

# Axioms for School Choice

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# Chapter 1

## Introduction

### 1.1 Background

Traditionally, economists study how to allocate resources, continuous or discrete, by the price system. However, in many cases, the usage of price system faces legal and ethical objections. Consider the following situation, for example, the allocation of public school seats to children, the allocation and exchange of human organs among patients, or the allocation of university seats to high school graduates based on their performances in examinations. Moreover, in many markets, resources are discrete and heterogeneous, simultaneously. The afore-mentioned features of such markets entail matching theory.

Generally, matching can be divided into two-sided and one-sided matching. In two-sided matching markets, two sets of agents, such as firms and workers, men and women, students and colleges, or hospitals and doctors, need to be matched with each other. In one-sided matching markets, a set of indivisible resources needs to be allocated to or exchanged among a set of agents.

The foundations for the framework of two-sided matching theory are laid by David Gale and Lloyd Shapley in 1962 when they published a paper discussing college admission and stability of marriage.<sup>1</sup> Gale and Shapley (1962) discover a deferred acceptance algorithm which is easy to understand and always leads to a stable matching between men and women. The Gale-Shapley deferred acceptance algorithm has had a fundamental influence on matching theory and market design. The theoretical framework of one-sided matching theory is initiated by Shapley and Scarf (1974), and Hylland and Zeckhauser (1979).

Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003a) lay the theoretical framework of school choice. School choice is the practice

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1. See Gale and Shapley (1962) for detail.

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where a set of public school seats needs to be allocated to a set of students. Students or their parents first report their preferences over schools, and schools have priorities, usually determined by law, over students. The school choice problem is different from the two-sided matching models initiated by Gale and Shapley (1962). In two-sided matching markets, agents on both sides of the markets are strategic identities. To be specific, in marriage problems, both men and women have strategic behavior, and in college admission problems, both students and colleges are strategic identities in the market. In the theoretical models of school choice problems, school seats are only considered pure public goods to be allocated among students, and schools are not strategic identities. That is, only students have strategic behavior and welfare considerations, and school seats are considered to be pure public objects to be allocated which have exogenously determined priority rankings for students.

Although the school choice model is different from two-sided matching model, inspirations from two-sided matching problems are still enlightening. In school choice problems, it is still possible to define stability, and apply the deferred acceptance algorithm. The important difference is that in school choice problems only the welfare of students is considered, and hence axioms are defined correspondingly only from the point of view of students. For instance, when defining stability for school choice problems, we use non-wastefulness and elimination of justified envy among students without taking account of schools. The same rule applies to definitions of Pareto efficiency, strategy-proofness, etc.

The practice of school choice influences a large population of students each year, which manifests its importance. The 2012 Nobel Prize in Economic Sciences went to Alvin Roth and Lloyd Shapley for their work on stable allocations and the practice of market design, which contains school choice as an important branch. Being rewarded the Nobel Prize also reflects the importance of school choice.

Prior to the seminal work of Abdulkadiroğlu and Sönmez (2003a), the Boston school choice mechanism has already been used in the US. The Boston procedure is not fair and easy to manipulate. Abdulkadiroğlu and Sönmez (2003a) then propose another two mechanisms: the student-optimal stable mechanism determined by the student-proposing deferred acceptance algorithm, and top trading cycles mechanism determined by the top trading cycles algorithm. By referring to literature on two-sided college problems, they propose three important dimensions to measure school choice mechanisms: stability, strategy-proofness, and Pareto efficiency. The three mechanisms and three axioms proposed by Abdulkadiroğlu and

Sönmez (2003a) deeply influenced both the theory and practice of school choice.

The three well-known school choice mechanisms proposed by Abdulkadiroğlu and Sönmez (2003a) have been investigated by economists theoretically, empirically, in lab experiments, etc. However, we find the axiomatic analysis of school choice mechanisms insufficient in the literature. Hence, this thesis provides more axiomatic analysis of school choice mechanisms.

## 1.2 Methodology

In this section, we will take a look at the methodology on which this thesis is based. The axiomatic method has been an important method of research in game theory and other branches of economics.<sup>2</sup> For a given domain of problems, if several solutions exist, and that some means should be found to distinguish between them, axiomatic approach will be helpful. Sometimes, we observe that there appears to be only one solution for the domain, and sometimes we observe that no proper solution is known. For example, in school choice problems, one famous axiomatization result is that the student-optimal stable mechanism is the unique mechanism satisfying stability and strategy-proofness.<sup>3</sup>

According to Thomson (2001), an axiomatic study takes the following steps. First, we specify the domain of the problems, and formulate a list of desirable properties of solutions for the domain, such as stability and Pareto efficiency for school choice mechanisms on all strict preference and priority domains. Second, we describe the families of solutions (possibly empty) satisfying various combinations of properties. For instance, in school choice problems, there exists no mechanism satisfying stability and Pareto efficiency at the same time<sup>4</sup>. Third, an analysis of logical relations between properties should be provided, which is an important way to assess the relative power of properties. In this thesis, we always prove independence of axioms after addressing an axiomatization result.

This thesis studies the school choice problem in terms of the axiomatic method. In general, we characterize school choice mechanisms on strict preference and priority domains, or constrained strict preference and priority domains. Chapters 3-4 characterize the student-optimal stable mechanism. Chapters 5-6 characterize the Boston mechanism. Chapter 7

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2. See Thomson (2001) for a survey.

3. See Alcade and Barberà (1994).

4. See lemma 3 in Balinski and Sönmez (1999).

## Chapter 1 Introduction

characterizes the probabilistic rank-consumption random assignment rule (PRC rule).

### 1.3 Outline of the Thesis

We first introduce papers, published or unpublished, on which this thesis is based.

Chapter 2: Yajing Chen, A new Pareto efficient school choice mechanism, *Economics Bulletin*, Volume 33, No. 1, pp. 271-277, 2013

Chapter 3: Yajing Chen, Axioms for school choice, *mimeo*, 2013

Chapter 4: Yajing Chen, Deferred acceptance and serial dictatorship, *The Waseda Journal of Political Science and Economics*, No. 358, pp. 50-55, 2013

Chapter 5: Yajing Chen, Characterizing the Boston mechanism, *G-COE GLOPE II Working Paper Series No. 52*, 2011

Chapter 6: Yajing Chen, When is the Boston mechanism strategy-proof?, *Mathematical Social Sciences*, *Conditionally Accepted*, 2013

Chapter 7: Yajing Chen, A new random assignment rule: axiomatization and equilibrium analysis, *mimeo*, 2013

School choice studies how to allocate public school seats to students based on schools' priority over students, with each student being assigned to one seat and each school is allocated to the number of students no more than its capacity. Formally, a school choice problem consists of five components: a set of students, a set of school types, a capacity vector of schools, a preference profile of students over schools, and a priority profile of schools over students. A school choice mechanism is a systematic way of finding a matching from schools to students for each problem.

The theoretical framework of school choice is established by Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003a). The latter authors discuss three well-known school choice mechanisms: the student-optimal stable mechanism (SOSM) determined by the student-proposing deferred acceptance algorithm, the top trading cycles mechanism (TTCM) determined by the top trading cycles algorithm, and the Boston mechanism

### 1.3 Outline of the Thesis

(BOSM) determined by the immediate acceptance algorithm. Abdulkadiroğlu and Sönmez (2003a) point out that the widely used BOSM has serious shortcomings, like being not stable and strategy-proof. They thus suggest to substitute this mechanism with the SOSM or TTCM, which do not suffer from incentive problems. The afore-mentioned three mechanisms are hotly debated by economists these years and will be the main research topic of this thesis.

Chapter 2 of this thesis introduces the basic model of school choice problem, basic axioms for school choice mechanisms, and definitions of main school choice mechanisms. Besides the three mechanisms mentioned in Abdulkadiroğlu and Sönmez (2003a), we introduce three more mechanisms: the school-optimal stable mechanism (SSOM) determined by the school-proposing deferred acceptance algorithm, the simple serial dictatorship (SSD) determined by the algorithm of serial dictatorship, and the recursive Boston mechanism (RBM) determined by the recursive immediate acceptance algorithm. Information about six school choice mechanisms to be discussed in this thesis is summarized in the following table 1.1 and 1.2.

Mechanism	Notation	Algorithm
SOSM	$\varphi^S$	Student-proposing Deferred Acceptance
TTCM	$\varphi^T$	Top Trading Cycles
BOSM	$\varphi^B$	Immediate Acceptance
SSOM	$\varphi^O$	School-proposing Deferred Acceptance
SSD	$\varphi^f$	Serial Dictatorship
RBM	$\varphi^R$	Recursive Immediate Acceptance

Table 1.1: School Choice Mechanisms

There is a well-known trade-off among fairness, Pareto efficiency, and strategy-proofness when choosing school choice mechanisms. It is known to all that the SOSM satisfies stability and strategy-proofness but violates Pareto efficiency, TTCM satisfies Pareto efficiency and strategy-proofness but violates stability, and BOSM satisfies only Pareto efficiency and violates strategy-proofness and fairness.

Mechanism	Stability	Strategy-proofness	Pareto Efficiency
SOSM	✓	✓	×
TTCM	×	✓	✓
BOSM	×	×	✓
SOOM	✓	×	×
SSD	×	✓	✓
RBM	×	×	✓

Table 1.2: School Choice Mechanisms and Three Properties

Table 1.1 and 1.2 show a new mechanism called the recursive Boston mechanism (RBM). Similar to the BOSM, the RBM violates stability and strategy-proofness, but satisfies Pareto efficiency. Moreover, Nash equilibrium outcomes of the preference revelation game induced by the RBM are all stable matchings. The RBM is first introduced and analyzed by Chen (2013c).

Chapter 3 of this thesis contributes to school choice literature along two lines. First, we propose several new axioms for school choice mechanisms related to stability, consistency, and monotonicity. These axioms are easy to be implemented in problems other than school choice. Five of them are crucial to our analysis. A mechanism satisfies mutual best if a student is always assigned his non-null favorite school if he has the highest priority for it. A mechanism satisfies strong top best if a student  $i$  is always assigned his non-null favorite school  $a$  if he has the  $q_a$  highest priority for it among all students who find this school acceptable, where  $q_a$  stands for the maximal number of students that can be admitted to school  $a$ . A mechanism satisfies strong group rationality if the mechanism never assigns a student  $i$  to a school worse than the non-null school  $a$  whenever  $i$  has the  $q_a$  highest priority for  $a$  among all students who find this school acceptable.

A mechanism satisfies weak consistency if whenever we remove a subset of students with their assignments and apply the mechanism to the smaller reduced problem, no remaining student is worse off. We say that a preference profile  $P'$  is a rank monotonic transformation of a preference profile  $P$  at a matching  $\mu$  if for all students, any school that is preferred to  $\mu$  under  $P'$  with the preference ranking  $k$  is also placed in the  $k^{th}$  preference ranking of  $P$ . A mechanism  $\varphi$  satisfies rank monotonicity if every student weakly prefers the matching  $\varphi(P')$  to the matching  $\varphi(P)$ , whenever  $P'$  is a rank monotonic transformation of  $P$  at  $\varphi(P)$ .

### 1.3 Outline of the Thesis

	SOSM	TTCM	BOSM	SSOM	SSD	RBM
Mutual Best	✓	✓	✓	✓	×	✓
Strong Top Best	✓	×	✓	✓	×	✓
Strong Group Rationality	✓	×	×	✓	×	×
Weak Consistency	✓	×	×	×	✓	×
Rank Monotonicity	✓	✓	✓	×	✓	✓

Table 1.3: School Choice Mechanisms and New Axioms

Second, chapter 3 provides new characterizations of the SOSM which is becoming the central mechanism. The following table 1.4 shows the main characterization results of chapter 3.

A school choice mechanism $\varphi$ is equivalent to the SOSM.
$\Updownarrow$
$\varphi$ is Pareto efficient subject to stability. (Gale and Shapley, 1962)
$\Updownarrow$
$\varphi$ is stable and strategy-proof. (Alcade and Barberà, 1994)
$\Updownarrow$
$\varphi$ is stable and respects improvements. (Balinski and Sönmez, 1999)
$\Updownarrow$
$\varphi$ is stable and weakly Maskin monotonic. (Kojima and Manea, 2010a)
$\Updownarrow$
$\varphi$ is non-wasteful, strongly top best, and IR monotonic. (Morrill, 2013)
$\Updownarrow$
$\varphi$ is stable and rank monotonic.
$\Updownarrow$
$\varphi$ is non-wasteful, strongly top best, and weakly Maskin monotonic.
$\Updownarrow$
$\varphi$ is non-wasteful, strongly group rational, and rank monotonic.
$\Updownarrow$
$\varphi$ is non-wasteful, mutually best, weakly consistent, and strategy-proof.
$\Updownarrow$
$\varphi$ is non-wasteful, mutually best, weakly consistent, and rank monotonic.
$\Updownarrow$
$\varphi$ is non-wasteful, mutually best, weakly consistent, and respects improvements.

Table 1.4: Axiomatic Analysis of SOSM

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Chapter 4 of this thesis characterizes the SOSM on constrained priority domains.<sup>5</sup> This chapter provides answers to the following question: when is the SOSM equivalent to SSD? To answer the question, we first define quota-acyclic priority structure. Quota-acyclic priority structure requires that according to the quota information of a problem, no disorder of students exists below a certain critical point of priority ranks. The critical point is the minimal quota of schools. The main result of this chapter is summarized in the following table 1.5.

The SOSM is equivalent to a special class of SSD.
↕
The special class of SSD is fair with respect to the priority structure.
↕
The priority structure is quota-acyclic.

Table 1.5: Constrained Axiomatic Analysis of SOSM

Chapter 5 of this thesis characterizes the BOSM determined by the student-proposing immediate acceptance algorithm. Four axioms are crucial to our analysis: respect of preference rankings, weak fairness, rank rationality, and rank monotonicity. A mechanism respects preference rankings if it is non-wasteful and rank-fair. Non-wastefulness requires that if a student prefers another school to his current assignment, then the quota of the preferred school has been fully occupied by other students. Rank-fairness means that if a student prefers the assignment of another student, then the later student should put the preferred school in a preference ranking not lower than the initial student. A matching is weakly fair if one student prefers the assignment of another student, and both of them put the preferred school in the same preference ranking, then the later student should have higher priority for the school than the initial student. A mechanism satisfies rank rationality if it never assigns a student  $i$  to a school worse than  $a$  whenever the following two conditions are satisfied: (1) The number of students, who put school  $a$  in a preference rankings higher than  $i$  does and find school  $a$  acceptable, is smaller than the capacity of this school; (2) Student  $i$  has the highest priority among all students, who put school  $a$  in preference rankings not lower than  $i$  does and find

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5. Characterizing school choice mechanisms on constrained domains has attracted much attention since Ergin (2002). See Kesten (2006), Ehlers and Erdil (2010), and Kojima and Manea (2010a) for more results.



school  $a$  acceptable. The main result of this chapter is summarized in the following table 1.6.

A school choice mechanism $\varphi$ is equivalent to the BOSM.
$\Updownarrow$
$\varphi$ respects preference rankings and is weakly fair.
$\Updownarrow$
$\varphi$ respects preference rankings, is rank rational and rank monotonic.

Table 1.6: Axiomatic Analysis of BOSM

Chapter 6 of this thesis studies the necessary and sufficient condition under which BOSM recovers desirable properties. This chapter shows that the BOSM is strategy-proof, if and only if it is fair, if and only if it is equivalent to SOSM, if and only if SOSM respects preference rankings, and if and only if the number of total seats at any two schools exceeds the number of students. Unlike the other school choice mechanisms, relative priority rankings do not matter in recovering desirable properties for the BOSM. Thus, the only way to recover strategy-proofness and fairness is increasing the number of seats in each school, which manifests the difficulty of having strategy-proof and fair Boston mechanism. The main result of this chapter is summarized in the following table 1.7.

The BOSM is strategy-proof.
$\Updownarrow$
The BOSM is fair.
$\Updownarrow$
The BOSM is equivalent to SOSM.
$\Updownarrow$
The SOSM respects preference rankings.
$\Updownarrow$
Number of total seats at any two schools exceeds number of students.

Table 1.7: Constrained Axiomatic Analysis of BOSM

Chapters 2-6 study school choice problem with strict priorities. However, in real-life problems, schools sometimes have coarse priorities over students.

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Chapter 7 of this thesis studies the problem of assigning  $n$  indivisible goods to  $n$  agents based on ordinal preferences of agents, known as the random assignment problem. Random assignment problem can be considered as a special case of school choice problem when schools are indifferent among all students. In real-life problems, the deterministic method such as the serial dictatorship, although being appealing for its simplicity, suffers from asymmetry. That is, deterministic methods usually treat agents unfairly. To restore fairness, randomization is commonplace in real-life problems. This chapter first proposes two new axioms for random assignment rules: sd-rank-fairness, and equal-rank envy-freeness. Second, this chapter proposes a new random assignment rule: the probabilistic rank-consumption rule (PRC rule). Third, this chapter characterizes the PRC rule by sd-rank-fairness, and equal-rank envy-freeness. Sd-rank-fairness is a refinement of ordinal efficiency, and equal-rank envy-freeness is a refinement of equal treatment of equals. Finally, this chapter shows that although the PRC rule is neither weakly strategy-proof nor weakly sd-envy-free, ordinal Nash equilibrium outcomes of the preference revelation game induced by the PRC rule are all weakly sd-envy-free. The characterization result of this chapter is summarized in the following table 1.8.

A random assignment rule $\varphi$ is equivalent to the PRC rule.
$\Updownarrow$
$\varphi$ satisfies <i>sd-rank-fairness</i> and <i>equal-rank envy-freeness</i> .

Table 1.8: Axiomatic Analysis of the PRC Rule

Chapter 8 concludes the thesis by summarizing the contribution and addressing possible directions for future extension.

## 1.4 Explanation of Abbreviation

Throughout the thesis, we will use abbreviations to represent the corresponding mechanisms. To emphasize here again, in chapters 2-6, SOSM represents the student-optimal stable mechanism, TTCM represents the top trading cycles mechanism, BOSM represents the the Boston mechanism, SSOM represents the school-optimal stable mechanism, SSD represents the simple serial dictatorship, and RBM represents the recursive Boston mechanism. Moreover, we denote SOSM, TTCM, BOSM, SSOM, SSD, and RBM by  $\varphi^S$ ,  $\varphi^T$ ,  $\varphi^B$ ,  $\varphi^O$ ,  $\varphi^f$ , and  $\varphi^R$ , respectively.

#### 1.4 Explanation of Abbreviation

In chapter 7, UA rule represents the uniform assignment rule, RSD represents random serial dictatorship, PR rule represents the probabilistic serial rule, and PRC rule represents the probabilistic rank-consumption rule. Moreover, we denote UA rule, RSD, PS rule, and PRC rule by  $\varphi^{ua}$ ,  $\varphi^{rsd}$ ,  $\varphi^{ps}$ , and  $\varphi^{prc}$ , respectively.

## 1.5 Explanation of Axioms

	Numbering	$\varphi^S$	$\varphi^T$	$\varphi^B$	$\varphi^O$	$\varphi^f$	$\varphi^R$
Non-wastefulness	Def. 2.1	✓	✓	✓	✓	✓	✓
Fairness	Def. 2.2	✓	×	×	✓	×	×
Stability	Def. 2.3	✓	×	×	✓	×	×
Pareto Efficiency	Def. 2.4	×	✓	✓	×	✓	✓
Strategy-proofness	Def. 2.5	✓	✓	×	×	✓	×
Group Strategy-proofness	Def. 2.6	×	✓	×	×	✓	×
Non-bossiness	Def. 2.7	×	✓	×	×	✓	×
Robust Stability	Def. 2.8	×	×	×	×	×	×
Respect of Improvements	Def. 2.9	✓	✓	✓	×	✓	✓
Mutual Best	Def. 3.1	✓	✓	✓	✓	×	✓
Strong Mutual Best	Def. 3.2	✓	×	✓	✓	×	✓
Top Best	Def. 3.3	✓	✓	✓	✓	×	✓
Strong Top Best	Def. 3.4	✓	×	✓	✓	×	✓
Top Rationality	Def. 3.5	✓	✓	×	✓	×	×
Strong Top Rationality	Def. 3.6	✓	×	×	✓	×	×
Group Rationality	Def. 3.7	✓	✓	×	✓	×	×
Strong Group Rationality	Def. 3.8	✓	×	×	✓	×	×
Consistency	Def. 3.9	×	×	×	×	✓	×
Weak Consistency	Def. 3.10	✓	×	×	×	✓	×
Maskin Monotonicity	Def. 3.11	×	✓	×	×	✓	×
Weak Maskin Monotonicity	Def. 3.12	✓	✓	×	×	✓	×
Population Monotonicity	Def. 3.13	✓	×	✓	✓	✓	✓
IR Monotonicity	Def. 3.14	✓	×	×	×	✓	×
Rank Monotonicity	Def. 3.15	✓	✓	✓	×	✓	✓
Strong Rank Monotonicity	Def. 3.16	×	✓	✓	×	✓	✓
IR Rank Monotonicity	Def. 3.17	✓	×	✓	×	✓	✓
Rank-fairness	Def. 5.1	×	×	✓	×	×	×
Respect of Preference Rankings	Def. 5.2	×	×	✓	×	×	×
Weak Fairness	Def. 5.3	✓	×	✓	✓	×	×
Rank Rationality	Def. 5.4	✓	×	✓	✓	×	×

Table 1.9: School Choice Mechanisms and Axioms

# Chapter 2

## Preliminaries

### 2.1 Model

A school choice problem has five components: a set of students, a set of schools, a quota vector of schools, a preference profile of students over schools, and a priority structure of schools over students. Formally, a school choice problem consists of

1. a finite set of students  $I$ ;
2. a finite set of school types  $O$ ;
3. a quota vector  $q = (q_a)_{a \in O}$  where  $q_a$  is the quota of school  $a$ ;
4. a preference profile of students  $P = (P_i)_{i \in I}$  where  $P_i$  denote the strict preference order of student  $i$  over schools and remaining unmatched;
5. a priority profile of schools  $\succ = (\succ_a)_{a \in O}$  where  $\succ_a$  is a strict priority of school  $a$  over the set of students.

We next explain the components of a school choice problem in detail. Let  $I$  and  $O$  denote finite sets of **students** and **school types**, respectively. A generic student is denoted  $i, j, k, l$ , or  $m$ , and a generic school type is denoted  $a, b$  or  $c$ . There is a null school type  $\emptyset$ . Let  $\tilde{O} = O \cup \{\emptyset\}$ . Each school  $a \in \tilde{O}$  has a positive capacity or quota  $q_a$  and  $q_\emptyset = |I|$ . Let  $q = (q_a)_{a \in O}$  be the **capacity vector** of all school types  $O$ . The set of school types  $O$  and the capacity vector  $q$  uniquely determine the set of **school copies**  $\mathbb{C}$ . For each  $\mathbb{D} \subset \mathbb{C}$ , let  $C(\mathbb{D})$  be the set of school types which have at least one copy in  $\mathbb{D}$  and  $q(\mathbb{D})$  be the corresponding capacity vector induced by  $\mathbb{D}$ .

A **matching** is a function  $\mu : I \rightarrow \tilde{O}$  where (1) for each  $i \in I$ ,  $\mu(i) \in \tilde{O}$ ; and (2) for each  $a \in O$ ,  $|\mu^{-1}(a)| \leq q_a$ . To simplify the notation, let  $\mu_i = \mu(i)$

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and  $\mu_a = \mu^{-1}(a)$ . For each  $J \subset I$ , let  $\mu_J$  be the set of school copies assigned to  $J$  under  $\mu$ . For each  $J \subset I$ , let  $\mu_{-J}$  be the **reduced matching** by removing  $J$  and  $\mu_J$ , and assignments of the other students remain the same.

Each student  $i \in I$  has a strict preference order  $P_i$  over  $\tilde{O}$ . Denote by  $\mathcal{P}$  the set of all such orders. Let  $R_i$  denote the weak part of  $P_i$ , that is,  $aR_ib$  if and only if  $aP_ib$  or  $a = b$ . For each  $P_i \in \mathcal{P}$  and  $a \in \tilde{O}$ , let  $U_a(i)$  be the set of schools strictly better than  $a$  for student  $i$ . Let  $P = (P_i)_{i \in I}$  denote the **preference profile** of all students. Let  $P_i(a)$  be the **preference ranking** of school  $a$  at  $P_i$ , i.e., if school  $a$  is the  $l^{\text{th}}$  choice of student  $i$  under  $P_i$ , then  $P_i(a) = l$ . Therefore, for any  $a, b \in \tilde{O}$ ,  $P_i(a) < P_i(b)$  if and only if  $aP_ib$ . A school  $a$  is **acceptable** to a student  $i$  if  $P_i(a) < P_i(\emptyset)$ . For any problem  $P$  and  $a \in O$ , let  $I_a^*$  be the set of students that find school  $a$  acceptable, i.e.,  $I_a^* = \{i | aP_i\emptyset\}$ . Let  $P_J = (P_i)_{i \in J}$  denote the preferences of any subset  $J \subset I$ . For each  $\mathbb{D} \subset \mathbb{C}$ , let  $P|_{C(\mathbb{D})}$  be the **projection** of  $P$  on  $C(\mathbb{D}) \cup \{\emptyset\}$ , i.e., for any  $a, b \in C(\mathbb{D}) \cup \{\emptyset\}$  and  $i \in I$ ,  $aP_ib$  if and only if  $aP_i|_{C(\mathbb{D})}b$ . We write  $\mu' R \mu$  if and only if  $\mu'_i R_i \mu_i$  for each  $i \in I$ . For each  $i \in I$  and  $P_i \in \mathcal{P}$ , let  $U_a(P_i)$  be the set of schools strictly preferred to  $a$  by student  $i$ .

Each school  $a \in O$  has a strict priority order  $\succ_a$  over  $I$ , whereas  $i \succ_a j$  means that student  $i$  has higher priority than student  $j$  at school  $a$ . Let  $\succ = (\succ_a)_{a \in O}$  denote the **priority profile** of all schools.<sup>6</sup> For each  $\mathbb{D} \subset \mathbb{C}$ , let  $\succ_{C(\mathbb{D})}$  be the priority profile of schools  $C(\mathbb{D})$ . For each  $J \subset I$ , let  $\succ|_J$  be the **projection** of  $\succ$  over  $J$ , i.e., for any  $a \in O$  and  $i, j \in J$ ,  $i \succ_a j$  if and only if  $i \succ_a|_J j$ . Let  $\succ_a(i)$  be the **priority ranking** of student  $i$  at school  $a$ . Similarly,  $\succ_a|_J(i)$  represents the priority ranking of student  $i$  at  $\succ_a|_J$ . For each  $k \in \{1, 2, \dots, |I|\}$  and  $a \in O$ , let  $\succ_a^{-1}(k)$  be the student who is ranked in the  $k^{\text{th}}$  place by school  $a$  at  $\succ_a$ . For each  $a \in O$  and  $\succ_a$ , let  $U_i(\succ_a)$  be the set of students who have higher priority than student  $i$ .

To sum up, a school choice **problem** is a five-tuple  $(I, O, q, P, \succ)$ . To simplify the notation, we denote a problem by the preference profile  $P$  most of the time. For any  $J \subseteq I$  and a matching  $\mu$ , let  $P_{-J}^\mu$  be the **reduced problem** by removing  $J$  and  $\mu_J$ . That is,  $P_{-J}^\mu$  denotes the following reduced problem  $(I \setminus J, C(\mathbb{C} \setminus \mu_J), q(\mathbb{C} \setminus \mu_J), P_{I \setminus J}|_{C(\mathbb{C} \setminus \mu_J)}, \succ_{C(\mathbb{C} \setminus \mu_J)}|_{I \setminus J})$ .

Let  $\mathcal{P}^{|I|}$  be the set of preference profiles and  $\mathcal{M}$  be the set of matchings. A school choice **mechanism**  $\varphi : \mathcal{P}^{|I|} \rightarrow \mathcal{M}$  maps the set of preference profiles to matchings. A  $P$ , student  $i$  is assigned to  $\varphi_i(P)$ , and school  $a$  is

6. In this thesis, we assume the priority of schools to be acceptant, i.e., a school will never reject a student as long as it still has vacant capacity. For more discussion of school choice mechanisms on full general priority domains, see Kojima and Ünver (2013) and Afacan (2013).

assigned to the set of students  $\varphi_a(P)$ . For any  $J \subseteq I$ , let  $\varphi_J(P)$  be the set of school copies assigned to  $J$ .

The school choice model is closely related to, but different from, the college admissions model of Gale and Shapley (1962). In the US, all college admission procedures are usually decentralized, and both students and colleges are strategic components. However, in some countries, such as China, Turkey, and Greece, the procedure of college admission is centralized. In such countries, colleges are not strategic, while only students are potentially strategic. School seats are considered pure objects to be allocated and consumed. The schools do not have their own preferences over students, but have priority orderings over students based on their examination scores, or other exogenously determined criteria. Correspondingly, axioms for school choice mechanisms to be addressed in the following section are different from axioms for two-sided matching mechanisms, though not totally independent with each other. To be specific, axioms for two-sided matching mechanisms are defined on both sides of the market, such as students and colleges, men and women, or companies and workers, while axioms for school choice mechanisms or priority-based allocation rules in general are defined only on students or agents, who constitute the strategic side of a market.

## 2.2 Basic Axioms

In this section, we will define properties of school choice mechanisms. We call these properties axioms.

**Definition 2.1.** A mechanism  $\varphi$  is **non-wasteful** if for each  $P \in \mathcal{P}^{|I|}$ ,  $i \in I$  and  $a \in \tilde{O}$ ,

$$aP_i\varphi_i(P) \Rightarrow |\varphi_a(P)| = q_a.$$

A mechanism  $\varphi$  satisfies non-wastefulness if whenever a student  $i$  prefers a school  $a$  to his own, then there is no empty seats left at school  $a$  under  $\varphi(P)$ .<sup>7</sup>

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7. Note that Balinski and Sönmez (1999) have defined stability of school choice mechanisms in a different way. They say that a mechanism is stable if and only if it is individually rational, non-wasteful, and fair. A mechanism  $\varphi$  is **individually rational** if for each  $P \in \mathcal{P}^{|I|}$  and  $i \in I$ ,  $\varphi_i(P)R_i\emptyset$ . However, non-wastefulness implies individual rationality in our setting because the null school is not scarce and the priority structure is acceptant.

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**Definition 2.2.** A mechanism  $\varphi$  is **fair** if for each  $P \in \mathcal{P}^{|I|}$ ,  $i \in I$  and  $a \in O$ ,

$$aP_i\varphi_i(P) \Rightarrow j \succ_a i, \forall j \in \varphi_a(P).$$

A mechanism is fair<sup>8</sup> whenever a student  $i$  prefers another student  $j$ 's school to his own, then student  $j$  has higher priority for his school than student  $i$ . Fairness is an important and natural axiom for school choice which requires that no pair of students have the incentive to deviate from the final outcome. However, this property is also rather strong since no mechanism satisfies fairness and efficiency, which is also an important criterion in school choice problems, simultaneously.

**Definition 2.3.** A mechanism  $\varphi$  is **stable** if it is non-wasteful and fair.

When designing mechanisms in matching markets, stability plays a central role in the theory. A mechanism is stable if no student and no pair of students have the incentive to deviate from the outcome produced by the mechanism. Note that stability in school choice problems are different from that in two-sided college admission problems. In school choice problems, schools are considered to be pure public goods to be allocated and have no say in the matching process. Therefore, stability concept only calculates the welfare of students.

**Definition 2.4.** A mechanism  $\varphi$  is **Pareto efficient** if for each  $P \in \mathcal{P}^{|I|}$ , there exists no other matching  $\mu$  such that  $\mu_i R_i \varphi_i(P)$  for each  $i \in I$ , and  $\mu_i P_i \varphi_i(P)$  for some  $i \in I$ .

A mechanism satisfies Pareto efficiency if for each problem and the matching determined by this mechanism and problem, no other matching can make all students weakly better off, while some students strictly better off. Also note here that Pareto efficiency is different from the efficiency concept in two-sided college admission problems. In two-sided college admission problems, stability is stronger than Pareto efficiency, and matchings in the core of two-sided matching markets by Gale and Shapley (1962) all satisfy Pareto efficiency<sup>9</sup>. But in school choice setting, Pareto efficiency is independent with stability and furthermore incompatible with stability.

**Definition 2.5.** A mechanism  $\varphi$  is **strategy-proof** if for each  $P \in \mathcal{P}^{|I|}$ ,  $i \in I$  and  $P'_i \in \mathcal{P}$ ,  $\varphi_i(P) R_i \varphi_i(P'_i, P_{I \setminus \{i\}})$ .

8. Also known as elimination of justified envy.

9. Sasaki and Toda (1992) characterize the core of two-sided matching markets by Pareto optimality, consistency, converse consistency, and anonymity.



A mechanism satisfies strategy-proofness if no student has the incentive to tell a lie, i.e., reporting the true preferences is the dominant strategy for each student. Dubins and Freedman (1981) and Roth (1982) show that the SOSM is strategy-proof.

**Definition 2.6.** A mechanism  $\varphi$  is **group strategy-proof** if for each  $P \in \mathcal{P}^{|I|}$ , there exists no  $J \subset I$  and  $P'_J \in \mathcal{P}^{|J|}$  such that  $\varphi_i(P'_J, P_{-J}) R_i \varphi_i(P)$  for each  $i \in J$ , and  $\varphi_i(P'_J, P_{-J}) P_i \varphi_i(P)$  for some  $i \in J$ .

A mechanism satisfies group strategy-proofness if no subset of students would gain by jointly misrepresenting their preferences. Takamiya (2001) shows that in school choice setting, group strategy-proofness is equivalent to the combination of strategy-proofness and non-bossiness<sup>10</sup>. Below is the formal definition of non-bossiness.

**Definition 2.7.** A mechanism  $\varphi$  is **non-bossy** if for each  $P \in \mathcal{P}^{|I|}$ ,  $i \in I$ , and  $P'_i \in (P)$ ,

$$\varphi_i(P'_i, P_{-i}) = \varphi_i(P) \Rightarrow \varphi(P'_i, P_{-i}) = \varphi(P).$$

A mechanism satisfies non-bossiness if no student can change the matching of the other students without changing his own by misreporting his preferences. Stability and non-bossiness are both important properties a mechanism designer cares. However, Kojima (2010) shows that stability and non-bossiness are incompatible in two-sided matching markets, and hence in school choice problems.<sup>11</sup>

**Definition 2.8.** A mechanism  $\varphi$  is **robustly stable** if the following conditions are satisfied:

- (1)  $\varphi$  is stable;
- (2)  $\varphi$  is strategy-proof;
- (3) There exists no  $i \in I$ ,  $a \in O$ ,  $P \in \mathcal{P}^{|I|}$ , and  $P'_i \in \mathcal{P}$  such that (i)  $a P_i \varphi_i(P)$ ; and (ii)  $\succ_a(i) < \succ_a(j)$  for some  $j \in \varphi_a(P'_i, P_{-i})$  or  $|\varphi_a(P'_i, P_{-i})| < q_a$ .

10. The concept of nonbossiness is first introduced by Satterthwaite and Sonnenschein (1981).

11. Matsubae (2010) proposes a weaker version of non-bossiness: non-damaging non-bossiness. A mechanism is non-damaging bossy if a student does not make the allocation of other students worse off without changing his own allocation. Formally, a mechanism  $\varphi$  is **non-damaging bossy** if for each  $i \in I$ ,  $P_i$  and  $P'_i$ ,  $\varphi_i(P'_i, P_{-i}) = \varphi_i(P)$  implies  $\varphi(P'_i, P_{-i}) R_i \varphi(P)$ . Matsubae (2010) shows that no mechanism exists satisfying stability and non-damaging non-bossiness simultaneously in two-sided matching markets. In school choice problems, we can see that a mechanism does exist satisfying stability and non-damaging non-bossiness simultaneously, and the SOSM is one example. Non-bossy clearly implies non-damaging bossiness.

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Kojima (2011) first introduces robust stability. A mechanism is robustly stable if it is stable, strategy-proof, and immune to a combined manipulation, where a student first misrepresents his preferences and then blocks the matching that is determined by the given mechanism. He proves that there is no robustly stable mechanism and SOSM satisfies robust stability if and only if the priority structure is Ergin-acyclic.

We say that a priority profile  $\succ'$  is an **improvement** for student  $i$  over a given profile  $\succ$  if  $\succ'$  satisfies the following three conditions:

- (i)  $\succ' \neq \succ$ ;
- (ii)  $i \succ_a j \Rightarrow i \succ'_a j, \forall a \in O, \forall j \in I \setminus \{i\}$ ;
- (iii)  $j \succ_a l \Leftrightarrow j \succ'_a l, \forall a \in O, \forall j, l \in I \setminus \{i\}$ .

**Definition 2.9.** A mechanism  $\varphi$  **respects improvements** if for each problem  $(P, \succ)$  and  $i \in I$ ,

$$\succ' \text{ is an improvement of student } i \text{ over } \succ \Rightarrow \varphi_i(P, \succ') R_i \varphi_i(P, \succ).$$

A mechanism respects improvements if whenever a student's standing in priorities improves, his assignment is expected not to be worse off.  $\succ'$  is an improvement for student  $i$  if the priority ranking of student  $i$  do not decrease for all schools and increase for some schools, while the relative priority ranking of other students remain the same. Respect of improvements is a natural requirement. It can be interpreted as follows: if a student gets a higher score in the examination, then he is not going to be assigned a school worse than his previous assignment. In other words, a mechanism respecting improvements does not punish a student for performing better in priorities.

## 2.3 Mechanisms

### 2.3.1 Student-optimal Stable Mechanism

Denote the student-optimal stable mechanism (SOSM) by  $\varphi^S$ . For each  $P \in \mathcal{P}^I$ ,  $\varphi^S$  is defined by setting  $\varphi^S(P)$  equal to the matching obtained by the following Gale-Shapley student-proposing deferred acceptance algorithm:

**Step 1:** Each student applies for his favorite school. If a student applies for the null school, he is tentatively assigned to the null school. For each

school  $a \in O$ , up to  $q_a$  applicants who have the highest priority for  $a$  are tentatively assigned to  $a$ . The remaining applicants are rejected.

⋮

**Step  $k$ :** Each student that is rejected in step  $k - 1$  applies for his next favorite school. If a student applies for the null school, he is tentatively assigned to the null school. For each school  $a \in O$ , up to  $q_a$  students who have the highest priority for  $a$  among the new applicants, and those tentatively assigned to it from an earlier step, are tentatively assigned to  $a$ . The remaining applicants are rejected.

The algorithm terminates when each student is tentatively assigned to a school. Each student is assigned to his final tentative school.

### 2.3.2 Top Trading Cycles Mechanism

Denote the top trading cycles mechanism (TTCM) by  $\varphi^T$ . For each  $P \in \mathcal{P}^{|I|}$ ,  $\varphi^T$  is defined by setting  $\varphi^T(P)$  equal to the matching obtained by the following student-proposing top trading cycles algorithm:

**Step 1:** Assign a counter to each school type. Its initial value is the capacity of that school. Each student points to his favorite school and each school points to the student who has the highest priority for it. If the null school is the favorite school of a student, then he forms a self-cycle. There is at least one cycle. Each student in a cycle is assigned the school he points to and is removed. The counter of each school in a cycle is reduced by one and if it becomes zero, the school type is also removed. The counters of all other schools remain the same.

⋮

**Step  $k$ ,  $k \geq 2$ :** Each remaining student points to his favorite school among the remaining schools and each remaining school points to the student who has the highest priority for it among the remaining students. If the null school is the favorite choice of a student, then he forms a self-cycle. There is at least one cycle. Each student in a cycle is assigned the school he points to and is removed. The counter of each school in a cycle is reduced by one and if it becomes zero, the school is also removed. The counters of all other schools remain the same.

The algorithm stops when all students have been removed. Each student who is temporarily not in a self-cycle but forms a cycle with a school is assigned that school.

### 2.3.3 The Boston Mechanism

Denote the Boston mechanism (BOSM) by  $\varphi^B$ . For each  $P \in \mathcal{P}^{|I|}$ ,  $\varphi^B$  is defined by setting  $\varphi^B(P)$  equal to the matching obtained by the following student-proposing immediate acceptance algorithm<sup>12</sup>. For each  $a \in \tilde{O}$ , let  $q_a^k$  be the reduced capacity of  $a$  in step  $k$ .

**Step 1:** Consider only the first choice of students. For each  $a \in O$ , up to  $q_a$  students whose first choice is  $a$  with the highest priority for it (all students if fewer than  $q_a$ ) are assigned to school  $a$  permanently. If a student puts the null school in the first preference ranking, then he is assigned the null school. Remove the set of students who are assigned a school in this step and their corresponding assignments.

⋮

**Step  $k$ :** Consider the  $k^{\text{th}}$  choice of the remaining students. For each school  $a \in O$  with  $q_a^k$  copies available, up to  $q_a^k$  students whose  $k^{\text{th}}$  favorite school is  $a$  with the highest priority for it (all students if fewer than  $q_a^k$ ) are assigned to school  $a$  permanently. If a student puts the null school in the  $k^{\text{th}}$  preference ranking, then he is assigned the null school. Remove the set of students who are assigned a school in this step and their corresponding matchings.

The algorithm terminates when all students have been removed.

### 2.3.4 School-optimal Stable Mechanism

Denote the school-optimal stable mechanism (SSOM) by  $\varphi^O$ . For each  $P \in \mathcal{P}^{|I|}$ ,  $\varphi^O$  is defined by setting  $\varphi^O(P)$  equal to the matching obtained by the following school-proposing deferred acceptance algorithm:

**Step 0:** Assign the null school to each student temporarily.

**Step 1:** Each school  $a$  applies for the students who have the highest  $q_a$  priority for it. Each student  $i \in I$  temporarily chooses one school which is most preferred by him among the applicants and the null school. The others are rejected.

⋮

**Step  $k$ :** For each school  $a \in O$ , if it is temporarily accepted by the number of students less than  $q_a$ , say  $q_a^k$ , in the previous steps, then it applies for

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12. The author thanks William Thomson for providing the term: immediate acceptance.

the next set of students who have the highest  $q_a - q_a^k$  priority for it. Each student temporarily chooses one school which is most preferred by him among the new applicants and the school tentatively assigned to him from an earlier step. The others are rejected.

The algorithm terminates when each non-null school seat is either temporarily assigned to some student or rejected by all the students. Each student is assigned to the school temporarily assigned to him in the final step.

### 2.3.5 Simple Serial Dictatorship

Let  $\mathcal{F}$  denote the set of all bijections from  $\{1, 2, \dots, |I|\}$  to  $I$ . We refer to each of these bijections as an order of students. That is, for any  $f \in \mathcal{F}$ , student  $f(1)$  is first and agent  $f(2)$  is second, and so on. For each  $f \in \mathcal{F}$  and  $k \in \{1, 2, \dots, |I|\}$ ,  $f(k)$  stands for the student who is ranked  $k^{\text{th}}$  in  $f$ . Given a subset of school copies  $\mathbb{D} \subseteq \mathbb{C}$ , the choice  $B_i(\mathbb{D})$  of student  $i$  is the best school for him among  $\mathbb{D}$ . Given an order of students  $f \in \mathcal{F}$  and  $P \in \mathcal{P}^{|I|}$ , denote  $\varphi^f(P)$  the matching of simple serial dictatorship (SSD) induced by  $f$ .  $\varphi^f(P)$  is determined by the following algorithm.

$$\begin{aligned}
 \varphi_{f(1)}^f(P) &= B_{f(1)}(\mathbb{C}), \\
 \varphi_{f(2)}^f(P) &= B_{f(2)}\left(\mathbb{C} \setminus \{\varphi_{f(1)}^f(P)\}\right), \\
 &\vdots \\
 \varphi_{f(k)}^f(P) &= B_{f(k)}\left(\mathbb{C} \setminus \bigcup_{l=1}^{k-1} \{\varphi_{f(l)}^f(P)\}\right), \\
 &\vdots \\
 \varphi_{f(|I|)}^f(P) &= B_{f(|I|)}\left(\mathbb{C} \setminus \bigcup_{l=1}^{|I|-1} \{\varphi_{f(l)}^f(P)\}\right).
 \end{aligned}$$

That is, the student who is ordered first gets his top possible choice, the student who is ordered second gets his top possible choice among what remains, and so on. Note that  $\varphi^f(P)$  totally ignores the priority of schools in each problem, and hence has no flavor of fairness at all.

### 2.3.6 Recursive Boston Mechanism

For each  $a \in \tilde{O}$ , let  $q_a^k$  be the reduced capacity of  $a$  in step  $k$ . Denote the recursive Boston mechanism (RBM)<sup>13</sup> by  $\varphi^R$ . For each  $P \in \mathcal{P}^{|I|}$ ,  $\varphi^R$  is defined by setting  $\varphi^R(P)$  equal to the matching obtained by the following recursive immediate acceptance algorithm:

**Step 1:** Consider only the first choice of students. For each school  $a \in O$ , up to  $q_a$  students whose first choice is  $a$  with the highest priority for it (all students if fewer than  $q_a$ ) are assigned to school  $a$  permanently. If a student puts the null school in the first preference ranking, then he is assigned the null school. Remove the set of students who are assigned a school in this step and their corresponding assignments.

⋮

**Step  $k$ :** Consider the **subproblem**<sup>14</sup> induced by the removal of students who get a seat in the previous steps and their assignments. For each school  $a \in O$  with  $q_a^k$  copies available, up to  $q_a^k$  students whose favorite school is  $a$  under the subproblem with the highest priority for it (all students if fewer than  $q_a^k$ ) are assigned to school  $a$  permanently. If a student puts the null school in the first preference ranking under the subproblem, then he is assigned the null school. Remove the set of students who are assigned a school in this step and their corresponding assignments.

The algorithm terminates when all students have been removed. The following example shows that the BOSM and RBM are different.

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13. This mechanism is first introduced by Chen (2013c). The RBM is similar to the well-known BOSM. While the BOSM considers the reduced problem of the original problem after removing students and their assignments in the previous step, RBM considers the subproblem. We show that RBM does not satisfy strategy-proofness and stability, but satisfies Pareto efficiency. Moreover, the set of Nash equilibrium outcomes of the preference revelation game induced by RBM is equivalent to the set of stable matchings with respect to the true preferences of students.
14. Given a problem, a matching, and a subset of students, a **reduced problem** is defined by removing the subset of students and their corresponding assignments under the given matching, while the preference profile remains the same. A **subproblem** with respect to the given matching and subset of students is defined by removing the subset of students and their corresponding assignments under the given matching, while the preference profile of the subproblem are defined over the set of schools with strictly positive capacity left. That is, the schools' capacities are reduced and the students' preferences are defined over the set of schools with strictly positive capacity left. While BM considers the reduced problem of the original problem after removing students and their assignments in the previous step, RBM considers the subproblem.

## 2.3 Mechanisms

**EXAMPLE 2.1** The problem  $P$  is defined as follows. Let  $I = \{i, j, l, m\}$ ,  $O = \{a, b, c\}$ , and the capacity of each school is one. The preferences of students and the priority orders of schools are listed below:

$P_i$	$P_j$	$P_l$	$P_m$	$\succ_a$	$\succ_b$	$\succ_c$
<u>a</u>	a	c	<u>c</u>	i	j	i
b	c	<u>b</u>	a	j	l	j
c	<u>b</u>	a	b	l	i	m
$\emptyset$	<u><math>\emptyset</math></u>	<u><math>\emptyset</math></u>	$\emptyset$	m	m	l

It is easy to calculate that  $\varphi^R(P)$  results in the above matching marked with boxes, and  $\varphi^B(P)$  results in the above underlined matching.

Note that BOSM and RBM both assign  $a$  to  $i$  and  $c$  to  $m$  in the first step. The difference occurs in step 2. In step 2, BOSM considers the second choices of  $j$  and  $l$ , and assigns  $b$  to  $l$  because  $j$  puts  $b$  in a lower preference ranking than  $l$  does. On the contrary, RBM considers the subproblem by removing  $a, c, i$  and  $m$ . In the updated preference profile,  $j$  and  $l$  both list  $b$  as their first choice. Because  $j$  has higher priority than  $l$  for  $b$ ,  $b$  is assigned to  $j$  under RBM.

Finally, we summarize the information given in this chapter in the following table.

	$\varphi^S$	$\varphi^T$	$\varphi^B$	$\varphi^O$	$\varphi^f$	$\varphi^R$
Non-wastefulness	✓	✓	✓	✓	✓	✓
Fairness	✓	×	×	✓	×	×
Stability	✓	×	×	✓	×	×
Pareto Efficiency	×	✓	✓	×	✓	✓
Strategy-proofness	✓	✓	×	×	✓	×
Non-bossiness	×	✓	×	×	✓	×
Group Strategy-proofness	×	✓	×	×	✓	×
Respect of Improvements	✓	✓	✓	×	✓	✓
Robust Stability	×	×	×	×	×	×

Table 2.1: School Choice Mechanisms and Basic Axioms





# Chapter 3

## Axioms for Deferred Acceptance

### 3.1 Introduction

School choice studies how to allocate public school seats to students based on schools' priority over students, with each student being assigned to one seat and each school is allocated to the number of students no more than its capacity. Formally, a school choice problem consists of a set of students, a set of school types, a capacity vector of schools, a preference profile of students over schools, and a priority profile of schools over students. A school choice mechanism is a systematic way of finding a matching from schools to students for each problem. The seminal work of Abdulkadiroğlu and Sönmez (2003a) discuss three well-known school choice mechanisms: the student-optimal stable mechanism (SOSM) determined by the Gale-Shapley student-proposing deferred acceptance algorithm, the student-optimal top trading cycles mechanism (TTCM) determined by the student-proposing top trading cycles algorithm, and the student-optimal Boston mechanism (BOSM) determined by the student-proposing immediate acceptance algorithm. The previous authors point out that the BOSM lacks strategy-proofness and fairness and suggest to substitute it with the other two mechanisms. In reality, the BOSM was replaced by SOSM in the Boston Public School System in 2005, and New Orleans Recovery School District became the first school district to adopt TTCM in 2012. Each year, a large population of students are influenced by which mechanisms their school districts choose, which manifests the significance of school choice problems.

This chapter studies the school choice problem in an axiomatic way by proposing new axioms for school choice mechanisms related to monotonicity, fairness, and consistency, and providing new characterizations of the SOSM. First, we define eight axioms weaker than stability. Three of them are crucial to our analysis. A mechanism satisfies *mutual best* if a student

### Chapter 3 Axioms for Deferred Acceptance

$i$  is always assigned his non-null favorite school  $a$  if he has the highest priority for it. A mechanism satisfies *strong top best* if a student  $i$  is always assigned his non-null favorite school  $a$  if he has the  $q_a$  highest priority for it among all students who find this school acceptable. A mechanism satisfies *strong group rationality* if the mechanism never assigns a student  $i$  to a school worse than the non-null school  $a$  whenever  $i$  has the  $q_a$  highest priority for  $a$  among all students who find this school acceptable. Clearly, stability implies strong group rationality, strong group rationality implies strong top best, and strong top best implies mutual best.

Next, we define weak consistency. Informally, a mechanism satisfies *weak consistency* if whenever we remove a subset of students with their assignments and apply the mechanism to the smaller reduced problem, no remaining student is worse off. In other words, all remaining students weakly prefer the new matching determined by the reduced problem to the original matching. The SOSM obviously satisfies weak consistency, while the TTCM and BOSM violate it. Consistency is a classical and important axiom in matching theory.<sup>15</sup> However, in one-sided school choice problems, it is difficult to figure out nice mechanisms satisfying consistency for all problems other than SSD.<sup>16</sup> Hence, consistency is too demanding for school choice. We thus propose weak consistency, which coincidentally is satisfied by SOSM. The interpretation of weak consistency shows although it is not as powerful as consistency, it is not a “negative” axiom.

We then define rank monotonicity. Like Maskin monotonicity and weak Maskin monotonicity, rank monotonicity restricts how a mechanism reacts to changes in the preferences of students. We say that a preference profile  $P'$  is a rank monotonic transformation of a preference profile  $P$  at a matching  $\mu$  if for all students, any school that is preferred to  $\mu$  under  $P'$  with the preference ranking  $k$  is also placed in the  $k^{\text{th}}$  preference ranking of  $P$ . A mechanism  $\varphi$  satisfies *rank monotonicity* if every student weakly prefers the matching  $\varphi(P')$  to the matching  $\varphi(P)$ , whenever  $P'$  is a rank monotonic transformation of  $P$  at  $\varphi(P)$ . If  $P'$  is a rank monotonic transformation of  $P$  at  $\varphi(P)$ , then the interpretation of the change in reported preferences from  $P$  to  $P'$  is that some students increase the preference ranking of their assignment  $\varphi_i(P)$ , and keep relative preference rankings of the schools

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15. See Thomson (2011) for an introduction.

16. Although the SOSM and TTCM are not consistent, they are consistent under some conditions. Ergin (2002) shows that the SOSM recovers consistency if and only if the priority structure is Ergin-acyclic. Kesten (2006) shows that the TTCM recovers consistency if and only if the priority structure satisfies another stronger acyclicity condition.

above their assignment under the new preference to be the same. Maskin monotonicity due to Maskin (1999) and weak Maskin monotonicity due to Kojima and Manea (2010a) both imply rank monotonicity. The SOSM, TTCM and BOSM all satisfy rank monotonicity.

Finally and importantly, this chapter provides new characterizations of the SOSM. Prior to our research, the SOSM has been characterized by constrained efficiency subject to stability in Gale and Shapley (1962), stability and strategy-proofness in Alcade and Barberà (1994), stability and respect of improvements in Balinski and Sönmez (1999), stability and weak Maskin monotonicity in Kojima and Manea (2010a), and non-wastefulness, strong top best (mutual best), and IR (individually rational) monotonicity for all substitutable priorities in Morrill (2013). Ehlers and Klaus (2012) characterize the SOSM for all responsive priorities. However, we find the previous studies insufficient in the following aspects. First, most of the results characterize by means of stability. Second, even if some of the results do not refer to stability, the authors characterize on restricted priority domains. Therefore, we seek to characterize the SOSM without stability for all priority structures.

Based on the new axioms we propose and the previous research, we prove that a school choice mechanism is equivalent to the SOSM if and only if it satisfies whichever of the following groups of axioms: stability, rank monotonicity; non-wastefulness, strong top best, weak Maskin monotonicity; non-wastefulness, strong group rationality, rank monotonicity; non-wastefulness, mutual best, weak consistency, strategy-proofness; non-wastefulness, mutual best, weak consistency, rank monotonicity; non-wastefulness, mutual best, weak consistency, respect of improvements. Our new characterizations identify the tradeoff between SOSM and TTCM, BOSM, SSD, and SSOM, which will provide references for the social planner to compare alternative mechanisms.

The characterizations of other school choice mechanisms are listed below. Abdulkadiroğlu and Che (2010) characterize the TTCM for the first time, by strategy-proofness, efficiency, and recursive respect of top priorities. Morrill (2011) characterizes the TTCM by strategy-proofness or weak Maskin monotonicity, together with efficiency, mutual best, and independence of irrelevant rankings. Abdulkadiroğlu and Che (2010) and Morrill (2011) both get their results when each school has only one seat available. Recently, Dur (2012) characterizes the TTCM by strategy-proofness, efficiency, and other three weak auxiliary axioms, when a school capacity greater than one is permitted. The BOSM is characterized by Kojima and Ünver (2013), Afacan (2013), and Chen (2011), respectively.

## 3.2 Axioms Related to Stability

In this section, we introduce eight axioms weaker than stability.

**Definition 3.1.** A mechanism  $\varphi$  satisfies **mutual best** if for each  $P \in \mathcal{P}^I$ ,  $i \in I$  and  $a \in O$ ,

$$P_i(a) = 1 \ \& \ \succ_a(i) = 1 \Rightarrow \varphi_i(P) = a.$$

A mechanism satisfies mutual best if a student is always assigned his non-null favorite school whenever he has the highest priority for it. Toda (2006) is the first to introduce mutual best. He considers mutual best in two-sided matching markets. In his work, mutually best requires a pair of mutually best agents (man and woman, or student and college) to be matched at every solution outcome.<sup>17</sup>

For any problem  $P$  and  $a \in O$ , let  $I_a^*$  be the set of students that find school  $a$  acceptable, i.e.,

$$I_a^* = \{i | a P_i \emptyset\}.$$

**Definition 3.2.** A mechanism  $\varphi$  satisfies **strong mutual best** if for each  $P \in \mathcal{P}^I$ ,  $i \in I$  and  $a \in O$ ,

$$P_i(a) = 1 \ \& \ \succ_a |_{I_a^*}(i) = 1 \Rightarrow \varphi_i(P) = a.$$

A mechanism satisfies strong mutual best if a student is always assigned his non-null favorite school whenever he has the highest priority for it among the set of students who find this school acceptable. It is obvious that strong mutual best implies mutual best, but the converse does not hold.  $\varphi^S$  and  $\varphi^B$  satisfy both mutual best and strong mutual best.  $\varphi^T$  satisfies mutual best and violates strong mutual best. Also note that strong mutual best is independent with fairness, although it is implied by stability.

**EXAMPLE 3.1:  $\varphi^T$  violates strong mutual best.** Let  $I = \{i, j, l\}$ ,  $O = \{a, b\}$ , and  $q_a = q_b = 1$ . The preference profile  $P$  and priority profile  $\succ$  are specified as follows:

$P_i$	$P_j$	$P_l$	$\succ_a$	$\succ_b$
$\boxed{a}$	$\boxed{b}$	$a$	$j$	$i$
$\emptyset$	$\emptyset$	$b$	$l$	$l$
		$\boxed{\emptyset}$	$i$	$j$

17. Note that Morrill (2013) also defines a version of mutual best. However, his definition is different from our definition. His definition of mutual best corresponds to strong top best (definition 3.4) in our research.

### 3.2 Axioms Related to Stability

$\varphi^T(P)$  is the above matching marked with boxes. It is easy to observe that  $P_i(a) = 1$ ,  $\succ_a |_{I_a^*}(l) = 1$ , but  $\varphi_i^T(P) = \emptyset \neq a$ . This shows that  $\varphi^T$  violates strong mutual best.

**Definition 3.3.** A mechanism  $\varphi$  satisfies **top best** if for each  $P \in \mathcal{P}^{|I|}$ ,  $i \in I$  and  $a \in O$ ,

$$P_i(a) = 1 \ \& \ \succ_a(i) \leq q_a \Rightarrow \varphi_i(P) = a.$$

A mechanism satisfies top best if a student is always assigned his non-null favorite school  $a$  whenever he has the  $q_a$  highest priority for it.

**Definition 3.4.** A mechanism  $\varphi$  satisfies **strong top best** if for each  $P \in \mathcal{P}^{|I|}$ ,  $i \in I$  and  $a \in O$ ,

$$P_i(a) = 1 \ \& \ \succ_a |_{I_a^*}(i) \leq q_a \Rightarrow \varphi_i(P) = a.$$

A mechanism satisfies strong top best if a student is always assigned his non-null favorite school  $a$  whenever he has the  $q_a$  highest priority for it among the set of students who find this school acceptable.<sup>18</sup>

**Definition 3.5.** A mechanism  $\varphi$  satisfies **top rationality** if for each  $P \in \mathcal{P}^{|I|}$ ,  $i \in I$  and  $a \in O$ ,

$$\succ_a(i) = 1 \Rightarrow \varphi_i(P) R_i a.$$

A mechanism satisfies top rationality if the mechanism will never assign student  $i$  to a non-null school worse than  $a$  for him whenever he has the highest priority for it.<sup>19</sup> Top rationality is a weaker version of fairness. However the Boston mechanism  $\varphi^B$  violates this axiom, as shown in the example below.

**EXAMPLE 3.2:  $\varphi^B$  violates top rationality.** Let  $I = \{i, j, l\}$ ,  $O = \{a, b\}$ , and  $q_a = q_b = 1$ . The preference profile and priority profile are specified as follows:

$P_i$	$P_j$	$P_l$	$\succ_a$	$\succ_b$
<span style="border: 1px solid black; padding: 2px;"><math>a</math></span>	<span style="border: 1px solid black; padding: 2px;"><math>b</math></span>	$a$	$i$	$l$
$\emptyset$	$\emptyset$	$b$	$l$	$i$
		<span style="border: 1px solid black; padding: 2px;"><math>\emptyset</math></span>	$j$	$j$

18. Morrill (2013) is the first to define strong top best and he calls this property mutual best. He claims that the TTCM satisfies strong top best. However, as shown in example 3.1, TTCM violates this axiom.

19. Top rationality is first introduced by Abdulkadiroğlu and Che (2010). In their paper, they call this axiom individual rationality for top students.

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$\varphi^B(P)$  is the above matching marked with boxes. It is easy to observe that  $\succ_b(l) = 1$ , but  $bP_l\varphi_l^B(P) = \emptyset$ . This shows that  $\varphi^B$  violates top rationality.

**Definition 3.6.** A mechanism  $\varphi$  satisfies **strong top rationality** if for each  $P \in \mathcal{P}^{|I|}$ ,  $i \in I$  and  $a \in O$ ,

$$\succ_a |_{I_a^*}(i) = 1 \Rightarrow \varphi_i(P)R_ia.$$

A mechanism satisfies strong top rationality if the mechanism will never assign student  $i$  to a non-null school worse than  $a$  for him whenever he has the highest priority for  $a$  among the set of students who find this school acceptable.

**Definition 3.7.** A mechanism  $\varphi$  satisfies **group rationality** if for each  $P \in \mathcal{P}^{|I|}$ ,  $i \in I$  and  $a \in O$ ,

$$\succ_a(i) \leq q_a \Rightarrow \varphi_i(P)R_ia.$$

A mechanism satisfies group rationality if the mechanism will never assign student  $i$  to a school worse than  $a$  for him whenever he has the  $q_a$  highest priority for  $a$ .<sup>20</sup>

**Definition 3.8.** A mechanism  $\varphi$  satisfies **strong group rationality** if for each  $P \in \mathcal{P}^{|I|}$ ,  $i \in I$  and  $a \in O$ ,

$$\succ_a |_{I_a^*}(i) \leq q_c \Rightarrow \varphi_i(P)R_ia.$$

A mechanism satisfies strong group rationality if the mechanism will never assign student  $i$  to a school worse than  $a$  for him whenever he has the  $q_a$  highest priority for  $a$  among the set of students who find this school acceptable.

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20. Top rationality is first introduced by Dur (2012).

### 3.3 Weak Consistency

	$\varphi^S$	$\varphi^T$	$\varphi^B$	$\varphi^O$	$\varphi^f$	$\varphi^R$
Mutual Best	✓	✓	✓	✓	×	✓
Strong Mutual Best	✓	×	✓	✓	×	✓
Top Best	✓	✓	✓	✓	×	✓
Strong Top Best	✓	×	✓	✓	×	✓
Top Rationality	✓	✓	×	✓	×	×
Strong Top Rationality	✓	×	×	✓	×	×
Group Rationality	✓	✓	×	✓	×	×
Strong Group Rationality	✓	×	×	✓	×	×

Table 3.1: School Choice Mechanisms and Stability-related Axioms

### 3.3 Weak Consistency

Recall that for each  $J \subset I$ ,  $P \in \mathcal{P}^{|I|}$  and  $\mu$  with respect to the problem  $P$ ,  $P_{-J}^\mu$  is the reduced problem by removing students  $J$  and their assignments  $\mu_J$  from the original problem. That is,

$$P_{-J}^\mu = (I \setminus J, C(C \setminus \mu_J), q(C \setminus \mu_J), P_{I \setminus J} |_{C(C \setminus \mu_J)}, \succ_{C(C \setminus \mu_J)} |_{I \setminus J}).$$

**Definition 3.9.** A mechanism  $\varphi$  satisfies **consistency** if for each  $P \in \mathcal{P}^{|I|}$  and  $J \subset I$ ,

$$\varphi_i(P_{-J}^{\varphi(P)}) = \varphi_i(P), \quad \forall i \in I \setminus J.$$

A mechanism  $\varphi$  satisfies consistency if for each problem  $P$  and the matching  $\varphi(P)$ , and then for each reduced problem by imagining the departure of a subset of students with their assignments under  $\varphi(P)$ , the mechanism, when applied to the reduced problem, will choose the same assignment with the assignment under the original matching  $\varphi(P)$  for each remaining student. In words, consistency requires that the removal of a subset of students with their assignments does not change the assignment of any remaining student. Consistency and its converse has played an important role in resource allocation problems.<sup>21</sup> As we all know, the three well-known mechanisms proposed by Abdulkadiroğlu and Sönmez (2003a) all violate consistency. Moreover, the only existing mechanism satisfying consistency is the simple serial dictatorship, which is in some sense trivial for school choice problems because it totally ignores schools' priorities.

21. See Thomson (2011) for a survey.

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Moreover, authors like Ergin (2002), Kesten (2006) and Chen (2013e) have figured out the conditions on priority structures under which the three main mechanisms: SOSM, TTCM, and BOSM recover consistency. However, those conditions are rather stringent, which are difficult to be satisfied. This further proves that consistency is a too demanding property for school choice mechanisms. We therefore weaken consistency and propose weak consistency.

**Definition 3.10.** A mechanism  $\varphi$  satisfies **weak consistency** if for each  $P \in \mathcal{P}^{|I|}$  and  $J \subset I$ ,

$$\varphi_i(P_{-J}^{\varphi(P)})R_i\varphi_i(P), \forall i \in I \setminus J.$$

A mechanism  $\varphi$  satisfies weak consistency if for each problem  $P$  and the matching  $\varphi(P)$ , and then for each reduced problem by imagining the departure of a subset of students with their assignments under  $\varphi(P)$ , the mechanism, when applied to the reduced problem, will choose an assignment not worse than the assignment under the original matching  $\varphi(P)$  for each remaining student. In words, weak consistency requires that the removal of a subset of students with their assignments does not make any remaining student worse off. Obviously, weak consistency is implied by consistency.  $\varphi^S$  satisfies weak consistency, while  $\varphi^T$  and  $\varphi^B$  violate it.

**Proposition 3.1.**  $\varphi^S$  satisfies weak consistency.

*Proof.* We proceed by contradiction. Suppose that  $\varphi^S$  is not weakly consistent. Thus, we can find a problem  $P$  and a subset  $J \subset I$  such that there exist at least one student  $i \in I \setminus J$  with

$$\varphi_i^S(P)P_i\varphi_i^S(P_{-J}^{\varphi^S(P)}).$$

It is obvious that the reduced matching  $\varphi_{-J}^S(P)$  is a stable matching under the reduced problem  $P_{-J}^{\varphi^S(P)}$ . Because  $\varphi^S(P_{-J}^{\varphi^S(P)})$  is the student-optimal stable matching which Pareto dominates any other stable matchings under  $P_{-J}^{\varphi^S(P)}$ , we have

$$\varphi_i^S(P_{-J}^{\varphi^S(P)})R_i\varphi_i^S(P),$$

which contradicts the previous conclusion that  $\varphi_i^S(P)P_i\varphi_i^S(P_{-J}^{\varphi^S(P)})$ .  $\square$

**EXAMPLE 3.3:  $\varphi^T$  and  $\varphi^B$  violate weak consistency.** Let  $I = \{i, j, l\}$ ,  $O = \{a, b, c\}$ , and  $q_a = q_b = q_c = 1$ . The preferences of students  $P = (P_i, P_j, P_l)$  and the priority orders of schools  $\succ = (\succ_a, \succ_b, \succ_c)$  are listed below:



### 3.3 Weak Consistency

$P_i$	$P_j$	$P_l$	$\succ_a$	$\succ_b$	$\succ_c$
<u>a</u>	<u>c</u>	<u>c</u>	$j$	$i$	$i$
$b$	$a$	<u>b</u>	$l$	$j$	$l$
$c$	<u>b</u>	$a$	$i$	$l$	$j$

$\varphi^T(P)$  is the above underlined matching, and  $\varphi^B(P)$  is the above matching marked with boxes.

Consider  $\varphi^T(P)$ . If we remove student  $i$  together with his assignment  $\varphi_i^T(P) = a$  from the problem, we get the reduced problem  $P_{-\{i\}}^{\varphi^T(P)}$ . Let  $P' = P_{-\{i\}}^{\varphi^T(P)}$ . Note that  $I' = \{j, l\}$ ,  $D' = \{b, c\}$ . The reduced preference profile and priority profile are as follows:

$P_j$	$P_l$	$\succ'_b$	$\succ'_c$
$c$	<u>c</u>	$j$	$l$
<u>b</u>	$b$	$l$	$j$

$\varphi^T(P')$  is the above underlined matching. We conclude that  $\varphi^T$  violates weak consistency because student  $j$  becomes worse off when student  $i$  leaves in advance.

Consider  $\varphi^B(P)$ . If we remove student  $l$  together with his assignment  $c$  from the problem, we get the following reduced problem  $P_{-\{l\}}^{\varphi^B(P)}$ . Let  $P'' = P_{-\{l\}}^{\varphi^B(P)}$ . Note that  $I'' = \{i, j\}$ ,  $D'' = \{a, b\}$ . The reduced preference profile and priority profile are as follows:

$P_i$	$P_j$	$\succ''_a$	$\succ''_b$
$a$	<u>a</u>	$j$	$i$
<u>b</u>	$b$	$i$	$j$

$\varphi^B(P'')$  is the above matching marked with boxes. We conclude that  $\varphi^B$  violates weak consistency because student  $i$  becomes worse off when student  $l$  leaves in advance.

**EXAMPLE 3.4:  $\varphi^O$  violates weak consistency.** Let  $I = \{i, j, l\}$ ,  $O = \{a, b, c\}$ , and  $q_a = q_b = q_c = 1$ . The preferences of students  $P = (P_i, P_j, P_l)$  and the priority orders of schools  $\succ = (\succ_a, \succ_b, \succ_c)$  are listed below:

$P_i$	$P_j$	$P_l$	$\succ_a$	$\succ_b$	$\succ_c$
<u>a</u>	<u>c</u>	$a$	$j$	$i$	$i$
$b$	$a$	<u>b</u>	$i$	$j$	$j$
$c$	$b$	$c$	$l$	$l$	$l$

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$\varphi^O(P)$  is the above matching marked with boxes. If we remove student  $l$  together with his assignment  $b$  from the problem, we get the following reduced problem  $P_{-\{l\}}^{\varphi^O(P)}$ . Let  $P' = P_{-\{l\}}^{\varphi^O(P)}$ . Note that  $I' = \{i, j\}$ ,  $D' = \{a, c\}$ . The reduced preference profile and priority profile are as follows:

$P'_i$	$P'_j$	$\succ'_a$	$\succ'_c$
$a$	$c$	$j$	$i$
$\boxed{c}$	$\boxed{a}$	$i$	$j$

$\varphi^O(P')$  is the above matching marked with boxes. We conclude that  $\varphi^O$  violates weak consistency because student  $i$  and  $j$  both become worse off when student  $l$  leaves in advance.

	$\varphi^S$	$\varphi^T$	$\varphi^B$	$\varphi^O$	$\varphi^f$	$\varphi^R$
Consistency	×	×	×	×	✓	×
Weak Consistency	✓	×	×	×	✓	×

Table 3.2: School Choice Mechanisms and Consistency-related Axioms

## 3.4 Axioms Related to Monotonicity

To introduce Maskin monotonicity, we first introduce monotonic transformation. We say that  $P'_i$  is a monotonic transformation of  $P_i$  at  $a \in \tilde{O}$  ( $P'_i$  m.t.  $P_i$  at  $a$ ) if any school that is ranked above  $a$  under  $P'_i$  is ranked above  $a$  under  $P_i$ , that is,

$$bP'_i a \Rightarrow bP_i a, \forall b \in \tilde{O}.$$

$P'$  is a **monotonic transformation** of  $P$  at a matching  $\mu$  ( $P'$  m.t.  $P$  at  $\mu$ ) if  $P'_i$  m.t.  $P_i$  at  $\mu_i$  for all  $i$ .

**Definition 3.11.** A mechanism  $\varphi$  satisfies **Maskin monotonicity** if for each  $P \in \mathcal{P}^{|I|}$ ,

$$P' \text{ m.t. } P \text{ at } \varphi(P) \Rightarrow \varphi(P') = \varphi(P).$$

**Definition 3.12.** A mechanism  $\varphi$  satisfies **weak Maskin monotonicity** if for each  $P \in \mathcal{P}^{|I|}$ ,

$$P' \text{ m.t. } P \text{ at } \varphi(P) \Rightarrow \varphi(P') R' \varphi(P).$$

### 3.4 Axioms Related to Monotonicity

A mechanism  $\varphi$  satisfies weak Maskin monotonicity if every student weakly prefers the matching  $\varphi(P')$  to the matching  $\varphi(P)$ , whenever  $P'$  is a monotonic transformation of  $P$  at  $\varphi(P)$ . A mechanism  $\varphi$  satisfies Maskin monotonicity if the matching  $\varphi(P')$  remains the same with the matching  $\varphi(P)$ , whenever  $P'$  is a monotonic transformation of  $P$  at  $\varphi(P)$ . Maskin monotonicity obviously implies weak Maskin monotonicity.

For each  $J \subset I$ , let  $P_J^\emptyset$  be the preference profile which ranks  $\emptyset$  as the favorite school for each student in  $J$ .

**Definition 3.13.** A mechanism  $\varphi$  satisfies **population monotonicity** if for each  $P \in \mathcal{P}^{|I|}$  and  $J \subset I$ ,

$$\varphi_i(P_J, P_{I \setminus J}^\emptyset) R_i \varphi_i(P), \forall i \in J.$$

Population monotonicity is an important solidarity property, which is one of the standard axioms in the study of variable population models.<sup>22</sup> When applied to school choice problems, it says that if the number of students decreases but schools remain fixed, then all remaining students should weakly gain together.

To introduce individually rational monotonicity, we first introduce individually rational monotonic transformation. We say that  $P'_i$  is an individually rational monotonic transformation of  $P_i$  at  $a \in \tilde{O}$  ( $P'_i$  r.m.t.  $P_i$  at  $a$ ) if any school that is ranked above both  $a$  and  $\emptyset$  under  $P'_i$  is also ranked above  $a$  under  $P_i$ , that is

$$b P'_i a \ \& \ b P'_i \emptyset \Rightarrow b P_i a, \forall b \in \tilde{O}.$$

$P'$  is an **individually rational (IR) monotonic transformation** of  $P$  at a matching  $\mu$  ( $P'$  i.r.r.m.t.  $P$  at  $\mu$ ) if  $P'_i$  i.r.r.m.t.  $P_i$  at  $\mu_i$  for all  $i$ .

**Definition 3.14.** A mechanism  $\varphi$  satisfies **IR monotonicity** if for each  $P \in \mathcal{P}^{|I|}$ ,

$$P' \text{ i.r.m.t. } P \text{ at } \varphi(P) \Rightarrow \varphi(P') R' \varphi(P).$$

IR monotonicity resembles Maskin monotonicity, but the two axioms are independent. A mechanism  $\varphi$  satisfies IR monotonicity if every student weakly prefers the matching  $\varphi(P')$  to the matching  $\varphi(P)$  with respect to  $P'$ , whenever  $P'$  is a rank monotonic transformation of  $P$  at  $\varphi(P)$ . Note that if  $P'$  is an IR monotonic transformation of  $P$  at  $\varphi(P)$ , then  $P'$  is also a monotonic transformation of  $P$  at  $\varphi(P)$ . Therefore, IR monotonicity

22. See Thomson (1983a) and Thomson (1983b).

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implies weak Maskin monotonicity. Moreover, IR monotonicity also implies population monotonicity.

To introduce the new monotonicity axioms, we first introduce new forms of monotonic transformation. We say that  $P'_i$  is a rank monotonic transformation of  $P_i$  at  $a \in \tilde{O}$  ( $P'_i$  r.m.t.  $P_i$  at  $a$ ) if any school that is ranked above  $a$  under  $P'_i$  is ranked in the same preference ranking under  $P_i$ , that is,

$$bP'_i a \Rightarrow P_i(b) = P'_i(b), \forall b \in \tilde{O}.$$

$P'$  is a **rank monotonic transformation** of  $P$  at a matching  $\mu$  ( $P'$  r.m.t.  $P$  at  $\mu$ ) if  $P'_i$  r.m.t.  $P_i$  at  $\mu_i$  for all  $i$ . Intuitively, if  $P'$  r.m.t.  $P$  at  $\mu$ , then some students “truncate” their preferences by moving their assignment under  $\mu$  upwards. It is easy to see that rank monotonic transformation has implication for a smaller set of preference profile pair  $(P, P')$  than monotonic transformation.

**Definition 3.15.** A mechanism  $\varphi$  satisfies **rank monotonicity** if for each  $P \in \mathcal{P}^{|I|}$ ,

$$P' \text{ r.m.t. } P \text{ at } \varphi(P) \Rightarrow \varphi(P') R' \varphi(P).$$

**Definition 3.16.** A mechanism  $\varphi$  satisfies **strong rank monotonicity** if for each  $P \in \mathcal{P}^{|I|}$ ,

$$P' \text{ r.m.t. } P \text{ at } \varphi(P) \Rightarrow \varphi(P') = \varphi(P).$$

Like weak Maskin monotonicity and Maskin monotonicity, rank monotonicity and strong rank monotonicity restrict how a mechanism reacts to changes in the preferences of students. A mechanism  $\varphi$  satisfies rank monotonicity if every student weakly prefers the matching  $\varphi(P')$  to the matching  $\varphi(P)$ , whenever  $P'$  is a rank monotonic transformation of  $P$  at  $\varphi(P)$ . A mechanism  $\varphi$  satisfies strong rank monotonicity if the matching  $\varphi(P')$  remains the same with the matching  $\varphi(P)$ , whenever  $P'$  is a rank monotonic transformation of  $P$  at  $\varphi(P)$ . Note that if  $P'$  is a rank monotonic transformation of  $P$  at  $\varphi(P)$ , then  $P'$  is also a monotonic transformation of  $P$  at  $\varphi(P)$ . Therefore, rank monotonicity is weaker than weak Maskin monotonicity and strong rank monotonicity is weaker than Maskin monotonicity. The SOSM, TTCM, and BOSM all satisfy rank monotonicity, and only SOSM among them violates strong rank monotonicity.

To introduce the next axiom, we say that  $P'_i$  is an individually rational (IR) rank monotonic transformation of  $P_i$  at  $a \in \tilde{O}$  ( $P'_i$  r.m.t.  $P_i$  at  $a$ ) if any

### 3.5 Characterizations of the Student-optimal Stable Mechanism

school that is ranked above both  $a$  and  $\emptyset$  under  $P'_i$  is ranked in the same preference ranking under  $P_i$ , that is,

$$bP'_i a \ \& \ bP'_i \emptyset \Rightarrow P_i(b) = P'_i(b), \ \forall b \in \tilde{O}.$$

$P'$  is an **IR rank monotonic transformation** of  $P$  at a matching  $\mu$  ( $P'$  i.r.r.m.t.  $P$  at  $\mu$ ) if  $P'_i$  r.m.t.  $P_i$  at  $\mu_i$  for all  $i$ .

**Definition 3.17.** A mechanism  $\varphi$  satisfies **IR rank monotonicity** if for each  $P \in \mathcal{P}^{|I|}$ ,

$$P' \text{ i.r.r.m.t. } P \text{ at } \varphi(P) \Rightarrow \varphi(P') R' \varphi(P).$$

A mechanism  $\varphi$  satisfies IR rank monotonicity if every student weakly prefers the matching  $\varphi(P')$  to the matching  $\varphi(P)$  under  $P'$  whenever  $P'$  is an IR rank monotonic transformation of  $P$  at  $\varphi(P)$ . Note that if  $P'$  is an IR rank monotonic transformation of  $P$  at  $\varphi(P)$ , then  $P'$  is also an IR monotonic transformation of  $P$  at  $\varphi(P)$ . Therefore, IR rank monotonicity is weaker than IR monotonicity. Among the three mechanisms proposed by Abdulkadiroğlu and Sönmez (2003a), only the TTCM violates IR rank monotonicity. It is also obvious that IR rank monotonicity implies both rank monotonicity and population monotonicity.

IR rank monotonicity resembles strong rank monotonicity, but the two axioms are independent.

	$\varphi^S$	$\varphi^T$	$\varphi^B$	$\varphi^O$	$\varphi^f$	$\varphi^R$
Maskin Monotonicity	×	✓	×	×	✓	×
Weak Maskin Monotonicity	✓	✓	×	×	✓	×
IR Monotonicity	✓	×	×	×	✓	×
IR Rank Monotonicity	✓	×	✓	×	✓	✓
Strong Rank Monotonicity	×	✓	✓	×	✓	✓
Rank Monotonicity	✓	✓	✓	×	✓	✓
Population Monotonicity	✓	×	✓	✓	✓	✓

Table 3.3: School Choice Mechanisms and Monotonicity-related Axioms

## 3.5 Characterizations of the Student-optimal Stable Mechanism

**Proposition 3.2.** (Alcalde & Barberà, 1994) A mechanism  $\varphi$  satisfies stability and strategy-proofness if and only if  $\varphi = \varphi^S$ .

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**Proposition 3.3.** (Balinski & Sönmez, 1999) *A mechanism  $\varphi$  satisfies stability and respect of improvements if and only if  $\varphi = \varphi^S$ .*

**Proposition 3.4.** (Kojima & Manea, 2010a) *A mechanism  $\varphi$  satisfies stability and weak Maskin monotonicity if and only if  $\varphi = \varphi^S$ .*

**Proposition 3.5.** (Morrill, 2013) *A mechanism  $\varphi$  satisfies non-wastefulness, strong top best<sup>23</sup> and IR monotonicity<sup>24</sup> for all substitutable priorities if and only if  $\varphi = \varphi^S$ .*

It is clear that the existing characterizations are not satisfactory due to the following reasons. First, proposition 2, 3, and 4 all characterize by stability. In other words, the authors identify stability as the unique feature of SOSM. Second, proposition 5 characterizes for the restricted priority domain. Third, except for stability, the previous authors do not figure out the main difference between SOSM and the other mechanisms like BOSM, TTCM, SSOM, SSD, and so on. Next, we provide six new characterizations of SOSM. All of our results are derived on full acceptant priority domain. Moreover, we weaken stability in most of the results. Third, except for stability, our characterizations reveal the unique feature of SOSM over the other mechanisms.

**Theorem 3.1.** *A mechanism  $\varphi$  satisfies stability and rank monotonicity if and only if  $\varphi = \varphi^S$ .*

*Proof.* It is obvious that  $\varphi^S$  satisfies both stability and rank monotonicity. For the only if part, fix a mechanism  $\varphi$  satisfying stability and rank monotonicity. To show that  $\varphi = \varphi^S$ , we need to show that for each  $P \in \mathcal{P}^{|I|}$ ,  $\varphi(P) = \varphi^S(P)$ . For each preference profile  $P \in \mathcal{P}^{|I|}$ , let  $P'$  be the truncation of  $P$  at  $\varphi^S(P)$ , i.e.,  $P'$  is a modified preference profile such that for each  $i \in I$  and  $a \in \tilde{O}$ ,

- (i)  $aR_i\varphi_i^S(P) \Rightarrow P'_i(a) = P_i(a)$ ; and
- (ii)  $P'_i(\emptyset) = P'_i(\varphi_i^S(P)) + 1$ , if  $\varphi_i^S(P) \neq \emptyset$ .

23. Note that in the original paper of Morrill (2013), the author uses mutual best instead of strong top best.

24. As individually rational (IR) monotonicity implies population monotonicity and weak Maskin monotonicity, it is easy to see that for all substitutable priorities, the SOSM is the unique mechanism satisfying non-wastefulness, strong top best, population monotonicity and weak Maskin monotonicity, which is exactly what theorem 1 of Morrill (2013) tells us. However, population monotonicity here is redundant. That is, there exists no mechanism which satisfies non-wastefulness, strong top best, and weak Maskin monotonicity, but violates population monotonicity. For more information, see theorem 3.6 in the current paper.

### 3.5 Characterizations of the Student-optimal Stable Mechanism

Kojima and Manea (2010a) establish that  $\varphi^S(P)$  is the unique stable allocation at  $P'$ . Because  $\varphi$  is a stable mechanism, we have  $\varphi(P') = \varphi^S(P)$ . By the construction of  $P'$  and  $\varphi(P') = \varphi^S(P)$ , we have  $P$  r.m.t.  $P'$  at  $\varphi(P')$ . As  $\varphi$  satisfies rank monotonicity, it follows that  $\varphi(P)R\varphi(P')$ , and naturally

$$\varphi(P)R\varphi^S(P).$$

Since  $\varphi(P)$  is a stable matching and  $\varphi^S(P)$  is the student-optimal stable matching which Pareto dominates any other stable matchings under  $P$ , we have

$$\varphi^S(P)R\varphi(P).$$

$\varphi(P)R\varphi^S(P)$  and  $\varphi^S(P)R\varphi(P)$  imply  $\varphi(P) = \varphi^S(P)$ .  $\square$

**Independence of axioms:**  $\varphi^O$  satisfies stability but violates rank monotonicity.  $\varphi^T$ ,  $\varphi^B$  and  $\varphi^f$  all satisfy rank monotonicity but violate stability.

**EXAMPLE 3.5:  $\varphi^O$  violates rank monotonicity.** Let  $I = \{i, j, l\}$ ,  $O = \{a, b, c\}$ , and  $q_a = q_b = q_c = 1$ . The preferences of students  $P = (P_i, P_j, P_l)$  and the priority orders of schools  $\succ = (\succ_a, \succ_b, \succ_c)$  are listed below:

$P_i$	$P_j$	$P_l$	$P'_j$	$\succ_a$	$\succ_b$	$\succ_c$
$\boxed{b}$	$c$	$b$	$c$	$i$	$j$	$l$
$a$	$\boxed{a}$	$a$	$a$	$j$	$i$	$i$
$c$	$\emptyset$	$\boxed{c}$	$b$	$l$	$l$	$j$
$\emptyset$		$\emptyset$	$\emptyset$			

$\varphi^O(P)$  is the above matching marked with boxes. Now we consider the preference profile  $P' = (P_i, P'_j, P_l)$ . Note that  $P'$  r.m.t.  $P$  at  $\varphi^O(P)$ . It is easy to calculate that  $\varphi^O(P_i, P'_j, P_l)$  is the following matching:

$i$	$j$	$l$
$a$	$b$	$c$

which makes both student  $i$  and  $j$  worse off. This shows that  $\varphi^O$  violates rank monotonicity.

**Theorem 3.2.** *A mechanism  $\varphi$  satisfies non-wastefulness, strong top best, and weak Maskin monotonicity if and only if  $\varphi = \varphi^S$ .*

*Proof.* It is obvious that  $\varphi^S$  satisfies non-wastefulness, strong top best, and weak Maskin monotonicity. We need only to show the only if part. Fix a mechanism  $\varphi$  that satisfies non-wastefulness, strong top best, and weak Maskin monotonicity. To show that  $\varphi = \varphi^S$ , by theorem 3.3, we

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need only to show that for each  $P$ ,  $\varphi(P)$  satisfies fairness. We proceed by contradiction. Suppose  $\varphi(P)$  is not fair, i.e., there exist  $i, j \in I$  and  $a \in O$  such that  $aP_i\varphi_i(P)$ ,  $i \succ_a j$ , and  $j \in \varphi_j(P)$ .

For each  $J \subseteq I$  and  $l \in J$ , let  $P_J^\varphi$  be the preference profile which ranks  $\varphi_l(P)$  as the most preferred school and  $\emptyset$  as the second preferred school if  $\varphi_l(P) \neq \emptyset$ . Let  $P_i^{a\varphi_i(P)}$  be the preferences for student  $i$ , which ranks  $a$  first and  $\varphi_i(P)$  second. Consider the following preference profile

$$P' = (P_i^{a\varphi_i(P)}, P_{I \setminus \{i\}}^\varphi).$$

Note that  $P'$  m.t.  $P$  at  $\varphi(P)$ .<sup>25</sup> As  $\varphi$  satisfies weak Maskin monotonicity, it follows that for arbitrary  $l \in I \setminus \{i\}$ ,  $\varphi_l(P') R'_l \varphi_l(P)$ . As  $\varphi_l(P)$  is the favorite school of student  $l$  under  $P'$ , we have

$$\varphi_l(P') = \varphi_l(P), \forall l \in I \setminus \{i\}.$$

From the construction of  $P' = (P_i^{a\varphi_i(P)}, P_{I \setminus \{i\}}^\varphi)$ , it is easy to see that under  $P'$ ,  $I_a^* = \varphi_a(P) \cup \{i\}$ . As  $i \succ_a j$  and  $|I_a^*| = |\varphi_a(P) \cup \{i\}| = q_a + 1$ , we have  $\succ_a |I_a^*(i)| \leq q_a$ . As  $\varphi$  satisfies strong top best,  $P_i^{a\varphi_i(P)}(a) = 1$  and  $\succ_a |I_a^*(i)| \leq q_a$  imply that

$$\varphi_i(P') = a.$$

Therefore,  $\varphi_a(P') = \varphi_a(P) \cup \{i\}$ . As  $aP_i\varphi_i(P)$  and  $\varphi$  is non-wasteful, it follows that  $|\varphi_a(P)| = q_a$  and  $i \notin \varphi_a(P)$ . Hence,

$$|\varphi_a(P')| = |\varphi_a(P) \cup \{i\}| = q_a + 1.$$

This shows that  $\varphi(P')$  allocates school  $a$  to  $q_a + 1$  students, which is a contradiction with the feasibility of  $\varphi$ .  $\square$

**Independence of axioms:**  $\varphi^T$  satisfies non-wastefulness and weak Maskin monotonicity but violates strong top best.  $\varphi^B$  satisfies non-wastefulness and strong top best but violates weak Maskin monotonicity. The following mechanism  $\varphi^\emptyset$  satisfies strong top best and weak Maskin monotonicity but violates non-wastefulness.

25. Here  $P'$  is a monotonic transformation but not a rank monotonic transformation of  $P$ . Therefore, weak Maskin monotonicity can not be relaxed to rank monotonicity in theorem 3.6.



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**Definition 3.18.** For each  $P \in \mathcal{P}^{|I|}$  and  $i \in I$ , let

$$\varphi_i^\emptyset(P) = \begin{cases} a, & \text{if } P_i(\emptyset) = 1, P_i(a) = 2, \succ_a(i) = 1, \text{ and } P_j(a) < P_j(\emptyset) \text{ for each } j \in I \setminus \{i\}, \\ \varphi_i^S(P), & \text{otherwise.} \end{cases}$$

**Theorem 3.3.** A mechanism  $\varphi$  satisfies non-wastefulness, strong group rationality, and rank monotonicity if and only if  $\varphi = \varphi^S$ .

*Proof.* It is obvious that  $\varphi^S$  satisfies non-wastefulness, strong group rationality, and rank monotonicity. We need only to show the only if part. Fix a mechanism  $\varphi$  that satisfies non-wastefulness, strong group rationality, and rank monotonicity. To show that  $\varphi = \varphi^S$ , by theorem 3.5, we need only to show that for each  $P$ ,  $\varphi(P)$  satisfies fairness. We proceed by contradiction. Suppose  $\varphi(P)$  is not fair, i.e., there exist  $i, j \in I$ ,  $a \in O$  such that  $a P_i \varphi_i(P)$ ,  $i \succ_a j$ , and  $j \in \varphi_j(P)$ .

For each  $J \subseteq I$  and  $l \in J$ , recall that  $P_J^\varphi$  is the preference profile which ranks  $\varphi_l(P)$  as the most preferred school and  $\emptyset$  as the second preferred school if  $\varphi_l(P) \neq \emptyset$ . Consider the following preference profile

$$P' = (P_i, P_{I \setminus \{i\}}^\varphi).$$

Note that  $P'$  r.m.t.  $P$  at  $\varphi(P)$ . As  $\varphi$  satisfies rank monotonicity, it follows that for arbitrary  $l \in I \setminus \{i\}$ ,  $\varphi_l(P') R'_l \varphi_l(P)$ . As  $\varphi_l(P)$  is the favorite school of student  $l$  under  $P'$ , we have

$$\varphi_l(P') = \varphi_l(P), \forall l \in I \setminus \{i\}.$$

From the construction of  $P' = (P_i, P_{I \setminus \{i\}}^\varphi)$ , it is easy to see that under  $P'$ ,  $I_a^* = \varphi_a(P) \cup \{i\}$ . As  $i \succ_a j$  and  $|I_a^*| = |\varphi_a(P) \cup \{i\}| = q_a + 1$ , we have  $\succ_a |I_a^*(i)| \leq q_a$  under  $P'$ . Because  $\varphi$  satisfies strong group rationality, it follows that  $\varphi_i(P') R_i a$ . By non-wastefulness of  $\varphi$ , for each  $b \in U_i(P_a)$ ,  $|\varphi_b(P')| = |\varphi_b(P)| = q_b$ . Therefore, it is impossible to assign student  $i$  to a school strictly better than  $a$ , i.e.,  $\varphi_i(P') \notin U_i(P_a)$ . Hence,  $a R_i \varphi_i(P')$ .  $\varphi_i(P') R_i a$  and  $a R_i \varphi_i(P')$  imply

$$\varphi_i(P') = a.$$

Therefore,  $\varphi_a(P') = \varphi_a(P) \cup \{i\}$ . As  $a P_i \varphi_i(P)$  and  $\varphi$  is non-wasteful, it follows that  $|\varphi_a(P)| = q_a$  and  $i \notin \varphi_a(P)$ . Hence,

$$|\varphi_a(P')| = |\varphi_a(P) \cup \{i\}| = q_a + 1.$$

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This shows that  $\varphi(P')$  allocates school  $a$  to  $q_a + 1$  students, which is a contradiction with the feasibility of  $\varphi$ .  $\square$

**Independence of axioms:**  $\varphi^\emptyset$  (definition 3.18) satisfies strong group rationality and rank monotonicity but violates non-wastefulness.  $\varphi^T$  and  $\varphi^B$  both satisfy non-wastefulness and rank monotonicity but violate strong group rationality.  $\varphi^O$  satisfies non-wastefulness and strong group rationality but violates rank monotonicity.

**Theorem 3.4.** *A mechanism  $\varphi$  satisfies non-wastefulness, mutual best, weak consistency, and strategy-proofness if and only if  $\varphi = \varphi^S$ .*

*Proof.* We have known that  $\varphi^S$  satisfies non-wastefulness, mutual best, weak consistency, and strategy-proofness. We need only to show the only if part. Fix a mechanism  $\varphi$  that satisfies non-wastefulness, mutual best, weak consistency, and strategy-proofness. To show that  $\varphi = \varphi^S$ , by theorem 3.1, we need only to show that  $\varphi$  satisfies fairness. To prove  $\varphi$  satisfies fairness, we use the following proposition.

**Proposition 3.6.** *If a mechanism  $\varphi$  satisfies non-wastefulness, mutual best, and weak consistency, then it satisfies fairness.*

*Proof.* Arbitrarily choose a preference profile  $P$ , we then prove that  $\varphi(P)$  is fair. We proceed by contradiction. Suppose  $\varphi(P)$  is not fair, i.e., there exist  $i, j \in I$ ,  $a \in O$  such that  $aP_i\varphi_i(P)$ ,  $i \succ_a j$ , and  $j \in \varphi_a(P)$ .

Consider the reduced problem  $P_{-[I \setminus \{i, j\}]}$  by removing  $I \setminus \{i, j\}$  and  $\varphi_{[I \setminus \{i, j\}]}(P)$  from the original problem. Thus, only student  $i$  and  $j$  remain. Let

$$P' = P_{-[I \setminus \{i, j\}]}^{\varphi(P)}.$$

It is easy to have that in the reduced problem,  $P'_i(a) = 1$ ,  $P'_i(\varphi_i(P)) = 2$ , and  $P'_j(a) = 1$ . Since  $i \succ_a j$  and  $i, j$  are the only remaining students in the reduced problem, we have student  $i$  has the highest priority for school  $a$  under the reduced problem  $P'$ . Moreover,  $a$  has only one capacity left. As  $P'_i(a) = 1$ ,  $i$  has the highest priority for  $a$ , and  $\varphi$  satisfies mutual best, we have

$$\varphi_i(P') = a.$$

By weak consistency of  $\varphi$ ,  $\varphi_j(P')R_j a$ . Since  $P'_j(a) = 1$ , it follows that

$$\varphi_j(P') = a.$$

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Hence, under  $P'$ ,

$$|\varphi_a(P')| = |\{i, j\}| = 2$$

and school  $a$  has only one capacity left. This shows that  $\varphi(P')$  allocates school  $a$  to the number of students more than its capacity, which is a contradiction with the feasibility of  $\varphi$ .  $\square$

$\square$

**Independence of axioms:**  $\varphi^\emptyset$  (definition 3.18) satisfies mutual best, weak consistency, and strategy-proofness but violates non-wastefulness.  $\varphi^f$  satisfies non-wastefulness, weak consistency, and strategy-proofness but violates mutual best.  $\varphi^T$  satisfies non-wastefulness, mutual best, and strategy-proofness but violates weak consistency. The following mechanism  $\varphi^W$  satisfies non-wastefulness, mutual best, and weak consistency but violates strategy-proofness.

**Definition 3.19.** For each  $P \in \mathcal{P}^{|I|}$  and  $i \in I$ , let

$$\varphi_i^W(P) = \begin{cases} \varphi_i^O(P), & \text{if } \varphi^O(P) \text{ is weakly consistent,} \\ \varphi_i^S(P), & \text{otherwise.} \end{cases}$$

**EXAMPLE 3.6:**  $\varphi^O(P)$  satisfies non-wastefulness, mutual best, and weak consistency but  $\varphi^O$  is not strategy-proof under  $P$ . Let  $I = \{i, j, l\}$ ,  $O = \{a, b, c\}$ , and  $q_a = q_b = q_c = 1$ . The preferences of students  $P = (P_i, P_j, P_l)$  and the priority orders of schools  $\succ = (\succ_a, \succ_b, \succ_c)$  are listed below:

$P_i$	$P_j$	$P_l$	$P_j''$	$\succ_a$	$\succ_b$	$\succ_c$
<span style="border: 1px solid black;">b</span>	$c$	$b$	$c$	$i$	$j$	$l$
$a$	<span style="border: 1px solid black;">a</span>	$a$	$\emptyset$	$j$	$i$	$i$
$c$	$\emptyset$	<span style="border: 1px solid black;">c</span>		$l$	$l$	$j$
$\emptyset$		$\emptyset$				

The school-optimal stable matching  $\varphi^O(P)$  is the above matching marked with boxes. It is easy to see that  $\varphi^O(P)$  satisfies non-wastefulness, mutual best, and weak consistency. If student  $j$  reports  $P_j''$ , then  $\varphi^O(P_i, P_j'', P_l) = \varphi^S(P_i, P_j'', P_l)$  is the following matching

$i$	$j$	$l$
$b$	$c$	$a$

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which makes student  $j$  better off. This shows that  $\varphi^O$  is not strategy-proof, even when it satisfies non-wastefulness, mutual best and weak consistency.

**Theorem 3.5.** *A mechanism  $\varphi$  satisfies non-wastefulness, mutual best, weak consistency, and respect of improvements if and only if  $\varphi = \varphi^S$ .*

*Proof.* Theorem 3.2 and proposition 3.2 complete the proof. □

**Independence of axioms:**  $\varphi^\emptyset$  (definition 3.18) satisfies mutual best, weak consistency, and respect of improvements but violates non-wastefulness.  $\varphi^f$  satisfies non-wastefulness, weak consistency, and respect of improvements but violates mutual best.  $\varphi^T$  and  $\varphi^B$  both satisfy non-wastefulness, mutual best, and respect of improvements but violate weak consistency.  $\varphi^W$  (definition 3.19) satisfies non-wastefulness, mutual best, and weak consistency but violates respect of improvements.

**EXAMPLE 3.7:  $\varphi^O(P)$  satisfies non-wastefulness, mutual best, and weak consistency but  $\varphi^O$  does not respect improvements under  $P$ .** Let  $I = \{i, j\}$ ,  $O = \{a, b\}$ , and  $q_a = q_b = 1$ . The preferences of students  $P = (P_i, P_j)$  and the priority orders of schools  $\succ = (\succ_a, \succ_b)$  are listed below:

$P_i$	$P_j$	$\succ_a$	$\succ_b$	$\succ'_a$
<u>a</u>	<u>b</u>	$i$	$i$	$j$
<u>b</u>	<u>a</u>	$j$	$j$	$i$
$\emptyset$	$\emptyset$			

$\varphi^O(P, \succ)$  is the above matching marked with boxes. It is easy to see that  $\varphi^O(P, \succ)$  satisfies non-wastefulness, mutual best, and weak consistency. Moreover,  $\succ' = (\succ'_a, \succ'_b)$  is an improvement of student  $j$  over  $\succ$ . However, the matching  $\varphi^O(P, \succ')$  is the above underlined matching, which punishes student  $j$  when he has higher priority for school  $a$ . This shows that  $\varphi^O$  does not respect improvements, even when  $\varphi^O(P, \succ)$  satisfies non-wastefulness, mutual best, and weak consistency.

**Theorem 3.6.** *A mechanism  $\varphi$  satisfies non-wastefulness, mutual best, weak consistency, and rank monotonicity if and only if  $\varphi = \varphi^S$ .*

*Proof.* Theorem 3.5 and proposition 3.2 complete the proof. □

**Independence of axioms:**  $\varphi^\emptyset$  (definition 3.18) satisfies mutual best, weak consistency, and rank monotonicity but violates non-wastefulness.  $\varphi^f$  satisfies non-wastefulness, weak consistency, and rank monotonicity but violates mutual best.  $\varphi^T$  and  $\varphi^B$  both satisfy non-wastefulness, mutual best,

and rank monotonicity but violate weak consistency.  $\varphi^W$  (definition 3.19) satisfies non-wastefulness, mutual best, and weak consistency but violates rank monotonicity.<sup>26</sup>

## 3.6 Conclusion

In this chapter we first propose a group of new axioms for school choice mechanisms related to stability, consistency, and monotonicity. Most of them are new not only to school choice problems, but also to the other resource allocation problems. An interesting direction for future research is to apply these axioms to problems other than school choice.

This chapter also characterizes the celebrated SOSM, which is becoming the central school choice mechanism.<sup>27</sup> Some of our results tighten the existing characterizations, and the other results find new ways to understand SOSM. Our characterizations show the tradeoff between SOSM and the other school choice mechanisms, which will certainly help the social planner to choose alternative mechanisms. This chapter characterizes the SOSM on full strict and acceptant priority and full strict preference domains. Future work is needed to characterize SOSM on more general priority and preference domains. Future work is also needed to characterize the other school choice mechanisms based on the new axioms provided in this paper.

Prior to our research, Kojima and Manea (2010a) characterize the SOSM for some acceptant substitutable priorities by two groups of axioms: non-wastefulness and IR monotonicity; non-wastefulness, population monotonicity, and weak Maskin monotonicity. While their axioms are all priority-free, their characterizations are not on full strict and acceptant priority domain. We prove in this chapter that to characterize the SOSM on full strict and acceptant priority domain, it is impossible not to use priority-related axioms. The reason is obvious. As we can observe, the simple serial dictatorship satisfies all priority-free axioms (section 3.3 and 3.4) satisfied by SOSM but violates all priority-related axioms (section 3.2). Hence, to characterize school choice mechanisms, especially the SOSM, by only priority-free axioms will eventually result in the SSD, which is a trivial mechanism in some sense for school choice problems.

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26. Example 3.5 shows that  $\varphi^O(P)$  satisfies non-wastefulness, mutual best, and weak consistency but  $\varphi^O$  violates rank monotonicity under  $P$ .

27. See Roth (2008) for a survey on Gale-Shapley deferred acceptance algorithm.



# Chapter 4

## Deferred Acceptance and Serial Dictatorship

### 4.1 Introduction

Including school choice and student placement, the allocation of indivisible resources is a commonly observed real-life phenomenon. Typical examples include the allocation of houses, working positions, offices or tasks, and so on. To pursue consistency of this thesis, we use students to represent agents, and schools to represent indivisible objects. In real-life problems, the simple serial dictatorship (SSD) is widely used for its simplicity, Pareto efficiency, and non-manipulability. However, when priorities are taken in, respect of priorities becomes a very important dimension to evaluate a mechanism. SSD becomes not so appealing because this mechanism totally ignores priorities of schools. Instead, as mentioned earlier, the student-optimal stable mechanism (SOSM) based on the Gale-Shapley student-proposing deferred acceptance algorithm, which always respects priorities, is becoming the central mechanism in priority-based allocation or school choice problems. Moreover, the SOSM matching Pareto dominates any other stable matchings, and is strategy-proof for students.

While both the SOSM and SSD have been studied widely, less attention has been devoted on studying relationship of these two mechanisms. In this chapter, we address the following question: When is the SOSM equivalent to the SSD? To answer the question, this chapter derives a condition on the priority structure that guarantees equivalence of the SOSM and SSD. We are not the first to discuss the relationship between SOSM and SSD in school choice setting. Balinski and Sönmez (1999) show that if priority orders for all schools are identical, then outcomes of the SOSM and SSD, for which the order of students is determined by the priority order of all schools, are equivalent. However, identical priority structure only serves as a sufficient

## Chapter 4 *Deferred Acceptance and Serial Dictatorship*

condition for the equivalence of these two competing mechanisms. This chapter generalizes the result of Balinski and Sönmez (1999) and identifies necessary and sufficient conditions on priorities to guarantee equivalence of the SOSM and SSD.

We define a new notion of quota-acyclic priority structure. Quota-acyclic priority structure requires that according to the quota information of a problem, no disorder of students exists below a certain critical point of priority ranks. The critical point is the minimal quota of schools. Intuitively, quota-acyclicity brings no restrictions as to the relative positions in members who are ranked in the first minimal quota place. However, it restricts ranks of a student across any two priority orders to be the same under the minimal quota place. Quota-acyclicity is stronger than both Kesten-acyclicity (hence Ergin-acyclicity) and strong X-acyclicity (hence X-acyclicity).<sup>28</sup> Another interesting property of quota-cyclic priority structure is that if there exists a school whose quota is exactly one, then quota-acyclic priority structure deteriorates into the one where each school possesses the same priority order.

Let SSD-P represent the special class of SSD where the order of students is determined by the priority order of any school. Our first result shows that matchings of the SOSM and SSD-P are equivalent, if and only if SSD-P is fair with respect to the priority profile, and if and only if the priority structure is quota-acyclic. Our result reveals that if the priority structure satisfies quota-acyclicity, then it is not necessary to use the relatively complicated procedure of deferred acceptance to calculate the outcome. Instead, to use the SSD directly determined by any priority order of a single school is enough to determine the final matching. In real-life problems, we expect the result to help students understand the procedure of school admission more easily, and save more computational work of the social planner.

The SOSM and SSD have been studied by many other economists prior to our research. Ergin (2002) proves that the SOSM recovers efficiency, consistency and group strategy-proofness if and only if the priority structure is Ergin-acyclic. Ehlers and Klaus (2004), on the basis of Ergin's conclusion, prove that the SOSM is equivalent to a mixed dictator-pairwise-exchange mechanism (MDPEM) under Ergin-acyclic priority structure. Later, Kesten (2006) finds that the SOSM and top trading cycles mechanism (TTCM) due to Abdulkadiroğlu and Sönmez (2003a) are equivalent if and only if the priority structure is Kesten-acyclic. When students are only allowed to submit a preference list containing a limited number of schools, the SOSM

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28. See definition 4.2, 4.3, 4.4, and 4.5 for detail.



## 4.2 Deferred Acceptance and Serial Dictatorship

is not strategy-proof any longer. Haeringer and Klijn (2009) study stability and efficiency of Nash equilibrium outcomes for the preference revelation games induced by the numerical constraint on students' preferences when the SOSM is used. Kojima and Manea (2010a), Morrill (2013), Ehlers and Klaus (2012) and Chen (2013a) characterize the SOSM, respectively. The SSD is investigated for divisible goods by Satterthwaite and Sonnenschein (1981) and for indivisible goods by Svensson (1999).

The rest of this chapter is organized as follows. Section 2 studies the equivalence of SOSM and SSD-P. Section 3 discusses relations between acyclicity notions on priority structures. Section 4 concludes.

## 4.2 Deferred Acceptance and Serial Dictatorship

In school choice problems, schools generally have a single priority order over students because the priorities are usually determined by examination scores of students. When each school has the same priority across each other, the analysis of school choice problem will be enormously simplified. Balinski and Sönmez (1999) first discuss the equivalence of the SOSM and SSD in student placement problem of Turkey and they show that when schools share one uniform priority order, then the SSD determined by this order is equivalent to the deferred acceptance matching determined by the priority structure constructed from the given priority order.

**Proposition 4.1.** (Balinski & Sönmez, 1999) *For each  $P$  and  $\succ$ , if  $f = \succ_a = \succ_b$  for any  $a, b \in O$ , then  $\varphi^S(P, \succ) = \varphi^f(P)$ .*

Proposition 4.1 tells us that if all schools have identical priority order over students, then the SOSM induced by the identical priority structure is equivalent to the SSD induced by the priority order of any individual school. However, identical priority structure only serves as a sufficient condition to guarantee equivalence of the SOSM and SSD. The following example 4.1 verifies this.

**EXAMPLE 4.1** Let  $I = \{i, j, l, m\}$ ,  $O = \{a, b\}$ , and  $q_a = q_b = 2$ . The priority profile is specified as follows:

$\succ_a$	$\succ_b$
$i$	$j$
$j$	$i$
$l$	$l$
$m$	$m$

## Chapter 4 Deferred Acceptance and Serial Dictatorship

It is easy to verify that for any possible preference profile of students, matchings determined by the SOSM and SSD-P are equivalent. However,  $\succ_a \neq \succ_b$ .

Motivated by the previous example 4.1, we try to extend the result of Balinski and Sönmez (1999) and identify necessary and sufficient condition on priorities under which the SOSM and SSD are equivalent. We first introduce a new notion of acyclicity for priority structures. Let  $M$  denote the minimal quota of schools, i.e.,  $M = \min\{q_a\}_{a \in O}$ . Given a student  $i$  and a school  $a$ , recall that  $U_i(\succ_a) = \{j \in I \mid \succ_a(j) < \succ_a(i)\}$  is the set of students who have higher priority than  $i$  for  $a$ .

**Definition 4.1.** Given a priority structure  $\succ$ , a **quota-cycle** is constituted of distinct  $a, b \in O$  and  $i, j \in I$  such that the following two conditions are satisfied:

Quota-cycle condition:  $\succ_a(i) < \succ_a(j)$  and  $\succ_b(j) < \succ_b(i)$  and

Quota-scarcity condition: there exists a set of students  $I_a \subseteq I \setminus \{i, j\}$  such that  $I_a \subset U_j(\succ_a)$  and  $|I_a| \geq M - 1$ .

The priority structure  $\succ$  is **quota-acyclic** if it has no quota-cycles.

Quota-acyclicity requires that according to the quota information of a problem, no disorder of students exists below the rank  $M$  in  $\succ$ . Next, we provide a characterization of quota-acyclicity showing that students in any position of the priority orders lower than the minimal quota should be the same across every school.

**Proposition 4.2.**  $\succ$  is quota-acyclic if and only if for any  $a, b \in O$  and  $k \in \{M + 1, M + 2, \dots, |I|\}$ ,  $\succ_a^{-1}(k) = \succ_b^{-1}(k)$ .<sup>29</sup>

*Proof.* We proceed by contradiction. For the if part, suppose on the contrary that for each  $a, b \in O$  and  $k \in \{M + 1, M + 2, \dots, |I|\}$ , we have  $\succ_a^{-1}(k) = \succ_b^{-1}(k)$ , but there exists a quota-cycle in  $\succ$ . Then there exists a set of students  $I_a \subseteq I \setminus \{i, j\}$  such that  $I_a \subset U_j(\succ_a)$  and  $|I_a| \geq M - 1$ . Therefore,  $\succ_a(j) > M$  and hence  $\succ_a(j) = \succ_b(j)$ . Since  $\succ_b(j) < \succ_b(i)$ , we also have  $\succ_a(i) = \succ_b(i)$ , which implies  $\succ_a(j) < \succ_a(i)$ , a contradiction.

For the only if part, suppose that for some  $a, b \in O$  and  $k \geq M + 1$ ,  $j = \succ_a^{-1}(k) \neq \succ_b^{-1}(k) = i$ . Let  $k$  be the maximum of such numbers. If  $k = \succ_a(j) < \succ_a(i) = k'$ , then by the definition of  $k$ ,  $i = \succ_a^{-1}(k') = \succ_b^{-1}(k')$ , which contradicts  $i = \succ_b^{-1}(k)$ . Therefore,  $\succ_a(i) < \succ_a(j)$ . By the same argument, we can show that  $\succ_b(j) < \succ_b(i)$ . Because  $M + 1 \leq k < \succ_a(j)$ , we can find  $I_a \subseteq I \setminus \{i, j\}$  such that  $I_a \subset U_a(j)$  and  $|I_a| = M - 1$ .  $\square$

29. Note that for each  $k \in \{1, 2, \dots, |I|\}$  and  $a \in O$ ,  $\succ_a^{-1}(k)$  is the student who is ranked in the  $k^{\text{th}}$  place at  $\succ_a$ .

## 4.2 Deferred Acceptance and Serial Dictatorship

Given  $\succ$  and  $a \in O$ , let  $L_a(M)$  be the set of students who are ranked in the last  $|I| - M$  positions of  $\succ_a$ . We say that  $L_a(M)$  belong to the lower class and the remaining students  $I \setminus L_a(M)$  belong to the upper class of  $\succ_a$ . Proposition 4.2 tells us that quota-acyclicity brings no restrictions as to the relative positions in upper class members. However, it restricts ranks of a lower class member across any two priority orders to be the same.

Another interesting property of quota-cyclic priority structure is that if there exists a school whose quota is exactly one, then quota-acyclic priority structure deteriorates into the one where each school possesses the same priority order. But this does not mean that quota-acyclicity is trivial because in real-life student placement problems, there is no university providing only one seat.

We are now ready to present the first result of this chapter.

**Theorem 4.1.** *For each  $P \in \mathcal{P}^{|I|}$  and  $a \in O$ ,*

- (i)  $\varphi^S(P, \succ) = \varphi^{\succ_a}(P)$ ;
- (ii)  $\varphi^{\succ_a}(P)$  is fair (stable) with respect to  $\succ$ ;
- (iii)  $\succ$  is quota-acyclic;

*Proof.* (iii)  $\Rightarrow$  (i): If  $\succ$  is quota-acyclic, then for each  $P \in \mathcal{P}^{|I|}$ , all members in  $I \setminus L_a(M)$  are assigned to their favorite schools. Therefore,  $\varphi^S(P, \succ) = \varphi^f(P)$  for each  $i \in I \setminus L_a(M)$ . By proposition 4.1,  $\varphi^S(P, \succ) = \varphi^f(P)$  for each  $i \in L_a(M)$ . Therefore, for each  $a \in O$ , we have  $\varphi^S(P, \succ) = \varphi^{\succ_a}(P)$ .

(iii)  $\Rightarrow$  (ii): If (iii) holds, then we have  $\varphi^S(P, \succ) = \varphi^{\succ_a}(P)$ . Because  $\varphi^S(P, \succ)$  is fair with respect to  $\succ$ ,  $\varphi^{\succ_a}(P)$  is also fair with respect to  $\succ$ .

(i)  $\Rightarrow$  (iii): We proceed by contradiction. Suppose that  $\varphi^S(P, \succ) = \varphi^{\succ_a}(P)$  for each  $a \in O$ , but  $\succ$  has a quota-cycle. Then, there exists a set of students  $I_a \subseteq I \setminus \{i, j\}$  such that  $I_a \subset U_j(\succ_a)$  and  $|I_a| \geq M - 1$ . Consider the following preference profile  $P$ :

$$\begin{aligned} P_{I_a \cup \{i, j\}} &: a_M, \emptyset \\ P_{others} &: \emptyset \end{aligned}$$

It is easy to verify that  $\varphi_j^{\succ_a}(P) = \emptyset$  and  $\varphi_j^{\succ_b}(P) = a_M$ . Therefore,  $\varphi_j^{\succ_a}(P) \neq \varphi_j^{\succ_b}(P)$ , a contradiction.

(ii)  $\Rightarrow$  (iii): We proceed by contradiction. Suppose that  $\varphi^{\succ_a}(P)$  is fair with respect to  $\succ$  for each  $a \in O$ , but  $\succ$  has a quota-cycle. Then, there exists a set of students  $I_a \subseteq I \setminus \{i, j\}$  such that  $I_a \subset U_j(\succ_a)$  and  $|I_a| \geq M - 1$ .

## Chapter 4 Deferred Acceptance and Serial Dictatorship

Consider again the preference profile  $P$ :

$$\begin{aligned} P_{I_a \cup \{i, j\}} &: a_M, \emptyset \\ P_{others} &: \emptyset \end{aligned}$$

It is easy to verify that  $\varphi_j^{\succ^a}(P) = \emptyset$  and  $\varphi_j^{\succ^b}(P) = a_M$ . Furthermore, there exists  $l \in I_a \cup \{i\}$  such that  $\varphi_l^{\succ^a}(P) = a_M$  and  $\varphi_l^{\succ^b}(P) = \emptyset$ . For  $a_M$ , either  $j \succ_{a_M} l$  or  $l \succ_{a_M} j$ . If  $j \succ_{a_M} l$ , then  $\varphi^{\succ^a}(P)$  is not fair with respect to  $\succ$ . If  $l \succ_{a_M} j$ , then  $\varphi^{\succ^b}(P)$  is not fair with respect to  $\succ$ . □

### 4.3 Relationship between Acyclicity Conditions

In this section, we compare quota-acyclicity with the other acyclicity notions.

**Definition 4.2.** *Given a priority structure  $\succ$ , an **Ergin-cycle** is constituted of distinct  $a, b \in O$  and  $i, j, l \in I$  such that the following two conditions are satisfied:*

*Ergin-cycle condition:  $\succ_a(i) <_{\succ_a}(j) <_{\succ_a}(l)$  and  $\succ_b(l) <_{\succ_b}(i)$  and*

*Ergin-scarcity condition: there exist distinct sets of students  $I_a, I_b \subseteq I \setminus \{i, j, l\}$  such that  $I_a \subset U_j(\succ_a)$ ,  $I_b \subset U_i(\succ_b)$ ,  $|I_a| = q_a - 1$ , and  $|I_b| = q_b - 1$ .*

*The priority structure  $\succ$  is **Ergin-acyclic** if it has no Ergin-cycles.*

Note that acyclicity is a joint property of the priority structure and the vector of quotas, although the latter will often be suppressed. Acyclicity becomes more restrictive as resources become more scarce. Ergin-acyclicity is first proposed by the seminal work of Ergin (2002) and has been investigated by many other authors. Kesten (2009) shows that a mechanism satisfies efficiency, coalitional strategy-proofness, and resource monotonicity if and only if it is the SOSM associated with an Ergin-acyclic priority structure. Kojima (2011) shows that there is a robustly stable mechanism in a market if and only if the priority structure of schools in that market is Ergin-acyclic. Moreover, if there is a robustly stable mechanism, then it coincides with the SOSM.

**Definition 4.3.** *For a priority structure  $\succ$ , a **Kesten-cycle** is constituted of distinct  $a, b \in O$  and  $i, j, l \in I$  such that the following two conditions are satisfied:*

*Kesten-cycle condition:  $\succ_a(i) <_{\succ_a}(j) <_{\succ_a}(l)$  and  $\succ_b(l) <_{\succ_b}(i), \succ_b(j)$  and*

### 4.3 Relationship between Acyclicity Conditions

**Kesten-scarcity condition:** *there exists a set of students  $I_a \subseteq I \setminus \{i, j, l\}$  such that  $I_a \subset U_i(\succ_a) \cup [U_j(\succ_a) \setminus U_l(\succ_b)]$  and  $|I_a| = q_a - 1$ .*

*The priority structure  $\succ$  is **Kesten-acyclic** if it has no Kesten-cycles.*

Kesten-acyclicity is first proposed by Kesten (2006). He shows that the TTCM and SOSM are equivalent, or TTCM recovers population monotonicity, resource monotonicity, or stability if and only if the priority structure satisfies Kesten-acyclicity. Later, Chen (2013f) shows that the TTCM recovers robust stability and IR monotonicity if and only if the priority structure is Kesten-acyclic.

**Definition 4.4.** *Given a priority structure  $\succ$ , an **X-cycle** is constituted of distinct  $a, b \in O$  and  $i, j \in I$  such that the following two conditions are satisfied:*

**X-cycle condition:**  $\succ_a(i) <_{\succ_a}(j)$  and  $\succ_b(j) <_{\succ_b}(i)$  and

**X-scarcity condition:** *there exist disjoint sets of students  $I_a, I_b \subseteq I \setminus \{i, j\}$  such that  $I_a \subset U_i(\succ_a)$ ,  $I_b \subset U_j(\succ_b)$ ,  $|I_a| = q_a - 1$  and  $|I_b| = q_b - 1$ .*

*The priority structure  $\succ$  is **X-acyclic** if it has no X-cycles.*

**Definition 4.5.** *Given a priority structure  $\succ$ , a **weak X-cycle** is constituted of distinct  $a, b \in O$  and  $i, j \in I$  such that the following two conditions are satisfied:*

**Weak X-cycle condition:**  $\succ_a(i) <_{\succ_a}(j)$  and  $\succ_b(j) <_{\succ_b}(i)$  and

**Weak X-scarcity condition:** *there exist disjoint sets of students  $I_a, I_b \subseteq I \setminus \{i, j\}$  such that  $I_a \subset U_j(\succ_a)$ ,  $I_b \subset U_i(\succ_b)$ ,  $|I_a| = q_a - 1$  and  $|I_b| = q_b - 1$ .*

*The priority structure  $\succ$  is **strongly X-acyclic** if it has no weak X-cycles.*

X-acyclicity and strong X-acyclicity are first introduced by Haeringer and Klijn (2009). They are necessary and sufficient conditions on the priorities to guarantee efficiency of either of Nash equilibrium outcomes under school choice mechanisms, when students are only allowed to submit a preference list containing a limited number of schools.

**Proposition 4.3.** *If a priority structure  $\succ$  is quota-cyclic, then it is Kesten-acyclic (hence Ergin-acyclic).*

**Proposition 4.4.** *If a priority structure  $\succ$  is quota-cyclic, then it is strongly X-acyclic (hence X-acyclic).*

We omit proofs of proposition 4.4 and 4.5 because the result is obvious. The following Figure 4.1 shows the relationship between quota-acyclicity and the other acyclicity notions. It is easy to verify that quota-acyclicity is stronger than all the existing acyclicity conditions.

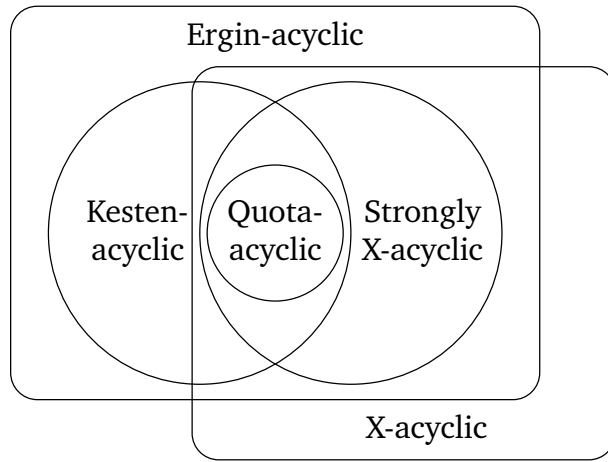


Figure 4.1: Relationship between Acyclicity Conditions

## 4.4 Conclusion

This chapter solves the following problem: when is the SOSM equivalent to SSD? We study the equivalence relation of the SOSM, determined by the Gale-Shapley student-proposing deferred acceptance algorithm, and SSD for which the order of students may be determined by the priority order of any school. Our conclusion shows that these two mechanisms are equivalent to each other if and only if the priority structure is quota-acyclic, which is a notion stronger than all the other existing priority structures.

Prior to our research, there existed several papers investigating conditions under which the SOSM are equivalent to some other school choice mechanisms. Kesten (2006) proves that the SOSM is equivalent to TTCM under Kesten-acyclic priority structure. Ehlers and Klaus (2004), on the basis of Ergin’s conclusion, prove that the SOSM is equivalent to a mixed dictator-pairwise-exchange mechanism (MDPEM) under Ergin-acyclic priority structure. Denote the current chapter as Chen (2013DASD). Related results are summarized in the following table 4.1. Future work is needed to study the equivalence relation of SSD and other mechanisms such as the TTCM and BOSM.

#### 4.4 Conclusion

Priority structure	SOSM=?	Literature
Ergin-acyclic	MDPEM	Ehlers and Klaus (2004)
Kesten-acyclic	TTCM	Kesten (2006)
Quota-acyclic	SSD-P	Chen(2013DASD)

Table 4.1: SOSM under Acyclic Priority Structures

Note that as long as there is one school with single supply of capacity, quota-acyclicity deteriorates into the shape where each school has identical priority order. But in real-life school choice problems, one school usually has multiple supply, which manifests that our result is useful and will help social planners in the process of assigning school seats to students.





# Chapter 5

## Axioms for Immediate Acceptance

### 5.1 Introduction

The Boston mechanism (BOSM), determined by the immediate acceptance algorithm, is a popular school choice mechanism around the world. Under this mechanism, students first report their preferences over schools to the social planner. Given the reported preferences, the social planner follows the immediate acceptance algorithm to assign students to schools. The immediate acceptance algorithm first allocates school seats to students who put that school in the first place of their preference list, then to those who put it in the second place if there is any remaining seat, and so forth.

Abdulkadiroğlu and Sönmez (2003a) find first that BOSM is not fair. Second, this mechanism is not strategy-proof and, worse still, is easy for students to manipulate. They thus suggest to substitute it with the other two mechanism which do not suffer from serious incentive problems: the student-optimal stable mechanism (SOSM) and top trading cycles mechanism (TTCM). Later, Ergin and Sönmez (2006) find that the set of Nash equilibrium outcomes of the preference revelation game induced by the BOSM is equivalent to the set of all stable matchings. Chen and Sönmez (2006) prove manipulable property of the BOSM in a lab experiment. Pathak and Sönmez (2008) show that if some sincere students report their true preferences over schools while the sophisticated others manipulate, then sophisticated students can become better off under the BOSM than in the SOSM by taking the advantage of their sincere companions.

Kojima and Ünver (2013) characterize the BOSM for the first time. Based on the results of Kojima and Ünver (2013), the current chapter provides two new characterizations of the BOSM in terms of two new axioms related to stability: weak fairness and rank rationality. A mechanism satisfies *weak fairness* if under the matching derived by this mechanism, a student prefers the assignment of another student and both of them put the preferred

school in the same preference ranking, then the later student has higher priority than the former student for the preferred school. A mechanism satisfies *rank rationality* if it never assigns a student  $i$  to a school worse than  $a$  whenever the following two conditions are satisfied: (1) the number of students, who put school  $a$  in a preference ranking higher than  $i$  does and find school  $a$  acceptable, is smaller than the capacity of this school; (2) Student  $i$  has the highest group of priorities for  $a$  among all students, who put school  $a$  in a preference ranking not lower than  $i$  does and find school  $a$  acceptable. Weak fairness and rank rationality are both weaker than stability, and thus satisfied by SOSM.

Our first characterization states that a mechanism is equivalent to the BOSM for all acceptant priorities if and only if it satisfies respect of preference rankings and weak fairness (theorem 5.1). Our second characterization states that a mechanism is equivalent to the BOSM for all acceptant priorities if and only if it satisfies respect of preference rankings, rank rationality, and rank monotonicity (theorem 5.2). As the SOSM satisfies weak fairness, rank rationality and rank monotonicity, our result reveals that respect of preference rankings is the unique axiom which distinguishes BOSM from SOSM.

Our results are different from those in Kojima and Ünver (2013) for the following reasons. First, we assume the priority structure to be part of the primitive, while Kojima and Ünver (2013) derive the priority structure as part of characterizations of the BOSM. Second, we assume acceptant priority structures, while Kojima and Ünver (2013) impose no restriction on priorities.

Afacan (2013) is another work characterizing the BOSM. He shows that a mechanism is outcome equivalent to the BOSM at every priority if and only if it respects both preference rankings and priorities<sup>30</sup> and satisfies individual rationality for schools<sup>31</sup>. In environments where each student is acceptable to every school, only one axiom, i.e., respecting both preference rankings and priorities is enough to characterize the BOSM. Afacan (2013) is different from our work in the following aspects. First, Afacan (2013) characterizes the BOSM on full strict priority domains, with the help of a new axiom: individual rationality for schools, while we only characterize on full strict and acceptant priorities. Second, Afacan (2013)'s main axiom: respect of both preference rankings and priorities, is equivalent to the

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30. A mechanism respects both preference rankings and priorities if it satisfies non-wastefulness, rank-fairness, and weak fairness.

31. A mechanism satisfies individual rationality for schools if no school seat is assigned to students who are unacceptable to this school.

combination of respect of preference rankings and weak fairness in our work. Finally, proofs for these two works are different.

## 5.2 Related Axioms

**Definition 5.1.** A mechanism  $\varphi$  is **rank-fair** if for each  $P \in \mathcal{P}^{|I|}$ ,  $i \in I$  and  $a \in O$ ,

$$aP_i\varphi_i(P) \Rightarrow P_j(a) \leq P_i(a), \forall j \in \varphi_a(P).$$

**Definition 5.2.** A mechanism  $\varphi$  **respects preference rankings** if it is non-wasteful<sup>32</sup> and rank-fair.

A mechanism respects preference rankings if it is non-wasteful and rank-fair. Non-wastefulness requires that if a student prefers another school to his current assignment, then the quota of the preferred school has been fully occupied by other students. Rank-fairness requires that if a student prefers the assignment of another student, then the latter student puts the preferred school in a preference ranking not lower than the former student. Kojima and Ünver (2013) prove that if a mechanism respects preference rankings, then it is Pareto efficient.<sup>33</sup>

**Definition 5.3.** A mechanism  $\varphi$  satisfies **weak fairness** if for any  $i, j \in I$  and  $P \in \mathcal{P}^{|I|}$ ,

$$\varphi_j(P)P_i\varphi_i(P) \& P_i(\varphi_j(P)) = P_j(\varphi_j(P)) \Rightarrow j \succ_{\varphi_j(P)} i.$$

Weak fairness requires that if a student prefers the assignment of another student and both of them put the preferred school in the same preference ranking, then the latter student should have higher priority for this school. Obviously, weak fairness is implied by fairness.

Recall that for each problem  $P$  and  $a \in O$ ,  $I_a^*$  represents the set of students that find school  $a$  acceptable, i.e.,

$$I_a^* = \{i | aP_i\emptyset\}.$$

32. See definition 2.1.

33. Note that Kojima and Ünver (2013) conclude that if a mechanism respects preference rankings, then it is constrained Pareto efficient. Constrained Pareto efficiency is the correspondence of Pareto efficiency under general priorities. If we assume acceptant priorities, then constrained Pareto efficiency is equivalent to Pareto efficiency. Therefore, in our setting, respect of preference rankings implies Pareto efficiency.

## Chapter 5 Axioms for Immediate Acceptance

For each student  $i$ , let  $I_{a,i}^*$  be the set of students that find school  $a$  acceptable and put  $a$  in a preference ranking higher than student  $i$  does, i.e.,

$$I_{a,i}^* = \{j | P_j(a) < P_i(a) \ \& \ a P_j \emptyset\}.$$

**Definition 5.4.** A mechanism  $\varphi$  satisfies **rank rationality** if for each  $P \in \mathcal{P}^{|I|}$ ,  $i \in I$  and  $a \in O$ ,

$$q_a - |I_{a,i}^*| > 0 \ \& \ \succ_a |_{\{I_a^* \setminus I_{a,i}^*\}}(i) \leq (q_a - |I_{a,i}^*|) \Rightarrow \varphi_i(P) R_i a.$$

We say that a mechanism satisfies rank rationality if it never assigns a student  $i$  to a school worse than the non-null school  $a$  whenever the following two conditions holds: (1) The number of students who find school  $a$  acceptable and put school  $a$  in a higher preference ranking than  $i$  does is smaller than the capacity of school  $a$ ; and (2) Student  $i$  has the highest group of priorities for  $a$  among the set of students who find school  $a$  acceptable and put school  $a$  in a preference ranking not higher than  $i$  does. The following proposition shows the relationship between stability-related axioms.

**Proposition 5.1.** *Stability (definition 2.3), strong top rationality (definition 3.6), and strong group rationality (definition 3.8) all imply rank rationality.*

	$\varphi^S$	$\varphi^T$	$\varphi^B$	$\varphi^O$	$\varphi^f$	$\varphi^R$
Respect of Preference Rankings	×	×	√	×	×	×
Rank-fairness	×	×	√	×	×	×
Weak Fairness	√	×	√	√	×	×
Rank Rationality	√	×	√	√	×	×

Table 5.1: School Choice Mechanisms and Boston-related Axioms

## 5.3 Characterizations of the Boston Mechanism

**Theorem 5.1.** *A mechanism  $\varphi$  respects preference rankings and satisfies weak fairness if and only if  $\varphi = \varphi^B$ .*

*Proof.* We proceed by contradiction. Fix a mechanism  $\varphi$  which satisfies respect of preference rankings and weak fairness. Suppose that  $\varphi \neq \varphi^B$ . Given a problem  $P$ , denote the matching  $\varphi^B(P)$  as  $\mu$ , and the matching  $\varphi(P)$  as  $\pi$ .

### 5.3 Characterizations of the Boston Mechanism

*Step 1:* Suppose that for some student  $i$  and school  $a$  such that  $\mu_i = a$  and  $P_i(a) = 1$ ,  $\pi_i \neq a$ . By weak fairness of  $\mu$ , we have that for any student  $j$  such that  $j \notin \mu_a$  and  $P_j(a) = 1$ ,  $i \succ_a j$ . Because  $\pi$  respects preference rankings, we have that under  $\pi$  school  $a$  is assigned to a student who also puts it in the first preference ranking, i.e.,  $a$  is assigned to a student  $j$  where  $j \notin \mu_a$  and  $P_j(a) = 1$ . By weak fairness of  $\mu$ , we have  $j \succ_a i$ , a contradiction.

*Step k:* Suppose that for some student  $i$  and school  $a$  such that  $\mu_i = a$  and  $P_i(a) = k$ ,  $\pi_i \neq a$ . By weak fairness of  $\mu$  and the previous  $k - 1$  steps, we have that for any student  $j$  such that  $j \notin \mu_a$  and  $P_j(a) = k$ ,  $i \succ_a j$ . Because  $\pi$  respects preference rankings and by the previous  $k - 1$  steps, we have that under  $\pi$  school  $a$  is assigned to a student who also puts it in the  $k^{\text{th}}$  preference ranking, i.e.,  $a$  is assigned to a student  $j$  where  $j \notin \mu_a$  and  $P_j(a) = k$ . By weak fairness of  $\mu$ , we have  $j \succ_a i$ , a contradiction.  $\square$

**Independence of axioms:** The null matching, where each student is assigned to the null school, trivially satisfies rank-fairness and weak fairness, but violates non-wastefulness. The SOSM  $\varphi^S$  satisfies non-wastefulness and weak fairness, but violates rank-fairness. The reversed Boston mechanism  $\varphi^V$  defined below satisfies non-wastefulness and rank-fairness, but violates weak fairness.

For any problem  $P \in \mathcal{P}^{|I|}$ , the **reversed Boston mechanism**, denoted by  $\varphi^V$ , determines a matching  $\varphi^V(P)$  through the following algorithm:

**Step 1:** Consider only the first choice of students. For each school  $a \in O$ , up to  $q_a$  students whose first choice is  $a$  with the **lowest priority** for it (all students if fewer than  $q_a$ ) are assigned to school  $a$  permanently. If a student puts the null school in the first preference ranking, then he is assigned the null school. Remove the set of students who are assigned a school in this step and their corresponding assignments.

$\vdots$

**Step k:** Consider the  $k^{\text{th}}$  choice of the remaining students. For each school  $a \in O$  with  $q_a^k$  copies available, up to  $q_a^k$  students whose  $k^{\text{th}}$  favorite school is  $a$  with the **lowest priority** for it (all students if fewer than  $q_a^k$ ) are assigned to school  $a$  permanently. If a student puts the null school in the  $k^{\text{th}}$  preference ranking, then he is assigned the null school. Remove the set of students who are assigned a school in this step and their corresponding matchings.

The algorithm terminates when all students have been removed.  $\varphi^V$

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differs from  $\varphi^B$  in solving conflicts. If more students than the quota of a school apply for this school,  $\varphi^B$  solves conflicts fairly, i.e., according to the priority order of this school. However,  $\varphi^V$  does not solve conflicts fairly. The following example might be helpful in demonstrating how  $\varphi^V$  and  $\varphi^B$  work.

**EXAMPLE 5.1** Let  $I = \{i, j, l\}$ ,  $C = \{a, b, c\}$ , and  $q_a = q_b = q_c = 1$ . The preference profile and priority profile are listed below:

$P_i$	$P_j$	$P_l$	$\succ_a$	$\succ_b$	$\succ_c$
$a$	$a$	$c$	$i$	$l$	$i$
$b$	$c$	$b$	$j$	$j$	$j$
$c$	$b$	$a$	$l$	$i$	$l$

$\varphi^B$  results in the following matching:

$i$	$j$	$l$
$a$	$b$	$c$

$\varphi^V$  results in the following matching:

$i$	$j$	$l$
$b$	$a$	$c$

**Remark:** Theorem 5.1 characterizes the BOSM by two intuitive axioms. A mechanism outcome is weakly fair (for a given priority ordering profile) under a certain preference profile if whenever one student envies the school of another student and they have the same ranking for this envied school, then the envied student has a higher priority at that school than the original student. A mechanism outcome respects preference rankings if a student would rather be matched with a school than his current assignment, then this school has no empty seats and no student who is assigned to this school have ranked it lower in his preferences than the original student. Respect of preference rankings is a stronger version of Pareto efficiency, and weak fairness is a weaker version of fairness. Proofs for this characterization are also intuitive and simple. Many scholars criticize theorem 5.1 as the two axioms we use are immediately an alternative definition of the BOSM. Indeed, the characterization in theorem 5.1 is too similar to the definition of the immediate acceptance algorithm, which undermines the theoretical important of this result.

Nevertheless, theorem 5.1 is meaningful first in figuring out the main difference between BOSM and SOSM. As we all know SOSM satisfies non-wastefulness and weak fairness, but violates only rank-fairness. This tells

### 5.3 Characterizations of the Boston Mechanism

us that it is rank-fairness that distinguishes BOSM from SOSM. Moreover, Kojima and Ünver (2013) characterize BOSM for some priorities by respect of preference rankings together with three other normative axioms. However, as we can see easily, the reversed Boston mechanism  $\varphi^V$  satisfies all axioms proposed by Kojima and Ünver (2013). Therefore, to characterize BOSM by only priority-free axioms will eventually result in the reversed Boston mechanism  $\varphi^V$ , which is undesirable at all.

As the afore-stated theorem 5.1 is disputable among researchers, we seek to characterize BOSM: (1) without using weak fairness; (2) with axioms not similar to the definition of BOSM; (3) for all possible priority structures instead of for some priorities. The following theorem 5.2 presents a second characterization of BOSM, by referring to respect of preference rankings, together with two weak axioms related to stability and monotonicity. This characterization is problem-free in being similar to the definition of the immediate acceptance algorithm.

**Theorem 5.2.** *A mechanism  $\varphi$  satisfies rank rationality, rank monotonicity, and respects preference rankings if and only if  $\varphi = \varphi^B$ .*

*Proof.* It is easy to see that  $\varphi^B$  satisfies rank rationality, rank monotonicity, and respects preference rankings. We only need to show the only if part. To prove  $\varphi = \varphi^B$ , we use the following result.

**Proposition 5.2.** *If  $\varphi$  satisfies rank rationality, rank monotonicity, and respects preference rankings, then  $\varphi$  satisfies weak fairness.*

*Proof.* Fix a mechanism  $\varphi$  that satisfies rank rationality, rank monotonicity, and respects preference rankings. Suppose that  $\varphi$  violates weak fairness, i.e., there exist  $P \in \mathcal{P}^I$ ,  $i, j \in I$ ,  $a \in O$  such that  $aP_i\varphi_i(P)$ ,  $P_i(a) = P_j(a)$ ,  $i \succ_a j$ , and  $j \in \varphi_a(P)$ .

For each  $J \subseteq I$  and  $l \in J$ , recall that  $P_J^\varphi$  is the preference profile which ranks  $\varphi_l(P)$  as the most preferred school and  $\emptyset$  as the second preferred school if  $\varphi_l(P) \neq \emptyset$ . Consider the following preference profile

$$P' = (P_i, P_j, P_{I \setminus \{i, j\}}^\varphi).$$

Note that  $P'$  r.m.t.  $P$  at  $\varphi(P)$ . As  $\varphi$  satisfies rank monotonicity, it follows that

$$\varphi_l(P') R'_l \varphi_l(P), \forall l \in I. \quad (5.1)$$

As  $\varphi_l(P)$  is the favorite school for each  $l \in I \setminus \{i, j\}$  under  $P'$ , we have

$$\varphi_l(P') = \varphi_l(P), \forall l \in I \setminus \{i, j\}. \quad (5.2)$$

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To calculate the assignment of  $i$  and  $j$ , we discuss about two cases.

**Case 1:**  $P_i(a) = P_j(a) = 1$ . From the construction of  $P'$ , we have that the set of students finding school  $a$  acceptable is  $I_a^* = \varphi_a(P) \cup \{i\}$ , and  $|I_a^*| = q_a + 1$ . Because  $i \succ_a j$  and  $|I_a^*| = q_a + 1$ , it follows that  $\succ_a |I_a^*| \leq q_a$ . As all students in  $I_a^*$  put school  $a$  in the first preference ranking, we have  $I_{a,i}^* = \emptyset$ . Therefore,  $q_a - |I_{a,i}^*| = q_a > 0$  and  $\succ_a |_{\{I_a^* \setminus I_{a,i}^*\}}(i) = \succ_a |I_a^*(i)| \leq q_a$ . Because  $\varphi$  satisfies rank rationality, we have  $\varphi(P') R'_i a$ . As  $P'_i(a) = P_i(a) = 1$ , it follows that

$$\varphi_i(P') = a. \quad (5.3)$$

Moreover, as  $P_i(a) = P_j(a) = 1$  and  $P_j(a) = P'_j(a)$ , by equation 5.1, it follows that

$$\varphi_j(P') = a. \quad (5.4)$$

Hence, by equations 5.2, 5.3, and 5.4,  $\varphi_a(P') = \varphi_a(P) \cup \{i\}$  and  $|\varphi_a(P')| = q_a + 1$ . This means that  $\varphi$  assigns school  $a$  to  $q_a + 1$  students, which contradicts its feasibility.

**Case 2:**  $P_i(a) = P_j(a) = k, k > 1$ . From the construction of  $P'$ , we have that the set of students finding school  $a$  acceptable is  $I_a^* = \varphi_a(P) \cup \{i\}$ , and  $|I_a^*| = q_a + 1$ . Moreover, it is easy to verify that under  $P'$ ,  $I_{a,i}^* = \varphi_a(P) \setminus \{j\}$ . Thus,  $I_a^* \setminus I_{a,i}^* = \{i, j\}$ . As  $i \succ_a j$ , we have  $q_a - |I_{a,i}^*| = 1$  and  $\succ_a |_{\{I_a^* \setminus I_{a,i}^*\}}(i) = 1$ . By rank rationality of  $\varphi$ ,  $\varphi_i(P') R'_i a$ . As  $\varphi$  respects preference rankings, it follows that  $\varphi_i(P') \notin U_a(P_i)$ . Thus,  $a R_i \varphi_i(P')$ .  $\varphi_i(P') R'_i a$  and  $a R_i \varphi_i(P')$  imply that

$$\varphi_i(P') = a. \quad (5.5)$$

As  $\varphi$  respects preference rankings, it follows that  $\varphi_j(P') \notin U_a(P_j)$ . Hence,  $a R_j \varphi_j(P')$ . By equation 5.1, we have  $\varphi_j(P') R'_j a$ .  $a R_j \varphi_j(P')$  and  $\varphi_j(P') R'_j a$  imply that

$$\varphi_j(P') = a. \quad (5.6)$$

Hence, by equations 5.2, 5.5, and 5.6,  $\varphi_a(P') = \varphi_a(P) \cup \{i\}$  and  $|\varphi_a(P')| = q_a + 1$ . This means that  $\varphi$  assigns school  $a$  to  $q_a + 1$  students, which contradicts its feasibility.  $\square$

Proposition 5.1 and theorem 5.1 then complete the proof.  $\square$

**Independence of axioms:**  $\varphi^S$  satisfies rank rationality and rank monotonicity, but does not respect preference rankings. The reversed Boston mechanism  $\varphi^V$  satisfies rank monotonicity and respects preference rankings, but violates rank rationality. The following example reveals a matching which respects preference rankings and satisfies rank rationality, but violates rank monotonicity.



## 5.4 Conclusion

**EXAMPLE 5.2** Let  $I = \{i, j, l, m\}$ ,  $C = \{a, b, c\}$ , and  $q_a = q_b = q_c = 1$ . The preferences profile and priority profile are listed below:

$P_i$	$P_j$	$P_l$	$P_m$	$\succ_a$	$\succ_b$	$\succ_c$
<u>a</u>	a	<u>c</u>	a	i	m	i
b	c	b	c	j	j	j
c	<u>b</u>	a	b	l	i	l
$\emptyset$	$\emptyset$	$\emptyset$	<u><math>\emptyset</math></u>	m	l	m

Let  $\varphi$  be a mechanism which respects preference rankings and satisfies rank rationality. Let  $\varphi(P)$  be the above matching marked with boxes. It is easy to verify that  $\varphi(P)$  respects preference rankings and satisfies rank rationality.

Next, we consider the following preference profile  $P' = \{P'_i, P'_j, P'_l, P'_m\}$  as follows:

$P'_i$	$P'_j$	$P'_l$	$P'_m$	$\succ_a$	$\succ_b$	$\succ_c$
<u>a</u>	a	<u>c</u>	a	i	m	i
$\emptyset$	c	$\emptyset$	c	j	j	j
c	b	a	<u>b</u>	l	i	l
b	<u><math>\emptyset</math></u>	b	$\emptyset$	m	l	m

From the construction of  $P'$ , we know that  $P'$  r.m.t.  $P$  at  $\varphi(P)$ . Now, under  $P'$ , the unique matching which respects preference rankings and satisfies rank rationality is the above underlined matching. Because  $\varphi$  respects preference rankings and satisfies rank rationality, we have that  $\varphi(P')$  is the same underlined matching.

Note that  $P'$  r.m.t.  $P$  at  $\varphi(P)$ , but  $\varphi_j(P)P_j\varphi_j(P')$ . This shows  $\varphi$  violates rank monotonicity under  $P$ .

## 5.4 Conclusion

The axiomatization of school choice mechanism has attracted much attention in the recent years. One of the pioneering work is Kojima and Manea (2010a) characterizing the SOSM with the help of several monotonicity axioms. Their work has intrigued a list of followers like Ehlers and Klaus (2012), Morrill (2013), and Chen (2013a) to give more characterizations of SOSM, Abdulkadirođlu and Che (2010), Morrill (2011), and Dur (2012) to give more characterizations of TTCM, and Kojima and Ünver (2013) and Afacan (2013) to give more characterizations of BOSM.

## Chapter 5 Axioms for Immediate Acceptance

Almost all of these papers consider acceptant priority domain. So does our research.

Our research contributes to the literature in two aspects. First, we propose two new axiom weaker than the classical concept of stability. One direction for future research is to extend these axioms to the fields other than school choice. Second, we give new characterizations of the BOSM, which is still a widely used mechanism so far. In chapter 3 of this thesis, we propose new monotonicity axioms such as rank monotonicity, strong rank monotonicity, and IR rank monotonicity, which are satisfied by the BOSM. Future work is called for to give further characterizations of the BOSM based on the afore-mentioned axioms. It is interesting to see that the recursive Boston mechanism  $\varphi^R$  and reversed Boston mechanism  $\varphi^V$  also satisfy those monotonicity axioms. Thus, in future, more attention should be paid to figuring out the difference between  $\varphi^B$ ,  $\varphi^R$ , and  $\varphi^V$ .

Although many researchers claim that respect of preference rankings is too similar to the definition of immediate acceptance algorithm, our analysis shows that to characterize BOSM, we also need other axioms: weak fairness; or rank rationality and rank monotonicity. Moreover, both of our theorems reveal that respect of preference rankings is the unique feature of BOSM over SOSM. It is easy to see that SOSM satisfies also weak fairness, rank rationality, and rank monotonicity. Another future direction to extend our research is to characterize SOSM by these new axioms.

# Chapter 6

## When is the Boston Mechanism Strategy-proof?

### 6.1 Introduction

In chapter 5 of this thesis, we characterize the Boston mechanism (BOSM) on full strict and acceptant priority domain by two groups of axioms: respect of preference rankings and weak fairness; respect of preference rankings, rank rationality, and rank monotonicity. In this chapter, we will study the necessary and sufficient condition under which the BOSM recovers desirable properties. As pointed out by Abdulkadiroğlu and Sönmez (2003a), the main difficulty with the BOSM is that it is neither strategy-proof nor fair.

Results of this chapter reveal that the BOSM recovers strategy-proofness if and only if it recovers fairness, if and only if it is equivalent to the SOSM, if and only if SOSM recovers respect of preference rankings, and if and only if the number of total seats at any two schools exceeds the number of students. If the number of total seats at any two schools exceeds the number of students, then SOSM respects preference rankings. Unlike the other school choice mechanisms, relative priority rankings do not matter in recovering desirable properties for the BOSM. Thus, the only way to recover strategy-proofness and fairness is increasing the number of seats in each school, which manifests the difficulty of having strategy-proof and fair BOSM.

Prior to the current research, the following papers have investigated the necessary and sufficient conditions under which the SOSM and TTCM satisfy more desirable properties. Ergin (2002) shows that the SOSM recovers efficiency, group strategy-proofness, or consistency if and only if the priority structure is Ergin-acyclic. Later, Kojima (2011) shows that the SOSM satisfies robust stability if and only if the priority structure is Ergin-

acyclic. Kesten (2006) shows that the TTCM is fair, resource monotonic, or population monotonic if and only if the priority structure is Kesten-acyclic. Chen (2013b) finds the necessary and sufficient condition under which the SOSM is equivalent to SSOM and SSD.

Kumano (2013) has proceeded a similar work identifying necessary and sufficient conditions under which the BOSM is strategy-proof or fair (stable). He proves that the BOSM is strategy-proof if and only if it is stable (fair), and if and only if the priority structure is strongly acyclic, assuming that the total number of schools is no smaller than two and the total number of students is no smaller than three. Results of the current chapter are more general than Kumano (2013)'s because we do not necessarily assume that the total number of students is no smaller than three. Moreover, under Kumano (2013)'s assumption, we prove that a priority structure is strongly acyclic if and only if the number of total seats at any two schools exceeds the number of students.

## 6.2 Strategy-proof Boston Mechanism

**Theorem 6.1.** *The following statements are equivalent:*

- (i)  $\varphi^B$  is strategy-proof;
- (ii)  $\varphi^B$  is fair;
- (iii)  $\varphi^B = \varphi^S$ ;
- (iv)  $\varphi^S$  respects preference rankings;
- (v) for all  $a, b \in O$  with  $a \neq b$ ,  $q_a + q_b \geq |I|$ .

*Proof.* (i)  $\Rightarrow$  (v), (ii)  $\Rightarrow$  (v), (iii)  $\Rightarrow$  (v) and (iv)  $\Rightarrow$  (v): Suppose that  $\varphi^B$  is strategy-proof, or fair, or equivalent to SOSM, or SOSM respects preference rankings, but (v) does not hold, i.e., there exist  $a, b \in O$  with  $a \neq b$ ,  $q_a + q_b \leq |I| - 1$ . Let  $i$  be the student such that for each  $l \in I \setminus \{i\}$ ,  $l \succ_a i$ . Because  $\succ_a$  is a strict order over  $I$ ,  $|U_i(\succ_a)| = |I| - 1$ .<sup>34</sup> Naturally, there is  $j \in I$  such that  $|U_j(\succ_b) \setminus \{i\}| = q_b$ . Let  $S_b = U_j(\succ_b) \setminus \{i\}$ . It is easy to verify that  $|S_b \cup \{j\}| = q_b + 1$  and  $S_b \cup \{j\} \subset U_i(\succ_a)$ . Because  $q_a + q_b \leq |I| - 1$ ,  $|U_i(\succ_a)| = |I| - 1$ ,  $|S_b \cup \{j\}| = q_b + 1$ , and  $S_b \cup \{j\} \subset U_i(\succ_a)$ , we conclude that  $|U_i(\succ_a) \setminus [S_b \cup \{j\}]| = (|I| - 1) - (q_b + 1) \geq q_a + q_b - (q_b + 1) = q_a - 1$ .

Since  $|U_i(\succ_a) \setminus [S_b \cup \{j\}]| \geq q_a - 1$ , we can take  $S_a \subseteq U_i(\succ_a) \setminus [S_b \cup \{j\}]$  such that  $|S_a| = q_a - 1$ . Consider the following preference profile  $P$ :

34. Recall that for each  $i \in I$  and  $a \in O$ ,  $U_i(\succ_a)$  represents the set of students who have higher than student  $i$  for school  $a$ , i.e.,  $U_i(\succ_a) = \{j | j \succ_a i\}$ .

## 6.2 Strategy-proof Boston Mechanism

$P_{S_a \cup \{i\}}$	$P_{I \setminus [S_a \cup \{i\}]}$
$a$	$b$
$b$	$a$
$\vdots$	$\vdots$

By the procedure of  $\varphi^B$ ,  $\varphi_i^B(P) = a$  and  $aP_j\varphi_j^B(P)$ .<sup>35</sup> Then,  $aP_j\varphi_j^B(P)$ ,  $i \in \varphi_a^B(P)$ , and  $j \succ_a i$ . This shows that  $\varphi^B$  is not fair.

Moreover, if student  $j$  reports  $P'_j$  where  $P'_j(a) = 1$  and  $P'_j(b) = 2$ , then  $\varphi_j^B(P'_j, P_{-j}) = a$ . Then,  $\varphi_j^B(P'_j, P_{-j})P_j\varphi_j^B(P)$ . This shows that  $\varphi^B$  is not strategy-proof.

Furthermore, it is easy to calculate that  $\varphi_i^S(P) \neq a = \varphi_i^B(P)$ . This shows that  $\varphi^B \neq \varphi^S$ .

Finally, it is easy to calculate that  $\varphi_i^S(P) \neq a$ , which shows that  $\varphi^S$  violates respect of preference rankings.

(v)  $\Rightarrow$  (i): Suppose that for all  $a, b \in O$  with  $a \neq b$ ,  $q_a + q_b \geq |I|$ , but  $\varphi^B$  is not strategy-proof. Then, there exist  $P \in \mathcal{P}^{|I|}$ ,  $i \in I$  and  $P'_i \in \mathcal{P}$  such that  $\varphi_i^B(P'_i, P_{-i})P_i\varphi_i^B(P)$ . Let  $\varphi_i^B(P'_i, P_{-i})$  and  $\varphi^B(P)$  be  $a$  and  $\mu$  respectively. We then prove that  $1 < P_i(a) < P_i(\mu_i)$ . Because  $aP_i\mu_i$ , it is natural that  $P_i(a) < P_i(\mu_i)$ . To prove  $P_i(a) > 1$ , suppose on the contrary that  $P_i(a) = 1$ . By the procedure of  $\varphi^B$ , we conclude that (1)  $|\mu_a| = q_a$ ; (2) for each  $l \in \mu_a$ ,  $l \succ_a i$ ; and (3) for each  $l \in \mu_a$ ,  $P_l(a) = 1$ . Therefore, given  $P_{-i}$ , there is no way that  $\varphi_i^B(P'_i, P_{-i}) = a$  even if student  $i$  ranks school  $a$  as his first choice. Thus,  $P_i(a) = 1$  does not hold. Hence, there is another school  $b$  with  $b \neq a$  such that  $P_i(b) = 1$ . Since  $P_i(b) < P_i(a) < P_i(\mu_i)$  and  $\varphi^B$  satisfies non-wastefulness, we have that  $|\mu_a| = q_a$ ,  $|\mu_b| = q_b$ ,  $\mu_a \cap \mu_b = \emptyset$ , and  $i \notin \{\mu_a \cup \mu_b\}$ . Therefore,  $|\mu_a \cup \mu_b \cup \{i\}| = q_a + q_b + 1 \leq |I|$ . Hence,  $q_a + q_b \leq |I| - 1$ , which shows that  $q_a + q_b \not\geq |I|$ .

(v)  $\Rightarrow$  (ii): Suppose that for all  $a, b \in O$  with  $a \neq b$ ,  $q_a + q_b \geq |I|$ , but  $\varphi^B$  is not fair. Thus, there exists  $P \in \mathcal{P}^{|I|}$  and  $i, j \in I$  such that  $\varphi_j^B(P)P_i\varphi_i^B(P)$  and  $i \succ_{\varphi_j^B(P)} j$ . Let  $\varphi^B(P) = \mu$  and  $\varphi_j^B(P) = a$ . By the procedure of  $\varphi^B$ , we know that  $P_i(a) > 1$ . To prove  $P_i(a) > 1$ , suppose on the contrary that  $P_i(a) = 1$ . By the procedure of  $\varphi^B$ , we conclude that (1)  $|\mu_a| = q_a$ ; and (2) for each  $l \in \mu_a$ ,  $l \succ_a i$ . Therefore, there exists no  $j \in \mu_a$  such that  $aP_i\mu_i$  and  $i \succ_a j$ . This shows that  $P_i(a) = 1$  does not hold. Since  $P_i(a) > 1$ , there is another school  $b$  with  $b \neq a$  such that  $P_i(b) = 1$ . Since  $P_i(b) < P_i(a) < P_i(\mu_i)$  and  $\varphi^B$  satisfies non-wastefulness, we have that  $|\mu_a| = q_a$ ,  $|\mu_b| = q_b$ ,  $\mu_a \cap \mu_b = \emptyset$ , and  $i \notin \{\mu_a \cup \mu_b\}$ . Therefore,  $|\mu_a \cup \mu_b \cup \{i\}| = q_a + q_b + 1 \leq |I|$ . Hence,  $q_a + q_b \leq |I| - 1$ , which shows

35. Note that  $\varphi_j^B(P) \notin \{a, b\}$ .

## Chapter 6 When is the Boston Mechanism Strategy-proof?

that  $q_a + q_b \not\leq |I|$ .

(v)  $\Rightarrow$  (iii): If (v) holds, i.e., for all  $a, b \in O$  with  $a \neq b$ ,  $q_a + q_b \geq |I|$ , then we have that for each problem  $P$ , the immediate acceptance algorithm will end in two steps and under  $\varphi^B$ , all students are assigned either their first or second choices, i.e., for each  $i \in I$ , we have  $P_i(\varphi_i^B(P)) \leq 2$ . Let us divide the set of students  $I$  into two sets  $I = I_1 \cup I_2$ , where  $I_1$  stands for the set of students assigned to their favorite schools, and  $I_2$  stands for the set of students assigned to their second preferred schools. Moreover, it is easy to imagine that students in  $I_2$  are all rejected by one single school. Without loss of generality, let us assume that they are all rejected by school  $a$ . Let us further divide  $I$  into three sets  $I = \{I_1 \setminus \beta_a(P)\} \cup \beta_a(P) \cup I_2$ .

By the procedure of SOSM, it is easy to calculate that for each  $i \in I_1 \setminus \beta_a(P)$ , we have  $\varphi_i^B(P) = \varphi_i^S(P)$  because no other student is going to compete with them. Furthermore, it is easy to calculate that for each  $i \in \beta_a(P)$ , we have  $\varphi_i^B(P) = \varphi_i^S(P)$  because students who are rejected under the BOSM still have no chance of being assigned to  $a$ . Finally, it is easy to calculate that for each  $i \in I_2$ , we have  $\varphi_i^B(P) = \varphi_i^S(P)$  because these students will also be rejected by school  $a$  under SOSM. Therefore, we conclude that  $\varphi^B = \varphi^S$ .

(v)  $\Rightarrow$  (iv): If (v) holds, then we have  $\varphi^B = \varphi^S$ . Because  $\varphi^B$  respects preference rankings, it follows that  $\varphi^S$  also respects preference rankings.  $\square$

### 6.3 Discussions on Outside Options

The current thesis assumes the existence of the null school which is not scarce and allows students to submit singleton preference lists. Kesten and Kurino (2013) consider two departures from our setting to investigate the strategic role of outside options. They first consider problems where students cannot submit singleton preference lists and second the environment where no outside options necessarily exist, i.e., the null school does not exist and  $|I| \leq \sum_{a \in O} q_a$ . Are the conclusions of our research applicable in the environments due to Kesten and Kurino (2013)? The answer is yes. It is easy to verify that our main result (theorem 6.1) still holds even if no singleton preference lists is allowed or no outside options necessarily exist because singleton preference lists and the null school do not appear in the proof of theorem 6.1. Future work is needed to further investigate the role played by outside options for school choice problems.

## 6.4 Conclusion

In this chapter, we identify the necessary and sufficient condition under which the BOSM recovers strategy-proofness and fairness, and BOSM is equivalent to SOSM. The main theorem reveals that to recover nice properties of BOSM is very difficult, which reflects a further disadvantage of this already disputable mechanism.

Many people might ask why BOSM is still widely used in practice if it is so undesirable. Actually, BOSM has a very big advantage over the SOSM and TTCM. That is, it is easy to understand and calculate. One can easily understand that when the number of students and schools become large, SOSM and TTCM become very complicated to calculate, while BOSM does not suffer from this problem. In old days when computers were not common, BOSM was the best choice for social planners mainly due to its simplicity. In modern days, however, we strongly recommend to replace BOSM with the other alternatives because calculation is not a binding constraint now. The computers can process very large problems with ease. Therefore, the alleged advantage of BOSM vanishes and it is not necessary to use such a mechanism which leads to strategic behavior of students and causes unfair matchings.

Indeed, SOSM and TTCM are not perfect compared with BOSM with the former violating Pareto efficiency and the later violating stability. However, this does not mean that BOSM should be given equal chance in practice. Our conclusion (theorem 1) shows that to recover desirable properties, BOSM needs more stringent conditions on priorities, which restricts its further use in future.





# Chapter 7

## Axioms for Random Assignment

### 7.1 Introduction

Chapters 2-6 study school choice problem with strict priorities. However, in real-life problems, schools sometimes have coarse priorities over students. This chapter discusses school choice model when schools are indifferent to all students. Because this corresponds to the model of random assignment, we use different notations and terms in this chapter.

This chapter studies the problem of randomly assigning  $n$  heterogeneous indivisible objects to  $n$  agents without monetary transfers and priority orders, when one agent can only receive one object. This problem, initiated by Hylland and Zeckhauser (1979), has a number of applications, such as dormitory allocation in universities, the assignment of tasks to workers, course bidding, kidney exchange, and so on. Without monetary transfers and priority orders, the deterministic approaches suffer from incompatibility between efficiency and fairness. To restore fairness, randomization is a common method.

Two competing random assignment rules have attracted much attention recently: the random serial dictatorship (RSD, also known as random priority) introduced by Abdulkadiroğlu and Sönmez (1998), and the probabilistic serial rule (PS rule) first introduced by Bogomolnaia and Moulin (2001). The random serial dictatorship (RSD) and its variants are widely used for assigning indivisible objects. The RSD first chooses the priority over agents at randomly with uniform probability. For the chosen priority order, the agent who is ordered first is assigned his top choice, the agent who is ordered second is assigned his top choice among what remain, and so on. The RSD is strategy-proof, ex post efficient, and easy to implement to real-life problems, i.e., the lottery assignment that induces the random assignment of the RSD is specified explicitly. Thus, it is very appealing.

However, as Bogomolnaia and Moulin (2001) pointed out, the RSD

violates sd-efficiency (We use the prefix "sd" for stochastic dominance in other expressions below. Sd-efficiency is also known as ordinal efficiency due to Bogomolnaia and Moulin (2001)). We say that a random assignment is sd-efficient if no other random assignment first-order stochastically dominates it for all agents. An alternative rule which satisfies sd-efficiency canonically, i.e., the probabilistic serial rule (PS rule), was proposed by Bogomolnaia and Moulin (2001). The outcome of the PS rule is computed via the following simultaneous eating algorithm (SEA). Between time 0 and 1, each agent eats his favorite objects first, and eating speeds are fixed in one across agents. Each agent switches to eat his most preferred object among the available objects when the supply of the object that he is currently eating is eaten away. The PS rule outperforms the RSD not only in that it satisfies sd-efficiency, but also in that it satisfies sd-envy-freeness. However, nice efficiency and fairness performance comes at a cost. The PS rule is weakly strategy-proof, but not strategy-proof.

Interestingly, although RSD and PS rule perform differently in finite assignment problems, when the market becomes infinitely large, Che and Kojima (2010) prove that these two rules asymptotically converge to each other. Therefore, RSD and PS rule can be considered as one rule in large assignment problems. Until now, appealing rules that are asymptotically different from the RSD and PS in large assignment problems are absent. The current paper proposes a new rule called the probabilistic rank-consumption (PRC) rule. The PRC rule first lets agents eat their favorite objects simultaneously and in a speed fixed in one until there is no quota left or agents get one copy of their favorite objects, second lets agents eat their second-choice objects simultaneously and in a speed fixed in one until there is no quota left or agents get one copy of their favorite objects, and so on.

The PRC rule satisfies two new axioms: sd-rank-fairness and equal-rank envy-freeness. Sd-rank-fairness means that whenever an agent gets a positive share of one object, all agents who put the object in higher preference ranks are satiated with this object. Sd-rank-fairness is a refinement of sd-efficiency. Equal-rank envy-freeness means that if two agents put an object in the same preference rank, then changing the assignments of the two agents for this object cannot increase the surplus at the same object for any of them. Equal-rank envy-freeness is a refinement of equal treatment of equals. Moreover, sd-rank-fairness and equal-rank envy-freeness are enough to characterize the PRC rule. However, better efficiency performance of the PRC rule comes at a cost, it is neither weakly strategy-proof nor weakly sd-envy-free (hence not envy-free).

Although showing no good incentive properties, ordinal Nash outcomes of the preference revelation game induced by the PRC rule are all weakly sd-envy-free, which stands in contrast with the PS rule<sup>36</sup>.

### 7.1.1 Related Literature

The following papers study the RSD and the relationship between RSD and the PS rule. Abdulkadiroğlu and Sönmez (1998) showed that the RSD is equivalent to the core from random endowment. Manea (2009) proved that sd-inefficiency of the RSD does not disappear even in large markets. Che and Kojima (2010) showed that the RSD and the PS rule are asymptotically equivalent. In the scheduling problem with opting out, Crès and Moulin (2001) showed that the PS rule is relatively better than RSD, but as the market size becomes large, these two rules serve as proxy of each other.

The following papers studies the PS rule. Bogomolnaia and Moulin (2001) first introduced the PS rule. Bogomolnaia and Moulin (2002) gave two characterizations of the PS rule. Kojima and Manea (2010b) studied incentives of the PS rule in large assignment problems. Katta and Sethuraman (2006), Yilmaz (2009), and Kojima (2009) consider the PS rule when weak preferences, and multi-unit demand are allowed, respectively. Kesten (2009) introduced two mechanisms and showed that these mechanisms are equivalent to the PS rule. Budish *et al.* (2013) generalized random assignment theory to situations when multi-unit allocations and several real-life constraints are allowed. Liu and Pycia (2012) showed that in large markets without transfers all efficient, symmetric, and asymptotically strategy-proof ordinal allocation mechanisms coincide asymptotically. Ekici and Kesten (2012) studied properties of Nash equilibrium outcomes of the PS rule.

The following papers studies sd-efficiency. Sd-efficiency was introduced by Bogomolnaia and Moulin (2001), and analyzed among others by Abdulkadiroğlu and Sonmez (2003b), McLennan (2002), and Manea (2008). Featherstone (2011) studied an efficiency concept stronger than sd-efficiency: rank efficiency, and proposed the rank-value rule to derive rank efficient random assignments.

Our paper also contributes in characterizing the random assignment rules. Axiomatizations of the PS rule has received much attention recently. Hashimoto *et al.* (2013), Bogomolnaia and Heo (2012) characterized the PS rule independently. Chambers (2004) characterized the uniform assignment rule by probabilistic consistency and equal treatment of equals.

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36. See Ekici and Kesten (2012) for more information

## 7.2 The Model

A random assignment problem is a triple  $(I, O, \succ)$  where  $I$  is a finite set of **agents**,  $O$  is a finite set of **objects** with  $|I| = |O| = n$ , and  $\succ = (\succ_i)_{i \in I} \in \mathcal{O}^n$  is the **ordinal preference profile** of agents. Let  $\succeq_i$  be the weak preference relation (i.e.,  $o \succeq_i o'$  means that  $o \succ_i o'$  or  $o = o'$ ) associated with  $\succ_i$ . Let  $\succ_i(o)$  be the **rank** of object  $o$  at  $\succ_i$ , i.e., if object  $o$  is the  $l^{\text{th}}$  choice of agent  $i$  under  $\succ_i$ , then  $\succ_i(o) = l$ . Symmetrically, let  $\succ_i(l)$  be the object that is the  $l^{\text{th}}$  choice of agent  $i$  under  $\succ_i$ . For each  $i \in I$  and  $\succ_i \in \mathcal{O}$  and each  $o \in O$ , denote by  $U(\succ_i, o) = \{o' \in O : o' \succ_i o\}$  the strict upper contour set of  $\succ_i$  at  $o$ , and  $\widehat{U}(\succ_i, o) = U(\succ_i, o) \cup \{o\}$  the weak upper contour set of  $\succ_i$  at  $o$ . Assume that  $\succ_i(o) = k$ , we also denote by  $U(\succ_i, k) = \{o' \in O : o' \succ_i o\}$  the strict upper contour set of  $\succ_i$  at  $o$ , and  $\widehat{U}(\succ_i, k) = U(\succ_i, k) \cup \{o\}$  the weak upper contour set of  $\succ_i$  at  $o$ .

A **deterministic assignment** is a bijection  $\mu : I \rightarrow O$ . It is represented by a permutation matrix  $D = (D_{i,o})_{i \in I, o \in O}$  (an  $n \times n$  matrix with entries 0 or 1 and exactly one nonzero entry per row and one per column). Let  $\mathcal{D}$  denote the set of all deterministic assignments. A **lottery assignment**  $L = \sum_{\mu \in \mathcal{D}} \lambda_\mu$ , with  $\lambda_\mu \geq 0$  and  $\sum_{\mu} \lambda_\mu = 1$ , is a probability distribution over  $\mathcal{D}$ , where  $\lambda_\mu$  is the probability weight of the deterministic assignment  $\mu$ .

For each agent  $i \in I$ , a **random allocation**  $R_i = (R_{i,o})_{o \in O}$  is a probability distribution over  $O$ . A random allocation  $R_i$  specifies for an agent the probabilities of receiving various objects: object  $o$  is received with probability  $R_{i,o}$ . Let  $\mathcal{R}$  denote the set of possible random allocations. A **random assignment** is represented by a bistochastic matrix  $R = (R_{i,o})$  such that  $\sum_{o \in O} R_{i,o} = 1 \forall o \in O$ , and  $\sum_{i \in I} R_{i,o} = 1 \forall i \in I$ . A random assignment specifies for all agents probabilities of receiving various objects:  $R_{i,o}$  is the probability that agent  $i$  receives object  $o$ . We refer to the vector  $R_i$  as agent  $i$ 's random allocation at  $R$ . By the well-known Birkhoff-von Neumann theorem (See Birkhoff (1946), and von Neumann (1953). Also see Budish *et al.* (2013) for generalizations of this theorem.), each bistochastic matrix can be represented by a lottery assignment.

Throughout the paper, we fix  $I$  and  $O$ , and a **problem** is denoted by a preference profile  $\succ \in \mathcal{O}^n$  where  $\mathcal{O}^n$  stands for the set of ordinal preference profiles. Let  $\mathcal{R}^n$  denote the set of all random assignments. A **rule** is a mapping  $\varphi : \mathcal{O}^n \rightarrow \mathcal{R}^n$ .

### 7.2.1 Axioms

A deterministic assignment  $D \in \mathcal{D}$  is **Pareto efficient** if  $\nexists D' \in \mathcal{D}$  such that  $D'_i \succeq_i D_i \forall i \in I$ , and  $D'_i \succ_i D_i$  for some  $i \in I$ .

For each agent  $i \in I$  and  $R_i, R'_i \in \mathcal{R}$ , we say that

(i)  $R_i$  **weakly stochastically dominates**  $R'_i$ , written as  $R_i sd(\succ_i) R'_i$ , if

$$\sum_{o' \in \hat{U}(\succ_i, o)} R_{i, o'} \geq \sum_{o' \in \hat{U}(\succ_i, o)} R'_{i, o'}, \forall o \in O;$$

(ii)  $R_i$  **stochastically dominates**  $R'_i$ , written as  $R_i SD(\succ_i) R'_i$ , if

$$\sum_{o' \in \hat{U}(\succ_i, o)} R_{i, o'} \geq \sum_{o' \in \hat{U}(\succ_i, o)} R'_{i, o'}, \forall o \in O \text{ and } R_i \neq R'_i.$$

**Ex post efficiency:** A rule  $\varphi$  is ex post efficient if for each  $\succ \in \mathcal{O}^n$ ,  $\varphi(\succ)$  admits a lottery decomposition over Pareto efficient deterministic assignments.

**Sd-efficiency:** A rule  $\varphi$  is sd-efficient if for each  $\succ \in \mathcal{O}^n$ ,  $\nexists R \in \mathcal{R}^n$  such that  $R_i sd(\succ_i) \varphi_i(\succ) \forall i$  and  $R \neq \varphi(\succ)$ .

**Equal treatment of equals:** A rule  $\varphi$  satisfies equal treatment of equals if for each  $\succ \in \mathcal{O}^n$  and  $i, j \in I$  such that  $\succ_i = \succ_j$ ,  $\varphi_i(\succ) = \varphi_j(\succ)$ .

**Weak sd-envy-freeness:** A rule  $\varphi$  is weakly sd-envy-free if for each  $\succ \in \mathcal{O}^n$ ,  $\nexists i, j \in I$  such that  $\varphi_j(\succ) SD(\succ_i) \varphi_i(\succ)$ .

**Sd-envy-freeness:** A rule  $\varphi$  is sd-envy-free if for each  $\succ \in \mathcal{O}^n$ ,  $\varphi_i(\succ) sd(\succ_i) \varphi_j(\succ) \forall i, j \in I$ .

**Weak strategy-proofness:** Let  $\bar{\succ}_i$  be the true preference of agent  $i$ . A rule  $\varphi$  is weakly strategy-proof if for each  $\succ_{-i} \in \mathcal{O}^{n-1}$ ,  $\nexists \succ_i \in \mathcal{O}$  such that  $\varphi_i(\succ_i, \succ_{-i}) SD(\bar{\succ}_i) \varphi_i(\bar{\succ}_i, \succ_{-i})$ .

**Strategy-proofness:** A rule  $\varphi$  is strategy-proof if for each  $\succ_i \in \mathcal{O}$ ,  $\succ_{-i} \in \mathcal{O}^{n-1}$ ,  $\varphi_i(\bar{\succ}_i, \succ_{-i}) sd(\bar{\succ}_i) \varphi_i(\succ_i, \succ_{-i})$ .

**Ordinal Nash equilibrium:** Under a rule  $\varphi$ ,  $\succ$  constitutes an ordinal Nash equilibrium if  $\forall i$ ,  $\nexists \succ'_i \in \mathcal{O}$  such that  $\varphi_i(\succ'_i, \succ_{-i}) SD(\succ_i) \varphi_i(\succ_i, \succ_{-i})$ .

Define the rank distribution of assignment  $R$  to be

$$N^R(k) = \sum_{i \in I} \sum_{o' \in \hat{U}(\succ_i, k)} (R_{i, o'})$$

$N^R(k)$  is the expected number of agents who get their  $k^{\text{th}}$  choice or better under assignment  $R$ . Given  $\succ \in \mathcal{O}^n$ , a random assignment  $R$  is *rank-dominated* by another random assignment  $R'$  at  $\succ$  if the rank distribution

of  $R'$  stochastically dominates that of  $R$ , that is,  $N^{R'}(k) \geq N^R(k)$  for all  $k$  and  $N^R \neq N^{R'}$ .

**Rank efficiency:** A rule  $\varphi$  is rank efficient if for each  $\succ \in \mathcal{O}^n$ ,  $\varphi(\succ)$  is not rank-dominated by any other assignments.

## 7.3 Three Existing Random Assignment Rules

### 7.3.1 Uniform Assignment Rule

The uniform assignment rule (UA rule) was introduced by Chambers (2004). For a given problem, the uniform assignment is defined as the random assignment which places equal probability on all deterministic assignments. The uniform assignment rule recommends the uniform assignment for all problems. Denote the UA rule as  $\varphi^{ua}$ .

**EXAMPLE 7.1** Let  $I = (1, 2, 3, 4)$ ,  $O = (A, B, C, D)$ . Preferences  $\succ = (\succ_1, \succ_2, \succ_3, \succ_4)$  are as follows:

Agent 1 :  $A \succ_1 B \succ_1 C \succ_1 D$ ; Agent 2 :  $A \succ_2 C \succ_2 D \succ_2 B$

Agent 3 :  $A \succ_3 C \succ_3 D \succ_3 B$ ; Agent 4 :  $B \succ_4 A \succ_4 C \succ_4 D$

The resulting UA assignment,  $\varphi^{ua}(\succ)$  is given by:

	A	B	C	D
$\varphi^{ua}(\succ)$ : 1	1/4	1/4	1/4	1/4
2	1/4	1/4	1/4	1/4
3	1/4	1/4	1/4	1/4
4	1/4	1/4	1/4	1/4

### 7.3.2 Random Serial Dictatorship

Let  $\mathcal{F}$  denote the set of all bijections from  $\{1, 2, \dots, |I|\}$  to  $I$ . We refer to each of these bijections as an ordering of agents. That is, for any  $f \in \mathcal{F}$ , agent  $f(1)$  is first and agent  $f(2)$  is second, and so on. Given any ordering  $f \in \mathcal{F}$ , the deterministic assignment derived by simple serial dictatorship, denoted  $\varphi_f^{ssd}(\succ)$ , is defined as follows: agent  $f(1)$  receives his most preferred object according to  $\succ_{f(1)}$ , agent  $f(2)$  receives his most preferred object among what remain according to  $\succ_{f(2)}$ , and so on. Define the random serial dictatorship (RSD), denoted  $\varphi^{rsd}$ , as

### 7.3 Three Existing Random Assignment Rules

$$\varphi^{rsd}(\succ) = \sum_{f \in \mathcal{F}} \frac{1}{n!} \varphi_f^{ssd}$$

**EXAMPLE 7.1** revisits. The resulting RSD assignment,  $\varphi^{rsd}(\succ)$  is given by:

	A	B	C	D
$\varphi^{rsd}(\succ) :$	1/3	1/4	1/6	1/4
1	1/3	1/4	1/6	1/4
2	1/3	0	3/8	7/24
3	1/3	0	3/8	7/24
4	0	3/4	1/12	1/6

#### 7.3.3 Probabilistic Serial Rule

The PS rule is one of the most well-known random assignment rule. Given a preference profile  $\succ \in \mathcal{O}^n$ , the PS rule, denoted  $\varphi^{ps}$ , determines the assignment  $\varphi^{ps}(\succ)$  by using the following simultaneous eating algorithm (SEA). Consider each unit of object as infinitely divisible. Suppose that agents eat the objects during a unit interval of time at a unit speed. The amount of object  $o$  that an agent eats represents the probability that she obtains for object  $o$ . During the time interval,  $t \in [0, 1]$ , the agents behave as follows.

- At time  $t = 0$ , each agent starts eating his favorite object. Each agent continues to eat the same object until either the time is up or the object that he is currently eating is exhausted, i.e., the sum of the amounts that each agent eats reaches 1.
- When object  $o$  is eaten away, each agent who has been eating  $o$  changes his behavior. He continues to eat his next preferred object which has a positive amount of quota left.
- At time  $t = 1$ , the process ends. For each agent  $i$  and object  $o$ ,  $\varphi_{i,o}^{ps}(\succ)$  is set to the amount of object  $o$  that  $i$  has eaten.

**EXAMPLE 7.1** revisits. The resulting PS assignment,  $\varphi^{ps}(\succ)$  is given by:

	A	B	C	D
$\varphi^{ps}(\succ) :$	1/3	1/3	1/12	1/4
1	1/3	1/3	1/12	1/4
2	1/3	0	5/12	1/4
3	1/3	0	5/12	1/4
4	0	2/3	1/12	1/4

## 7.4 Two New Axioms

In this section, we first introduce a new axiom that is stronger than sd-efficiency: sd-rank-fairness. Sd-rank-fairness requires that if the assignment of an agent for an object is greater than zero, then all agents who put the object in higher preference ranks get a surplus of one at this object, i.e., are satiated at this object. A rule satisfying sd-rank-fairness tries to assign an object to agents who put it in a preference rank as high as possible, and only when assigning the object to agents with higher preference rank for this object is impossible, does it consider assigning it to agents who put the object in lower preference ranks.

**Definition 7.1.** A rule  $\varphi$  satisfies **sd-rank-fairness** if for each  $\succ \in \mathcal{O}^n$ ,  $i \in I$  and  $o \in O$ ,  $\varphi_{i,o}(\succ) > 0$  implies that  $\sum_{o' \in \hat{U}(\succ_j, o)} \varphi_{j,o'}(\succ) = 1$  for each  $j$  such that  $\succ_j(o) < \succ_i(o)$ .

Bogomolnaia and Moulin (2001) showed that sd-efficiency is a refinement of ex post efficiency. Featherstone (2011) proposed a new efficiency notion: rank efficiency. Rank efficiency is a refinement of both sd-efficiency and ex post efficiency. While sd-rank-fairness and rank efficiency are **independent** concepts, it is also true that sd-rank-f  
-efficiency and ex post efficiency.

**Proposition 7.1.** If a rule  $\varphi$  satisfies sd-rank-fairness, then  $\varphi$  also satisfies sd-efficiency; however, the converse need not hold.

*Proof.* The assignment derived by the PS rule in example 7.1 in section 7.3 shows that the converse need not hold. Now, we need to prove that sd-rank-fairness implies sd-efficiency. Suppose that there exists an assignment  $R$  that satisfies sd-rank-f  
-efficiency. By lemma 3 of Bogomolnaia and Moulin (2001), there exist  $i_1, i_2, \dots, i_k$  and  $o_1, o_2, \dots, o_k$  such that  $o_2 \succ_{i_1} o_1$  and  $R_{i_1, o_1} > 0$ ;  $o_3 \succ_{i_2} o_2$  and  $R_{i_2, o_2} > 0$ ;  $\dots$ ;  $o_1 \succ_{i_k} o_k$  and  $R_{i_k, o_k} > 0$ . Let  $K = \text{Max}_{t \in \{1, 2, \dots, k\}} \succ_{i_t}(o_t)$ . Without loss of generality, suppose that  $\succ_{i_1}(o_1) = K$ . Consider agent  $i_k$  for whom  $o_1 \succ_{i_k} o_k$  and  $R_{i_k, o_k} > 0$ . Because  $R_{i_k, o_k} > 0$ ,  $\sum_{o \in \hat{U}(\succ_{i_k}, o_1)} \varphi_{i_k, o}(\succ) < 1$ , i.e., agent  $i_k$  is not satiated at object  $o_1$ . By the definition of  $K$ ,  $\succ_{i_k}(o_1) < \succ_{i_k}(o_k) \leq \succ_{i_1}(o_1) = K$ . Then, we have that  $\succ_{i_k}(o_1) < \succ_{i_1}(o_1)$  and agent  $i_k$  is not satiated at object  $o_1$  but  $R_{i_1, o_1} > 0$ , which contradicts sd-rank-fairness of  $R$ .  $\square$



<b>Efficiency concepts</b>	Ex post efficiency	$\Rightarrow$ Sd-efficiency $\Leftarrow$	$\Rightarrow$ Sd-rank-fairness $\Leftarrow$
<b>Rules</b>	RSD	PS rule	PRC rule

Table 7.1: R

, sd-efficiency is equivalent to ex post efficiency, while sd-rank-fairness is equivalent to both. Formally,

**Proposition 7.2.** *Let  $x$  be a deterministic assignment. Then,*

- (i)  $x$  is ex post efficient implies that it is sd-efficient; however, the converse need not hold.
- (ii)  $x$  is sd-rank-fair implies that it is sd-efficient; however, the converse need not hold.

*Proof.* Part (i) comes directly from proposition 8 of Featherstone (2011). We need only to prove part (ii), i.e., sd-rank-fairness implies sd-efficiency in the deterministic environment. In the deterministic environment, sd-rank-fairness deteriorates into the following shape: a deterministic assignment  $\mu$  satisfies sd-rank-fairness if  $\mu_i = o$  means that for any  $j$  such that  $\succ_j(o) < \succ_i(o)$ ,  $\mu_j \in \widehat{U}(\succ_j, o)$ . Thus, for any  $j$  such that  $\succ_j(o) < \succ_i(o)$ ,  $j$  does not envy agent  $i$ . Thus, for any  $l \in I$ ,  $\mu_i \succ_l \mu_l$  implies that  $\succ_i(\mu_i) \leq \succ_l(\mu_i)$ , where  $\mu$  is a deterministic assignment. The previous statement is equivalent to respect of preference rankings due to Kojima and Ünver (2013). Proposition 2 of Kojima and Ünver (2013) tells us that respect of preference rankings implies ex post efficiency. We thus complete the proof.  $\square$

Next, we define the second new axiom: equal-rank envy-freeness. An assignment satisfies equal-rank envy-freeness, if two agents put an object in the same preference rank, then changing the assignments of the two agents for this object cannot increase the surplus at the same object for any of them.

**Definition 7.2.** *A rule  $\varphi$  satisfies **equal-rank envy-freeness** if for each  $\succ \in \mathcal{O}^n$ ,  $o \in O$ , and  $i, j \in I$  such that  $\succ_i(o) = \succ_j(o)$ ,*

$$\min\left(\sum_{o' \in U(\succ_i, o)} \varphi_{i, o'}(\succ) + \varphi_{j, o}(\succ), 1\right) \leq \sum_{o' \in \widehat{U}(\succ_i, o)} \varphi_{i, o'}(\succ).$$

**Proposition 7.3.** *If a rule  $\varphi$  satisfies equal-rank envy-freeness, then  $\varphi$  also satisfies equal treatment of equals.*

We omit proof for proposition 7.3 because the result is obvious.

In this section, we introduce two new axioms for random assignment rules: sd-rank-fairness and equal-rank envy-freeness. Unfortunately, sd-rank-fairness is incompatible with strategy-proofness. Indeed, the cost of strategy-proofness can be quite high, not only in random assignment problems. However, this does not mean that sd-rank-fairness should be dismissed. Sd-rank-fairness is a stronger efficiency axiom. Of course, now it is customary in mechanism design theory to impose incentive constraint first, and investigate other aspect of criteria later. However, if strategy-proofness is not emphasized in some situation, then it is interesting to investigate how a rule performs when other constraints are imposed first. In this situation, sd-rank-fairness is an important axiom to measure a rule because of its good efficiency implication.

## 7.5 The Probabilistic Rank-consumption Rule (PRC Rule)

In this section, we describe a new rule called the probabilistic rank-consumption rule, denoted by  $\varphi^{prc}$ . For each  $\succ \in \mathcal{O}^n$ , the PRC assignment  $\varphi^{prc}(\succ)$  is determined by the following **probabilistic rank-consumption algorithm**.

- Step 1, only the first choices of the agents are considered. Each agent starts consuming the object that he prefers most **simultaneously** at a speed fixed in one across agents. An agent stops consuming his favorite object until there is no quota left, or he gets one copy of this object. Remove the agents who get one copy of their favorite objects.
- Step  $k$ , only the  $k^{th}$  choices of the remaining agents are considered. Each remaining agent starts consuming his  $k^{th}$  preferred object **simultaneously** at a speed fixed in one across agents. An agent stops consuming his  $k^{th}$  preferred object until there is no quota left, or the agent gets a total surplus of one at this object. Remove the agents who get a surplus of one at their  $k^{th}$  preferred object.

Given  $\succ \in \mathcal{O}^n$  and  $i \in I$  and  $o \in O$  where  $\succ_i(o) = k$ , let  $I(o, k) = \{j : \succ_j(o) = k\}$ , i.e.,  $I(o, k)$  is the set of agents who put object  $o$  in the

### 7.5 The Probabilistic Rank-consumption Rule (PRC Rule)

same preference rank  $k$ . Formally, the PRC algorithm is defined as follows. Let  $t_i(0) = 0$  for each  $i \in I$ .

$$t_i(k) = \text{Max}\{t \in [0, 1] | t - t_i(k-1) + \sum_{j \in [I(o,k) \setminus \{i\}]} \varphi_{j,o}(\succ) \leq 1\} \quad (7.1)$$

$$\varphi_{i,o}(\succ) = t_i(k) - t_i(k-1) \quad (7.2)$$

$$\min\left(\sum_{o' \in U(\succ_{i,o})} \varphi_{i,o'}(\succ) + \varphi_{j,o}(\succ), 1\right) \leq \sum_{o' \in \hat{U}(\succ_{i,o})} \varphi_{i,o'}(\succ) \quad (7.3)$$

**EXAMPLE 7.1** revisits. Let  $I = (1, 2, 3, 4)$ ,  $O = (A, B, C, D)$ . Preferences  $\succ = (\succ_1, \succ_2, \succ_3, \succ_4)$  are as follows:

Agent 1 :  $A \succ_1 B \succ_1 C \succ_1 D$ ; Agent 2 :  $A \succ_2 C \succ_2 D \succ_2 B$

Agent 3 :  $A \succ_3 C \succ_3 D \succ_3 B$ ; Agent 4 :  $B \succ_4 A \succ_4 C \succ_4 D$

The PRC rule finds the assignment through the following procedure. In the first step, agent 1, 2, 3 start consuming object  $A$  simultaneously in a speed fixed in one across each other, and each of them get  $1/3$  of object  $A$ , while agent 4 starts eating object  $B$  until he consumes  $B$  away. Agent 4 is then removed from the economy. In the second step, agent 1 does not consume any object because the quota of object  $B$  is zero now, while agent 2 and 3 consume object  $C$  at the same time and in the same speed until there is no object  $C$  left, and each of them get  $1/2$   $C$ . In the third step, agent 2, 3 start eating object  $D$  simultaneously and in a speed fixed in one. Agent 2 and 3 stop when their total surplus at object  $D$  reaches one, and each of them get  $1/6$  of object  $D$ , while agent 1 consumes nothing because object  $C$  has been consumed way in the previous step. In the fourth step, agent 1 consumes object  $D$  until his surplus at object  $D$  reaches one and there is no object  $D$  left. Agent 1 gets  $2/3$  object  $D$ . The resulting PRC assignment,  $\varphi^{prc}(\succ)$ , is given as follows:

	A	B	C	D	
$\varphi^{prc}(\succ) :$	1	1/3	0	0	2/3
	2	1/3	0	1/2	1/6
	3	1/3	0	1/2	1/6
	4	0	1	0	0

Chapter 7 Axioms for Random Assignment

It is obvious to see that the PRC rule satisfies both sd-rank-fairness and equal-rank envy-freeness. However, the PRC rule are vulnerable to preference manipulation of agents. In other words, agents have strong incentives not to report their true ordinal preferences.

**Proposition 7.4.**  $\varphi^{prc}$  is not weakly strategy-proof (and hence not strategy-proof).

*Proof.* **EXAMPLE 7.1** revisits. If agent 1 reports different preference orders  $\succ'_1: A \succ'_1 C \succ'_1 B \succ'_1 D$ , or  $\succ''_1: A \succ''_1 C \succ''_1 D \succ''_1 B$  instead of  $\succ_1$ , then the PRC assignment is given by:

	A	B	C	D
$\varphi^{prc}(\succ_1, \succ_{-1}) = \varphi^{prc}(\succ'_1, \succ_{-1}) :$	1	1/3	0	1/3
	2	1/3	0	1/3
	3	1/3	0	1/3
	4	0	1	0

which is preferred by agent 1 to  $\varphi^{prc}(\succ)$ . □

**Proposition 7.5.**  $\varphi^{prc}$  is not weakly sd-envy-free (and hence not sd-envy-free).

*Proof.* It is easy to verify that the assignment corresponding to example 7.1

	A	B	C	D
$\varphi^{prc}(\succ) :$	1	1/3	0	2/3
	2	1/3	0	1/2
	3	1/3	0	1/2
	4	0	1	0

is not weakly sd-envy-free because  $\varphi_2^{prc}(\succ)SD(\succ_1)\varphi_1^{prc}(\succ)$  and  $\varphi_3^{prc}(\succ)SD(\succ_1)\varphi_1^{prc}(\succ)$ . □

**Proposition 7.6.**  $\varphi^{prc}$  is not rank-efficient.

*Proof.* It is easy to verify that the following assignment

	A	B	C	D
$R :$	1	1	0	0
	2	0	0	1/2
	3	0	0	1/2
	4	0	1	0

rank dominates  $\varphi^{prc}(\succ)$  at  $\succ$ . □

## 7.6 Characterization of the PRC Rule

Next, we propose a characterization of the PRC rule. Bogomolnaia and Moulin (2001) showed that no random assignment rule satisfies strategy-proofness, sd-efficiency, and equal treatment of equals at the same time. The following result tells us that if we strengthen sd-efficiency and equal treatment of equals, and abandon strategy-proofness, we can achieve another rule: the PRC rule.

**Theorem 7.1.** *A rule  $\varphi$  satisfies sd-rank-fairness and equal-rank envy-freeness if and only if  $\varphi = \varphi^{prc}$ .*

*Proof.* Suppose that there exists another rule  $\varphi$  that satisfies sd-rank-fairness and equal-rank envy-freeness, but is different from the PRC rule  $\varphi^{prc}$ . Given a problem  $\succ$ , denote the random assignment  $\varphi^{prc}(\succ)$  and  $\varphi(\succ)$  as  $R$  and  $S$ , respectively. Suppose that  $R \neq S$ . We prove the theorem through the following steps.

**Step 1:** Suppose that for some  $i, o$  such that  $\succ_i(o) = 1$ ,  $R_{i,o} \neq S_{i,o}$ . There are two subcases.

**Step 1.1,**  $S_{i,o} < R_{i,o}$ . Because  $R_{i,o} \leq 1$ ,  $S_{i,o} < 1$ . By equal-rank envy-freeness of  $S$ ,  $S_{j,o} \leq R_{j,o}$  for each  $j \in I(o, 1)$  (Note that  $I(o, k) = \{j : \succ_j(o) = k\}$ ). Therefore, there exists agent  $l \in I$  such that  $\succ_l(o) > 1$  and  $S_{l,o} > 0$ .  $\succ_i(o) = 1$ ,  $S_{i,o} < 1$ ,  $\succ_l(o) > 1$ , and  $S_{j,o} > 0$  contradict sd-rank-fairness of  $S$ .

**Step 1.2,**  $S_{i,o} > R_{i,o}$ . The logic is similar with step 1.1 and we omit detailed proof here.

**Step k:** Suppose that for some  $i, o$  such that  $\succ_i(o) = k$ ,  $R_{i,o} \neq S_{i,o}$ . There are two subcases.

**Step k.1,**  $S_{i,o} < R_{i,o}$ . Because  $\sum_{o' \in \widehat{U}(\succ_{i,o})} R_{i,o'} \leq 1$ ,  $\sum_{o' \in \widehat{U}(\succ_{i,o})} S_{i,o'} < 1$ . By equal-rank envy-freeness of  $S$ ,  $S_{j,o} \leq R_{j,o}$  for any  $j \in I(o, k)$ . Therefore, there exists agent  $l \in I$  such that  $\succ_l(o) > k$  and  $S_{l,o} > 0$ .  $\succ_i(o) = k$ ,  $\sum_{o' \in \widehat{U}(\succ_{i,o})} S_{i,o'} < 1$ ,  $\succ_l(o) > k$ , and  $S_{l,o} > 0$  contradict sd-rank-fairness of  $S$ .

**Step k.2,**  $S_{i,o} > R_{i,o}$ . The logic is similar with step k.1 and we omit detailed proof here.  $\square$

**Independence of the axioms:** The uniform assignment rule  $\varphi^{ua}$  due to Chambers (2004) satisfies equal-rank envy-freeness, but violates sd-rank-fairness. The rule which satisfies equations 7.1 and 7.2 but violates equation 7.3 satisfies sd-rank-fairness, but violates equal-rank envy-freeness.

## 7.7 Equilibrium Analysis of the PRC Rule

**Theorem 7.2.** *If  $\succ$  constitutes an ordinal Nash equilibrium of  $\varphi^{prc}$ , then  $\varphi^{prc}(\succ)$  is weakly sd-envy-free with respect to  $\succ$ .*

*Proof.* Suppose that  $\succ$  constitutes an ordinal Nash equilibrium of  $\varphi^{prc}$ , but  $\varphi^{prc}(\succ)$  is not weakly sd-envy-free with respect to  $\succ$ . Let  $R = \varphi^{prc}(\succ)$ . Then, there exist  $i, j \in I$  such that  $R_j SD(\succ_i) R_i$ . To be specific, there exists  $o_m \in O$  such that  $R_{j, o_m} > R_{i, o_m}$ ,  $\sum_{o' \in \hat{U}(\succ_i, o_m)} R_{i, o'} < 1$ , and  $\sum_{o' \in \hat{U}(\succ_j, o_m)} R_{j, o'} < 1$ . Let  $\succ_j(o_m) = k$  and  $\succ_i(o_m) = k'$ . We get the following results:

- (i)  $k > 1$ ;
- (ii)  $k' > k$ .
- (iii)  $R_{i, o_m} = 0$ ;
- (iv)  $R_{i, \succ_i(k)} = 0$ ;

We are now ready to show the shape of  $\succ_i$  and  $\succ_j$ . If  $k = 2$ ,  $\succ_i(1) = \succ_j(1)$  and  $R_{i, \succ_i(1)} = R_{j, \succ_j(1)} > 0$ . If  $k \geq 3$ , for all  $l \in \{1, \dots, k-1\}$ , either  $\succ_i(l) = \succ_j(l)$  or  $\succ_i(l) \neq \succ_j(l)$  and  $R_{i, \succ_i(l)} = R_{i, \succ_i(l)} = 0$ .

As to the ranks from  $k$  to  $k'$ , we discuss about two cases.

**Case 1,**  $k' = k + 1$ . Let  $\succ'_i$  be the preference order where

- (i)  $\forall l \in \{1, \dots, k-1\}$ ,  $\succ'_i(l) = \succ_i(l)$ ;
- (ii)  $\forall l \in \{k, \dots, n-1\}$ ,  $\succ'_i(l) = \succ_i(l+1)$ .

By the definition of the probabilistic rank-consumption algorithm,  $\varphi_i^{prc}(\succ'_i, \succ_{-i}) SD(\succ_i) R_i$ . Therefore,  $\succ$  does not constitute an ordinal Nash equilibrium of  $\varphi^{prc}$ , which contradicts our assumption.

**Case 2,**  $k' = k + x$ ,  $x \geq 2$ . We discuss about two subcases.

**Case 2.1,**  $\exists t \in \{k+1, \dots, k'-1\}$  such that  $R_{i, \succ_i(t)} > 0$  and  $R_{i, \succ_i(l)} = 0$   $\forall l \in \{k, \dots, t-1\}$ . Let  $\succ'_i$  be the preference order where

- (i)  $\forall l \in \{1, \dots, k-1\}$ ,  $\succ'_i(l) = \succ_i(l)$ ;
- (ii)  $\forall l \in \{k, \dots, n-t+k\}$ ,  $\succ'_i(l) = \succ_i(l+t-k)$ .

By the definition of the probabilistic rank-consumption algorithm,  $\varphi_i^{prc}(\succ'_i, \succ_{-i}) SD(\succ_i) R_i$ . Therefore,  $\succ$  does not constitute an ordinal Nash equilibrium of  $\varphi^{prc}$ , which contradicts our assumption.

**Case 2.2,**  $R_{i, \succ_i(l)} = 0$  for  $\forall l \in \{k, \dots, k'\}$ , i.e.,  $\nexists t \in \{k+1, \dots, k'-1\}$  such that  $R_{i, \succ_i(t)} > 0$  and  $R_{i, \succ_i(l)} = 0$  for  $\forall l \in \{k, \dots, t-1\}$ . Let  $\succ'_i$  be the preference order where

- (i)  $\forall l \in \{1, \dots, k-1\}$ ,  $\succ'_i(l) = \succ_i(l)$ ;
- (ii)  $\forall l \in \{k, \dots, n-k'+k\}$ ,  $\succ'_i(l) = \succ_i(l+k'-k)$ .

By the definition of the probabilistic rank-consumption algorithm,  $\varphi_i^{prc}(\succ'_i, \succ_{-i}) SD(\succ_i) R_i$ . Therefore,  $\succ$  does not constitute an ordinal Nash equilibrium of  $\varphi^{prc}$ , which contradicts our assumption.

## 7.8 Concluding Remarks

Therefore, if  $\succ$  constitutes an ordinal Nash equilibrium of  $\varphi^{prc}$ ,  $\varphi^{prc}(\succ)$  is weakly sd-envy-free with respect to  $\succ$ .  $\square$

**Theorem 7.3.** Let  $\succ$  be the true preferences of agents. For each  $n \leq 3$  and  $\succ \in \mathcal{O}^n$ ,  $\succ$  constitutes an ordinal Nash equilibrium of  $\varphi^{prc}$ .

*Proof.* It is enough to prove the case of  $n = 3$ . Consider the following problem:  $I = \{1, 2, 3\}$ ,  $O = \{A, B, C\}$ . Consider three types of preference profiles  $P^1$ ,  $P^2$ , and  $P^3$ :

$P_1^1$	$P_2^1$	$P_3^1$		$P_1^2$	$P_2^2$	$P_3^2$		$P_1^3$	$P_2^3$	$P_3^3$
A	B	C		A	B	B		A	A	A
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$

If  $\succ = P^1$ , then the PRC rule  $\varphi^{prc}$  will assign each agent to their favorite object with certainty and naturally  $P^1$  constitutes an ordinal Nash equilibrium of  $\varphi^{prc}$ .

If  $\succ = P^2$ , then in the first step of the PRC rule, agent 3 will be assigned object  $B$  with probability one and removed from the problem, and agent 1 and 2 each get half  $A$ . Next, no matter what preferences agent 1 and 2 have, each of them will be assigned to object  $C$  with half probability. Therefore,  $P^2$  constitutes an ordinal Nash equilibrium of  $\varphi^{prc}$ .

If  $\succ = P^3$ , then in the first step of PRC rule, each agent will be assigned to their favorite object with equal probability  $1/3$ . Moreover, each agent will be assigned to their second choices with positive probability. Therefore, if the other two agents do not change their preferences, no single agent can be strictly better off for every cardinal utility consistent with their ordinal preferences over the set of objects. Therefore,  $P^3$  constitutes an ordinal Nash equilibrium of  $\varphi^{prc}$ .

We thus complete the proof.  $\square$

## 7.8 Concluding Remarks

The following table compares the PRC rule with the other random assignment rules. We can easily observe from the table that there is an obvious tradeoff between fairness and efficiency between the UA rule and the RSD. There is a tradeoff between incentive properties and fairness and efficiency between the RSD and the PS rule. Finally, there is an obvious tradeoff between incentive properties and efficiency between the PS rule

and the PRC rule, while the tradeoff between incentive properties and fairness is vague.

	$\varphi^{ua}$	$\varphi^{rsd}$	$\varphi^{ps}$	$\varphi^{prc}$
Weak Strategy-proofness	✓	✓	✓	×
Strategy-proofness	✓	✓	×	×
Equal treatment of equals	✓	✓	✓	✓
Weak Sd-envy-freeness	✓	✓	✓	×
Sd-envy-freeness	✓	×	✓	×
Equal-rank envy-freeness	✓	×	×	✓
Ex post efficiency	×	✓	✓	✓
Sd-efficiency	×	×	✓	✓
Sd-rank-fairness	×	×	×	✓
Rank efficiency	×	×	×	×

Table 7.2: Random Assignment Rules and Axioms

This chapter contributes to the literature of random assignment problems mainly on three aspects. First, we propose a new random assignment rule called the probabilistic rank-consumption rule (PRC rule). Recently, there is growing body of literature that studies the PS rule, proposed by Bogomolnaia and Moulin (2001). The PS rule outperforms the RSD because it is sd-efficient, which is a refinement of ex post efficiency. However, better efficiency performance comes at a cost: the PS is weakly strategy-proof, but not strategy-proof. The PRC rule proposed in this chapter outperforms both the PS rule and the RSD because it satisfies sd-rank-fairness, which is a refinement of both sd-efficiency and ex post efficiency. However, we show that better efficiency performance of the PRC rule also comes at a cost: the PRC rule is not even weakly strategy-proof, let alone be strategy-proof. To further understand the tradeoff between efficiency and strategy-proofness, more theoretical and empirical works, such as experiments and simulations, are needed. moreover, the current chapter only discusses the case where the number of agents are equivalent to the number of objects and agents have strict preferences. More works are need to extend the PRC rule to more general environments. This chapter studies the PRC rule only in finite random assignment problems, and does not discuss it in large assignment problems. Recently, the research about large market properties of random assignment rules are very popular. We leave these as open questions.

Second, the present chapter introduces a new efficiency concept: sd-



## 7.8 Concluding Remarks

rank-fairness. A random assignment satisfies sd-rank-fairness, if the probability of assigning an object to an agent being not zero means that all agents who put the object in higher preference ranks are stochastically satiated with respect to this object. The PRC rule satisfies sd-rank-fairness. Sd-rank-fairness is a refinement of both ex post efficiency and sd-efficiency. Parallel to our research, Feasterstone (2011) proposed a family of rank-value rules, which satisfies rank efficiency: also a refinement of both sd-efficiency and ex post efficiency. However, the rank-value rules are interpreted as a linear program that maximize a welfare sum of agents, given an assumption about agents's cardinal utilities (identical among each other), which is more difficult to understand and calculate than rank-consumption rules because to get a rank efficient assignment with respect to a given ordinal preference profile, different cardinal preferences consistent to the given ordinal one may derive different rank efficient assignments. On the contrary, given an ordinal preference profile, a random assignment satisfying sd-rank-fairness with respect to the given preference is sd-rank-fair with respect to any cardinal preferences consistent with the initial ordinal preference.

Furthermore, this chapter provides a characterization of the PRC rule. We introduce another new axiom: equal-rank envy-freeness. A random assignment satisfies equal-rank envy-freeness, if two agents put an object in the same preference rank, then changing the assignments of the two agents for this object cannot increase the surplus at the same object for any of them. The PRC rule satisfies equal-rank envy-freeness. Equal-rank envy-freeness is a refinement of equal treatment of equals. Our characterization result shows that the a rule satisfies sd-rank-fairness and equal-rank envy-freeness if and only if it is the PRC rule. Future work is needed to characterize generalized version of the PRC rule.

Finally, we prove that all ordinal Nash equilibrium outcomes of the preference revelation games induced by the PRC rule are weakly sd-envy-free. Although the PRC rule commits strong strategic behavior of agents, it turns out to have nice equilibrium properties.



# Chapter 8

## Conclusion

The final chapter summarizes the contribution of this thesis and then concludes by addressing possible further topics. This thesis studies the school choice problem in an axiomatic way. Chapters 3 to 6 provide axiomatizations of school choice mechanisms on full or restricted priority domains. Chapter 7 provides axiomatizations of the probabilistic rank-consumption random assignment rule, which is a generalization of the Boston school choice mechanism to random environments.

As a foundation of this thesis, chapter 2 introduces basic model of school choice problems, basic axioms for school choice mechanisms, and six basic school choice mechanisms related to our discussion throughout the thesis. The theoretical framework of school choice is established by Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003a), and has been elaborated by many followers. Although seeming redundant, we still want to recall the five components of a school choice problem: a set of students, a set of school types to be allocated, a quota or capacity vector of schools, a preference profile of students over school types, and a priority profile of schools over students.

Among the six mechanisms introduced in this thesis, the first three, i.e., the student-optimal stable mechanism (SOSM), the top trading cycles mechanism (TTCM), and the Boston mechanism (BOSM) are initiated by the seminal work of Abdulkadiroğlu and Sönmez (2003a). The school-optimal stable mechanism (SSOM) and simple serial dictatorship (SSD) are introduced to prove independence of axioms in our axiomatic analysis. Finally, the recursive Boston mechanism (RBM) is a new one in the literature. It is first mentioned by my own work. As an unexploited mechanism, more future work is being expected.

Chapter 3 contributes to the literature along two lines. First, we propose new axioms related to stability, consistency, and monotonicity for school choice mechanisms. Most of these axioms are new in and outside this field. They are expected to be generalized to other research problems. Take

## *Chapter 8 Conclusion*

consistency and weak consistency as an example. In the literature of resource allocation problems, consistency is playing a central or an important role. This property is appealing and wide-spread, but too demanding for school choice. We then propose a weaker version: weak consistency, which requires that no remaining student is going to be worse off if a subset of their colleagues is removed with their assignments. Weak consistency is novel not only for school choice problems, but also for other resource allocation problems, which shows the potentiality of this axiom. We look forward to seeing more work to further explore new axioms first defined in this thesis.

Chapter 3 then provides new characterizations of the celebrated SOSM. As the SOSM is becoming the central school choice mechanism, our analysis is of great theoretical and application importance. We prove that a school choice mechanism is equivalent to the SOSM if and only if it satisfies whichever of the following groups of axioms: stability, rank monotonicity; non-wastefulness, strong top best, weak Maskin monotonicity; non-wastefulness, strong group rationality, rank monotonicity; non-wastefulness, mutual best, weak consistency, strategy-proofness; non-wastefulness, mutual best, weak consistency, rank monotonicity; non-wastefulness, mutual best, weak consistency, respect of improvements. Our characterizations show the tradeoff between SOSM and the other school choice mechanisms, which will certainly help the social planner to choose alternative mechanisms. Moreover, we are the first to characterize SOSM on full strict and acceptant priority domain, and thus provide the strongest characterization result so far.

The SOSM is a central mechanism in practice, and stability is a natural concept in the field. Traditionally, stability, Pareto efficiency, and strategy-proofness are three basic criteria to judge a mechanism, and SOSM is appealing because it satisfies stability and strategy-proofness. Chapter 3 proposes new criteria to judge a school choice mechanism, which provides the social planner with more dimensions to compare mechanisms in practice. Moreover, although there is a growing body of literature that studies the SOSM, nobody has characterized it on all priority domain. Prior to our research, some other market designers have characterized the SOSM for some priority structure. However, in real-life problems, we can not exclude any priority structure. Therefore, the comparison between SOSM and the other school choice mechanisms for full priority domain is vague. Our result in Chapter 3 solves the puzzle and tells the social planner that which axioms distinguish SOSM from the other mechanisms. Therefore, if the social planner has a set of objectives to achieve, it becomes clearer

about which mechanism to choose now.

Chapter 4 characterizes the SOSM on restricted priority domains. There is a long list of previous literature on this topic. But nobody has studied the equivalence of SOSM and simple serial dictatorship (SSD). Let SSD-P represent the SSD where the order of students is determined by the priority order of any school. In this chapter, I find that for any preference profile of students, the SOSM is equivalent to SSD-P, if and only if SSD-P is fair, and if and only if the priority structure satisfies quota-acyclicity. Quota-acyclic priority structure requires that according to the quota information of a problem, no disorder of students exists below a certain critical point of priority ranks. The critical point is the minimal quota of schools. Quota-acyclicity is a rather strong restriction of priorities.

In real-life school choice problems, one school usually has multiple supply, which manifests that our result is useful and will help social planners in the process of assigning school seats to students. To be specific, if the priority structure is quota-acyclic, then our result tells the social planner that it is enough to use relatively simple serial dictatorship to find a stable matching, instead of using the SOSM. This will save a lot of time and energy to explain the procedure for social planners and understand the procedure for students.

Chapter 5 gives axiomatizations of the BOSM. To do so, we first introduce two new axioms related to stability: weak fairness and rank rationality. Weak fairness and rank rationality are both weaker than stability, and thus satisfied by SOSM. Our first characterization shows that the BOSM is the unique mechanism satisfying respect of preference rankings and weak fairness for all acceptant priorities. Our second characterization shows that the BOSM is the unique mechanism satisfying respect of preference rankings, rank rationality, and rank monotonicity. As the SOSM satisfies both rank rationality and rank monotonicity, our result reveals that respect of preference rankings is the unique axiom which distinguishes BOSM from SOSM.

In Chapter 6, we study the BOSM in restricted priority domains. Our main result shows that the BOSM is strategy-proof, if and only if it is fair, if and only if it is equivalent to the SOSM, and if and only if the number of total seats at any two schools exceeds the number of students. As the condition we identify guaranteeing nice properties of BOSM is rather stringent, we have that it is almost impossible to have BOSM with nice properties, which further argues against the already disputing mechanism. Moreover, we find that if the number of total seats at any two schools exceeds the number of students, then SOSM respects preference rankings.

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Chapter 7 considers the problem of randomly assigning  $n$  indivisible objects to  $n$  agents based on ordinal preferences of agents. For the aforementioned random assignment problem, we propose a new rule called the probabilistic rank-consumption rule (PRC rule). We introduce two new axioms: sd-rank-fairness, and equal-rank envy-freeness. Sd-rank-fairness is a refinement of sd-efficiency. Equal-rank envy-freeness is a refinement of equal treatment of equals. Sd-rank-fairness and equal-rank envy-freeness are enough to characterize the PRC rule. Although the PRC rule is neither weakly strategy-proof nor weakly sd-envy-free, ordinal Nash equilibrium outcomes of the preference revelation game induced by the PRC rule are all weakly sd-envy-free.

The PRC rule that we propose in Chapter 7 is a natural rule easy to understand and implement in practice. Unfortunately, this rule is not even weakly strategy-proof. This does not mean, however, that the market designer should freely dismiss it. The PRC rule satisfies a natural axiom: sd-rank-fairness, which is stronger than the well-known ordinal efficiency. PRC rule and sd-rank-fairness are both simple and natural, and both can serve as competitive alternatives to the rank-value mechanisms and rank efficiency proposed by Featherstone (2011).

After the seminal paper of Abdulkadiroğlu and Sönmez (2003a), school choice has become a hotly debated topic in public policy and school choice mechanism reform in the United States and around the world. In reality, the SOSM is becoming the central mechanism. More and more school districts in the US are adopting the SOSM, and 28 out of 31 provinces in China use variations of the SOSM to assign high school graduates to universities. The practice of school choice and student placement influence a large population of elementary school and high school students. Our theoretical research first provides the social planner with new criteria to evaluate a mechanism. Traditionally, we compare school choice mechanisms mainly by means of stability, Pareto efficiency, and strategy-proofness. This thesis proposes several new axioms and thus more dimensions to evaluate mechanisms. Second, this thesis reveals the advantage of SOSM over the other mechanisms. Therefore, we provide new theoretical foundations for the wide adoption of SOSM. In 2012, New Orleans Recovery School District became the first school district to adopt the TTCM. Theorem 3.2 tells the social planner that the main difference between TTCM and SOSM is strong top best, which is an axiom weaker than stability. Thus, if stability is not stressed, then maybe TTCM is a nice choice, too.

The allocation of public school seats to children is a wide-spread practical problem around the world. Countries like Japan are paying more and

more attention to figuring out better mechanisms to assign students to school seats. Moreover, many countries in the world like China, Turkey and Germany, use centralized student placement system to allocate students to universities. Our result will certainly help social planners of these countries, too.

Yet, a lot of open questions are to be solved in school choice problems.

First, one interesting direction to extend this thesis is to do empirical studies. One of the pioneering work in this direction is Wu and Zhong (2013). They compare the student placement system in China by empirical test on data from a top-ranked Chinese college. They find that although students admitted under the pre-exam Boston mechanism have lower college entrance exam scores than students admitted through other mechanisms, such as serial dictatorship, on average, they exhibit similar or even better college academic performance. China's college admission system is one of the world's largest matching systems. Around 10 million high school graduates applied to 2,300 or so higher education institutions in 2009.<sup>37</sup> The practice of priority-based matching problem influences a large population of students in the world, which reveals its theoretical and practical importance. While theoretical boom of this field is intrigued these years, empirical research is relatively insufficient. In 2012, the Nobel Prize on Economics Science went to Alvin E. Roth and Lloyd S. Shapley. This will certainly further stimulate the development of matching theory and hence school choice problem. We expect more scholars to study school choice problem from the empirical point of view.

Second, this thesis studies school choice mechanisms on strict preference and priority domains. More work is expected to study the generalized model, i.e., to allow indifference in preference and priorities. When priorities are weak, Erdil and Ergin (2008) and Ehlers (2006) find that all potential student-optimal stable matching results can be computed by using a different tie-breaking method and then apply the student-optimal deferred acceptance algorithm to the corresponding problem. Abdulkadiroğlu *et al.* (2009) show that the SOSM with arbitrary tie-breaking method is Pareto efficient subject to strategy-proofness. Future work is called for to do axiomatic analysis of school choice mechanisms under generalized preference and priority domains.

Third, future work is called for to further investigate the recursive Boston mechanism (RBM) and its random correspondence. RBM is a totally new mechanism in both school choice and random assignment problems.

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37. See Wu and Zhong (2013).

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This mechanism seems more natural than the classical BOSM.

Forth, future work is called for to characterize the top trading cycles school choice mechanism (TTCM), which seems appealing and is also used in real-life problems. There already exist several papers in this direction. However, none of the results are intuitive and appealing enough. The study of top trading cycles algorithm due to Shapley and Scarf (1974) has a long tradition. Pápai (2000) characterizes the hierarchical exchange rule, which contains TTCM as a special case. Pycia and Ünver (2011) introduce and characterize another generalized version of TTCM. We are looking forward to seeing more appealing result on characterizing the top trading cycles school choice mechanism, which is becoming more and more popular today.

Finally, in real-life school choice problems, one major concern is about racial and ethnic segregation. It is interesting to integrate racial and ethnic segregation into the theoretical framework of school choice. Other concerns such as inequality and education, public education and government spending are also interesting directions to extend the school choice model discussed in this thesis. We look forward to seeing more future work in these directions.



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