effect.

Second, even if financial authorities add 1% to the money growth rate, the rise in the long-run inflation rate is smaller than 1% because of the rise in the long-run growth rate g_y^* [See (5.26)]. That is, the impact of money growth on the long-run inflation rate is weakened by endogenizing growth. Moreover, for high productivity R&D, which is captured by large values of η, the inflation rate might even decrease.

As for dual BGPs, we can prove the following proposition in a similar way to that of Proposition 5.4.

Proposition 5.5 Let dual BGPs exist. At the BGP with lower employment level, an increase in the money growth rate raises employment and the balanced-growth rate. Whereas, at the BGP with a higher employment level, an increase in the money growth rate depresses employment and the balanced-growth rate.

5.3.5 No-growth steady state

There exists a different steady state from the BGP at which no labor is allocated to R&D and long-run growth never occurs. We refer to such a steady state as the no-growth steady state. At the no-growth steady state, since the free-entry condition (5.8) does not hold with an equality, (5.19), (5.20), and (5.21) are not fulfilled, and $\ell_n = 0$ and $\ell = \ell_x$ hold instead of them.

The value of each variable at this steady state is derived as follows: [26]

$$R^0 = \theta + \rho, \quad \pi^0 = \omega^0 = \theta, \quad \chi^0 = \frac{\ell_x^0}{\alpha\zeta}, \quad \ell^0 = \ell_x^0 = \left[\frac{1}{\delta_\ell}\left(\frac{\gamma\theta\rho}{\upsilon} + \alpha\zeta\right)\right]^{\frac{1}{1+\psi}}.$$

If and only if $\theta \leq \theta_1$, the no-growth steady state, (R^0, χ^0, ω^0), exists. [27] When $\theta \leq \theta_1$ and there is no BGP, the no-growth steady state, (R^0, χ^0, ω^0), is a unique steady state of the economy. If two BGPs exist as shown in

[26] When θ is negative, γ must be sufficiently small and satisfy $\gamma < -\alpha\zeta\upsilon/(\theta\rho)$ to obtain the steady state with the positive labor supply.

[27] For $\theta > \theta_1$, ℓ_x^0 is greater than $\underline{\ell}$, and the free-entry condition (5.8) is not fulfilled. From (5.3) and (5.10), $\dot{V}/V = \omega^0 = \theta$ and $V = \Pi/\rho$ hold at this steady state, and substituting the latter equation into (5.8) yields $\ell_x^0 \leq \underline{\ell}$.

Proposition 5.3, there are three steady states in all, and global indeterminacy arises among them.

5.4 Local determinacy of balanced growth paths

To examine local stability, we linearize the system (5.16) through (5.18) around the fixed point, (R^*, χ^*, ω^*).

$$\begin{bmatrix} \dot{R} \\ \dot{\chi} \\ \dot{\omega} \end{bmatrix} = \mathbf{J} \begin{bmatrix} R - R^* \\ \chi - \chi^* \\ \omega - \omega^* \end{bmatrix}, \quad \text{where} \quad \mathbf{J} \equiv \begin{bmatrix} \theta + \rho & 0 & 0 \\ \chi^* & 0 & -\chi^* \\ -\Gamma(\ell^*) & -\zeta\rho\Omega'(\ell^*) & \rho + \Gamma(\ell^*) \end{bmatrix},$$

where

$$\Omega'(\ell) = \frac{\upsilon}{\gamma\rho}\left[\delta_\ell(1+\psi)\ell^\psi - \frac{1}{[\chi^*(\ell)]^2}\frac{\rho}{\zeta\eta}\right], \tag{5.27}$$

and

$$\Gamma(\ell^*) \equiv \frac{\upsilon}{\gamma\eta}\left[\frac{1}{\chi^*(\ell^*)} - \delta_\ell(1+\psi)(\ell^*)^\psi\right] = \frac{\upsilon\ell^*}{[\chi^*(\ell^*)]^2\zeta\gamma\eta} - \frac{\rho}{\eta}\Omega'(\ell^*).$$

One of three eigenvalues of the Jacobian matrix, $\mathbf{J}$, is $\theta + \rho > 0$; the other two eigenvalues are equal to the eigenvalues of the following sub matrix:

$$\mathbf{J}_1 \equiv \begin{bmatrix} 0 & -\chi^* \\ -\zeta\rho\Omega'(\ell^*) & \rho + \Gamma(\ell^*) \end{bmatrix}.$$

Here, $\operatorname{tr}\mathbf{J}_1 = \rho + \Gamma(\ell^*)$ and $\det\mathbf{J}_1 = -\zeta\rho\chi^*\Omega'(\ell^*)$ hold.

5.4.1 The unique balanced growth path

First, we study the dynamic property of the unique BGP. Since $\Omega'(\ell^*) > 0$ holds, $\det\mathbf{J}_1$ is negative. Therefore, $\mathbf{J}_1$ has two real eigenvalues with opposite signs. As a result, the Jacobian matrix, $\mathbf{J}$, has one negative real root and two positive real roots. Since R, χ, and ω are non-predetermined variables, the fixed point is locally indeterminate.

5.4.2 The dual balanced growth paths

Next, we analyze the case of the dual equilibria, which is argued in Proposition 5.3. Let ℓ_1^* and ℓ_2^* denote the roots of $\theta = \Omega(\ell)$ and $\ell_1^* < \ell_2^*$. Then $\Omega'(\ell_1^*) < 0$ and $\Omega'(\ell_2^*) > 0$ hold as shown in Figure 5.2(c). For ℓ_1^*, $\Gamma(\ell_1^*) > -(\rho/\eta)\Omega'(\ell_1^*) > 0$ holds. Therefore, $\operatorname{tr} \mathbf{J}_1 > 0$ and $\det \mathbf{J}_1 > 0$ hold, so that both roots of J_1 have positive real parts. Since all eigenvalues of $\mathbf{J}$ have positive real parts, this fixed point is locally determinate.

On the other hand, regarding ℓ_2^*, since $\det \mathbf{J}_1 < 0$, $\mathbf{J}_1$ or $\mathbf{J}$ has one negative real root. Therefore, the fixed point is locally indeterminate.

To sum up, we have shown the following:

Proposition 5.6 (a) A unique BGP is locally indeterminate.

(b) Let dual BGPs exist. Then, the BGP with a lower employment level is locally determinate, whereas the BGP with a higher employment level is locally indeterminate.

Even if the BGP is locally determinate, there are two BGPs and a no-growth steady state. Therefore, global indeterminacy remains.

5.5 Conclusions

This chapter developed a R&D-based endogenous growth model by introducing money growth and a price adjustment process. This study assumed that nominal wage is adjusted stickily because of adjustment cost and derived the new Keynesian Phillips curve, under which money is not superneutral even in the long-run.

When the money growth rate is sufficiently high, the economy has a unique balanced growth path, and can sustain long-run positive growth based on sustained R&D. Furthermore, faster money growth brings greater employment and faster economic growth along a unique balanced growth path. In contrast, under some parameter restrictions, when the money growth rate is sufficiently low, there is no balanced growth path, and the economy is trapped in a no-growth steady state. These results suggest that money growth may be an important factor for long-run economic growth. However, the unique balanced growth path is always locally indeterminate without depending on a monetary policy. The following chapter extends the monetary endogenous growth model with nominal wage stickiness, and considers this issue.

Appendix

5.A Dynamical optimization of households

Let us define the Hamiltonian function of the optimal problem (5.11) as follows:

$$
\begin{aligned}
\mathrm{H} = \log c_j + \delta_m \log m_j - \frac{\delta_\ell}{1+\psi}\left[\left(\frac{W_j}{W}\right)^{-\frac{1}{1-\zeta}}\ell\right]^{1+\psi} - \frac{\gamma}{2}\omega_j^2 \\
+ \lambda_1\left[ra_j + \frac{W_j}{p}\left(\frac{W_j}{W}\right)^{-\frac{1}{1-\zeta}}\ell - c_j - Rm_j\right] + \lambda_2\omega_j W_j,
\end{aligned}
$$

where λ_1 and λ_2 are co-state variables of a_j and W_j. A set of necessary conditions for optimality can be written as follows:

$$\frac{\partial \mathrm{H}}{\partial c_j} = \frac{1}{c_j} - \lambda_1 = 0, \tag{5.28}$$

$$\frac{\partial \mathrm{H}}{\partial m_j} = \frac{\delta_m}{m_j} - \lambda_1 R = 0, \tag{5.29}$$

$$\frac{\partial \mathrm{H}}{\partial \omega_j} = -\gamma\omega_j + \lambda_2 W_j = 0, \tag{5.30}$$

$$\dot{\lambda}_1 = \rho\lambda_1 - \frac{\partial \mathrm{H}}{\partial a_j} = (\rho - r)\lambda_1, \tag{5.31}$$

$$\dot{\lambda}_2 = \rho\lambda_2 - \frac{\partial \mathrm{H}}{\partial W_j} = \rho\lambda_2 - \left[\frac{\delta_\ell \ell_j^{1+\psi}}{(1-\zeta)W_j} - \frac{\zeta}{1-\zeta}\lambda_1\frac{\ell_j}{p} + \lambda_2\omega_j\right]. \tag{5.32}$$

Furthermore, the transversality condition is given by $\lim_{t\to\infty} \lambda_1(t)a_j(t)e^{-\rho t} = 0$.

Derivation of (5.12) From (5.28) and (5.29), we get $R = \delta_m c_j/m_j$. In addition, from (5.28) and (5.31), we get $-\dot{c}_j/c_j = \rho - r$. Substituting $c = c_j$, $m = m_j$, $\forall j$, and $r = R - \pi$ into these equations yields (5.12).

Derivation of (5.13) and (5.15) From (5.30), $\lambda_2 = \gamma\omega_j/W_j$ and $\dot{\lambda}_2 = \gamma\dot{\omega}_j/W_j - \lambda_2\omega_j$ hold. Substituting these equations and $\lambda_1 = 1/c_j$ into (5.32), we obtain

$$\gamma\dot{\omega}_j = \rho\gamma\omega_j - \frac{\delta_\ell \ell_j^{1+\psi}}{1-\zeta} + \frac{\zeta W_j \ell_j}{(1-\zeta)c_j p}.$$

When $\gamma > 0$, we can divide both sides by γ. Since $W_j = W$, $\omega_j = \omega$, $\ell_j = \ell$, $c_j = c$, $8j$, (5.13) holds. On the other hand, when $\gamma = 0$, we obtain $\zeta\frac{(W_j/p)}{c_j} = \delta_\ell \ell_j^{\psi}$. Therefore, (5.15) holds.

5.B Derivation of the law of motion

5.B.1 Derivation of (5.19) and (5.21)

From the free-entry condition (5.8), $\dot{V}/V = \omega - \eta\ell_n$. From (5.10), (5.3), and (5.8), $\dot{V}/V = R - (\Pi/V) = R - (1-\alpha)\alpha^{-1}\ell_x\eta$. Eliminating $\dot{V}/V$ from the two equations above, we obtain

$$\ell_n = \frac{\omega - R}{\eta} + \frac{1-\alpha}{\alpha}\ell_x,$$

and substituting $\ell_x = \alpha\zeta\chi$, we get (5.21). Moreover, substituting (5.21) and (5.20) into the labor market clearing condition, $\ell = \ell_x + \ell_n$, yields (5.19).

5.B.2 Derivation of (5.16)

From (5.12),

$$\frac{\dot{R}}{R} = \frac{\dot{c}}{c} - \frac{\dot{m}}{m} = R - \rho - \theta.$$

5.B.3 Derivation of (5.17)

From (5.4),

$$\frac{\dot{y}}{y} = \frac{1-\alpha}{\alpha}\frac{\dot{N}}{N} + \frac{\dot{\ell}_x}{\ell_x}.$$

From the Euler equation (5.12) and the final goods market clearing condition, $y = c$,

$$R - \rho - \pi = \frac{1-\alpha}{\alpha}\eta\ell_n + \frac{\dot{\ell}_x}{\ell_x}.$$

Using (5.22) and $\dot{\chi}/\chi = \dot{l}_x/l_x$, we obtain (5.17).

5.B.4 Derivation of (5.18)

From (5.4) and (5.6), $(\ell/c)w = (\ell/y)w = \alpha\ell/\ell_x$ holds, and substituting this equation into (5.13) yields

$$\underline{\omega} = \omega\rho + \left[\alpha\zeta\frac{\ell}{\ell_x} - \delta_\ell \ell^{1+\psi}\right]\frac{\upsilon}{\gamma}.$$

Using $l_x = \alpha\zeta\chi$, we get (5.18).

Chapter 6

Indeterminacy in an R&D-based Endogenous Growth Model with Nominal Wage Stickiness

6.1 Introduction

In this chapter, we continue to study the new long-run dynamic model with price stickiness and a price adjustment process. The previous chapter focused on how money growth affects steady-state growth. Expanding on this topic, this chapter gives particular attention to how money growth affects the determinacy property of the steady states and economic stabilization.

We will expand the model constructed in Chapter 5 as follows. First, we introduce capital accumulation into the model (note that labor had been the only production factor in our preceding model). In response to this addition, the specification of R&D is changed from *knowledge-driven specification* to *lab equipment specification*.[1] Second, we assume finite-lived patent instead of infinitely-lived patent discussed in the previous chapter. Furthermore, we adopt the discrete-time version of the dynamic model because it works well with the assumption of temporary patent protection. These modified assumptions more appropriately explain the actual economy.

With regard to steady-state growth, we obtain results similar to those in

[1]For details on the differences between these specifications, please refer to Chapter 3.

the previous chapter; that is, we find a positive correlation between money growth and long-run economic growth. Our new contribution is with respect to the determinacy property of the steady state of the monetary endogenous growth model. By investigating the local dynamics within the neighborhoods of the steady states, we show that changes in money growth rates have an influence on determinacy of the equilibrium path. Under the specific parameters, whether the balanced growth path is determinate or indeterminate varies depending upon the money growth rate; therefore, policy makers can eliminate volatility in the economy through their decisions. However, faster money growth causes faster balanced growth, although the balanced growth path is more likely to be indeterminate; that is, the policy trade-off may exist between growth promotion and economic stabilization. In Inoue and Tsuzuki's (2011) model, a monetary policy had no influence on the determinacy property of the steady state and the long-run growth rate. These results are original ones derived from our study.

Many studies have analyzed the determinacy of equilibrium in the context of the monetary endogenous growth theory, for example, Itaya and Mino (2007, 2003), Mino and Itaya (2004), and Suen and Yip (2005). Our study examines a new channel attributed to nominal rigidities, through which the monetary policy influences economic growth and volatility.

The rest of this chapter is organized as follows. The next section presents the model used in our theoretical investigation. Section 6.3 derives the law of motion and the steady states, which characterize the equilibrium path of the economy. Section 6.4 examines the local determinacy of the steady states. Section 6.5 concludes the chapter.

6.2 Model

We consider a discrete-time dynamic model. Time is indexed by $t = 0, 1, 2, \ldots$. The economy is inhabited by many infinitely-lived households under monopolistic competition in the labor market, and there are rigidities of nominal wage. There is a single final good taken as a numeraire, which is produced using intermediate goods and labor. It is supplied competitively and can be consumed and invested. A new variety of intermediate goods is invented by allocating capital for R&D activities. Inventors are able to enjoy a one-period monopoly through temporary patent protection. The available intermediate goods are produced by multiple intermediate firms using capital. As a monetary policy rule, we use the k-percent rule under which financial authorities expand money supply at a constant rate.

6.2.1 Employment agency

The manufacturing sectors regard each household's labor as an imperfect substitute for any other household's labor. To simplify the analysis, we assume that an employment agency combines differentiated labor forces into a composite labor force according to the Dixit-Stiglitz function:

$$L_t = \left[\int_0^1 L_{j,t}^{\zeta} dj\right]^{\frac{1}{\zeta}}, \quad \zeta \in (0,1),$$

where $L_{j,t}$ denotes differentiated labor supplied by household $j \in [0,1]$, and L_t denotes the composite labor force. The number of households is normalized to 1. $1/(1-\zeta)(>1)$ is the elasticity of substitution between each pair of differentiated labor inputs.

Cost minimization of the employment agency yields the following demand functions for differentiated labor j:

$$L_{j,t} = \left(\frac{W_{j,t}}{W_t}\right)^{-\frac{1}{1-\zeta}} L_t,$$

where $W_{j,t}$ denotes the nominal wage rate of labor force j, and W_t denotes the nominal wage rate of the composite labor force, which is given by

$$W_t = \left[\int_0^1 W_{j,t}^{-\frac{\beta}{1-\beta}} dj\right]^{-\frac{1-\zeta}{\zeta}}.$$

6.2.2 Final goods producer

The final goods firm produces the quantity y_t according to the Dixit-Stiglitz function as follows:

$$y_t = L_t^{1-\alpha} \int_0^{N_t} x_{i,t}^{\alpha} di, \quad \alpha \in (0,1), \tag{6.1}$$

where L_t is the amount of composite labor, $x_{i,t}$ is the quantity of intermediate goods indexed by $i \in [0, N_t]$, and $1/(1-\alpha)$ (>1) represents the elasticity of substitution between every pair of intermediate goods. N_t is the number of available intermediate goods in period t that represents the technology level of the economy.

The nominal profit of the representative final goods firm is given by

$$P_t y_t - L_t W_t - \int_0^{N_t} p_{i,t} x_{i,t} di,$$

where P_t is the price of the final goods, and $p_{i,t}$ is the price of the intermediate goods i.

Profit maximization yields the following equations:

$$W_t = P_t(1-\alpha)\frac{y_t}{L_t}, \tag{6.2}$$

$$\frac{p_{i,t}}{P_t} = \alpha L_t^{1-\alpha} x_{i,t}^{\alpha-1}. \tag{6.3}$$

(6.3) is the inverse demand function for each intermediate goods firm i.

6.2.3 Intermediate goods firms

Each intermediate good is produced using one unit of capital. The nominal profit of the intermediate goods firm i is given by

$$\Pi_{i,t} = (p_{i,t} - R_{t-1})x_t,$$

where R_{t-1} is the nominal price of capital.

Because of the temporary patent protection, the \old" intermediate goods, $[0, N_{t-1}]$, are competitively supplied, whereas the \new" intermediate goods, which are invented in period t, $(N_{t-1}, N_t]$, are monopolistically supplied. Therefore, the price of the intermediate goods i is derived as

$$p_{i,t} = \begin{cases} R_{t-1}, & \text{for } i \text{ 2 } [0, N_{t-1}], \\ \frac{1}{\alpha} R_{t-1}, & \text{for } i \text{ 2 } (N_{t-1}, N_t]. \end{cases}$$

The monopoly profit earned by the intermediate firm i 2 $(N_{t-1}, N_t]$ is $\Pi_t = \frac{1-\alpha}{\alpha} R_{t-1} x_{mt}$. All intermediate goods enter symmetrically into the production of the final goods, i.e., $x_{i,t} = x_{ct}$ for i 2 $[0, N_{t-1}]$ and $x_{i,t} = x_{mt}$ for i 2 $(N_{t-1}, N_t]$. By using (6.3), we can easily show $x_{ct} = \alpha^{-\frac{1}{1-\alpha}} x_{mt}$.

6.2.4 R&D

A new variety of intermediate goods is invented by allocating $1/\eta$ units of capital for R&D activities. Each inventor enjoys a one-period monopoly and earns a profit of Π_t. Therefore, in equilibrium, the following free-entry condition must be satisfied:

$$\Pi_t \leq R_{t-1}/\eta, \quad \text{with an equality whenever } N_t > N_{t-1}.$$

The breakeven point of x_{mt} is derived as

$$\bar{x}_{mt} \equiv \frac{\alpha}{1-\alpha}\eta^{-1}. \tag{6.4}$$

Finally, the capital market clears when

$$k_{t-1} = (x_{mt} + \eta^{-1})(N_t - N_{t-1}) + x_{ct}N_{t-1}, \tag{6.5}$$

where k_{t-1} denotes the amount of capital accumulated in period $t-1$ and available in period t. All capital is depreciated in one period. The available capital is utilized by R&D, producing monopolistic intermediate goods and competitive intermediate goods, as shown on the right-hand side of (6.5).

6.2.5 Households

Household j possesses nominal money balances, $M_{j,t+1}$, and the capital stock, $P_t k_{j,t}$. The capital stock $P_t k_{j,t}$ yields returns at rate R_t. Thus, the budget constraint in nominal terms of household j is given by

$$A_{j,t} = P_t k_{j,t} + M_{j,t+1} = M_{j,t} + R_{t-1}P_{t-1}k_{j,t-1} + W_{j,t}L_{j,t} - P_t c_{j,t} + P_t\tau_t,$$

where $A_{j,t}$ represents the nominal assets to household j, $L_{j,t}$ represents labor supplied elastically by household j, and $c_{j,t}$ represents consumption of household j. $P_t\tau_t$ is nominal transfer income from the financial authorities in a lump-sum fashion. Clearing the final goods market requires

$$y_t = k_t + c_t, \quad \text{where } c_t \equiv \int_0^1 c_{j,t}dj. \tag{6.6}$$

We can rewrite the budget constraint in real terms as

$$a_{j,t} = r_{t-1}a_{j,t-1} - (R_{t-1} - 1)m_{j,t} + \frac{W_{j,t}}{P_t}L_{j,t} - c_{j,t} + \tau_t,$$

where $\pi_t = (P_t/P_{t-1}) - 1$ represents the inflation rate, $r_{t-1} \equiv R_{t-1}/(1+\pi_t)$ represents the real interest rate, $m_{j,t} \equiv M_{j,t}/P_t$ represents real money balances, and $a_{j,t} \equiv A_{j,t}/P_t$ represents the stock of assets in real terms.

Household j obtains utility from consumption, $c_{j,t}$, and real money balances $M_{j,t+1}/P_t$, and it encounters disutility from the labor supply, $L_{j,t}$, and wage negotiations.[2] Therefore, the instantaneous utility function of household j is given by

$$u\left(c_{j,t}, \frac{M_{j,t+1}}{P_t}, L_{j,t}, \omega_{j,t}\right) = \log c_{j,t} + \delta_m \log \frac{M_{j,t+1}}{P_t} - \delta_L \frac{L_{j,t}^{1+\psi}}{1+\psi} - \frac{\gamma}{2}\omega_{j,t}^2,$$

where $\psi > 0$ is the elasticity of the marginal disutility of labor supply. $\gamma > 0$ denotes the scale of the nominal wage adjustment cost from wage negotiations and $\omega_{j,t} \equiv (W_{j,t}/W_{j,t-1}) - 1$. If $\gamma = 0$, the nominal wage is flexible; however, if $\gamma > 0$, the nominal wage is sticky. $\delta_m (> 0)$ and $\delta_L (> 0)$ are scale parameters.

Summarizing the above, household j faces the following dynamical optimization problem:

$$\begin{aligned}
&\max_{c_{j,t}, m_{j,t}, \omega_{j,t}} \sum_{t=0}^{1} \beta^t \left[\log c_{j,t} + \delta_m \log \frac{M_{j,t+1}}{P_t} - \delta_L \frac{L_{j,t}^{1+\psi}}{1+\psi} - \frac{\gamma}{2}\omega_{j,t}^2\right], \\
&\text{subject to} \quad a_{j,t} = \tau_t + r_{t-1}a_{j,t-1} - (R_{t-1} - 1)m_{j,t} + \frac{W_{j,t}}{P_t}L_{j,t} - c_{j,t}, \\
&\qquad W_{j,t} = (1 + \omega_{j,t})W_{j,t-1}, \\
&\qquad L_{j,t} = \left(\frac{W_{j,t}}{W_t}\right)^{-\frac{1}{1-\zeta}} L_t,
\end{aligned} \tag{6.7}$$

where β 2 $(0, 1)$ is the discount factor. Since all households behave symmetrically according to the same equations, $W_{j,t} = W_t$, $c_{j,t} = c_t$, $L_{j,t} = L_t$, and $m_{j,t} = m_t$ hold. When $\gamma > 0$, the solution to the optimization problem above is characterized by the Euler equations and the wage versions of the NKPC

[2] In this study, we assume the so-called cash-when-I'm-done (CWID) timing, which supposes that the money balances held by a household at the end of period t (beginning of period $t+1$) enter the utility function in period t.

as follows:

$$\frac{c_{t+1}}{c_t} = \beta r_t, \tag{6.8}$$

$$\delta_m \frac{c_t}{m_t} = \beta(R_{t-1} - 1), \tag{6.9}$$

$$\Omega_{t+1} = \frac{1}{\beta}\Omega_t + \frac{\zeta}{1-\zeta} L_t \frac{w_t}{c_t} - \delta_L L_t^{1+\psi} \frac{1}{1-\zeta}, \tag{6.10}$$

where $m_t \equiv \int_0^1 m_{j,t} dj$ is real money balances for the entire economy, and $\Omega_t \equiv \beta\gamma\omega_t(1+\omega_t)$. The transversality condition for the households is given by

$$\lim_{T \to \infty} \beta^T \frac{a_{T+1}}{c_T} = 0. \tag{6.11}$$

6.2.6 Money growth

We assume that financial authorities expand money supply M at a constant rate of $\theta \geq 0$; that is, the monetary policy is given by $(M_{t+1}/M_t) - 1 = \theta$. Therefore, we obtain the following equation:

$$\frac{m_{t+1}}{m_t} = \frac{1+\theta}{1+\pi_{t+1}}. \tag{6.12}$$

All seigniorage is transferred to households; that is, $P_t\tau_t = M_{t+1} - M_t$ holds.

6.2.7 Equilibrium

By using (6.5), we obtain the following equation:

$$\frac{N_t - N_{t-1}}{N_{t-1}} = \max\{0, \alpha^{-\frac{\alpha}{1-\alpha}}(\kappa_{t-1} - 1)\}, \tag{6.13}$$

where κ_{t-1} is defined as

$$\kappa_{t-1} \equiv \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\eta \frac{k_{t-1}}{N_{t-1}}.$$

The positive amount of capital is allocated for R&D and technological progress occurs if and only if $\kappa_{t-1} > 1$; that is, the economy has a sufficient stock of capital relative to its technological level.

By using (6.1), (6.4), (6.13), and $x_{ct} = \alpha^{-\frac{1}{1-\alpha}} x_{mt}$, we obtain the total output as

$$\frac{y_t}{k_{t-1}} = \ell_t^{1-\alpha} \xi(\kappa_{t-1})^{-(1-\alpha)}, \tag{6.14}$$

where $\xi(\kappa) \equiv \min\{1, \kappa\}$ and $\ell_t \equiv \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\eta L_t$.

6.3 Equilibrium paths

6.3.1 Law of motion

When the nominal wage is sticky ($\gamma > 0$), the equilibrium path is characterized by the transversality condition (6.11) and the following equations:[3]

$$R_t = \frac{1+\theta}{1+\theta-\beta(R_{t-1}-1)}, \tag{6.15}$$

$$\kappa_t = \begin{cases} (1-\alpha\beta\chi_{t-1})\frac{r_{t-1}}{\alpha}\kappa_{t-1}, & \text{if } \kappa_{t-1} \leq 1, \\ \frac{1-\alpha\beta\chi_{t-1}}{1+\alpha^{-\frac{\alpha}{1-\alpha}}(\kappa_{t-1}-1)}\frac{r_{t-1}}{\alpha}\kappa_{t-1}, & \text{if } \kappa_{t-1} > 1, \end{cases} \tag{6.16}$$

$$\chi_t = \frac{\alpha\beta\chi_{t-1}}{1-\alpha\beta\chi_{t-1}}, \tag{6.17}$$

$$\Omega(\omega_{t+1}) = \frac{1}{\beta}\Omega(\omega_t) + \frac{\zeta(1-\alpha)}{(1-\zeta)\alpha\beta}\frac{1}{\chi_{t-1}} - \Gamma_2 r_{t-1}^{\frac{1+\psi}{1-\alpha}}\xi(\kappa_{t-1})^{1+\psi}, \tag{6.18}$$

$$r_t = \left[\frac{R_t}{1+\omega_{t+1}}\frac{1-\alpha\beta\chi_{t-1}}{\alpha}\frac{\xi(\kappa_{t-1})}{\xi(\kappa_t)}\right]^{1-\alpha} r_{t-1}, \tag{6.19}$$

where $\chi_t \equiv c_t/k_t$, $\Omega(\omega_t) \equiv \beta\gamma\omega_t(1+\omega_t)$, and

$$\Gamma_2 \equiv \frac{\delta_\ell}{(1-\zeta)[\alpha^{\frac{1+\alpha}{1-\alpha}}(1-\alpha)\eta]^{1+\psi}} > 0.$$

However, the non-predetermined variables, R and χ, satisfy the following equations for any $t \geq 0$:[4]

$$R_t = R^* \equiv \frac{1+\theta}{\beta}, \quad \chi_t = \chi^* \equiv \frac{1-\alpha\beta}{\alpha\beta}.$$

[3] Full derivations are given in Appendix 6.A.

[4] See Appendix 6.A.1.

Therefore, the law of motion, (6.15) through (6.19), can be simplified as follows:

$$\begin{aligned} \kappa_t &= \begin{cases} \beta r_{t-1}\kappa_{t-1}, & \text{if } \kappa_{t-1} \leq 1, \\ \frac{\beta r_{t-1}}{1+\alpha^{-\frac{\alpha}{1-\alpha}}(\kappa_{t-1}-1)}\kappa_{t-1}, & \text{if } \kappa_{t-1} > 1, \end{cases} \\ \Omega(\omega_{t+1}) &= \frac{1}{\beta}\Omega(\omega_t) + \Gamma_1 - \Gamma_2 r_{t-1}^{\frac{1+\psi}{1-\alpha}}\xi(\kappa_{t-1})^{1+\psi}, \\ r_t &= \left[\frac{1+\theta}{1+\omega_{t+1}}\frac{\xi(\kappa_{t-1})}{\xi(\kappa_t)}\right]^{1-\alpha} r_{t-1}, \end{aligned} \tag{6.20}$$

where

$$\Gamma_1 \equiv \frac{\zeta(1-\alpha)}{(1-\zeta)(1-\alpha\beta)} > 0.$$

6.3.2 Steady states

6.3.2.1 Balanced growth path

When the parameters satisfy $\Omega(\theta) > \frac{\beta}{1-\beta}\left[\Gamma_2\left(\frac{1}{\beta}\right)^{\frac{1+\psi}{1-\alpha}} - \Gamma_1\right]$, the law of motion, (6.20), has the following fixed point:[5]

$$\omega^* = \theta, \quad \kappa^* = 1 + \alpha^{\frac{\alpha}{1-\alpha}}(\beta r^* - 1), \quad r^* = \left[\frac{\left(\frac{1}{\beta}-1\right)\Omega(\theta) + \Gamma_1}{\Gamma_2}\right]^{\frac{1-\alpha}{1+\psi}}.$$

At this fixed point, y, N, c, and k continue to grow at a constant rate, $g^* \equiv \beta r^* - 1 > 0$. We shall define this steady state as the **balanced growth path (BGP)**.

The inflation rate along the BGP is given by

$$1 + \pi^* = \frac{R^*}{r^*} = \frac{1+\theta}{\beta r^*} = \frac{1+\theta}{1+g^*}.$$

The amount of employment is $L^* = \frac{(r^*)^{\frac{1}{1-\alpha}}}{(1-\alpha)\eta\alpha^{\frac{1+\alpha}{1-\alpha}}}$.

[5] This condition is equivalent to $\beta r^* > 1$.

6.3.2.2 No-growth steady state

In contrast, when $\Omega(\theta) \leq \frac{\beta}{1-\beta}\left[\Gamma_2\left(\frac{1}{\beta}\right)^{\frac{1+\psi}{1-\alpha}} - \Gamma_1\right]$ holds, the law of motion, (6.20), has the following fixed point:[6]

$$\omega^0 = \theta, \quad \kappa^0 = \left[\left(1 - \frac{1}{\beta}\right)\Omega(\theta) - \Gamma_1 + \Gamma_2\left(\frac{1}{\beta}\right)^{\frac{1+\psi}{1-\alpha}}\right]^{\frac{1}{1+\psi}}, \quad r^0 = \frac{1}{\beta}.$$

At this fixed point, R&D never occurs, and the economy does not grow. We shall refer to the fixed point, $(\kappa^0, r^0, \omega^0)$, as the no-growth steady state. The inflation rate at the no-growth steady state is given by $\pi = \theta$, and the amount of employment is given by $L^0 = \frac{\kappa^0}{\beta^{\frac{1}{1-\alpha}}\alpha^{\frac{1+\alpha}{1-\alpha}}(1-\alpha)\eta}$.

Because we assume that the money growth rate, θ, is non-negative, $\Omega(\theta) = \beta\gamma\theta(1+\theta) > 0$ and $\Omega'(\theta) = \beta\gamma(2\theta + 1) > 0$ hold. Therefore, we can summarize the above results in the following way:

Proposition 6.1 Let the rate of money growth, θ, be non-negative.

(a) When $\bar{\Gamma} \equiv \Gamma_2\left(\frac{1}{\beta}\right)^{\frac{1+\psi}{1-\alpha}} \leq \Gamma_1$ holds, the BGP, $(\kappa^*, \omega^*, r^*)$, uniquely exists for any positive values of θ.

(b) Let $\bar{\Gamma} > \Gamma_1$ hold. The BGP, $(\kappa^*, \omega^*, r^*)$, uniquely exists for $\theta > \theta_1$, whereas for $\theta \leq \theta_1$, the BGP does not exist, and the no-growth steady state, $(\kappa^0, \omega^0, r^0)$, is a unique steady state. θ_1 is the positive root of the following quadratic equation:

$$\Omega(\theta_1) - \frac{\beta}{1-\beta}\left(\bar{\Gamma} - \Gamma_1\right) = 0, \tag{6.21}$$

which uniquely exists as long as $\bar{\Gamma} > \Gamma_1$.

Proposition 6.1 establishes that the economy has a BGP for sufficiently high rates of money growth. Once the equilibrium path reaches the BGP, the economy will be able to sustain long-run positive growth. In contrast, when $\bar{\Gamma} > \Gamma_1$ holds, for low rates of money growth, the BGP does not exist,

[6] This condition is derived from $\kappa^0 \leq 1$.

and the economy cannot sustain growth. In such case, it is trapped in a no-growth steady state in the long run as shown in the next section.

When the BGP exists, the following proposition can be verified.

Proposition 6.2 Let $\theta > \max\{\theta_1, 0\}$ hold; that is, a unique BGP exists. In response to a permanent increase in the money growth rate, θ, the economy experiences greater employment and faster economic growth along the BGP.

It is easy to prove this proposition by using $\partial L^*/\partial r^* > 0$, $\partial g^*/\partial r^* > 0$ and $\partial r^*/\partial \theta > 0$. In this model, nominal wage stickiness remains at the steady state, and money is not superneutral, even in the long run. Faster money growth causes greater employment and faster economic growth along the BGP.

6.4 Dynamics

6.4.1 Determinacy of no-growth steady states

With regard to local determinacy of the no-growth steady state, we can verify the following proposition.

Proposition 6.3 The no-growth steady state is locally indeterminate if it exists.

proof. See Appendix 6.B. □

The trajectories converge toward the no-growth steady state for the initial conditions with κ_0 that belong to the neighborhoods within the no-growth steady state. However, the equilibrium paths which converge toward the steady state exist continuously. Our model has no mechanism to choose between them, and thus the equilibrium path is indeterminate.

6.4.2 Determinacy of balanced growth paths

Local determinacy property of the BGP is investigated in the following way.

Proposition 6.4 Let $\bar{\bar{\Gamma}} \equiv \Gamma_2 \left(\frac{\alpha^{-\frac{\alpha}{1-\alpha}} - 1}{\beta} \right)^{\frac{1+\psi}{1-\alpha}} > \max\{\bar{\Gamma}, \Gamma_1\}$ hold.[7] For $\theta \in (\max\{0, \theta_1\}, \theta_2)$, the BGP is locally determinate. In contrast, for $\theta > \theta_2$, the BGP is locally indeterminate.[8] θ_2 is a root of the following quadratic equation:

$$\Omega(\theta_2) - \frac{\beta}{1-\beta} \left(\bar{\bar{\Gamma}} - \Gamma_1 \right) = 0,$$

which uniquely exists and is larger than $\max\{0, \theta_1\}$.

proof. See Appendix 6.C. □

Proposition 6.4 establishes that the money growth rate influences not only economic growth but also the determinacy property of the BGP and the volatility of the economy. If the condition of Proposition 6.4 is satisfied, adjusting the money growth rate to the appropriate interval makes the determinate BGP possible.[9] Recall from Chapter 5, the unique BGP for positive rates of money growth is always indeterminate.[10] On the other hand, in this chapter's model, policy makers have a tool to eliminate economic volatility. However, we should note that faster money growth brings a higher balanced growth rate, whereas it makes the BGP indeterminate and the economy volatile. In other words, policy makers may face a trade-off between implementing growth enhancing policies and economic stabilization policies.[11]

These effects on money growth are purely attributed to nominal wage stickiness. A small value of γ diminishes the impact of money growth on

[7] The conditions $\bar{\bar{\Gamma}} > \bar{\Gamma}$ is necessary and sufficient to hold $\theta_2 > \theta_1$. The condition $\bar{\bar{\Gamma}} > \Gamma_1$ is necessary and sufficient to hold $\theta_2 > 0$. $\bar{\bar{\Gamma}} > \bar{\Gamma}$ is satisfied if and only if $\alpha > 1/2$, which is an adequate value. $\bar{\bar{\Gamma}} > \Gamma_1$ is more likely to satisfy for smaller values of η, β, and ζ, and larger values of δ_ℓ. When $\bar{\bar{\Gamma}} > \max\{\bar{\Gamma}, \Gamma_1\}$ does not hold, the BGP is locally indeterminate if it exists.

[8] When $\theta > \theta_2$, $\beta r^* > \alpha^{-\frac{\alpha}{1-\alpha}} - 1$ holds.

[9] Matsuyama (1990) have shown the opposite results; that is, indeterminacy is more likely to arise for low rates of money growth.

[10] Even if the determinate BGP exists, it is accompanied by another indeterminate BGP, and global indeterminacy arises.

[11] The efficient rate of money growth is defined as the money growth rate that maximizes households' utility along the BGP. For the plausible range of parameter values, we can numerically verify that both cases in which the BGP is determinate or indeterminate may arise when the financial authorities apply the efficient rate of money growth.

economic growth and determinacy property. In a flexible-price economy, a change in the money growth rate has no effect on economic growth and determinacy property.

6.5 Conclusions

This study has developed an R&D-based endogenous growth model by introducing exogenous money growth and nominal wage stickiness and investigated how money growth affects long-run economic growth and determinacy property of the steady state. In our model, money is not superneutral in the long run, and its growth has influences on both long-run growth rates and determinacy of the steady states.

When the money growth rate is sufficiently high, a unique balanced growth path exists, along which the economy can continue to grow in the long run based on sustained R&D. Furthermore, faster money growth results in faster balanced growth. In contrast, under some restricted parameters, when the money growth rate is sufficiently low, balanced growth path does not exist, and the economy is trapped in a no-growth steady state. We analyzed the local determinacy of each steady state. The no-growth steady state is locally indeterminate without depending on money growth rate as long as it exists. On the other hand, the determinacy of the balanced growth path depends on the money growth rate. For low rates of money growth, the balanced growth path is locally determinate; however, for high rates of money growth, it becomes locally indeterminate. Summarizing the above results, we conclude that a policy trade-off may exist between growth promotion and economic stabilization.

Appendix

6.A Derivation of the law of motion

Derivation of (6.15) Combining (6.8), (6.9), and (6.12) gives

$$\frac{R_t - 1}{R_{t-1} - 1} = \beta \frac{R_t}{1 + \theta},$$

which is equivalent to (6.15).

Derivation of (6.16) In equilibrium, $P_t y_t = R_{t-1}P_{t-1}k_{t-1} + W_t L_t$ holds. Combining with (6.2), we obtain

$$\frac{y_t}{k_{t-1}} = \frac{R_{t-1}}{\alpha(1+\pi_t)} = \frac{r_{t-1}}{\alpha}. \tag{6.22}$$

Substituting (6.8) and (6.22) into the clearing condition of the final goods market (6.6) yields

$$k_t = \frac{r_{t-1}}{\alpha}(1-\alpha\beta\chi_{t-1})k_{t-1}. \tag{6.23}$$

Multiplying both sides by $\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\eta\frac{1}{N_t}$ and using (6.13) yield (6.16).

Derivation of (6.17) Dividing both sides of the Euler equation (6.8) by k_t yields

$$\frac{c_t}{k_t} = \beta r_{t-1}\frac{c_{t-1}}{k_{t-1}}\frac{k_{t-1}}{k_t}.$$

Substituting (6.23) into the above equation, we obtain (6.17).

Derivation of (6.18) Substituting (6.2), (6.8), and (6.22) into NKPC (6.10) yields

$$\Omega_{t+1} = \frac{1}{\beta}\Omega_t + \frac{\zeta}{1-\zeta}\frac{1-\alpha}{\alpha\beta\chi_{t-1}} - \frac{\delta_L}{(1-\zeta)[\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\eta]^{1+\psi}}\ell_t^{1+\psi}. \tag{6.24}$$

From (6.14) and (6.22), we get

$$\ell_t = \left(\frac{r_{t-1}}{\alpha}\right)^{\frac{1}{1-\alpha}}\xi(\kappa_{t-1}). \tag{6.25}$$

Substituting (6.25) into (6.24) gives (6.18).

Derivation of (6.19) From (6.2), (6.22), and (6.23),[12]

$$1+\omega_{t+1} = \frac{W_{t+1}}{W_t} = \frac{L_t}{L_{t+1}}R_t\frac{1-\alpha\beta\chi_{t-1}}{\alpha}.$$

On the other hand, from (6.25),

$$\frac{L_t}{L_{t+1}} = \frac{\ell_t}{\ell_{t+1}} = \left(\frac{r_t}{r_{t-1}}\right)^{\frac{1}{1-\alpha}}\frac{\xi(\kappa_t)}{\xi(\kappa_{t-1})}.$$

Summarizing the above equations yields (6.19).

12 $\frac{y_{t+1}}{y_t} = \frac{r_t k_t}{r_{t-1}k_{t-1}} = \frac{1+\pi_t}{1+\pi_{t+1}}\frac{R_t}{R_{t-1}}\frac{k_t}{k_{t-1}}$.

6.A.1 Simplification of the law of motion

Rewriting (6.15) gives

$$\frac{1}{R_t} - \frac{\beta}{1+\theta} = R_{t-1}\left(\frac{1}{R_{t-1}} - \frac{\beta}{1+\theta}\right).$$

Because R_{t-1} is larger than 1, R diverges to infinity if $R_t \neq (1+\theta)/\beta$. Therefore, $R_t = R^* \equiv (1+\theta)/\beta$ must hold for any $t \geq 0$.

Similarly, rewriting (6.17) yields

$$\frac{1}{\chi_t} - \frac{\alpha\beta}{1-\alpha\beta} = \frac{1}{\alpha\beta}\left(\frac{1}{\chi_{t-1}} - \frac{\alpha\beta}{1-\alpha\beta}\right).$$

Because $1/\alpha\beta$ is larger than 1, χ diverges to infinity if $\chi_t \neq (1-\alpha\beta)/\alpha\beta$. Therefore, $\chi_t = \chi^* \equiv (1-\alpha\beta)/\alpha\beta$ must hold for any $t \geq 0$.

6.B Proof of Proposition 6.3

To prove local indeterminacy, we linearize the system (6.20) around the fixed point, $(\kappa^0, \omega^0, r^0)$:

$$\begin{bmatrix} \kappa_t - \kappa^0 \\ \omega_{t+1} - \omega^0 \\ r_t - r^0 \end{bmatrix} = \mathbf{J}^0 \begin{bmatrix} \kappa_{t-1} - \kappa^0 \\ \omega_t - \omega^0 \\ r_{t-1} - r^0 \end{bmatrix}.$$

$\mathbf{J}^0$ is the Jacobian matrix. Let us define the following implicit function:

$$f(\omega_{t+1}, \omega_t, r_{t-1}, \kappa_{t-1}) \equiv \Omega(\omega_{t+1}) - \frac{1}{\beta}\Omega(\omega_t) - \Gamma_1 + \Gamma_2 r_{t-1}^{\frac{1+\psi}{1-\alpha}} \xi(\kappa_{t-1})^{1+\psi}.$$

Applying an implicit function theorem, we obtain

$$\begin{aligned} &\frac{\partial \omega_{t+1}}{\partial \kappa_{t-1}} = -\frac{f_{\kappa_{t-1}}}{f_{\omega_{t+1}}} = -\frac{\Gamma_2 r_{t-1}^{\frac{1+\psi}{1-\alpha}} \xi'(\kappa_{t-1})}{\Omega'(\omega_{t+1})}, \quad \frac{\partial \omega_{t+1}}{\partial \omega_t} = -\frac{f_{\omega_t}}{f_{\omega_{t+1}}} = \frac{\frac{1}{\beta}\Omega'(\omega_t)}{\Omega'(\omega_{t+1})}, \\ &\frac{\partial \omega_{t+1}}{\partial r_{t-1}} = -\frac{f_{r_{t-1}}}{f_{\omega_{t+1}}} = -\frac{\frac{1+\psi}{1-\alpha}\Gamma_2 r_{t-1}^{\frac{1+\psi}{1-\alpha}-1}}{\Omega'(\omega_{t+1})}. \end{aligned} \tag{6.26}$$

By combining (6.26) and $\xi^{0}(\kappa^0) = 1$ for $\kappa^0 < 1$, the Jacobian matrix is derived as

$$\mathbf{J}^0 = \begin{bmatrix} 1 & 0 & \beta\kappa^0 \\ j_{21}^0 & 1/\beta & j_{23}^0 \\ -\chi j_{21}^0 & -\chi/\beta & 1-\chi j_{23}^0 \end{bmatrix},$$

where

$$j_{21}^0 \equiv -\frac{\Gamma_2 \beta^{-\frac{1+\psi}{1-\alpha}}}{\beta\gamma(1+2\theta)} < 0, \quad j_{23}^0 \equiv -\frac{\frac{1+\psi}{1-\alpha}\Gamma_2\beta^{-\frac{1+\psi}{1-\alpha}+1}}{\beta\gamma(1+2\theta)}, < 0, \quad \chi \equiv \frac{1-\alpha}{(1+\theta)\beta} > 0.$$

The eigenvalues of $\mathbf{J}^0$, denoted as λ_n^0, $n = \{1, 2, 3\}$, are obtained by solving the following characteristic equation:

$$\begin{aligned} P^0(\lambda^0) \equiv & |\mathbf{J}^0 - \lambda^0 \mathbf{I}| \\ = & -(\lambda^0)^3 + \left(2 - \chi j_{23}^0 + \frac{1}{\beta}\right)(\lambda^0)^2 \\ & + \left[-1 - \frac{2}{\beta} + \chi(j_{23}^0 - \beta\kappa^0 j_{21}^0)\right]\lambda^0 + \frac{1}{\beta} = 0. \end{aligned}$$

The three-dimensional system, (6.20), has one predetermined variable, κ, and two non-predetermined variables, ω and r. If both roots have a modulus of less than 1, then the no-growth steady state is locally indeterminate. From $\lim_{\lambda\to\infty} P^0(\lambda^0) = -\infty$ and $P(1/\beta) = -[(1-\beta)/\beta^2]\chi j_{23}^0 - \kappa^0\chi j_{21}^0 > 0$, there is at least one real root that is larger than $1/\beta$. We define this real root as λ_3^0. As for the other two roots, we will consider the following two cases.

Case of complex roots If the characteristic equation, $P^0(\lambda^0) = 0$, has complex roots, $\lambda_1^0 \equiv a + bi$ and $\lambda_2^0 \equiv a - bi$, where a and b are non-negative real numbers, they would satisfy the following equation:

$$\prod_{n=1}^{3} \lambda_n^0 = (a^2 + b^2)\lambda_3^0 = \frac{1}{\beta}.$$

Since λ_3^0 is larger than $1/\beta$, $a^2 + b^2$ is smaller than 1. Therefore, the complex roots have a modulus of less than one, and thus, the no-growth steady state is locally indeterminate.

Case of real roots Some algebra shows that[13]

$$
\begin{aligned}
(P^0)'(0) &= -1 - \frac{2}{\beta} + \chi(j^0_{23} - \beta\kappa^0 j^0_{21}) < 0,\\
(P^0)'(1) &= -\chi j^0_{23} - \chi\beta\kappa^0 j^0_{21} > 0,\\
\lim_{\lambda^0 \to \infty} (P^0)'(\lambda^0) &= -\infty < 0.
\end{aligned}
$$

That is, the cubic function, $P^0(\lambda^0)$, has a local minimum point in $(0,1)$ and a local maximum point in $(1,\infty)$. Taking into account $P(1) = -\chi\beta\kappa^0 j^0_{21} > 0$, we can verify that if the characteristic equation has three real roots, two of these belong to $(0,1)$. □

6.C Proof of Proposition 6.4

Similar to the previous section, we linearize the system (6.20) around the fixed point, $(\kappa^*, \omega^*, r^*)$:

$$
\begin{bmatrix} \kappa_t - \kappa^* \\ \omega_{t+1} - \omega^* \\ r_t - r^* \end{bmatrix} = \mathbf{J}^* \begin{bmatrix} \kappa_{t-1} - \kappa^* \\ \omega_t - \omega^* \\ r_{t-1} - r^* \end{bmatrix}.
$$

Using (6.26) and $\xi'(\kappa^*) = 0$, we obtain the Jacobian matrix as follows:

$$
\mathbf{J}^* = \begin{bmatrix} -(\alpha^{-\frac{\alpha}{1-\alpha}} - 1)/(\beta r^*) & 0 & \kappa^*/r^* \\ 0 & 1/\beta & j^*_{23} \\ 0 & -\chi r^* & 1 - \chi\beta r^* j^*_{23} \end{bmatrix},
$$

where

$$
j^*_{23} \equiv -\frac{\frac{1+\psi}{1-\alpha}\Gamma_2 (r^*)^{\frac{1+\psi}{1-\alpha}-1}}{\beta\gamma(2\theta+1)} < 0.
$$

One of the three eigenvalues of the Jacobian matrix, $\mathbf{J}^*$, is given by $\lambda^*_3 \equiv -(\alpha^{-\frac{\alpha}{1-\alpha}} - 1)/(\beta r^*)$; the other two eigenvalues are equal to those of the following sub matrix:

$$
\hat{\mathbf{J}}^* \equiv \begin{bmatrix} 1/\beta & j^*_{23} \\ -\chi r^* & 1 - \chi\beta r^* j^*_{23} \end{bmatrix}.
$$

[13] Since $\kappa^0 < 1$, $j^0_{23} - \beta\kappa^0 j^0_{21} < j^0_{23} - \beta j^0_{21} = \beta j^0_{21}(\frac{1+\psi}{1-\alpha} - 1) < 0$.

$\mathrm{tr}\,\hat{\mathbf{J}}^*$ and $\det \hat{\mathbf{J}}^*$ are derived as

$$\mathrm{tr}\,\hat{\mathbf{J}}^* = \frac{1}{\beta} + 1 - \chi\beta r^* j_{23}^*, \quad \det \hat{\mathbf{J}}^* = \frac{1}{\beta}.$$

Because j_{23}^* is negative, $1 < \det \hat{\mathbf{J}}^* < \mathrm{tr}\,\hat{\mathbf{J}}^* - 1 = 1/\beta - \chi\beta r^* j_{23}^*$ holds. Therefore, $\hat{\mathbf{J}}^*$ has real eigenvalues λ_1^* 2 $(0, 1)$ and λ_2^* 2 $(1, 1\ )$.[14]

The three-dimensional system, (6.20), has one predetermine variable, κ, and two non-predetermined variables, ω and r. Local determinacy of the BGP, $(\kappa^*, \omega^*, r^*)$, depends on the absolute value of λ_3^*. If $\alpha^{-\frac{\alpha}{1-\alpha}} - 1 > \beta r^*$ holds, $\lambda_3^* < -1$ and the BGP is locally determinate. On the other hand, if $\alpha^{-\frac{\alpha}{1-\alpha}} - 1 < \beta r^*$, λ_3^* 2 $(-1, 0)$ and the BGP is locally indeterminate. □

[14] See Azariadis (1993, Chap. 6) for further details.

Chapter 7

Conclusions

This study provided theoretical analyses of economic growth and business cycles based on the framework of the R&D-based endogenous growth model.

Chapters 2, 3, and 4 extended the endogenous growth cycle model on the basis of Deneckere and Judd (1992) and Matsuyama (1999, 2001). Chapter 2 analyzed the patent policy so as to promote long-run growth and stabilize economic fluctuations. Chapter 3 examined the endogenous growth cycle model, which had two specifications of R&D technology, heavily used in endogenous growth studies on the basis of the variety-expanding framework. Chapter 4 investigated the endogenous fluctuation in a non-scale growth model. Its analysis focused on the issue of cyclicality of R&D investment.

Chapters 5 and 6 proposed a new long-run model of macroeconomics, which involves endogenous sustained growth based on sustained R&D, exogenous money growth, and a price adjustment process. Chapter 5 analyzed the simple monetary endogenous growth model without capital accumulation and the limit of patent length. Chapter 6 extended the model by introducing capital accumulation and the finite-lived patent. Using these models, we examined how money growth affects the steady-state growth and determinacy property of the steady states.

This study concludes by focusing on the following two issues: (1) the behavior of R&D activities over business cycles and (2) the consistency between policies to promote growth and stabilize the economy.

Behavior of R&D activities over business cycles

This study analyzed how R&D activities behave over business cycles by extending the R&D-based growth models into growth cycle models.

One of the results of our study is to clarify how each R&D is carried out in a fluctuating economy which included both knowledge-driven and lab equipment R&D. We proved that when productivities of both R&D specifications are not extensive, the balanced growth path becomes unstable and the equilibrium path continues to fluctuate. Along this fluctuating equilibrium path, R&D continues to occur without stopping and both R&D technologies are alternately or periodically used. Focusing on the period-2 cycles, knowledge-driven R&D is carried out procyclically, whereas lab-equipment R&D is carried out coutercyclically.

We investigated a non-scale growth model with endogenous fluctuations and proved that a parameter set existes that establishes R&D investment as being procyclical. Despite the findings of earlier theoretical studies, empirical evidence does not support the prediction that R&D investment is countercyclical. We considered an R&D-based growth model with endogenous fluctuations, modifying the variety-expanding model of Matsuyama (1999, 2001) to introduce population growth and a negative externality that affects productivity of R&D. Using both the overlapping generations framework and the infinitely-lived agent framework, this modification makes R&D investment procyclical throughout the sustained business cycles.

Consistency between policies to promote growth and to stabilize the economy

This study analyzed two phenomena that were examined separately in macroeconomic literature, long-run growth and short-run cycles, in a unified setup. We found that growth promotion policies conflict with economic stabilization policies; that is, a policy trade-off exists in both real and monetary endogenous growth models.

In the real endogenous growth model, we focused our attention on a patent policy. Changes in patent breadth affect the economy through monopoly prices and patentees' market share. A broader patent stimulates firms to conduct higher levels of R&D activity through higher monopoly prices and higher patentees' market share. Therefore, extending patent breadth promotes technological progress and economic growth. On the other hand, for sufficiently broad patents, patentees' market share equals 1 and the extending patent may trigger instability of the balanced growth path. Therefore, the long-run average growth promotion policies may conflict with economic stabilization policies.

To analyze the monetary growth model that considers a price adjustment process, we assumed that nominal wages are adjusted sluggishly because of adjustment costs and derived a new Keynesian Phillips curve. First, we showed that financial authorities are required to maintain a high rate of money growth to achieve sustained and faster economic growth. However, in the extended model that includes capital accumulation and finite-lived patent, the determinacy property of the balanced growth path is dependent on the rate of money growth. For high rates of money growth, the balanced growth path is locally indeterminate; however, for low rates of money growth, it becomes locally determinate. Therefore, financial authorities can bring the determinate balanced growth path by keeping the money growth within a moderate rate. However, these monetary policies do not maximize the long-run growth rate. Again, a policy trade-off may exist between growth promotion and economic stabilization.

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