Essays on Equality of Opportunity in Welfare Economics

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Preface

This dissertation is a collection of studies in the theory of opportunity (in)equality. Following the introduction chapter, there are three essays, each of which has been published or is under consideration for publication in academic journals and also publicly available as below.

Chapter 2: ECINEQ Working Paper 2021-572, http://www.ecineq.org/2021/01/20/separable-utility-and-taste-independence (Also available at SSRN: https://ssrn.com/abstract=3734415).

Chapter 3: *Economics Letters*, Volume 194, September 2020, 109414, https://doi.org/10.1016/j.econlet.2020.109414 (Published by Elsevier B.V. via a CC BY-NC-ND 4.0 license).

Chapter 4: Available at SSRN: https://ssrn.com/abstract=3792137.

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CHAPTER 1

Introduction

1.1. Background

With the cherished desire of Arthur Cecil Pigou ([1920] 1932, p. vii), we consider that economics is a study for the *bettering of human life* and it is our task to explore. For that purpose, normative propositions—"ought" statements—are necessary. Pareto efficiency (Pareto, [1906] 2014; Hicks, 1939) is "usually the only prescriptive criterion taught in undergraduate economics classes and routinely appearing in economic policy analyses" (Konow and Schwettmann, 2016, p. 86), but it is not sufficient, in particular, for distributional issues. "In spite of the reluctance of many economists to view normative issues as part and parcel of their discipline, normative economics now represents an impressive body of literature" (Fleurbaey, 2016). Theory of *equality of opportunity* constitutes a significant portion of it, and we scrutinize this burgeoning subject. Moreover, we deal with growing empirical studies that provide us with descriptive propositions—"is" statements. They are divided into three streams. The first is the research of individual moral preferences using self-reported surveys or experiments. The second is the measurement of (in)equality of opportunity based on the normative theories. The third is the literature on social mobility, where "equality of opportunity" is often mentioned, although there is an ambiguity about the relationship. In this chapter, we review what has been elucidated in both theoretical and empirical literature regarding equality of opportunity and identify what has not yet been examined.

According to Ferreira (2019, 2020), there are three motivations (for measuring inequality of opportunity): normative arguments, evidence on preferences, and political salience. Firstly, political philosophy of personal responsibility underlies normative economic theories of equal opportunity. A leading group of position in the current opportunity egalitarian paradigm is called "luck egalitarianism" or "responsibility-sensitive egalitarianism." In short, it claims that individuals should be compensated for what they are not responsible for, but should not be compensated for what they are responsible for. We discuss the normative debates in detail in Section 1.2.

Secondly, the impetus for equality of opportunity also stems from the positive analysis of justice theories (Konow, 2003), which corresponds to the aforementioned first stream of empirical studies. That is, evidence on preference of individual valuation of "fairness" has been well-established by experiments of ultimatum game (Güth et al., 1982) and dictator game (Kahneman et al., 1986) and by questionnaire studies such as empirical social choice (Yaari and Bar-Hillel, 1984; Gaertner and Schokkaert, 2012).[1] People's attitudes towards the responsibility cut—the distinction between the factors that individuals should responsible for or not—are investigated. For example, the importance of whether outcomes are due to luck or efforts is argued in experimental literature (e.g., Hoffman and Spitzer, 1985). Moreover, the *accountability principle* is introduced, which requires that "fair allocation ... vary in proportion to the relevant variables which he can influence ..., but not according to those which he cannot reasonably

[1]See also Fehr and Gächter (2000b), Camerer (2003), Fehr and Schmidt (2006), Levitt and List (2007), and Konow and Schwettmann (2016) for surveys. Social sentiments such as altruism and reciprocity (and punitive action toward unfairness) have been extensively studied (e.g., Fehr and Schmidt, 1999; Fehr and Gächter, 2000a; Fehr and Fischbacher, 2003; Henrich et al., 2001, 2004; Gintis et al., 2005), and they are underpinnings for the classical explanation of voluntary redistribution (e.g., Buchanan and Tullock, 1962; Hochman and Rodgers, 1969; Zeckhauser, 1971; Eichenberger and Oberholzer-Gee, 1998). See also Mueller (2003, chap. 3) on this matter.

influence" (Konow, 1996, p. 14), which is supported by several experiments (e.g., Konow, 2000, 2001; Konow et al., 2020). It is congruent with the argument in political philosophy. There is also evidence that what is deemed to be a fair allocation depends on how it is determined (e.g., Cappelen et al., 2007, 2010).[2]

Moreover, the principle is also consistent with observational studies; that is, there is a strong correlation between preference for redistribution and beliefs that poverty is due more to bad luck than to lack of effort (Fong, 2001), and people's current social position strongly affects fairness views (Hvidberg et al., 2020). Also, perception for equal opportunity determines support for redistribution (e.g., Alesina and Angeletos, 2005; Alesina and La Ferrara, 2005), and such perception for equal opportunity is shaped by social mobility, which we discuss in Section 1.4 (e.g., Piketty, 1995, 2020; Alesina et al., 2012, 2018; Stantcheva, 2020). That is, "children's chances of rising up in the income distribution" (Chetty, 2021, p. 8) can influence the redistribution politics.

Thirdly, therefore, equality of opportunity is pursued as a time-honored political salience:

> We know that equality of individual ability has never existed and never will, but we do insist that equality of opportunity still must be sought (Franklin D. Roosevelt, Address at Little Rock, Arkansas, June 10, 1936).

Inequality (of outcomes) and inequality of opportunities are important political issues and the correlation between the two are shown by Corak (2013).[3]

[2]There is plurality in the location of responsibility cut (Schokkaert and Devooght, 2003), and such differences are due to family background (Almås et al., 2010, 2017). People are shown to focus on both *ex ante* opportunities and *ex post* outcomes (Cappelen et al., 2013).

[3]Corak (2013) shows the correlation between inequality and intergenerational income mobility, and the graph is coined as "Great Gatsby Curve" after Fitzgerald (1925) by Krueger (2012). See also Corak (2012, 2016) and Krueger (2015).

It gains the attention of the then-chairman of the U.S. Council of Economic Advisors:

> The rise in inequality in the United States over the last three decades has reached the point that inequality in incomes is causing an unhealthy division in opportunities, and is a threat to our economic growth (Alan Krueger, Center for American Progress, January 12, 2012).

Furthermore, based on such studies, the following political statements are born:

> (W)e know that people's frustrations run deeper than these most recent political battles. ... it's rooted in the fear that their kids won't be better off than they were. ... they experience in a very personal way the relentless, decades-long trend ... that is a dangerous and growing inequality and lack of upward mobility that has jeopardized middle-class America's basic bargain—that if you work hard, you have a chance to get ahead (Barack Obama, THEARC, December 4, 2013).

With these motives in mind, we survey the literature on how to define, to approach, and to measure (in)equality of opportunity. The remainder of this chapter is organized as follows. In Section 1.2 and Section 1.3, we describe concepts of and approaches to equality of opportunity. Then, we investigate social mobility literature and point out its problems in Section 1.4. Finally, we clarify the neglected issues, which we explore in the following chapters in Section 1.5.

1.2. Concepts

1.2.1. Political philosophy. When we consider redistribution, we need to "defend taking from some to give to others ... on grounds of 'justice"' (Friedman, 1962, p. 195). Equality of opportunity can be a candidate for a principle of justice, or a normative criterion. The ancient but well-established principle is *Equal treatment of equals*, which is discussed by Aristotle ([c. 350 BCE] 2009):

> The just, therefore, involves at least four terms; for the persons for whom it is in fact just are two, and the things in which it is manifested, the objects distributed, are two. And the same equality will exist between the persons and between the things concerned; for as the latter—the things concerned—are related, so are the former; if they are not equal, they will not have what is equal, but this is the origin of quarrels and complaints—when either equals have and are awarded unequal shares, or unequals equal shares (p. 84–85, 1131 a21–24).

This principle is still emphasized in the literature (e.g., Moulin, 2003), but the concept of equality of opportunity has been the subject of much debate and refinement in political philosophy.

According to Arneson (2018b), there are four conceptions of equal opportunity, which are currently endorsed. First concept is libertarian equality of opportunity by Lockean libertarian (Nozick, 1974), which is based on the idea of self-ownership: "every man has a property in his own person; this no body has any right to but himself" (Locke, [1689] 1764, p. 216). Because of its "inflated view of private property rights" (Arneson, 2018b, p. F153), any redistribution is impossible under this doctrine.

Second concept is called formal equality of opportunity, or "careers open to talent," which is found in arguments by classical liberals, or libertarians:

> No arbitrary obstacles should prevent people from achieving those positions for which their talents fit them and which their values lead them to seek. Not birth, nationality, color, religion, sex, nor any other irrelevant characteristic should determine the opportunities that are open to a person—only his abilities (Friedman and Friedman, 1980, p. 132).

Third concept is Rawlsian fair equality of opportunity, or "leveling the playing field," which is literally claimed by Rawls ([1971] 1999),

> by adding to the requirement of careers open to talents the further condition .... The thought here is that the positions are to be not only open in a formal sense, but that all should have a fair chance to attain them. ... In all sectors of society there should be roughly equal prospects of culture and achievement for everyone similarly motivated and endowed. The expectations of those with the same abilities and aspirations should not be affected by their social class (p. 63).

On the one hand, formal equality of opportunity, which "is an essential component of liberty" (Friedman and Friedman, 1980, p. 132), requires to eliminate such as discrimination and prejudice. On the other hand, fair equality of opportunity requires more affirmative actions to equalize different circumstances of people, which is required together with the difference principle, or maximin criterion, in the Rawls's second principle of justice.

Fourth concept is luck-egalitarian equality of opportunity, "which has become the prominent theory of distributive justice" (Hirose, 2015, p. 41).[4] It originates from criticism of Rawls's theory of justice by Dworkin (1981a,b, 2000) and is developed by Arneson (1989), Cohen (1989) and others.[5] The basic argument is that "we are responsible for the consequence of the choices we make out of those convictions or preferences or personality" (Dworkin, 2000, p. 7).[6] There is a very diverse range of ideas within luck egalitarianism. "The most general definition of luck egalitarianism, one that is sufficiently broad to include almost all versions of luck egalitarianism" is:

> Inequality is bad or unjust if it reflects the differential effects of brute luck. Inequality is not bad or unjust if it reflects the differential effects of option luck" (Hirose, 2015, p. 45).

Here, according to Dworkin (1981b):

> Option luck is a matter of how deliberate and calculated gambles turn out—whether someone gains or loses through accepting an isolated risk he or she should have anticipated and might have declined. Brute luck is a matter of how risks fall out that are not in that sense deliberate gambles (p. 293).

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[4]See Arneson (2008, 2011, 2013, 2018a), Lippert-Rasmussen (2015, 2018), and Hirose (2015, Chap. 2) for more details.

[5]"The general view inspired by Dworkin's accomplishment has become known as 'responsibility-sensitive egalitarianism', 'equality of fortune', or more commonly, 'luck egalitarianism' " (Knight, 2009, p. 1). The name "luck egalitarianism" is coined by Anderson (1999). However, Dworkin himself claims that "the name 'luck egalitarianism' which sometimes been used to describe resource equality is a misnomer" (Dworkin, 2002, p. 107).

[6]Dworkin (1981a,b) is also considered as a first response to the debate of "equality of what?" initiated by Sen (1980).

The basic ideas of luck-egalitarian equality of opportunity, or responsibility-sensitive egalitarianism, are accepted in economics literature (e.g., Roemer, 1993; Fleurbaey, 1994), and several approaches are developed.

1.2.2. Economic theories of just (re)distribution. Before proceeding to the economics discussion on equality of opportunity, we briefly review how redistribution policies are justified in economics literature. Utilitarianism has been dominant criterion.[7] Samuelson ([1948] 1964) puts it:[8]

> Economists who think the utilities of different persons can be added together to form a total social utility speak of taxing to produce maximum total utility .... Thus, if each extra dollar brings less and less satisfaction to a man, and if the rich and poor are alike in their capacity to enjoy satisfaction, a dollar taxed away from a millionaire and given to a median-income person is supposed to add more to total utility than it subtracts (p. 163).[9]

Thus, taxation and redistribution are justified by the principle of "equi-marginal sacrifice" (Edgeworth, 1897, p. 565); that is, "(i)n the utilitarian discussion of income distribution, equality of income is derived from the maximization conditions if it is further assumed that individuals have

[7]Utilitarianism is initiated by Bentham (1789) and advocated by Mill (1863), Sidgwick (1874), Edgeworth (1881), Marshall (1890), and Pigou ([1920] 1932).

[8]"Utilitarianism played a key role in the development of economic theory and provided key concepts such as utility and welfare that were used to analyze individual behaviors and social situations. It remains the normative basis for many economic policy judgments in the academic literature" (Fleurbaey and Zuber, 2021, p. 370). See also Sen and Williams (1982), Blackorby et al. (2002), and Riley (2008).

[9]The statement appears from the 6th edition (1964). In the 12th edition (1980), when the collaboration with William D. Nordhaus starts, the beginning of the expression changes to "Some economists like to think ...." In the 13th edition (1989) and the 14th edition (1992), they use the phrase: "Economists following in the utilitarian tradition ... used to argue that ...." Then, it does not appear from the 15th edition (1995) to the last 19th edition (2009). These changes may reflect the trend of mainstream economics.

the same utility functions, each with diminishing marginal utility" Arrow (1971, p. 409).

After Rawls ([1971] 1999), "the maximin criterion ... has received some attention" (Rawls, 1974). The most significant blow, however, comes from the criticism by Robbins ([1932] 1935) saying that "(t)here is no way of comparing the satisfactions of different people" (p. 140). "New" welfare economics departs from Pigou's "old" welfare economics by being constructed on the basis of ordinal and interpesonally non-comparable utility information. [10] So does justification of redistribution.

Since Arrow ([1951] 1963) considers that "interpersonal comparison of utilities has no meaning and, in fact, that there is no meaning relevant to welfare comparisons in the measurability of individual utility" (p. 9), the *extended sympathy* approach is pioneered by Arrow ([1951] 1963, Chap. 8, Sec. 4) and Suppes (1957, 1966).[11] which coincides with the observation by Smith ([1759] 2009):

> As we have no immediate experience of what other men feel, we can form no idea of the manner in which they are affected, by conceiving what we ourselves should feel in the like situation. ... By imagination we place ourselves in his situation, we conceive ourselves enduring all the same torments, we enter as it were into his body, and become in some measure the same person with him, and thence form some idea of his sensations, and even feel something which, though weaker in degree, is not altogether unlike them (pp. 13–14).

[10] See Suzumura (2002).

[11] See also Arrow (1977), Kolm (1997), Suzumura (1983), and Suzumura (1997).

It is formulated as "it is better in my judgement to be put in your position in social state *x* than to be put in somebody else's position in social state *y*" (Suzumura, 1997, p. 202). In fact, it is precisely the same as the *no-envy*, or *envy-free*, approach, which was introduced by Foley (1967), Kolm (1969, [1971] 1998), and Varian (1974, 1975).[12] It asks "each agent to put himself in the position of each of the other agents to determine if that is a better or worse position than the one he is now in" (Varian, 1975, p. 241) in resource allocation literature.[13] "If no individual prefers the bundle of good enjoyed by another person to his own, then that allocation is called *equitable*. If an allocation is both Pareto optimal and equitable, then it is called *fair*" (Sen, 1986, p. 1108).

This has been the dominant definition of *fairness* in economics literature. However, as Dworkin (1981a,b) "kicked off an extensive research on *responsibility and compensation*" (Suzumura, 2000b, p. 10), reexamination of fairness has been started, and is defined in view of equality of opportunity. Several fairness principles are proposed and characterized by the principles of reward and compensation, which are indeed derived from no-envy (cf. Fleurbaey, 2008; Fleurbaey and Maniquet, 2011a).[14] Above all, it is revealed that several incompatibilities between principles of reward and compensation are due to a deeper divide of the *ex ante* and the *ex post* perspectives of compensation (Fleurbaey and Peragine, 2013), and it is coined as opportunity paradox by Fleurbaey (2019).

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[12] According to Suzumura (2016b, p. 54), Tinbergen (1946) is attributed to the origin of the basic concept, and according to Arrow, John R. Hicks should be added to this list (Arrow et al., 2011, p. 25). According to Suzumura (2016a), there is a difference between the two approaches: the former is comparative assessment approach and the latter is transcedental institutionalism. The contrasting positions are proposed by Sen (2006, 2009).

[13] See, for example, Young (1994), Brams and Taylor (1996), Moulin and Thomson (1997), Moulin (2003), and Thomson (2008, 2011, 2016a,b, 2019).

[14] "In the literature, the pairwise laissez-faire objective has been variably called the responsibility principle, the natural reward principle, or the libberal reward principle" (Fleurbaey and Maniquet, 2018, p. 1046).

1.3. Approaches

1.3.1. Direct approaches: opportunity sets and procedural fairness. First group of approaches is to treat opportunities explicitly. Social states are ranked according to the evaluation of opportunity sets (e.g., Pattanaik and Xu, 1990; Gravel, 1994; Kranich, 1996, 1997; Ok, 1997; Ok and Kranich, 1998; Herrero et al., 1998; Weymark, 2003; Savaglio and Vannucci, 2007).[15] Moreover, procedural fairness is also directly characterized (e.g., Suzumura and Xu, 2003a,b, 2004, 2009).[16]

However, "the informational requirements for applying the direct approach to the measurement of inequality of opportunity—involving the observation of full choice sets– are very demanding ... that approach has never been applied empirically" (Ferreira and Peragine, 2016, p. 766).

1.3.2. Indirect approaches: responsibility and compensation. Second group of approaches is to formalize equality of opportunity indirectly. "These approaches are more structural and *consequentialist* in nature (Ferreira and Peragine, 2016, p. 753).[17] They focus on the consequences obtained by observable individual characteristics and circumstances.

Given the opportunity paradox, and other incompatibilities between principles of reward and compensation, we need to consider compromised solutions by weakening axioms of either principle of compensation or reward. Moreover, according to Fleurbaey (2008, p. 199), there are *liberal* and *utilitarian* approaches, both of which place "responsibility-sensitive egalitarianism between full egalitarianism and libertarianism: if individuals were

[15]See Barberà et al. (2004) for comprehensive survey. Moreover, the capability approach (Sen, 1980) can be included in this approach. See, for example, Gotoh and Yoshihara (2003), and, for surveys, Basu and Lòpez-Calva (2011), Foster (2011), and Suzumura (2020).

[16]See Suzumura (1999); Sen (2000); Suzumura (2011).

[17]Indeed, Suzumura and Yoshihara (2000) points out the formalizations, or approaches, is "extremely consequentialistic" (p. 180).

not responsible for anything, equality of well-being should be sought." In addition, the former considers "if [individuals] were fully responsible for their characteristics, no redistribution would need to take place;" the latter considers "if individuals were fully responsible for their characteristics, we should be indifferent to the distribution of well-being and only be interested in the sum total." Following Fleurbaey (2011), we can summarize four approaches to equal opportunity as Table 1.[18]

	Liberal	Utilitarian
Compensation	Egalitarian-equivalent	Mean-of-mins
Reward	Conditional equality	Min-of means

TABLE 1. Four approaches

We present these four approaches according to Fleurbaey (2008). Let $N = \{1, ..., n\}$ be the population, and every individual $i \in N$ is endowed with two kinds of characteristics: y_i, for which they are not responsible (circumstance), and z_i for which they are. A profile of characteristics is $(y_N, z_N) = ((y_1, ..., y_n), (z_1, ..., z_n))$. The set of y_i and z_i are denoted Y and Z, respectively. Individual i's well-being is denoted by u_i and is determined by a function u which is the same for all individuals:

$$u_i = u(x_i, y_i, z_i), \tag{1.1}$$

where $x_i \in X \subset \mathbb{R}$ is the quantity of money transfer to which individual i is submitted. The function u is assumed to be continuous and increasing in x_i over X.

[18]They do *not* correspond to Arneson's (2018a) four conceptions. See also Fleurbaey (2008, 2009), Fleurbaey and Maniquet (2011a,c, 2012), Roemer (2012a), Roemer and Trannoy (2015, 2016).

An economy is denoted $e = ((y_N, z_N), \Omega)$, where Ω is an aggregate endowment. Let $\mathcal{D}$ be the domain of economies $e = ((y_N, z_N), \Omega)$ under consideration. An *allocation rule* is a correspondence $S(e)$ for all $e \in \mathcal{D}$.

We focus on the "distribution" case and the "TU (transferable utility)" case so that we suppose quasilinear utility function:

$$u_i = x_i + v(y_i, z_i), \tag{1.2}$$

where x_i is always nonnegative (i.e., transfer).

The *liberal* approaches are established by Fleurbaey (1994, 1995a,b,c); Bossert (1995); Bossert and Fleurbaey (1996); Fleurbaey (2007) and Fleurbaey and Maniquet (1996b,a, 1997, 1999, 2005, 2008). By weakening no-envy, they propose axioms that stick to one of two principles to satisfy only weak conditions reflecting the other one.

Conditional equality respects *reward principle* and it defines "a reference values of responsibility characteristics and give priority (according to the leximin criterion) to individuals who, with their current resources and circumstances and this reference value of responsibility characteristics, would be the worst-off" (Fleurbaey, 2008, p. 61).

Conditional equality: Let $\tilde{z} \in Z$ be the reference. $\forall e \in \mathcal{D}$, $\forall x_N \in S_{CE}(e)$, $\forall i \in N$,

$$x_i = -v(y_i, \tilde{z}) + \frac{1}{n}\sum_{j \in N} v(y_j, \tilde{z}) + \frac{\Omega}{n}. \tag{1.3}$$

The aim is to obtain a situation in which $u(x_i, y_i, \tilde{z})$ has the same value for all $i \in N$.

Egalitarian-equivalent respects *compensation principle*, and it defines "a reference type of circumstances and give priority (leximin) to individuals whose current level of well-being would be obtained with the least resources if their circumstances were of the reference type (and their responsibility characteristics unchanged)" (Fleurbaey, 2008, p. 63).[19]

Egalitarian-equivalent: Let $\tilde{y} \in Y$ be the reference. $\forall e \in \mathcal{D}, \forall x_N \in S_{EE}(e), \forall i \in N$,

$$x_i = -v(y_i, z_i) + v(\tilde{y}, z_i) + \frac{1}{n} \sum_{j \in N} (v(y_j, z_j) - v(\tilde{y}, z_j)) + \frac{\Omega}{n}. \quad (1.4)$$

The next two criteria is *utilitarian* approach proposed by Van de gaer (1993) and Roemer (1993, 1998), respectively. Let the various y and z classes be denoted $N_y = \{i \in N | y_i = y\}$ and $N_z = \{i \in N | z_i = z\}$, with $n_y = |N_y|$, $n_z = |N_z|$.

Min of means approach respects *reward principle*, and it gives 'priority (according to the leximin criterion) to circumstance classes which are the worst-off in terms of average well-being" (Fleurbaey, 2008, p. 201).

Min of means:

$$\frac{1}{n_y} \sum_{i \in N_y} x_i = \bar{v}(e) - \frac{1}{n_y} \sum_{i \in N_y} v(y, z_i) \quad (1.5)$$

where $\bar{v}(e)$ is average well-being over the whole population.

Mean of mins approach respects *compensation principle*, and it "maximize(s) the average well-being over the whole population that would be obtained if every individual's well-being were put at the minimum observed in her own responsibility class" (Fleurbaey, 2008, p. 201).

Mean of mins:

$$\frac{1}{n} \sum_{z} n_z C_z = \bar{v}(e) \quad (1.6)$$

where C_z is a value such that for all $i \in N_z$, $u_i = C_z$.

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[19]Egalitarian equivalence is due to (Pazner and Schmeidler, 1978).

1.3.3. Measurement. There are empirical studies that correspond the four approaches, which are summarized in Table 2.

	Liberal	Utilitarian
Compensation	Fairness gap	Within tranches
Reward	Direct unfairness	Between types

TABLE 2. Four measures

We present these four measures according to Ferreira and Peragine (2016). Suppose there are n *types* of non-responsibility characteristics, indexed by $i = 1,...,n$, and m *tranches* of responsibility characteristics, indexed by $j = 1,...,m$. Let $v_{ij} = v(y_i, z_j)$ The population can be represented by a matrix $[V_{ij}]$:

	z_1	z_2	z_3	$\cdots$	z_m
y_1	v_{11}	v_{12}	v_{13}	$\cdots$	v_{1m}
y_2	v_{21}	v_{22}	v_{23}	$\cdots$	v_{2m}
y_3	v_{31}	v_{32}	v_{33}	$\cdots$	v_{3m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
y_n	v_{n1}	v_{n2}	v_{n3}	$\cdots$	v_{nm}

TABLE 3. The population with n types and m tranches of individuals

Let there be associated an $n \times m$ dimensional matrix $[P_{ij}]$, where each element p_{ij} gives the proportion of total population with non-responsibility characteristics i and responsibility characteristics j.

The measurement of inequality of opportunity is two steps. First, the actual distribution $[V_{ij}]$ is transformed into a counterfactual distribution $[\tilde{V}_{ij}]$ that reflects only and fully the unfair inequality in $[V_{ij}]$, while all the fair inequality is removed. Second, a measures of inequality is applied to $[\tilde{V}_{ij}]$.

The counterfactual distributions of *Between types* and *Direct unfairness* does not contain anyh inequality within types.

First measure, *Between types*, corresponds to *Min-of-means*.[20]

Between types ($\tilde{V}_{BT}$): Let $\mu_i = \sum_{j=1}^{m} p_{ij}v_{ij}$, for all $i \in \{1,...,n\}$ and $j \in \{1,...,m\}$,

$$\tilde{v}_{ij} = \mu_i. \tag{1.7}$$

	z_1	z_2	z_3
y_1	μ_1	μ_1	μ_1
y_2	μ_2	μ_2	μ_2
y_3	μ_3	μ_3	μ_3

TABLE 4. Between types inequality ($n = m = 3$)

The types of $\tilde{V}_{BT}$ are replications of the same outcome: the mean. The counterfactual distribution $\tilde{V}_{BT}$, therefore, does not reflect any inequality within types.

Second measure, *Direct unfairness*, corresponds to *Conditional equality*.[21]

Direct unfairness ($\tilde{V}_{DU}$): take $\tilde{e}$ as the reference effort. Then

$$\tilde{v}_{ij} = v(y_i, \tilde{z}) \tag{1.8}$$

for all $i \in \{1,...,n\}$ and $j \in \{1,...,m\}$.

	z_1	z_2	z_3
y_1	v_{11}	v_{11}	v_{11}
y_2	v_{21}	v_{21}	v_{21}
y_3	v_{31}	v_{31}	v_{31}

TABLE 5. Direct unfairness ($\tilde{z} = 1$, $n = m = 3$)

[20]*Between types* is used, for example, in Peragine (2002), Bourguignon et al. (2007), Checchi and Peragine (2010), and Ferreira and Gignoux (2011).

[21]*Direct unfairness* and *Fairness gap* are used, for example, in Devooght (2008), Fleurbaey and Schokkaert (2009), Almås et al. (2011).

In the counterfactual distributions of *Within tranches* and *Fairness gap*, all inequalities between tranches are removed.

Third measure, *Within tranches*, corresponds to *Mean-of-mins*.[22]

Within tranches ($\tilde{V}_{WTR}$): For all $i \in \{1,...,n\}$ and $j \in \{1,...,m\}$,

$$\tilde{v}_{ij} = v(y_i, z_j)/\nu_j. \tag{1.9}$$

	z_1	z_2	z_3
y_1	v_{11}/ν_1	v_{12}/ν_2	v_{13}/ν_3
y_2	v_{21}/ν_1	v_{22}/ν_2	v_{23}/ν_3
y_3	v_{31}/ν_1	v_{32}/ν_2	v_{33}/ν_3

TABLE 6. Within tranches inequality ($n = m = 3$)

Each tranche is obtained by rescaling original incomes by a constant ($1/\nu_j$). Therefore, $\tilde{V}_{WTR}$ reflects all of the original inequality within tranches.

Fourth measure, *Fairness gap*, corresponds to *Egalitarian-equivalent*.

Fairness gap ($\tilde{V}_{FG}$): take $\tilde{y}$ as the reference circumstance.

$$\tilde{v}_{ij} = v(y_i, z_j)/v(\tilde{y}_i, z_j) \tag{1.10}$$

for all $i \in \{1,...,n\}$ and $j \in \{1,...,m\}$.

	z_1	z_2	z_3
y_1	1	1	1
y_2	v_{21}/v_{11}	v_{22}/v_{12}	v_{23}/v_{13}
y_3	v_{31}/v_{11}	v_{32}/v_{12}	v_{33}/v_{13}

TABLE 7. Fairness gap ($\tilde{y} = 1$, $n = m = 3$)

Each tranche of $\tilde{V}_{FG}$ is divided by a reference outcome. Therefore, it reflects relative inequality within each tranche.

[22] *Within tranches* is used, for example, in Checchi and Peragine (2010) and Aaberge et al. (2011).

The measures presented here, and applied in practice, rely on approaches derived compromised allocation rules. Therefore, these measures themselves are necessarily also compromises. If reasonable escapes from the paradox were to be found theoretically, they can also contribute to the empirical measurement of inequality of opportunity.

1.4. Social mobility

1.4.1. Concepts of mobility. In a *Survey on Consensus among Economists* conducted by American Economic Association, it asks AEA members about their agreement or disagreement with 46 economic propositions and six demographic questions including "The US economy provides sufficient opportunities for social mobility." We can see that it is widely recognized that opportunities and social mobility are related conception in economics profession. It originates in sociology,[23] and there are also numerous economics literature on social and economic mobility, and they mention (in)equality of opportunity.[24]

One of the most general definition of social mobility is "movements by specific entities between periods in socioeconomic status indicators" (Behrman, 1999, p. 72).[25] In terms of status, income has received the most attention. We can distinguish four concepts of income mobility: positional change, individual income growth, reduction of longer-term inequality, and income risk (Jenkins, 2011; Jäntti and Jenkins, 2015).

[23] See, for example, Boudon (1973), Bartholomew ([1967] 1982), and Torche (2015) for studies in sociological literature.

[24] For comprehensive surveys, see Atkinson et al. (1988), Creedy (1994), Maasoumi (1998), Fields and Ok (1999), Jäntti and Jenkins (2015).

[25] See also Fields (1999), Fields (2008), and Fields (2019).

1.4.2. Intergenerational mobility, opportunity, and education. Among others, there are enormous literature on empirical analysis of intergenerational income mobility. According to Clark (2014), "all social mobility is governed by a simple underlying law, independent of social structure and government policy:

$$x_{t+1} = bx_t + e_t, \tag{1.11}$$

where x_t is the underlying social status of a family in generation t, e_t is a random component, and b is in the region 0.7–0.8. ... (T)his law of mobility implies that on average, the status of the descendants will move towards the mean for the society generation by generation" (p. 212).

It is indeed a "controversial" (p. ix) statement, but it coincides with classical regression model of intergenrational mobility (Becker and Tomes, 1979, 1986; Solon, 2004).[26] Causal relationship between parents and children is investigated,[27] and it is often discussed with related to education and family background (e.g., Chetty et al., 2018; Chetty and Hendren, 2018a,b),[28] The "American dream" is referred to as upward absolute intergenerational income mobility (Chetty et al., 2017; Chetty, 2021), and the comparison between Scandinavian countries and the U.S is also explored (e.g., Aaberge et al., 2002; Landersø and Heckman, 2017; Heckman and Landersø, 2021).

Theses empirical literature often mention equality of opportunity. They are indeed important to for public discussion of redistribution and education policies, but the evaluations are not on the basis of the concepts of equality of opportunity that are discussed in political philosophy.

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[26]See Guner (2015) and Becker et al. (2018).

[27]See, for example, Solon (1999), Piketty (2000), Solon (2002), Solon (2008), Burkhauser et al. (2011), Fox et al. (2016), and Solon (2018) for surveys on intergenerational mobility.

[28]See Mulligan (1997), Björklund and Jäntti (2011), Björklund and Salvanes (2011), Heckman and Mosso (2014) for surveys on education and mobility.

1.4.3. Normative discussion. "Does a society that embraces and fulfills equality of opportunity (rightly interpreted) necessarily provide social mobility? In a word, No" (Arneson, 2015).[29] For example, the "perfect mobility" of intergenerational income positional change would indicate the complete reversal of positions; that is, children of the top decile of parents will be the bottom and children of the bottom will be the top. The children's generation cannot considered to have equal opportunities in a sense because their income is decided by their parents' generation, as well as the "zero mobility" situation. Low mobility may indicate the lack of opportunity, but the relationship between social mobility and opportunity equality is ambiguous.[30] The similar observation is offered by Shorrocks (1978a) and Kanbur and Stiglitz (1986, 2016).

To evaluate mobility normatively, many indices have been proposed with and without axiomatic characterizations (e.g., Hart, 1976; Shorrocks, 1976, 1978a,b; Sommers and Conlisk, 1979; Conlisk, 1989, 1990; Shorrocks, 1993; Markandya, 1982, 1984; King, 1983; Atkinson, 1980, 1983b; Dardanoni, 1993; Cowell, 1985; Chakravarty et al., 1985; Fields and Ok, 1996). Recently, as if in response to increasing empirical literature using big data but without normative analysis such as axiomatic characterization (e.g., Chetty et al., 2014b,a, 2017, 2018; Bergman et al., 2019), axiomatizations of mobility indices that can incorporate ordinal data are proposed (e.g., D'Agostino and Dardanoni, 2009a,b; Amiel et al., 2015; Bossert et al., 2016, 2018; Chen and Cowell, 2017; Cowell and Flachaire, 2018). Still, they do not embody responsibility-sensitive egalitarianism, on which we rely as a normative criterion to measure (in)equality of opportunity.

[29]For critiques for using mobility for measuring equality of opportunity, see also Roemer (2004), Roemer (2012b), Dardanoni et al. (2006), Kanbur (2018, 2019).

[30]We also need to consider whether equalization of opportunity reduce social mobility, see, for example, Conlisk (1974).

1.5. Neglected issues

Based on the discussions so far, we point out issues that is neglected in the literature. One is on the normative theory; the other is on social mobility.

First, to address the opportunity paradox, compromised solutions have been proposed (cf. Fleurbaey, 2019). There are, however, alternative ways to escape the incompatibility such that we do not need to compromise either principles. We explore the means of preference domain restriction (cf. Gaertner, 2002; Le Breton and Weymark, 1996, 2011), which have not attracted attentions, in Chapter 2. Meanwhile, we do not restrict our attention to non-comparable utility because "(t)he standard approach of 'social welfare functions' because of its concentration on individual orderings only (without any use of interpersonal comparisons of levels and intensities) fails to provide a framework for distributional discussions" (Sen, [1973] 1997, p. 23).

Second, the literature on social mobility has two issues. There is an incompatibility between "perfect mobility" and "equality of opportunity" pointed out by Shorrocks (1978a) and Kanbur and Stiglitz (1986, 2016); normative assessment of social mobility should incorporate principles of luck egalitarianism, or responsibility-sensitive egalitarianism. We discuss the former in Chapter 3 and the latter in Chapter 4.

Now, we begin exploration to shedding light on these neglected issues in the following chapters.

CHAPTER 2

Taste-independence: an escape route from the opportunity paradox

2.1. Introduction

We demonstrate that there is an escape route from one of the eight paradoxes in welfare economics listed in Fleurbaey (2019): the opportunity paradox. This paradox refers to the tension that arises when considering redistribution policies to achieve opportunity equality that accommodates individual responsibility, as explored in a flourishing stream of theoretical and empirical literature (cf. Ferreira and Peragine, 2016). "The general structure of such theories relies on a distinction between responsibility characteristics and circumstance characteristics. Inequalities due to the former are deemed acceptable, unlike inequalities due to the latter" (Fleurbaey, 2019, p. 674). Incompatibilities between such principles have been shown since Fleurbaey (1994, 1995c); however, according to Fleurbaey and Peragine (2013), "the well documented conflicts between the compensation principles and various reward principles are but an aspect of a broader conflict between *ex ante* and *ex post* perspectives" (p. 126). Thus, it is worthwhile to unravel the logical incompatibility regarding equality of opportunity at this "deeper" level (p. 119). Focusing on utility that is quasilinear in consumption, we show that there is no incompatibility when the preference domain is restricted to separable utility with respect to consumption and labor supply.

Here, we briefly illustrate the opportunity paradox following Fleurbaey (2019, sec. 6), which can be seen in the (ℓ, c) space in Figure 2.1, where ℓ

is labor supply and c is consumption. As is conventional in the fair taxation literature (e.g., Fleurbaey and Maniquet, 2006), we treat labor supply and wage rate as responsibility and circumstance characteristics, respectively. Also, as is often the case with the *ex post* perspective of compensation, we temporarily require the reduction of consumption (or disposable income) inequality, not utility inequality.[1]

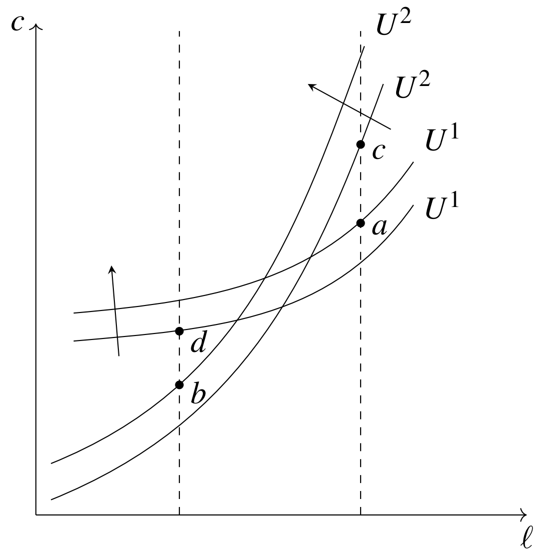

FIGURE 2.1. Crossing indifference curves of four individuals.

Figure 2.1 shows the allocations of four individuals. Suppose points a, b, c, and d represent labor supply and consumption at their maximized utility. The two types of preferences are exhibited by the indifference curves U^1 and U^2, and the arrows on them indicate the direction of preferences. Individuals at points a and c share the same responsibility characteristic, labor supply, as do points b and d. As the *ex post* perspective of compensation requires a reduction of consumption inequalities between individuals with the same responsibility characteristic, labor supply, regardless of their

[1]Fleurbaey (2008, Ch. 9, Sec. 9.5) and Fleurbaey (2019, Sec. 6) draw a similar, but not identical, figure showing opportunity sets, setting the vertical axis as the outcome to show opportunity paradoxes. The example given in Fleurbaey (2019) seems to imply that the outcome may be income.

preferences, it is desirable to reduce the difference between points a and c as well as between points b and d. However, these changes widen the gaps between the indifference curves of U^1 and of U^2. This is undesirable according to the *ex ante* perspective of compensation, which requires a reduction in the gaps in "the opportunities offered to individuals (as measured by the possible well-being levels achieved with given circumstances for the various values of responsibility characteristics)" (Fleurbaey, 2008, p. 239). Therefore, the *ex post* and the *ex ante* perspectives of compensation are incompatible in general.

To address this issue, we first consider compensation that is aimed at reducing inequalities between utilities, not incomes.[2] We suppose utilities to be interpersonally comparable for evaluating social orderings because "(t)he eschewal of interpersonal comparisons of utilities eliminates the possibility of taking note of inequality of utilities" (Sen, [1970] 2017, p. 17).[3] Namely, we focus on interpersonally comparable utility as the informational basis. Nevertheless, the paradox remains. Considering utility equalization, the *ex ante* perspective of compensation also requires a reduction in the gaps between the indifference curves of U^1 and those of U^2 because the *ex ante* perspective of compensation requires reducing the opportunity gaps between individuals.[4] Furthermore, although we cannot compare utilities

[2]As Fleurbaey and Maniquet (2018) mention, there is a "possibility, more respectful of individual preferences, ... to take utility as the outcome (assuming there is a comparable measure of utility)" (p. 1045).

[3]For a detailed justification of the construction of individual well-being measures that respect individual preferences and depend on the bundles of goods consumed by the individuals, especially with nonclassical goods such as labor supply, see Fleurbaey and Maniquet (2019).

[4]The graphical movement is the same as the previous argument; that is, for the *ex ante* approach, we require that individuals sharing the same preferences have equal opportunity to enjoy the same utility, but they can choose their labor/leisure time and income. It may seem odd that the outcome, or utility, is constant even if the responsibility characteristic, or labor supply, changes. Although they share the same utility, they may have different incomes, labor supply, and leisure time, and these are opportunities that individuals can choose as a result of their maximization behavior. Further, by considering taste for work,

between points a and c, or points b and d, there is a possibility that the *ex post* perspective of compensation may, in general, require reduction between them, just as when we consider income equality. That is, the paradox can still occur in the unrestricted domain;[5] therefore, secondly, we propose a property of preference that we call *taste-independence.* It assumes that the maximized utility levels are the same for individuals with the same wage rate but different tastes for work. It also reflects respect for individual responsibility; hence, the taste for work is also considered a responsibility characteristic. We argue that labor supply and taste for work are *ex post* and *ex ante* responsibility characteristics, respectively. When we consider utility as individual well-being or outcome, this distinction of responsibility characteristics enables us to clearly define the *ex ante* perspective of compensation and to plainly discuss the opportunity paradox using the concept of opportunity sets in a responsibility–outcome space. Focusing on utility that is quasilinear in consumption, we demonstrate that the opportunity paradox does not occur when utilities are taste-independent.[6]

The remainder of this chapter is organized as follows. We present the formal settings of the model in section 2.2. We define the preference domain restrictions in section 2.3. We present the axioms and the main result in section 2.4. We provide concluding remarks in section 2.5.

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which is considered to be *ex ante* responsibility characteristic, as argued in the following discussion, is not described by the horizontal axis; thus, the indifference curves only represent correspondences between *ex post* responsibility characteristic and income along the same outcome, or utility. See also footnotes 14 and Section 2.4.3 for further discussion regarding opportunity sets.

[5]Indeed, Fleurbaey and Peragine (2013) show the general incompatibility.

[6]According to Fleurbaey and Maniquet (2018), "(t)here are two main views on utilities. According to the first view, ... utilities are empirical objects that only need to be measured and can be used as the inputs of a social welfare function, ... According to the second view, utilities themselves, not just the social welfare function, are normative indexes that need to be constructed" (pp. 1034–1035). Regarding the first view, "(o)ne can distinguish two main approaches that adopt this view. In the first approach, utilities refer to the subjective self-assessments of well-being" (p. 1035). We adopt this approach and consider restricting such utility to be separable in consumption and labor supply.

2.2. The model

An *economy* E is composed of a finite set of individuals partitioned into a finite number of *taste types* and *wage rate types*. The set of taste type is $N(E) = \{1,...,n\}$, with $n \geq 2$, and the set of wage rate type is $M(E) = \{1,...,m\}$, with $m \geq 2$.

Individuals have conceivably different *tastes* for work, denoted by attribute parameter θ^i. The taste type is indicated by a superscript. Let c be *consumption* and ℓ be *labor supply* ($0 \leq \ell \leq 1$). The *preference* of an individual with taste θ^i is represented by an identical real-valued parametric *utility function* u;[7] that is,

$$U^i \equiv u(c,\ell;\theta^i), \text{ for } i \in N(E). \tag{2.1}$$

The utility function is increasing in c and decreasing in ℓ, and it is smooth and quasi-concave. We restrict it to be separable and taste-independent, as defined in the next section.

Let w_j be *wage rate*, and let T be *transfers* if positive and *taxes* if negative. The wage rate type is indicated by a subscript. The *disposable income* is determined by

$$w_j\ell + T, \text{ for } j \in M(E). \tag{2.2}$$

[7]Note that preference heterogeneity can be considered by using the taste parameter θ^i. See, for example, Deaton and Muellbauer (1980, Ch. 9) and Fleurbaey and Hammond (2004, Sec. 6.2). Utilities in our model can be interpreted as Harsanyi's (1977) concept of extended utility functions to warrant interpersonal comparison. See also Harsanyi (1953, 1955, 1982, 2008).

Individuals with taste θ^i and wage rate w_j maximize, or optimize, their utility subject to their disposable income:

$$\max_{0\leq\ell\leq 1} u(c,\ell;\theta^i) \tag{2.3}$$

$$\text{s.t. } c = w_j\ell + T. \tag{2.4}$$

The utility of individuals with taste θ^i and wage rate w_j is denoted by

$$U^i_j \equiv u(w_j\ell^i_j + \tau(w_j,\ell^i_j,\theta^i),\ell^i_j;\theta^i), \text{ for } i \in N(E),\ j \in M(E), \tag{2.5}$$

where ℓ^i_j is the corresponding labor supply. The transfers/taxes $\tau(w_j,\ell^i_j,\theta^i) = T$ are determined by the government as a *redistribution policy function* τ of w_j, ℓ^i_j, and θ^i.

Let $\bar{\ell}^i_j$ be the labor supply optimally chosen by individuals with taste θ^i and wage rate w_j, and let $\tau(w_j,\bar{\ell}^i_j,\theta^i)$ be the corresponding transfers/taxes.[8] Moreover, let $\bar{\ell}_j$ indicate the labor supply optimally chosen by an individual with wage rate w_j but with an arbitrary taste parameter, and let $\tau(w_j,\bar{\ell}_j)$ indicate transfers, determined by the government as a function of w_j and $\bar{\ell}_j$. We focus on such allocations obtained by individuals' utility maximization behavior when we consider the *ex post* perspective of compensation.

A *social ordering function* defines, for every economy E in domain D, an ordering $\succ_{(E)}$ over all possible maximized individual utilities, where $\tau \succ_{(E)} \tau'$ means that redistribution policy $\tau(\cdot)$ is socially better than policy $\tau'(\cdot)$. The domain D over which these social ordering functions $\succ_{(E)}$ are defined is the set of economies that satisfies the abovementioned conditions. Finally, we evaluate the orderings and do not restrict our attention to the tax and transfer policies that satisfy the government's budget balance.

[8]Individual optimal labor supply $\bar{\ell}^i_j$ and the corresponding government transfers/taxes $\tau(w_j,\bar{\ell}^i_j,\theta^i)$ are determined by simultaneously solving the optimization problems of the individuals and the government's redistribution policy function.

2.3. Preference domain restriction

2.3.1. Taste-independence. We introduce a property of preference such that the maximized utility levels of individuals with the same wage rate w_j but different tastes for work θ^i are evaluated as the same.[9] Taste-independence is regarded as respect for the individuals' freedom to choose the labor supply–consumption bundle (ℓ, c) at a given wage rate. Moreover, it respects individual responsibility; that is, individuals with low income due to high disutility of work would enjoy the same level of utility as individuals with high labor supply and high income because they have more leisure time.[10] In short, we can treat individuals with the same circumstance characteristic, or wage rate.[11] Thus, taste-independent utility is defined as follows.

DEFINITION 2.1 (Taste-independence). A utility is *taste-independent* if two individuals' utilities are the same whenever they maximize over the same budget set $w_j\ell + T$, where wage rate w_j and transfers/taxes T are given, regardless of their taste parameter θ^i.

In the (ℓ, c) space in Figure 2.2, the indifference curves of two preferences are depicted. Both are tangent to the same line with slope w_j, which means that individuals on points a and b share the same wage rate. Taste-independence then assumes the utilities obtained at points a and b are the same.[12]

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[9]The following statements provide some justification for the property on the set of primitive preferences that are in the economy (cf. footnote 6).

[10]We should often consider the high marginal disutility of work, such as that of individuals with (mental) illnesses or disabilities that do not depreciate their wage rates. In such cases, we need utility functions other than separable ones to derive desirable policies for those individuals.

[11]In fact, this is one of the standard requirements of responsibility, or reward, principles.

[12]The same concept is introduced by Henry de Frahan and Maniquet (2021) as "responsibility for one's preferences (the requirement that the social welfare function should treat identically agents with the same wage, independently of their preferences)" (p. 1).

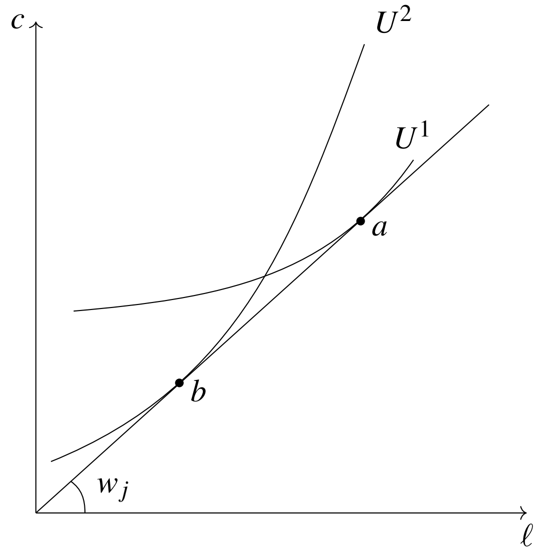

FIGURE 2.2. Taste-independence between two individuals with the same wage rate but different tastes for work: individuals represented by points a and b enjoy the same utility level.

2.3.2. Quasilinearity in consumption. We focus on utility functions that are *quasilinear in consumption* for analytical simplicity. It helps us to concentrate on the situations in which the *ex post* perspective of compensation applies. In general, through lump-sum transfers, individuals' maximization behavior may change their labor supply, and that immediately precludes a comparison between the utilities of individuals with the same labor supply. Without this assumption, we only consider situations when individuals' labor supply are invariant by lump-sum transfers, but these situations can be captured by quasilinearity in consumption.

The quasilinear utility function has the property that the marginal rate of substitution (MRS) between ℓ and c depends only on ℓ, which ensures that indifference curves of the same preference can all be obtained from any one of them by arbitrary translations parallel to the horizontal (labor supply) axis (Mas-Colell et al., 1995, p. 45). This enables us to consider lump-sum transfers/taxes without changing individual labor supply. The quasilinear assumption is often used for the specification in the optimal income tax literature (e.g., Salanié, 2011, pp. 101–107). Roemer (1996, pp. 297–301) uses quasilinear utility functions to examine the redistribution mechanism for equality of opportunity. More recently, Saez and Stantcheva (2016) assumes utility functions to be quasilinear in consumption to "rule out income effects on earnings which greatly simplifies optimal tax formulas" (p. 26).

2.4. Compatibility theorem

2.4.1. Axioms. We follow the axioms of Fleurbaey and Peragine (2013), but we modify them in our settings. The *ex post* approach to compensation focuses on inequality of utilities as a consequence of individuals' utility maximization behavior. This approach aims to reduce utility inequality between individuals, regardless of their taste parameter, but with the same *ex post* responsibility characteristic: labor supply.[13]

Ex Post Compensation. For all $E \in D$, $\tau \succ_{(E)} \tau'$ if there are $i, j \in M(E)$, such that $\bar{\ell}_i = \bar{\ell}_j$,

$$\begin{aligned} &u(w_i\bar{\ell}_i + \tau'(w_i, \bar{\ell}_i), \bar{\ell}_i; \cdot) > u(w_i\bar{\ell}_i + \tau(w_i, \bar{\ell}_i), \bar{\ell}_i; \cdot) \\ &\geq u(w_j\bar{\ell}_j + \tau(w_j, \bar{\ell}_j), \bar{\ell}_j; \cdot) > u(w_j\bar{\ell}_j + \tau'(w_j, \bar{\ell}_j), \bar{\ell}_j; \cdot), \end{aligned} \tag{2.6}$$

and $u(w_k\bar{\ell}_k + \tau(w_k, \bar{\ell}_k), \bar{\ell}_k; \cdot) = u(w_k\bar{\ell}_k + \tau'(w_k, \bar{\ell}_k), \bar{\ell}_k; \cdot)$ for all $k \in M(E) \setminus \{i, j\}$.

[13] It corresponds to the *Ex Post Compensation* axiom in Fleurbaey and Peragine (2013), "which says that it is good to reduce inequalities in outcomes between two cells sharing the same effort level but having unequal circumstances" (p.122), where a "cell is a set of individuals with the same characteristics" (p. 121). We just restrict our attention to utility as the outcome in Fleurbaey and Peragine (2013) (cf. footnotes 1 and 2). Also, our axiom is in fact stronger than those of Fleurbaey and Peragine (2013) because they constantly require strict inequality signs. Indeed, stronger axioms are better for our purpose of showing compatibility.

As defined in section 2.2, the labor supply $\bar{\ell}_i$ and $\bar{\ell}_j$ are optimally chosen by individuals facing wage rates w_i and w_j, but with an arbitrary taste parameter. Transfers/taxes are determined by the government as a function τ of w_i and $\bar{\ell}_i$ as well as w_j and $\bar{\ell}_j$. Inequality can occur due to a difference in circumstance characteristics, wage rates, and the respective transfers/taxes. This axiom focuses on the different allocations of $\tau(w_i, \bar{\ell}_i)$ and $\tau'(w_i, \bar{\ell}_i)$, and also $\tau(w_j, \bar{\ell}_j)$ and $\tau'(w_j, \bar{\ell}_j)$, and it tries to achieve utility equality between individuals with the same labor supply $\bar{\ell}_i = \bar{\ell}_j$ by changing transfers/taxes.

The *ex ante* approach to compensation aims to reduce utility inequality between individuals, regardless of their labor supply, but with the same *ex ante* responsibility characteristic: taste for work.[14]

EX ANTE COMPENSATION. For all $E \in D$, $\tau \succ_{(E)} \tau'$ if there are $i, j \in N(E)$, $i, j \in M(E)$,

$$u(w_i\ell_i^i + \tau'(w_i, \ell_i^i, \theta^i), \ell_i^i; \theta^i) > u(w_i\ell_i^i + \tau(w_i, \ell_i^i, \theta^i), \ell_i^i; \theta^i)$$
$$\geq u(w_j\ell_j^i + \tau(w_j, \ell_j^i, \theta^i), \ell_j^i; \theta^i) > u(w_j\ell_j^i + \tau'(w_j, \ell_j^i, \theta^i), \ell_j^i; \theta^i) \qquad (2.7)$$

[14] It corresponds to the *Strong Ex Ante Compensation* axiom in Fleurbaey and Peragine (2013), which "seeks situations in which two types are clearly unequal in terms of the perspectives offered by their circumstances and the respective transfer policies" (p. 122), where "a type is a set of individuals with the same circumstances" (p. 121); that is, they evaluate the budget sets faced by individuals with different circumstances. Comparing individuals with the same taste for work is an alternative way to achieve the identical goal while considering utility as an outcome and introducing parametric utility functions. This is because both of the axioms aim to equalize "the opportunities offered to individuals (as measured by the possible well-being levels achieved with given circumstances for the various values of responsibility characteristics" (Fleurbaey, 2008, p. 239), where the responsibility characteristic is labor supply. In this chapter, the well-being levels, or outcomes, are considered to be utilities, and they can vary between individuals with the same circumstance but with different preferences; thus, we need to compare individuals with the same taste for work. It is also stronger than those of Fleurbaey and Peragine (2013) because they constantly require strict inequality signs.

and

$$u(w_i\ell_i^j + \tau'(w_i, \ell_i^j, \theta^j), \ell_i^j; \theta^j) > u(w_i\ell_i^j + \tau(w_i, \ell_i^j, \theta^j), \ell_i^j; \theta^j)$$
$$\geq u(w_j\ell_j^j + \tau(w_j, \ell_j^j, \theta^i), \ell_j^j; \theta^j) > u(w_j\ell_j^j + \tau'(w_j, \ell_j^j, \theta^j), \ell_j^j; \theta^j), \quad (2.8)$$

and $u(w_k\ell_k^k + \tau(w_k, \ell_k^k, \theta^k), \ell_k^k; \theta^k) = u(w_k\ell_k^k + \tau'(w_k, \ell_k^k, \theta^k), \ell_k^k; \theta^k)$ for all $k \in N(E) \setminus \{i, j\}$ and $k \in M(E) \setminus \{i, j\}$.

As defined in section 2.2, the labor supply ℓ_i^i, ℓ_j^i, ℓ_i^j, and ℓ_j^j are those of individuals facing wage rates w_i, w_j, and taste parameters θ^i, θ^j.[15] Transfers/taxes are determined by the government as a function τ of these variables and types. Again, inequality can occur due to a difference in circumstance characteristics, wage rates, and the respective transfers/taxes. This axiom focuses on the different allocations of the transfers/taxes and tries to achieve utility equality between individuals, regardless of their labor supply ℓ_i^i, ℓ_j^i, ℓ_i^j, or ℓ_j^j, but within the same taste parameter θ^i or θ^j.

2.4.2. Statement of the main result. We provide the compatibility theorem, which is an escape route from the opportunity paradox. The first step is that the *ex post* perspective of compensation, **Ex Post Compensation**, requires a reduction in the inequality of utilities, not consumption or disposable incomes. Thus, we have to compare the utility levels, the "height" of the utility functions, which cannot be observed in the indifference curves. The second step is that using the separable—and taste-independent—domain restriction resolves this difficulty.

We provide an intuitive proof of the compatibility theorem using Figure 2.3. Points a, b, c, and d are all displayed at the same places as in Figure 2.1. Point a represents the allocation optimally chosen by an individual

[15]Note that superscripts and subscripts, i, j, k, identify the types of individuals.

with wage rate w_j and taste parameter θ^1, point b represents the allocation of an individual with w_j and θ^2, and point f represents the allocation of an individual with w_i and θ^1, where $w_i < w_j$. Points d and e represent the allocations of individuals with w_i and θ^1 who receive the respective lump-sum transfers.[16]

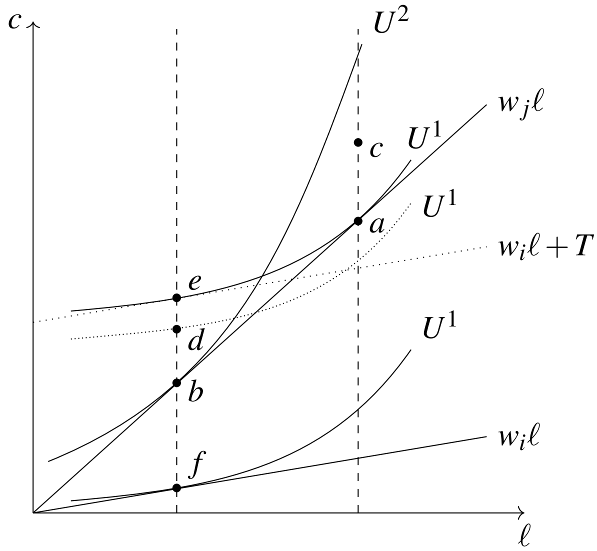

FIGURE 2.3. No opportunity paradox by introducing separable utility and taste-independence: both **Ex Post Compensation** and **Ex Ante Compensation** require that the individual on point d be compensated to e.

Suppose points a and b are laissez-faire allocations of individuals with wage rate w_j; that is, they are on the same budget line $w_j\ell$. By *taste-independence*, a and b exhibit the same utility levels for individuals with θ^1 and θ^2. Furthermore, points a and e have the same utility levels for individuals with θ^1 because the points are on the same indifference curve. Therefore, **Ex Post Compensation** requires lump-sum transfers to the individual on point d to e, not to b, to realize the same utility level between individuals on points b and d, which does not violate **Ex Ante Compensation**.[17] In this

[16]They have the same labor supply due to the assumption of quasilinearity in consumption.

[17]It may seem unreasonable to accept a transfer policy to change point d to e because it expands income inequality. However, consider, for instance, a proportional income tax

example, point d is actually the point at which an individual on point f, as the laissez-faire allocation, is given a certain amount of lump-sum transfers, which **Ex Post Compensation** deems insufficient.[18] The same argument applies to point c. Theorem 2.1 shows that this is not an exceptional case.

THEOREM 2.1. ***Ex Post Compensation*** *and* ***Ex Ante Compensation*** *are compatible when the parametric utility functions are taste-independent and quasilinear in consumption.*

PROOF. We prove that we can obtain allocations required by both **Ex Post Compensation** and **Ex Ante Compensation** in general.[19] There are two types of individuals (i.e., wage rate type and taste type) in our model; thus, the minimal society we need to consider consists of 2×2 (types of) individuals.

First, we construct all possible transfers/taxes required by **Ex Post Compensation**. Second, we show that arbitrary transfers/taxes required by **Ex Ante Compensation** are compatible with them.

We consider four individuals indicated by 1 to 4. They have wage rate w_i or w_j ($w_i \neq w_j$) and taste parameter θ^i or θ^j ($\theta^i \neq \theta^j$). As a result of their maximization behavior, the four individuals have labor supply $\bar{\ell}_i^i$, $\bar{\ell}_i^j$, $\bar{\ell}_j^j$, or $\bar{\ell}_j^i$, and receive lump-sum transfers/taxes denoted by $T_1, T_2, T_3, T_4 \neq 0$, respectively.

that changes the wage rate of individuals with preference θ^1 at points d, e, or f as the laissez-faire allocation. Their maximization behavior moves those points to a. Also, note that the outcome in Fleurbaey and Peragine (2013) can be utility (cf. footnotes 1, 2, and 13); hence, their *ex post* perspective of compensation would require the same policy if the outcome is utility that is separable in consumption and labor supply.

[18]These lump-sum transfers are implementable due to the assumption of quasilinearity in consumption.

[19]It does not depend on any particular social welfare function.

The maximized utility of each individual is indicated by

$$U_1^* \equiv U_i^i = u(w_i\bar{\ell}_i^i + T_1, \bar{\ell}_i^i; \theta^i), \quad U_2^* \equiv U_i^j = u(w_i\bar{\ell}_i^j + T_2, \bar{\ell}_i^j; \theta^j), \quad (2.9)$$

$$U_3^* \equiv U_j^j = u(w_j\bar{\ell}_j^j + T_3, \bar{\ell}_j^j; \theta^j), \quad U_4^* \equiv U_j^i = u(w_j\bar{\ell}_j^i + T_4, \bar{\ell}_j^i; \theta^i). \quad (2.10)$$

Moreover, assume, without loss of generality,

$$\bar{\ell}_i^i = \bar{\ell}_j^j, \; \bar{\ell}_i^j = \bar{\ell}_j^i. \quad (2.11)$$

By the property of *quasilinearity*, the following laissez-faire allocations exist:

$$u(w_i\bar{\ell}_i^i, \bar{\ell}_i^i; \theta^i), \; u(w_i\bar{\ell}_i^j, \bar{\ell}_i^j; \theta^j), \quad (2.12)$$

$$u(w_j\bar{\ell}_j^j, \bar{\ell}_j^j; \theta^j), \; u(w_j\bar{\ell}_j^i, \bar{\ell}_j^i; \theta^i). \quad (2.13)$$

By *taste-independence*, which is assured by Theorem **??**,

$$u(w_i\bar{\ell}_i^i, \bar{\ell}_i^i; \theta^i) = u(w_i\bar{\ell}_i^j, \bar{\ell}_i^j; \theta^j), \quad (2.14)$$

$$u(w_j\bar{\ell}_j^j, \bar{\ell}_j^j; \theta^j) = u(w_j\bar{\ell}_j^i, \bar{\ell}_j^i; \theta^i). \quad (2.15)$$

Also, by *quasilinearity*, there exist $\tilde{T}_1$, $\tilde{T}_2$, $\tilde{T}_3$, and $\tilde{T}_4$ to compensate between individuals with the same taste (e.g., by suitable lump-sum transfers, the government can compensate the individual represented by point f to e in Figure 2.3), such that

$$\tilde{U}_1 \equiv u(w_i\bar{\ell}_i^i + \tilde{T}_1, \bar{\ell}_i^i; \theta^i) = u(w_j\bar{\ell}_j^i, \bar{\ell}_j^i; \theta^i), \quad (2.16)$$

$$\tilde{U}_2 \equiv u(w_i\bar{\ell}_i^j + \tilde{T}_2, \bar{\ell}_i^j; \theta^j) = u(w_j\bar{\ell}_j^j, \bar{\ell}_j^j; \theta^j), \quad (2.17)$$

$$\tilde{U}_3 \equiv u(w_j\bar{\ell}_j^j + \tilde{T}_3, \bar{\ell}_j^j; \theta^j) = u(w_i\bar{\ell}_i^j, \bar{\ell}_i^j; \theta^j), \quad (2.18)$$

$$\tilde{U}_4 \equiv u(w_j\bar{\ell}_j^i + \tilde{T}_4, \bar{\ell}_j^i; \theta^i) = u(w_i\bar{\ell}_i^i, \bar{\ell}_i^i; \theta^i). \quad (2.19)$$

By transitivity, combining (2.14), (2.16), and (2.17),

$$\begin{aligned}\tilde{U}_1 \equiv u(w_i\bar{\ell}^i_i + \tilde{T}_1, \bar{\ell}^i_i; \theta^i) &= u(w_j\bar{\ell}^i_j, \bar{\ell}^i_j; \theta^i) \\ &= u(w_j\bar{\ell}^j_j, \bar{\ell}^j_j; \theta^j) = u(w_i\bar{\ell}^j_i + \tilde{T}_2, \bar{\ell}^j_i; \theta^j) \equiv \tilde{U}_2, \end{aligned} \tag{2.20}$$

and combining (2.15), (2.18), and (2.19),

$$\begin{aligned}\tilde{U}_3 \equiv u(w_j\bar{\ell}^j_j + \tilde{T}_3, \bar{\ell}^j_j; \theta^j) &= u(w_i\bar{\ell}^j_i, \bar{\ell}^j_i; \theta^j) \\ &= u(w_i\bar{\ell}^i_i, \bar{\ell}^i_i; \theta^i) = u(w_j\bar{\ell}^i_j + \tilde{T}_4, \bar{\ell}^i_j; \theta^i) \equiv \tilde{U}_4. \end{aligned} \tag{2.21}$$

Now, consider applying axioms between the four individuals with U_1^*, U_2^*, U_3^*, and U_4^* using equations (2.20) and (2.21). **Ex Post Compensation** requires that the inequality of utilities be eliminated if either U_1^* or U_3^* is higher than the other. In such a case, the following transfer or tax policy changes, (α) or (β), can cause individuals 1 and 3 to enjoy the same utility level.

(α): changes T_1 to $\tilde{T}_1$ and T_3 to 0, which result in the same utility level $\tilde{U}_1 = \tilde{U}_2$ for individuals 1 and 3.

(β): changes T_1 to 0 and T_3 to $\tilde{T}_3$, which result in the same utility level $\tilde{U}_3 = \tilde{U}_4$ for individuals 1 and 3.

In the same way, **Ex Post Compensation** requires that the inequality of utilities be eliminated if either U_2^* or U_4^* is higher than the other. In such a case, the following transfer or tax policy changes, (γ) or (δ), can cause individuals 2 and 4 to enjoy the same utility level.

(γ): changes T_2 to $\tilde{T}_2$ and T_4 to 0, which result in the same utility level $\tilde{U}_1 = \tilde{U}_2$ for individuals 2 and 4.

(δ): changes T_2 to 0 and T_4 to $\tilde{T}_4$, which result in the same utility level $\tilde{U}_3 = \tilde{U}_4$ for individuals 2 and 4.

Meanwhile, **Ex Ante Compensation** requires that the inequality of utilities be eliminated if either U_1^* or U_4^* is higher than the other. Either of the following combinations of policy changes, (α) and (γ), or (β) and (δ), can cause individuals 1 and 4 to enjoy the same utility level.

- (α) change T_1 to $\tilde{T}_1$ and (γ) change T_4 to 0, which result in the same utility level $\tilde{U}_1 = \tilde{U}_2$ for individuals 1 and 4.
- (β) change T_1 to 0 and (δ) change T_4 to $\tilde{T}_4$, which result in the same utility level $\tilde{U}_3 = \tilde{U}_4$ for individuals 1 and 4.

Furthermore, **Ex Ante Compensation** requires that the inequality of utilities be eliminated if either U_2^* or U_3^* is higher than the other. Either of the following combinations of policy changes, (α) and (γ), or (β) and (δ), can cause individuals 2 and 3 to enjoy the same utility level.

- (γ) change T_2 to $\tilde{T}_2$ and (α) change T_3 to 0, which result in the same utility level $\tilde{U}_1 = \tilde{U}_2$ for individuals 2 and 3.
- (δ) change T_2 to 0 and (β) change T_3 to $\tilde{T}_3$, which result in the same utility level $\tilde{U}_3 = \tilde{U}_4$ for individuals 2 and 3.

Therefore, combinations of transfer or tax policy changes, (α) and (γ), or (β) and (δ), result in allocations that satisfy the requirements of both **Ex Post Compensation** and **Ex Ante Compensation**. □

REMARK 2.1. **Ex Post Compensation** through lump-sum transfers/taxes only covers situations where individuals have utilities that are quasilinear in consumption (i.e., income effects are zero) or when income and substitution effects are offset. Once the labor supply varies according to transfers/taxes, we can no longer apply **Ex Post Compensation**, which compares individuals with the same labor supply. The assumption of quasilinearity with respect to consumption assures the existence of comparable allocations; otherwise, we cannot discuss the logical relationship between **Ex Ante**

Compensation and **Ex Post Compensation**. Therefore, quasilinearity is needed so that we can concentrate solely on resolving the opportunity paradox, and it is not the "crucial" assumption for our result.[20]

REMARK 2.2. In the proof, we demonstrate that we can achieve (perfect) equality of utilities between individuals by showing that **Ex Ante Compensation** and **Ex Post Compensation** requires the same direction of transfers/taxes, but this may require the government to implement a large amount of transfers/taxes. The theorem, however, incorporates some medium level or small amount of reductions in inequality, and transfers/taxes could also be "sufficiently small" (Fleurbaey and Peragine, 2013, p. 126–127). This fact may be useful when we take into account the government's budget constraint, which we do not in the present chapter.

2.4.3. Reinterpretation in a responsibility–outcome space. We demonstrate how the opportunity paradox can be escaped in a responsibility–outcome space. Since we consider the outcome to be utility, Figure 2.1 is not a responsibility–outcome space; we need to clarify the opportunity sets of individuals as well as the relationship between the existing literature, such as Fleurbaey (2019).[21] As a result, we provide a graphical alternative proof of Theorem 2.1. For simplicity, in the following discussion, we assume that opportunity gaps are only due to differences in wage rates, not the respective transfers/taxes.

In our framework, the outcome axis represents utility U_j^i, while the responsibility axis should represent two responsibility characteristics: taste

[20] "All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive. A 'crucial' assumption is one on which the conclusions do depend sensitively, and it is important that crucial assumptions be reasonably realistic" (Solow, 1956, p. 65).

[21] See footnotes 1, 2, and 4.

for work θ^i as *ex ante* and labor supply ℓ as *ex post* responsibility characteristics. First, by fixing θ^i, we explore what an *ex post* responsibility–outcome, or labor supply–utility, space looks like. An example of the utility functions of three individuals with the same taste but different wage rates is presented in Figure 2.4. These curves represent opportunity sets of individuals with wage rates $w_1 > w_2 > w_3$, and the same taste θ^3. Each utility is U_1^3, U_2^3, and U_3^3, respectively. As we assume maximization behavior by individuals, points a, b, and c are supposed be achieved.

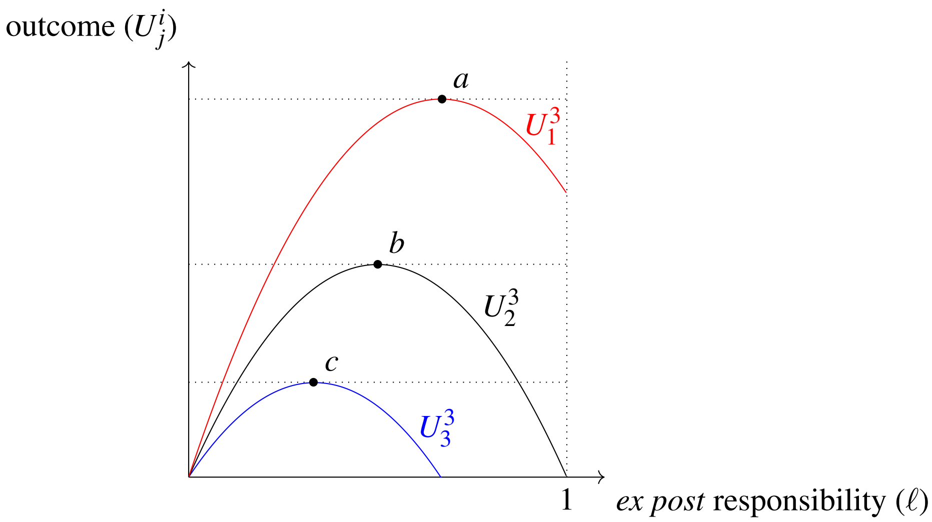

FIGURE 2.4. Opportunity sets of individuals (fixing the *ex ante* responsibility characteristic, or taste for work).

Then, we consider that the *ex ante* responsibility characteristic, or taste parameter θ^i, also varies. If taste-independence is satisfied, the utility functions of individuals with two different wage rates and three different tastes for work can be drawn like Figure 2.5. Individuals with wage rate w_1 have three different tastes, θ^1, θ^2, and θ^3. Their utilities are U_1^1, U_1^2, and U_1^3, respectively. Additionally, individuals with wage rate w_3 have three different tastes, θ^3, θ^4, and θ^5. Their utilities are represented by U_3^3, U_3^4, and U_3^5, respectively. Note that U_1^3 and U_3^3 remain the same as Figure 2.4, and they have the same taste, but they have different wage rates. Figure 2.5 shows how, given wage rates w_1 and w_3, the two responsibility characteristics, ℓ and θ^i, determine utility, or outcome. Utility functions are continuous with respect to taste parameter θ^i; thus, utilities with the same wage rate but different tastes can densely exist. Therefore, opportunity sets, which describe the correspondence between responsibility characteristics and outcome, can be represented by the envelope curves (lines) of utilities for each wage rate when maximization behavior is supposed. The area below the lines are included if maximization behavior is not supposed.

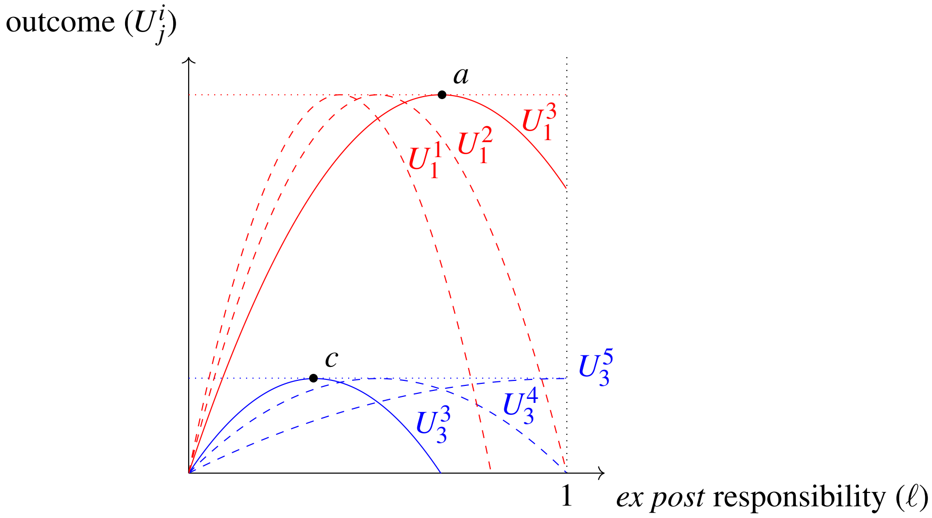

FIGURE 2.5. Opportunity sets of individuals whose utilities are taste-independent: dashed curves represent the corresponding sets to the change of the *ex ante* responsibility characteristic, or taste for work.

The argument so far can be described simply in Figure 2.6. We observe that the outcome is constant according to responsibility characteristics. This suggests that individuals with the same wage rate have equal opportunity to enjoy the same utility, even if they choose different labor/leisure time and incomes on their own responsibility. This is exactly what the taste-independence property implies, and the reason why it can be recognized as a responsibility requirement.

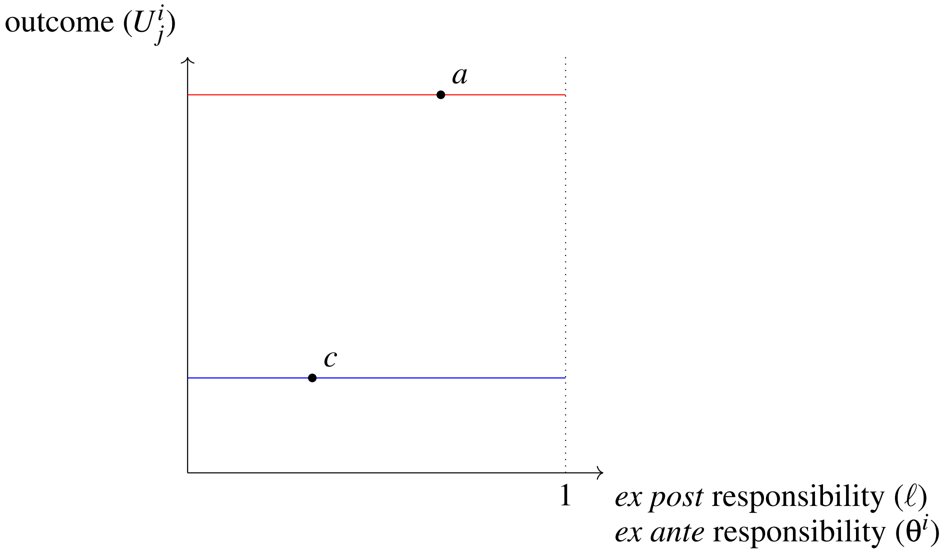

FIGURE 2.6. Opportunity sets of individuals whose utilities are taste-independent.

Now, we can illustrate how the taste-independence property works for escaping from the opportunity paradox. **Ex Ante Compensation** aims to reduce the utility inequality between individuals with the same taste for work, such as points a and c, which share the same taste for work θ^3. *Anywhere* such comparable individuals (i.e., those with same taste for work but different wage rates) exist, including a and c on the two lines, **Ex Ante Compensation** requires a reduction in the gaps of the two lines, or opportunity sets. In other words, the *ex ante* perspectives of compensation require that individuals with the same taste for work but different circumstances, have equal opportunity to enjoy the same utility, but can choose their labor/leisure time and income.[22]

[22]See also footnote 4.

Meanwhile, **Ex Post Compensation** aims to reduce the utility inequality between individuals with the same labor supply but different circumstances. Again, *anywhere* such comparable individuals (i.e., those with the same labor supply) exist on the two lines, **Ex Post Compensation** always requires a reduction in the gaps of the two lines. Therefore, **Ex Ante Compensation** and **Ex Post Compensation** require the same direction of transfers/taxes; that is, they are compatible.

If taste-independence is not satisfied, the opportunity sets are, in general, no longer horizontal lines but arbitrary nonlinear curves. Hence, responsibility–outcome sets, or opportunity sets, can be described as in Fleurbaey (2008, ch.9, sec. 9.5), Fleurbaey (2019, sec. 6), and Figure 2.1 as well as Figure 2.7 in the present chapter.[23] In such cases, the *ex ante* and *ex post* perspectives of compensation can conflict. For reference, we restate the same proof as Fleurbaey (2008) in Figure 2.7 using our axioms **Ex Ante Compensation** and **Ex Post Compensation**.[24]

We now consider that opportunity gaps are due to both different wage rates and the respective transfers/taxes. Solid curves A, B, C, and D represent four opportunity sets. Individuals on A and B share the same wage rate but different transfers/taxes, so do those on C and D. The nonlinearity of the curves indicates that the utilities are not taste-independent; that is, each of the curves represents the maximized utility of individuals with the same wage rate but different tastes for work. **Ex Ante Compensation** seeks

[23]That is, outcome, or utility, varies according to the two responsibility characteristics, or labor supply and taste for work, without taste-independence. See also footnote 4.

[24]Fleurbaey (2008) uses the axiom *Opportunity Dominance*, which implies our **Ex Ante Compensation**, and is setting the vertical axis as well-being. Figure 9.4 in Fleurbaey (2008, p. 238) corresponds to Figure 2.7 in this chapter, and Figure 5 in Fleurbaey (2019, p. 675) corresponds to Figure 2.1 in this chapter. Since, in Figure 2.1, we consider an labor supply–income, or *ex post* responsibility–income, space, Figure 2.7 is more suited for describing the opportunity sets. See also footnote 4.

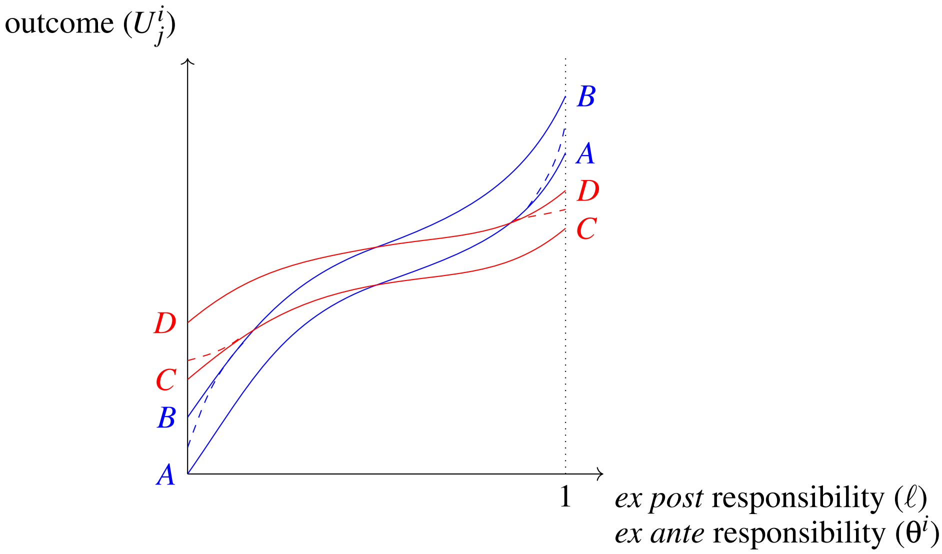

FIGURE 2.7. Crossing opportunity sets of individuals whose utilities are not taste-independent (reproduced from Fleurbaey, 2008, p. 238, Figure 9.4).

individuals with the same taste, and we can discuss this by arbitrarily picking some allocations to compare. However, since the difference between A and B is due the respective transfers/taxes, anywhere individuals with the same taste exist on curves A and B, **Ex Ante Compensation** requires a reduction in the gaps between curves A and B. The same is true for curves C and D. For example, modifications of curves, described by the dashed curves, can be required by **Ex Ante Compensation**, but they are generally incompatible with **Ex Post Compensation** because the utility inequalities between individuals on B and C with low labor supply as well as individuals on A and D with high labor supply are widened.

2.5. Concluding remarks

"(I)n the process of finding some meaningful escape routes from these logical impasses, we are brought to much richer understanding on what makes several social values mutually compatible than otherwise" (Suzumura, 2002, p. 25).

In this chapter, we introduced taste-independence as a property of preferences while taking individual responsibility into account. We then showed that *ex post* and *ex ante* perspectives of compensation, which focus on utilities, are compatible when the utility functions are taste-independent and quasilinear in consumption. In fact, quasilinearity is not an essential assumption, in the sense that it only makes the *ex post* perspective of compensation apply when there are lump-sum transfers/taxes. Without quasilinearity, we need to consider situations where the labor supply optimally chosen by individuals are invariant to lum-sum transfers/taxes, and these situations can be captured by quasilinear assumption.

In practice, for instance, when deriving the optimal income taxation formula, restrictions of the preference domain, such as quasilinearity, are commonly required. That is, we often solve the maximization problem of the Bergson-Samuelson (Bergson, 1938; Samuelson, [1947] 1983) social welfare function (SWF), whose inputs are separable utilities, subject to incentive compatibility constraints and the government's budget constraint. We face ethical conflicts such as the opportunity paradox when constructing the SWF itself; thus, maximizing it derives a solution (e.g., income taxation formula) that is, at any rate, a compromise of either the *ex ante* or the *ex post* perspectives of compensation. However, since we solve the maximization problem with the preference restriction after all, we can derive the SWF on a restricted domain where there is no compatibility (i.e., taste-independent and quasilinear utility). Therefore, we opened up the possibility obtaining a solution that is not a compromise of both the *ex ante* and the *ex post* perspectives of compensation.

Our result by preference domain restriction, however, can be interpreted as follows. Since taste-independence means that the partial differential coefficient of the utility function evaluated at the optimal solution with respect to the taste parameter is always zero, the indirect utility function derived from the original utility function u is constant with respect to θ^i. It is indeed a situation where the indirect utility function itself do not vary according to the types of individuals, or every individual has the same utility function. If the utility functions of the two individuals are taste-independent only when their (indirect) utility functions are the same, then the result of this chapter may imply that, even if we allow for interpersonal comparability of utility in the sense of taste-independence, the *ex ante* and the *ex post* perspectives of compensation are incompatible. It suggests that our contribution still demonstrates an impossibility between the principles of responsibility and compensation.

We conclude by suggesting directions for future research. As declared in footnote 6, we adopted an approach in which utility is a subjective measure of well-being, but such a welfarist approach has "serious weaknesses" (Fleurbaey and Maniquet, 2018, p. 1035). Further explorations are needed to accommodate critiques such as the "expensive tastes" (e.g., Dworkin, 1981a,b) and the "tamed housewife" arguments (Sen, 1985a,b).[25]

Moreover, as mentioned in footnotes 10 and 17, it is worthwhile to extend our theory to address issues regarding the heterogeneous preferences of individuals with disabilities, such as Cuff (2000) and Boadway et al. (2002), and curbing inequality of incomes, such as (Roemer et al., 2003).

[25] *Welfarism* is defined as "requiring that the goodness of a state of affairs be a function only of the utility information regarding that state" (Sen, 1987, p. 39). Our results may imply that there are limitations to using only utility information because just assuming separability leads to expansion of inequality of income through lump-sum transfers/taxes (cf. footnote 17).

Furthermore, applications to income taxation and transfer policies, relating to the existing fair taxation literature (e.g., Schokkaert et al., 2004; Fleurbaey and Maniquet, 2006, 2007, 2011b,c; Jacquet and Van de gaer, 2011; Lockwood and Weinzierl, 2015; Saez and Stantcheva, 2016; Fleurbaey and Maniquet, 2018), are expected.

Finally, we should analyze the functions that satisfies taste-independence as well as other preference domain restrictions to resolve the opportunity paradox. Furthermore, other related axioms of the *ex ante* perspective of compensation, or various reward principles, may be compatible with the *ex post* perspective of compensation when taste-independence is satisfied. In other words, we should explore the necessary and sufficient conditions for escaping the opportunity paradox, and such investigations will lead us to a more profound comprehension of the theoretical possibilities beyond the difficulties regarding equality of opportunity.

CHAPTER 3

Entropic mobility index as a measure of (in)equality of opportunity

3.1. Introduction

As stated in Jäntti and Jenkins (2015), "greater mobility is socially desirable because equality of opportunity is a principle that is widely supported, regardless of attitudes to inequality of outcomes" (p. 815). For this reason, the empirical literature that measures mobility derives implications for equality of opportunity (e.g., Chetty et al., 2017). However, as Shorrocks (1978b) and Kanbur and Stiglitz (1986, 2016) noted, there is a conflict between greater mobility and equality of opportunity; that is, the "diagonals view" and the "equality of opportunity view" of transition matrices cannot hold simultaneously. The perfect mobility, or when the transition matrix is antidiagonal, is desirable according to the diagonals view. On the other hand, it is desirable when each element of the transition matrix is the same by the equality of opportunity view. According to Cowell and Flachaire (2019), mobility concepts themselves follow a "more movement, move mobility" principle; however, "more movement" does not necessarily mean "more equality of opportunity."

To accommodate the conceptual gap between mobility and equality of opportunity, we propose a normative mobility index that includes a requirement of opportunity equality. The index adheres to the principle of "more the equality of opportunity, more the value of the mobility index" by applying entropy to transition matrices. Information entropy (Shannon,

1948) is "a unique, unambiguous criterion for the 'amount of uncertainty' represented by a discrete probability distribution" (Jaynes, 1957a, p. 622).[1] We modify quantum entropy (von Neumann, 1955), which is a generalized form of information entropy, to derive a new mobility index.[2] Using this index, we can evaluate social mobility from the perspective of equality of opportunity.

The remainder of this chapter is organized as follows. We present the basic definitions in Section 3.2. We provide a mobility index and its properties in Section 3.3. We conclude in Section 3.4.

3.2. Basic definitions

We describe mobility by a stochastic matrix following Atkinson (1983b). Suppose there are two periods and n classes of income or some other social status for $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers. For $k = 1,...,n$, let m_1^k be the relative number of observations in class k in period-1. The marginal discrete distribution in period-1 is indicated by the vector $\mathbf{m}_1 = (m_1^1, m_1^2, ..., m_1^n)$, and correspondingly in period-2. Thus, the pattern of mobility may be represented by the $n \times n$ *transition matrix* $\mathbf{A}$, where $\mathbf{m}_2 = \mathbf{m}_1\mathbf{A}$.

For $i, j = 1,...,n$, let $a_{i,j}$ be i-th row and j-th column element of $\mathbf{A}$. We focus on changes in relative positions, or pure exchange mobility, so that $\mathbf{A}$ is *doubly stochastic*, that is, $\sum_{i=1}^n a_{i,j} = \sum_{j=1}^n a_{i,j} = 1$. Typical element $a_{i,j}$ is the relative frequency of observations with income or status class i in period-1 and class j in period-2.

[1] Information entropy was introduced to economics by Theil (1967), which is, for example, used for inequality measurement.

[2] Although we investigate the entropy of a transition matrix, our index is different from the entropy of Markov chains (Khinchin, 1957) or the applications of the generalized entropy measure such as Shorrocks (1978a) and Tsui (2009).

For $i = 1, ..., n$, let λ_i denote *eigenvalue* of $\mathbf{A}$.[3] The bar notation $|\cdot|$ is used to denote absolute values. Transpose of a vector is denoted by superscript T.

We consider opportunities to be equal when each element of the transition matrix $\mathbf{A}$ is the same, that is, for all $i, j = 1, ..., n$, $a_{i,j} = 1/n$. Moreover, opportunities are at least equal both when perfectly immobile and perfectly mobile; that is, both when the transition matrix is diagonal (or identical) and antidiagonal. Namely, we adopt the "equality of opportunity view."

Finally, we recall the following classical results.

PROPOSITION 3.1 (Marcus and Minc, 1964). [4]

(1) *The vector* $[1, 1, ..., 1]^T$ *is an eigenvector of* $\mathbf{A}$ *corresponding to the eigenvalue* 1.
(2) *Every eigenvalue* λ_i *of* $\mathbf{A}$ *satisfies* $|\lambda_i| \leq 1$.

3.3. Mobility index

We introduce a mobility index. Let f_i be defined as

$$f_i = \frac{|\lambda_i|^\alpha}{\sum_{i=1}^{n} |\lambda_i|^\alpha}, \tag{3.1}$$

where α is real valued, which can be interpreted as a parameter of sensitivity to deviations from the equal opportunity situation.[5]

We define the mobility index $\phi(\mathbf{A})$ as follows.

[3] There are n eigenvalues including algebraic multiplicity.

[4] See Marcus and Minc (1964, p. 133), 5.13.2 and 5.13.3, respectively. They can be regarded as special cases of Perron-Frobenius theorem (cf. Nikaido, 1968; Lancaster, 1968).

[5] This generalization is inspired by Rényi (1961).

DEFINITION 3.1 (Mobility index). The mobility index is given by a function ϕ defined on a transition matrix $\mathbf{A}$:

$$\phi(\mathbf{A}) = \sum_{i=1}^{n} f_i \ln f_i \tag{3.2}$$

with the convention

$$0 \ln 0 = 0.^6 \tag{3.3}$$

This index takes a maximum value when opportunities are equal in the sense that each probability of the transition matrix is the same (i.e., $\forall i, j = 1, \ldots, n$, $a_{i,j} = 1/n$). Moreover, it takes a minimum value when opportunities are least unequal (i.e., $\mathbf{A}$ is diagonal or antidiagonal).

We examine these properties and how the parameter α works in the two-dimensional case (i.e., the case when there are two classes of income or status) in Example 3.1.

EXAMPLE 3.1. We consider the 2×2 transition matrix:

$$\mathbf{A} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}, \tag{3.4}$$

where p is real valued, and $0 \leq p \leq 1$. The values of the mobility index $\phi(\mathbf{A})$ are plotted in Figure 3.1.

We can observe that the index value is a maximum when each element of the transition matrix is the same (i.e., $\forall i, j = 1, 2$, $a_{i,j} = 1/2$). Moreover, it is always nonpositive; that is, the smaller value of this index represents the more "inequality" of opportunity. Furthermore, it is clear that the sensitivity of the index to deviations from the maximum varies according to the parameter α. Example 3.1 shows that our index has desirable properties as

[6] We use natural logarithm for analytical simplicity, but the base does not matter.

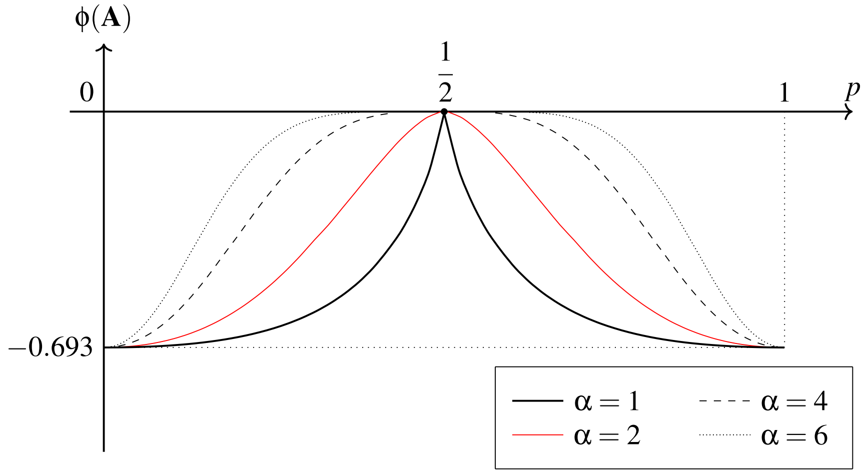

FIGURE 3.1. The values of the mobility index in the two-dimensional case.

a measure of opportunity (in)equality in the two-dimensional case. Before proving its general properties, we need the following lemma.

LEMMA 3.1. *Each of the following statements (i)-(iii) holds.*

(i) *If* $\mathbf{A}$ *is diagonal, or identical, that is,*

$$a_{i,j} = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}, \tag{3.5}$$

then all eigenvalues of $\mathbf{A}$ *are* 1.

(ii) *If* $\mathbf{A}$ *is antidiagonal, that is,*

$$a_{i,j} = \begin{cases} 0 & (i+j \neq n+1) \\ 1 & (i+j = n+1) \end{cases}, \tag{3.6}$$

then all eigenvalues of $\mathbf{A}$ *are* 1.

(iii) *If each element of* $\mathbf{A}$ *is the same, that is,* $a_{i,j} = 1/n$ *for all* $i, j = 1, \dots, n$*, then the only one eigenvalue of* $\mathbf{A}$ *is* 1*, and the other eigenvalues are all* 0.

PROOF. (i) Let det$[\cdot]$ denote a determinant of a matrix. We have

$$\det[\lambda I - \mathbf{A}] = \det \begin{bmatrix} \lambda_1 - a_{1,1} & & & \mathbf{0} \\ & \lambda_2 - a_{2,2} & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n - a_{n,n} \end{bmatrix} \tag{3.7}$$

$$= (\lambda_1 - a_{1,1})(\lambda_2 - a_{2,2}) \cdots (\lambda_n - a_{n,n}) \tag{3.8}$$

$$= (\lambda_1 - 1)(\lambda_2 - 1) \cdots (\lambda_n - 1) = 0 \tag{3.9}$$

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1. \tag{3.10}$$

Therefore, $\lambda_i = 1$ for all $i = 1, ..., n$.

(ii) We consider raising $\mathbf{A}$ to the power 2:

$$\mathbf{A}^2 = \begin{bmatrix} \mathbf{0} & & & a_{1,n} \\ & & \cdot^{\cdot^{\cdot}} & \\ & a_{n-1,2} & & \\ a_{n,1} & & & \mathbf{0} \end{bmatrix}^2 \tag{3.11}$$

$$= \begin{bmatrix} a_{1,n}a_{n,1} & & & \mathbf{0} \\ & a_{2,n-1}a_{n-1,2} & & \\ & & \ddots & \\ \mathbf{0} & & & a_{n,1}a_{1,n} \end{bmatrix} \tag{3.12}$$

$$= \begin{bmatrix} 1 & & & \mathbf{0} \\ & 1 & & \\ & & \ddots & \\ \mathbf{0} & & & 1 \end{bmatrix}. \tag{3.13}$$

Eigenvalues of $\mathbf{A}^2$ are λ_i^2, and from (i), they are all 1. Therefore, we have $\lambda_i = 1$ for all $i = 1, ..., n$.

(iii) Let $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ be eigenvector. By definition, $\mathbf{Ax} = \lambda_i \mathbf{x}$, thus,

$$\begin{bmatrix} 1/n & 1/n & \cdots & 1/n \\ 1/n & 1/n & \cdots & 1/n \\ \vdots & \vdots & \ddots & \vdots \\ 1/n & 1/n & \cdots & 1/n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \lambda_i \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{3.14}$$

Comparing the i-th row of the both sides, we obtain

$$(1/n)x_1 + (1/n)x_2 + \cdots + (1/n)x_n = \lambda_i x_i \tag{3.15}$$

Adding up for $i = 1, ..., n$,

$$n[(1/n)x_1 + (1/n)x_2 + \cdots + (1/n)x_n] = \lambda_i x_1 + \lambda_i x_2 + \cdots + \lambda_i x_n \tag{3.16}$$

$$(x_1 + x_2 + \cdots + x_n) = \lambda_i (x_1 + x_2 + \cdots + x_n). \tag{3.17}$$

Therefore, we have either

$$\lambda_i = 1 \tag{3.18}$$

or

$$x_1 + x_2 + \cdots + x_n = 0. \tag{3.19}$$

In the latter case, $0 = \lambda_i x_i$, thus, $\lambda_i = 0$. Therefore, we have an eigenvalue 1, and the other eigenvalues are 0.

□

We have the following two theorems and a corollary as the properties of our mobility index.

THEOREM 3.1. *The mobility index* $\phi(\boldsymbol{A})$ *takes a maximum value if each element of* $\boldsymbol{A}$ *is the same, that is,* $a_{i,j} = 1/n$ *for all* $i, j = 1, \ldots, n$.

PROOF. First, we prove $\phi(\mathbf{A})$ is nonpositive. Since, by the definition (3.1), $0 \leq f_i \leq 1$, we have $f_i \ln f_i \leq 0$ for all $i = 1, \ldots, n$. Hence,

$$\phi(\mathbf{A}) \leq 0. \tag{3.20}$$

Next, by Lemma 3.1 (iii), if $a_{i,j} = 1/n$ for all $i, j = 1, \ldots, n$, then the only one eigenvalue of $\mathbf{A}$ is 1, and the other eigenvalues are 0. Thus,

$$\phi(\mathbf{A}) = 1 \ln 1 + 0 \ln 0 + \cdots + 0 \ln 0 = 0. \tag{3.21}$$

Since $\phi(\mathbf{A})$ is always nonpositive, it is a maximum. □

COROLLARY 3.1. *The mobility index* $\phi(\boldsymbol{A})$ *takes a negative value if* $\boldsymbol{A}$ *has eigenvalues other than* 0 *or* 1.

PROOF. As we have shown, $\phi(\mathbf{A})$ is nonpositive. By Proposition 3.1, a transition matrix $\mathbf{A}$ has an eigenvalue 1, and the absolute value of the other eigenvalues are less than 1. Let $\lambda_1 = 1$, by the given conditions, $|\lambda_j| < 1$ for $j = 2, \ldots, n$. Then, there exists j such that $0 < f_j < 1$, which leads to $f_j \ln f_j < 0$. Therefore, we have $\phi(\mathbf{A}) < 0$. □

THEOREM 3.2. *The mobility index* $\phi(\boldsymbol{A})$ *takes a minimum value if* $\boldsymbol{A}$ *is diagonal or antidiagonal.*

PROOF. We consider the following constrained minimization problem:[7]

$$\min \sum_{i=1}^{n} f_i \ln f_i \tag{3.22}$$

$$\text{s.t.} \sum_{i=1}^{n} f_i = 1. \tag{3.23}$$

[7]The proof is an application of the principle of maximum entropy (Jaynes, 1957a).

Using the method of Lagrange multipliers,

$$L = \sum_{i=1}^{n} f_i \ln f_i - \lambda \left(\sum_{i=1}^{n} f_i - 1 \right). \tag{3.24}$$

We obtain the first-order condition (FOC):

$$\frac{\partial L}{\partial f_i} = \ln f_i + 1 - \lambda = 0 \tag{3.25}$$

$$f_i = e^{\lambda - 1}. \tag{3.26}$$

The FOC (3.26) indicates that each f_i is the same constant. Since, by the definition (3.1), $\sum_{i=1}^{n} f_i = 1$, we have

$$f_i = \frac{1}{n}. \tag{3.27}$$

By Lemma 3.1 (i) and (ii), if **A** is diagonal or antidiagonal, then all eigenvalues are 1. Hence,

$$f_i = \frac{|\lambda_i|^\alpha}{\sum_{i=1}^{n} |\lambda_i|^\alpha} = \frac{1^\alpha}{1^\alpha + 1^\alpha + \cdots + 1^\alpha} = \frac{1}{n}. \tag{3.28}$$

Therefore, if **A** is diagonal or antidiagonal, then the first-order condition of the minimization problem is satisfied.

Since $f_i > 0$, the second-order condition represents a minimum:

$$\frac{\partial^2 L}{\partial f_i^2} = \frac{1}{f_i} > 0. \tag{3.29}$$

□

3.4. Conclusion

We provided a new mobility index that includes a requirement for equality of opportunity and demonstrated some desirable properties. The index enables us to assess social mobility from the equality of opportunity perspective. It is maximized when each element of the transition matrix is the same and is minimized when the transition matrix is diagonal or antidiagonal.

The form of the index, $\phi(\mathbf{A}) = \sum_{i=1}^{n} f_i \ln f_i$, can be recognized as that of entropy. Therefore, existing axiomatic characterizations (e.g., Shannon, 1948; Shannon and Weaver, [1949] 1998; von Neumann, 1955; Jaynes, 1957a,b; Rényi, 1961; Aczél et al., 1974) can be applied to justify it as long as we ignore the meaning of f_i, which is no longer a probability (of a transition matrix). For future research, meaningful axiomatic characterizations—particularly from a normative perspective of the connection between social mobility and equality of opportunity—are expected to further explore the benefits and limitations of this index.

Moreover, although we focused on pure exchange mobility, the distinction of exchange and structural mobility can be a problem for empirical applications. As Fields and Ok (1999, p. 589) mention,“the bistochasticity of the transition matrix ‘does not imply that the distribution is unchanging over time, and that analyses based on quantile transition matrices may confound exchange and structural mobility”’ (Atkinson et al., 1992, p. 15). Therefore, the decomposition into exchange and structural components is a remaining issue.

CHAPTER 4

Mobility measures for the responsibility cut

4.1. Introduction

In the current opportunity egalitarian paradigm, the distinction between the sources of inequality is significant, that is, we need to distinguish whether individuals should be responsible for the outcomes or not. This prominent theory of distributive justice stems from seminal works of Dworkin (1981a,b) which suppose that "we are responsible for the consequence of the choices we make out of those convictions or preferences or personality" (Dworkin, 2000, p. 7). Equality of opportunity is considered to be achieved when inequality that does not stem from individual responsibility is compensated for, and inequality due to individual responsibility is not compensated for.

Meanwhile, social mobility is often used as a proxy measure of equality of opportunity (e.g., Chetty et al., 2014a,b, 2017) as we have mentioned in Chapter 3. The importance of process over outcomes is argued such as in Stiglitz (1999) who states "(u)nequal outcomes that serve a social function, are arrived at fairly, or are a consequence of individual exercise of responsibility are more acceptable than those that are not" (p. 46). We can indeed take note of individual responsibility by focusing on process rather than outcomes; however, "there may be an ambiguity about the connection between social welfare, equality of opportunity and social mobility. The phrase 'equality of opportunity' is perhaps too easily used in the literature on social mobility" (Fleurbaey, 2008, p. 230). In fact, the process, or social mobility, includes both factors that individuals should be responsible for and should not be responsible for.

In this chapter, we consider the distinction of social mobility because "if social mobility is understood in terms of equality of opportunity, one should rely on a notion of social welfare that embodies basic principles of responsibility-sensitive egalitarianism" (Fleurbaey, 2008, p. 231). We propose an additive decomposability property of mobility by (stochastically) independent factors. In addition to sources of inequality, we argue that sources of mobility should be distinguished; thus, it is meaningful to construct mobility measures that deal with such distinctions by their additive decomposability property.

The remainder of this chapter is organized as follows. We present the definitions and preparatory results in Section 4.2. We provide the axioms and demonstrate the axiomatic characterizations in Section 4.3. We conclude in Section 4.4.

4.2. Definitions

We use $\mathbb{N}$ and $\mathbb{R}$ to denote the set of positive integers and the set of real numbers, respectively. The set of all positive (resp. nonzero) real numbers is $\mathbb{R}_{++} = \{x > 0 | x \in \mathbb{R}\}$ (resp. $\mathbb{R}^* = \{x \gtrless 0 | x \in \mathbb{R}\}$). As in Chapter 3, we describe mobility by the $n \times n$ doubly stochastic matrix for $n \in \mathbb{N}$.

The set of eigenvalues of a square matrix $\mathbf{A}$ is denoted by $\sigma(\mathbf{A})$. The bar notation $|\cdot|$ is used to denote absolute values. Transpose of a vector or a matrix is denoted by superscript $\mathbf{T}$. Kronecker product is indicated by $\otimes$; for example, for 2×2 matrices

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}, \tag{4.1}$$

Kronecker product of **A** and **B** is

$$\mathbf{A}\otimes\mathbf{B}\equiv\begin{bmatrix} a_{1,1}\mathbf{B} & a_{1,2}\mathbf{B} \\ a_{2,1}\mathbf{B} & a_{2,2}\mathbf{B} \end{bmatrix}=\begin{bmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{bmatrix}. \tag{4.2}$$

A *block matrix* is a matrix that is defined using smaller matrices, called blocks. For example,

$$\mathbf{A}=\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}, \tag{4.3}$$

where $A_{1,1}$, $A_{1,2}$, $A_{2,1}$, and $A_{2,2}$ are themselves matrices, is a block matrix. A matrix **A** of the form

$$\mathbf{A}=\begin{bmatrix} A_{1,1} & \star & \cdots & \star \\ & A_{2,2} & & \vdots \\ & & \ddots & \star \\ \mathbf{0} & & & A_{k,k} \end{bmatrix}, \tag{4.4}$$

where $i=1,\ldots,k$ and all blocks below the block diagonals are zero is a *block upper triangular*. A block upper triangular matrix in which all the diagonal blocks are 1×1 or 2×2 is said to be *upper quasitriangular*.

A square matrix **P** is a *permutation matrix* if exactly one element in each row and column is equal to 1 and all other elements are 0. A matrix **A** is *permutation equivalent* to **B** if there is a permutation matrix **P** such that $\mathbf{A}=\mathbf{P}^{\mathbf{T}}\mathbf{B}\mathbf{P}$. Moreover, **A** is called *reducible* if $\mathbf{P}^{\mathbf{T}}\mathbf{A}\mathbf{P}$ is an upper quasitriangular matrix; otherwise **A** is called *irreducible*.

4.3. Factor-decomposable mobility indices

4.3.1. Axioms. We introduce axioms for a mobility index that is a function $\phi : \mathscr{A} \to \mathbb{R}$, where $\mathscr{A}$ is a set of $n \times n$ nonnegative square matrices. The first axiom requires that if two transition matrices are mutually independent, then Kronecker product of them is the sum of each value of indices.

AXIOM 1 (Decomposability of independent factors). For transition matrices **A** and **B**, each of which are generated from two independent factors,

$$\phi(\mathbf{A} \otimes \mathbf{B}) = \phi(\mathbf{A}) + \phi(\mathbf{B}). \tag{4.5}$$

EXAMPLE 4.1. For example, two independent factors generate transition matrices **A** and **B**.

$$\mathbf{A} = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.6}$$

Kronecker product of **A** and **B** is

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 2/3 & 0 & 1/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \end{bmatrix}. \tag{4.7}$$

Suppose that we observe (4.7) as a mobility matrix, and it is a composition of **A** as a non-responsibility factor and **B** as a responsibility factor. It seems that “0” elements of (4.7) are undesirable, but they are indeed due to individual responsibility. Thus, we require the index value of (4.7) to be additively decomposed into two factors so that we can consider reducing opportunity inequality due to the non-responsibility factor based on the index values.

The second axiom requires the value of index is invariant to change of basis.

AXIOM 2 (Permutation equivalence). If transition matrices **A** and **B** are permutation equivalent; that is, for permutation matrices **P**,

$$\mathbf{A} = \mathbf{P}^{\mathbf{T}}\mathbf{B}\mathbf{P}, \tag{4.8}$$

then

$$\phi(\mathbf{A}) = \phi(\mathbf{B}). \tag{4.9}$$

EXAMPLE 4.2. $\phi(\mathbf{A}) = \phi(\mathbf{B})$ holds for transition matrices **A** and **B**,

$$\mathbf{A} = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 0 & 2/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1/3 & 2/3 & 0 \\ 2/3 & 0 & 1/3 \\ 0 & 1/3 & 2/3 \end{bmatrix}, \tag{4.10}$$

where

$$\mathbf{P}^{\mathbf{T}}\mathbf{B}\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 0 \\ 2/3 & 0 & 1/3 \\ 0 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 0 & 2/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = \mathbf{A}. \tag{4.11}$$

Suppose the rows (columns) 1, 2, 3 be the first, second, and third class of income, respectively. On the one hand, the transition matrix **A** describes the situation where 2/3 of the first class will be the same and 1/3 will be the second class in the next period; **B** describes the situation where 1/3 of the first class will be the same and 2/3 will be the second class in the next period. In this respect, **B** may be better because it is more mobile. On the other hand, **A** describes the situation where 1/3 of the third class is persistent and 2/3 will be the second class in the next period; **B** describes the situation where 2/3 of third class is persistent and 1/3 will be the second class in the next period. This time, **A** is more mobile. That is, there is a trade-off, a kind of symmetric situation, between **A** and **B** because they are only "permuted." The index values should be complete order, and we therefore treat them the same.

The third axiom requires the index values to be the same in the "no mobility" situations.

AXIOM 3 (Symmetry). The index values of $n \times n$ matrices that are permutation matrices are the same.

EXAMPLE 4.3. For transition matrices **A**, **B**, and **C**,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad (4.12)$$

$\phi(\mathbf{A}) = \phi(\mathbf{B}) = \phi(\mathbf{C})$.

Note that the axioms *Permutation equivalence* and *Symmetry* are independent. If **A** and **B** satisfy *Permutation equivalence*, then they are similar matrices, that is, they have the same properties such as eigenvalues, trace, determinant, etc. However, any different permutation matrices share such properties; for example, it is easy to confirm that **A**, **B**, and **C** in (4.12) have different traces, $\text{tr}(\mathbf{A}) = 3$, $\text{tr}(\mathbf{B}) = 0$, and $\text{tr}(\mathbf{C}) = 1$. Therefore, different permutation matrices are not permutation equivalent.

The fourth axiom requires the index is a continuous function of elements of a transition matrix.

AXIOM 4 (Continuity). $\phi(\mathbf{A})$ is a continuous function of $a_{i,j} \in \mathbf{A}$.

The fifth axiom requires the value of index is maximized when individuals have equal probabilities.

AXIOM 5 (Equalization of life chances). The index value is maximized when all elements of the $n \times n$ transition matrix $\mathbf{A}$ are the same; that is, $a_{i,j} = 1/n$.

Equalization of life chances is introduced by Van de gaer et al. (2001). In chapter 3, following Shorrocks (1978b), we consider it as a requirement of equality of opportunity to resolve the incompatibility between greater mobility and equality of opportunity (cf. Kanbur and Stiglitz, 2016). Our purpose in this chapter, however, is to incorporate responsibility-sensitive egalitarianism into mobility evaluation for measuring (in)equality of opportunity. Hence, this is just one of the possible desirable properties for mobility indices.

The sixth and seventh axioms require that the index to be constant regardless of dimensions of transition matrices.

AXIOM 6 (Minimum invariance to dimensions). The minimum value of the index of $n \times n$ matrix is constant for any $n \in \mathbb{N}$

AXIOM 7 (Maximum invariance to dimensions). The maximum value of the index of $n \times n$ matrix is constant for any $n \in \mathbb{N}$

These invariance properties to dimensions require the values of mobility indices be identical if the actual mobility in a society is the same regardless of how to make categories at least at the maximum or minimum.

The last eighth axiom requires indices to be the same value if the mobility inside a block is the same as the mobility inside another block. We can evaluate and compare some parts of a transition matrix in the same standard.

AXIOM 8 (Block consistency). For blocks of $\mathbf{A} \otimes \mathbf{B}$,

$$\phi\left(a_{1,1} \cdot \mathbf{B}\right) = \cdots = \phi\left(a_{i,j} \cdot \mathbf{B}\right) = \cdots = \phi\left(a_{n,n} \cdot \mathbf{B}\right), \tag{4.13}$$

where $a_{i,j} \in \mathbf{A}$ for all $i = 1, ..., n$, and $j = 1, ..., n$.

EXAMPLE 4.4. We reproduce (4.12) with partitions:

$$\mathbf{A} \otimes \mathbf{B} = \left[\begin{array}{cc|cc} 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ \hline 2/3 & 0 & 1/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \end{array}\right]. \tag{4.14}$$

The mobility inside each block is the same as $\mathbf{B}$, where

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.15}$$

Hence, the values of each block should be the same.

4.3.2. Fundamental results. We derive mobility indices, each of which is a function $\phi : \mathscr{A} \to \mathbb{R}$, where $\mathscr{A}$ is a set of $n \times n$ nonnegative square matrices.

THEOREM 4.1. *If the mobility index satisfies Decomposability of independent factors, Permutation equivalence, Symmetry, and Continuity, then it is represented by a function* $\phi : \mathscr{A} \to \mathbb{R}$ *such that*

$$K \log\left(\sum_{i=1}^{n} |\lambda_i|^{\alpha}\right), \tag{4.16}$$

where $\lambda_i \in \sigma(\mathbf{A})$, $\mathbf{A} \in \mathscr{A}$, *and* $K, \alpha \in \mathbb{R}$.

LEMMA 4.1. *If the index satisfies Permutation equivalence, then it is represented by a function* $\varphi : \sigma(\cdot) \to \mathbb{R}$ *such that*

$$\phi(\mathbf{A}) = \varphi(\sigma(\mathbf{A})). \tag{4.17}$$

PROOF. By *Permutation equivalence*, for

$$\mathbf{A} = \mathbf{P}^{\mathbf{T}}\mathbf{B}\mathbf{P}, \tag{4.18}$$

$$\phi(\mathbf{A}) = \phi(\mathbf{B}). \tag{4.19}$$

For a doubly stochastic matrix **B**, the following real Shur form exists:[1]

$$\mathbf{B} = \mathbf{S}^{-1}\mathbf{U}\mathbf{S}, \tag{4.20}$$

where **S** is a nonsingular matrix, and **U** is an upper quasitriangular matrix

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_3 \\ \mathbf{0} & \mathbf{U}_2 \end{bmatrix}. \tag{4.21}$$

Since

$$\det[\mathbf{U} - \lambda_i \mathbf{I}] = \det[\mathbf{U}_1 - \lambda_i \mathbf{I}] \cdot \det[\mathbf{U}_2 - \lambda_i \mathbf{I}], \tag{4.22}$$

eigenvalues of **U** are given by the following.[2]

$$\sigma(\mathbf{U}) = \sigma(\mathbf{U}_1) \cup \sigma(\mathbf{U}_2). \tag{4.23}$$

We consider the cases when (i) **B** is reducible and (ii) **B** is irreducible.

[1]See, for example, Horn and Johnson (2012, p. 103, Theorem 2.3.4).

[2]See Marcus and Minc (1964, p. 23, 2.15.1) and Silvester (2000).

(i) When $\mathbf{B}$ is reducible, $\mathbf{P^T B P} = \mathbf{U}$, or

$$\mathbf{B} = \mathbf{P U P^T}. \tag{4.24}$$

Comparing (4.20) and (4.24), we have $\mathbf{S}^{-1} = \mathbf{P}$ and $\mathbf{S} = \mathbf{P^T}$. Since permutation matrices $\mathbf{P}$ and $\mathbf{P^T}$ are orthogonal, by substituting (4.24) into (4.18),

$$\mathbf{A} = \mathbf{P^T B P} = \mathbf{P P^T U P^T P} = \mathbf{U}, \tag{4.25}$$

and by (4.19) and (4.25), the following equation holds:

$$\phi(\mathbf{A}) = \phi(\mathbf{U}) = \phi(\mathbf{B}). \tag{4.26}$$

Now, for the Shur form (4.20), $\mathbf{B} = \mathbf{S}^{-1}\mathbf{U}\mathbf{S}$, the "diagonal blocks [of $\mathbf{U}$] are completely determined by the eigenvalues [of $\mathbf{B}$]" (Horn and Johnson, 2012, p. 103). Moreover, by Lemma 3 of Perfect and Mirsky (1965),[3] if a doubly stochastic matrix $\mathbf{B}$ is reducible, then $\mathbf{P^T B P}(=\mathbf{U})$ is a direct sum of doubly stochastic matrices; that is,

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_3 \\ \mathbf{0} & \mathbf{U}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{U}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_2 \end{bmatrix}. \tag{4.27}$$

Thus, the matrix $\mathbf{U}$, which is a direct sum of $\mathbf{U}_1$ and $\mathbf{U}_2$, is determined by the eigenvalues of $\mathbf{B}$. This implies that there exists a function φ such that

$$\phi(\mathbf{U}) = \varphi(\sigma(\mathbf{B})). \tag{4.28}$$

[3]See also Marcus and Minc (1964, p. 123, 5.3.1) for reducible matrix.

Also, by (4.23), the set of eigenvalues of $\mathbf{U}$ is the same as those of $\mathbf{B}$:

$$\sigma(\mathbf{U}) = \sigma(\mathbf{B}); \tag{4.29}$$

thus, we have

$$\phi(\mathbf{U}) = \varphi(\sigma(\mathbf{U})) = \varphi(\sigma(\mathbf{B})). \tag{4.30}$$

By (4.26),

$$\phi(\mathbf{A}) = \phi(\mathbf{U}) = \varphi(\sigma(\mathbf{U})) = \varphi(\sigma(\mathbf{B})) = \phi(\mathbf{B}). \tag{4.31}$$

Since $\mathbf{A}$ and $\mathbf{B}$ are permutation equivalent, $\sigma(\mathbf{A}) = \sigma(\mathbf{B})$.[4] Hence,

$$\phi(\mathbf{A}) = \varphi(\sigma(\mathbf{A})) = \varphi(\sigma(\mathbf{B})) = \phi(\mathbf{B}); \tag{4.32}$$

that is, when $\mathbf{B}$ is reducible, we must have a function φ, as the index, such that $\varphi : \sigma(\mathbf{A}) \to \mathbb{R}$.

(ii) When $\mathbf{A}$ is irreducible, $\sigma(\mathbf{A}) = \sigma(\mathbf{B})$ also holds because $\mathbf{A}$ and $\mathbf{B}$ are permutation equivalent. Hence, the following equation also holds:

$$\phi(\mathbf{A}) = \varphi(\sigma(\mathbf{A})) = \varphi(\sigma(\mathbf{B})) = \phi(\mathbf{B}). \tag{4.33}$$

In general, $\mathbf{A}$ can be reducible or irreducible; therefore, if ϕ satisfies *Permutation equivalence*, then there exists a function $\varphi : \sigma(\cdot) \to \mathbb{R}$ such that

$$\phi(\mathbf{A}) = \varphi(\sigma(\mathbf{A})). \tag{4.34}$$

[4] See, for example, Horn and Johnson (2012, p. 58, Corollary 1.3.4 (a)).

□

LEMMA 4.2. *If a function* $\varphi : \sigma(\cdot) \to \mathbb{R}$ *satisfies Decomposability of independent factors and Continuity, then it is represented by*

$$\varphi(\lambda_1, ..., \lambda_n) = K \log\left(\left|\sum_{i=1}^{n} \lambda_i^{\alpha}\right|\right), \tag{4.35}$$

where $\lambda_i \in \sigma(\mathbf{A})$ *and* $\sum_{i=1}^{n} \lambda_i^{\alpha} \in \mathbb{R}^*$.

PROOF. By *Decomposability of independent factors*,

$$\varphi(\sigma(\mathbf{A} \otimes \mathbf{B})) = \varphi(\sigma(\mathbf{A})) + \varphi(\sigma(\mathbf{B})). \tag{4.36}$$

First, we consider $\mathbb{R}^n$ as a product of lower product spaces:

$$\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q, \tag{4.37}$$

where $p \in \mathbb{N}$, $q \in \mathbb{N}$, and $p + q = n$. Every $\mathbf{x} \in \mathbb{R}^n$ can be represented as $\mathbf{x} = (\mathbf{x}_p, \mathbf{x}_q)$, with $\mathbf{x}_p \in \mathbb{R}^p$, $\mathbf{x}_q \in \mathbb{R}^q$, and if $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y} = (\mathbf{y}_p, \mathbf{y}_q)$, $\mathbf{y}_p \in \mathbb{R}^p$, $\mathbf{y}_p \in \mathbb{R}^p$, then

$$\mathbf{x} + \mathbf{y} = (\mathbf{x}_p, \mathbf{x}_q) + (\mathbf{y}_p, \mathbf{y}_q) = (\mathbf{x}_p + \mathbf{y}_p, \mathbf{x}_q + \mathbf{y}_q). \tag{4.38}$$

By Theorem 5.5.1 of Kuczma (2009, p. 138–139), if $\xi : \mathbb{R}^n \to \mathbb{R}$ is an additive function and $\mathbb{R}^n$ has decomposition (4.37), then there exist additive functions $\xi_p : \mathbb{R}^p \to \mathbb{R}$ and $\xi_q : \mathbb{R}^q \to \mathbb{R}$ such that[5]

$$\xi(\mathbf{x}) = \xi(\mathbf{x}_p, \mathbf{x}_q) = \xi_p(\mathbf{x}_p) + \xi_q(\mathbf{x}_q). \tag{4.40}$$

[5]For example, for $\mathbf{x}_p = (p_1, p_2, p_3)$, $\mathbf{x}_q = (q_1, q_2)$,

$$\xi(p_1, p_2, p_3, q_1, q_2) = p_1 + p_2 + p_3 + q_1 + q_2 = \xi_p(p_1, p_2, p_3) + \xi_q(q_1, q_2). \tag{4.39}$$

Put $\zeta_p(\mathbf{x}_p) = \log \xi_p(\mathbf{x}_p)$ and $\zeta_q(\mathbf{x}_p) = \log \xi_q(\mathbf{x}_p)$, then

$$\zeta_p(\mathbf{x}_p) + \zeta_q(\mathbf{x}_q) = \log \xi_p(\mathbf{x}_p) + \log \xi_q(\mathbf{x}_q) \tag{4.41}$$

$$= \log(\xi_p(\mathbf{x}_p) \times \xi_q(\mathbf{x}_q)). \tag{4.42}$$

Since ξ_p and ξ_q are additive functions,

$$\log(\xi_p(\mathbf{x}_p) \times \xi_q(\mathbf{x}_q)) = \log \xi(\mathbf{x}_p \otimes \mathbf{x}_q) \tag{4.43}$$

$$= \zeta(\mathbf{x}_p \otimes \mathbf{x}_q), \tag{4.44}$$

where $\zeta : \mathbb{R}^{p \times q} \to \mathbb{R}$, which is a composition of logarithmic function and additive function.[6] Summarizing the above, we have the following claim.

CLAIM 1. *If a function $\xi : \mathbb{R}^{p+q} \to \mathbb{R}$ is additive, then $\zeta : \mathbb{R}^{p \times q} \to \mathbb{R}$ is a composite function of logarithmic and additive functions.*

Now, since eigenvalues of $\mathbf{A} \otimes \mathbf{B}$ are $\lambda_i \mu_j$,[7] we have the following functional equation (cf. Aczél, 1966).[8]

$$F_{nm}(\lambda_1\mu_1, \lambda_1\mu_2, \lambda_2\mu_1, \lambda_2\mu_2, ..., \lambda_n\mu_m) = F_n(\lambda_1, ..., \lambda_n) + F_m(\mu_1, ..., \mu_m), \tag{4.49}$$

where $F_{nm} : \mathbb{R}^{n \times m} \to \mathbb{R}$, $F_n : \mathbb{R}^n \to \mathbb{R}$, and $F_m : \mathbb{R}^m \to \mathbb{R}$.

---

[6] For example, for $\mathbf{x}_p = (p_1, p_2, p_3)$, $\mathbf{x}_q = (q_1, q_2)$,

$$\zeta_p(p_1, p_2, p_3) + \zeta_q(q_1, q_2) = \log(p_1 + p_2 + p_3) + \log(q_1 + q_2) \tag{4.45}$$

$$= \log((p_1 + p_2 + p_3) \times (q_1 + q_2)) \tag{4.46}$$

$$= \log(p_1q_1 + p_1q_2 + p_2q_1 + p_2q_2 + p_3q_1 + p_3q_2) \tag{4.47}$$

$$= \zeta(p_1q_1, p_1q_2, p_2q_1, p_2q_2, p_3q_1, p_3q_2). \tag{4.48}$$

[7] See, for example, Mac Duffee (1933, p. 84, Corollary 43.81), Marcus and Minc (1964, p. 24, 2.15.11), and Horn and Johnson (1991, p. 245, Theorem 4.2.12).

[8] The equation (4.49) is not recognized as one of the Cauchy's functional equations, "because of the operation of multiplication occurring in the argument" (Kuczma, 2009, p. 343). Lemma 4.2 can be regarded as an extension of Theorem 5.5.1, 5.5.2, 13.1.2, and 13.1.5 of Kuczma (2009, pp. 139–140, 344, 348).

Moreover, for $\mathbf{t}_{nm} = (t_1, ..., t_{nm})$, put $G(\mathbf{t}_{nm}) = F(e^{\mathbf{t}_{nm}})$. We have by (4.49) for arbitrary $\mathbf{u}_n = (u_1, ..., u_n)$ and $\mathbf{v}_m = (v_1, ..., v_m)$,

$$G_n(\mathbf{u}_n) + G_m(\mathbf{v}_m) = F_n(e^{\mathbf{u}_n}) + F_m(e^{\mathbf{v}_m}) \tag{4.50}$$

$$= F_{nm}(e^{\mathbf{u}_n} e^{\mathbf{v}_m}) \tag{4.51}$$

$$= F_{nm}(e^{\mathbf{u}_n + \mathbf{v}_m}) \tag{4.52}$$

$$= G_{nm}(\mathbf{u}_n + \mathbf{v}_m). \tag{4.53}$$

That is, G_{nm} is additive. Here, G_{nm} corresponds to ξ, and F_{nm} corresponds to ζ in the previous argument. Hence, by Claim 1, F_{nm} is a composite function of logarithmic and additive functions, and for $\sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j > 0$, $\sum_{i=1}^{n} \lambda_i > 0$ and $\sum_{j=1}^{m} \mu_j > 0$,

$$KF_{nm}\left(\sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_i^{\alpha} \mu_j^{\alpha}\right) = KF_n\left(\sum_{i=1}^{n} \lambda_i^{\alpha}\right) + KF_m\left(\sum_{j=1}^{m} \mu_j^{\alpha}\right). \tag{4.54}$$

Furthermore, for $\sum_{i=1}^{n} \lambda_i = 1$,

$$F_n(1) = \log(1) = 0. \tag{4.55}$$

For $\sum_{i=1}^{n} \lambda_i = \sum_{j=1}^{m} \mu_j - 1$, by (4.49),

$$\phi(1) = \phi(-1) + \phi(-1) = -2\phi(-1). \tag{4.56}$$

Thus,

$$\phi(-1) = F_n(-1) = F_m(-1) = 0. \tag{4.57}$$

For $\sum_{i=1}^{n} \lambda_i < 0$ and $\sum_{j=1}^{m} \mu_j = -1$, by (4.54) and (4.57),

$$KF_{nm}\left(\left|\sum_{i=1}^{n} \lambda_i^{\alpha}\right|\right) = KF_{nm}\left(-\sum_{i=1}^{n} \lambda_i^{\alpha}\right) \tag{4.58}$$

$$= KF_n\left(\sum_{i=1}^{n} \lambda_i^{\alpha}\right) + F_m(-1) \tag{4.59}$$

$$= F_n\left(\sum_{i=1}^{n} \lambda_i\right). \tag{4.60}$$

We obtain, for $\sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_i \mu_j \in \mathbb{R}^*$, $\sum_{i=1}^{n} \lambda_i \in \mathbb{R}^*$ and $\sum_{j=1}^{m} \mu_j \in \mathbb{R}^*$,

$$KF_{nm}\left(\left|\sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_i^{\alpha}\mu_j^{\alpha}\right|\right) = KF_n\left(\left|\sum_{i=1}^{n} \lambda_i^{\alpha}\right|\right) + KF_m\left(\left|\sum_{j=1}^{m} \mu_j^{\alpha}\right|\right). \tag{4.61}$$

Continuity yields, by Theorem 5.5.2 of Kuczma (2009, p. 139),

$$KF_{nm} = K\log\left(\left|\sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_i^{\alpha}\mu_j^{\alpha}\right|\right), \tag{4.62}$$

$$KF_n = K\log\left(\left|\sum_{i=1}^{n} \lambda_i^{\alpha}\right|\right), \tag{4.63}$$

$$KF_m = K\log\left(\left|\sum_{j=1}^{m} \mu_j^{\alpha}\right|\right). \tag{4.64}$$

□

PROOF OF THEOREM 4.1. By Lemma 4.1 and 4.2, for $\sum_{i=1}^{n} \lambda_i \in \mathbb{R}^*$ and $\sum_{j=1}^{m} \mu_j \in \mathbb{R}^*$, there exist additive function $F : \mathbb{R}^{n\times m} \to \mathbb{R}$ such that

$$K\log(|\lambda_1^{\alpha}\mu_1^{\alpha} + \lambda_1^{\alpha}\mu_2^{\alpha} + \lambda_2^{\alpha}\mu_1^{\alpha} + \cdots + \lambda_n^{\alpha}\mu_m^{\alpha}|) = K\log(|\lambda_1^{\alpha} + \cdots + \lambda_n^{\alpha}|) + K\log(|\mu_1^{\alpha} + \cdots \mu_m^{\alpha}|). \tag{4.65}$$

By *Symmetry*, the indices should be constant regardless of the sign of each eigenvalues; thus, we have

$$K \log(|\lambda_1|^{\alpha} + \cdots + |\lambda_n|^{\alpha}), \tag{4.66}$$

for any $|\lambda_i|^{\alpha} \in \mathbb{R}$. □

THEOREM 4.2. *If the mobility index is represented by (4.16), then it satisfies (i) Decomposability of independent factors, (ii) Permutation equivalence, (iii) Symmetry, and (iv) Continuity.*

PROOF OF THEOREM 4.2. We show that the index (4.16) satisfies each axiom.

(i) Let $\mathbf{A}$ and $\mathbf{B}$ be two matrices where $\lambda_i \in \sigma(\mathbf{A})$ and $\mu_j \in \sigma(\mathbf{B})$. Kronecker product of $\mathbf{A}$ and $\mathbf{B}$ is $\lambda_i \mu_j$.

(ii) Since $\mathbf{A}$ is permutation equivalent to $\mathbf{B}$, we have $\sigma(\mathbf{A}) = \sigma(\mathbf{P}^{\mathbf{T}}\mathbf{B}\mathbf{P})$.[9] Thus, $\phi(\mathbf{A}) = \phi(\mathbf{P}^{\mathbf{T}}\mathbf{B}\mathbf{P})$.

(iii) For any diagonal and antidiagonal matrix, the absolute values of the eigenvalues are 1.[10] Hence, the index values of diagonal and antidiagonal matrices with the same dimensions are the same.

(iv) By "the facts that the (complex) roots of a polynomial depend continuously on the coefficients of the polynomial and that the eigenvalues of a matrix depend continuously on the entries of the matrix" (Uherka and Sergott, 1977), λ_i is continuous on $a_{i,j} \in \mathbf{A}$.[11]

[9] See, for example, Horn and Johnson (2012, p. 58, Corollary 1.3.4 (a)).

[10] See Lemma 3.1 (i) (ii).

[11] See, for example, Takagi (1930, pp. 56–57, Theorem 2.7) and Franklin (1968, p. 191–192, Theorem 1) for the proof using Rouché's theorem (e.g., Takagi, 1930, pp. 55–56, Theorem 2.6), and see also Horn and Johnson (2012, p. 122, Theorem 2.4.9.2) for the proof using Shur's unitary triangularization theorem. Uherka and Sergott (1977) provide an elementary proof.

The continuity of ϕ follows from the continuity of logarithmic functions. □

THEOREM 4.3. *For the mobility index represented by (4.16), (i) if $K < 0$, then it satisfies Equalization of life chances and Maximum invariance to dimensions; (ii) if $K > 0$, then it satisfies Minimum invariance to dimensions.*

PROOF OF THEOREM 4.3. We show that the index (4.16) satisfies each statement.

(i) If, and only if, all elements of the transition matrix are the same, the only one eigenvalue is 1 and the other eigenvalues are 0,[12] and the index value is $K \log 1 = 0$. In other cases, since $K < 0$, the index value is negative; thus, it satisfies *Equalization of life chances*. Moreover, the maximum value is always 0 for any dimensions; hence, it satisfies *Maximum invariance to dimensions*.

(ii) Since $K > 0$, the index value is always nonnegative, and the minimum value is 0; thus, it satisfies *Minimum invariance to dimensions*. □

THEOREM 4.4. *If the mobility index satisfies Decomposability of independent factors, Permutation equivalence, Symmetry, Continuity, and Maximum (resp. Minimum) invariance to dimensions, then it is represented by a function $\phi : \mathscr{A} \to \mathbb{R}$ such that*

$$K \log \left(\sum_{i=1}^{n} |\lambda_i|^{\alpha} \right), \tag{4.67}$$

where $\lambda_i \in \sigma(\mathbf{A})$, $\mathbf{A} \in \mathscr{A}$, $K < 0$ (resp. $K > 0$), and $\alpha \in \mathbb{R}$.

[12]For necessity, see Lemma 3.1 (iii), and sufficiency follows from the continuity of the eigenvalues on the elements of the matrix.

PROOF OF THEOREM 4.4. By Theorem 4.1, the functional form (4.16) is derived from the first four axioms. Since, doubly stochastic matrix always has 1 as eigenvalue,[13] $|\lambda_1|^\alpha + \cdots + |\lambda_n|^\alpha \geq 1$. Moreover, if, and only if, all elements of the transition matrix are the same, then the only one eigenvalue is 1 and the other eigenvalues are 0. *Equalization of life chances* requires this case to be maximum, and *Maximum invariance to dimensions* requires the index value of this case is constant regardless of n. Equation (4.16) yields $K \log 1 = 0$ in this case, $\log(|\lambda_1|^\alpha + \cdots + |\lambda_n|^\alpha) > 0$. Therefore, we have the index as the equation (4.1) with $K < 0$. □

COROLLARY 4.1. *Decomposability of independent factors, Permutation equivalence, Symmetry, Continuity implies Equalization of life chances.*

PROOF OF COROLLARY 4.1. It is obvious from Theorem 4.3 and 4.4. □

From the results of this section, we have the following factor-decomposable mobility index.

DEFINITION 4.1 (index I).

$$K \log(|\lambda_1|^\alpha + \cdots + |\lambda_n|^\alpha), \tag{4.68}$$

where $\alpha, K \in \mathbb{R}$. We refer to it as "index I $(-)$" if $K < 0$ and "index I $(+)$" if $K > 0$.

4.3.3. Incompatibility. We provide a difficulty in our requirements, or axioms.

[13]See Marcus and Minc (1964, p. 133, 5.13.1) and Gantmacher (1959, p. 100).

THEOREM 4.5. *Maximum invariance to dimensions and Minimum invariance to dimensions are incompatible for the index satisfying Decomposability of independent factors, Permutation equivalence, Symmetry, and Continuity.*

PROOF OF THEOREM 4.5. By Theorem 4.3, if $K < 0$, the index (4.16) satisfies *Maximum invariance to dimensions*, and if $K > 0$ it satisfies *Minimum invariance to dimensions*. Since $K \neq 0$ and cannot be both negative and positive, the theorem follows. □

Due to the incompatibility between *Maximum invariance to dimensions* and *Minimum invariance to dimensions*, we provide an index as a counterpart of index I $(-)$.

THEOREM 4.6. *If, and only if, the mobility index satisfies (i) Decomposability of independent factors, (ii) Permutation equivalence, (iii) Symmetry, (iv) Continuity, (v) Equalization of life chances, and (vi) Minimum invariance to dimensions, then it is represented by a function* $\phi : \mathscr{A} \to \mathbb{R}$ *such that*

$$K \log \left(\frac{n}{|\lambda_1|^{\alpha} + \cdots + |\lambda_n|^{\alpha}} \right), \tag{4.69}$$

where $\lambda_i \in \sigma(\mathbf{A})$, $\mathbf{A} \in \mathscr{A}$, $\alpha \in \mathbb{R}$, *and* $K > 0$.

PROOF OF THEOREM 4.6 (SUFFICIENCY). *Equalization of life chances* requires the index value is maximum when all elements of the transition matrix are the same; that is, the only one eigenvalue is 1 and the other eigenvalues are 0. If, and only if, the transition matrix is diagonal or antidiagonal, all absolute values of eigenvalues are 1.[14] Since $|\lambda_i| \leq 1$ for doubly

[14]For necessity, see Lemma 3.1 (i) (ii), and sufficiency follows from the continuity of the eigenvalues on the elements of the matrix.

stochastic matrix,[15] and *Minimum invariance to dimensions* requires that $K\log(|\lambda_1|^\alpha+\cdots+|\lambda_n|^\alpha)=K\log(1+\cdots+1)=K\log(n)$ be minimum and constant with respect to n. We consider the following operation.

By Theorem 4.1, we have equation (4.16) and

$$K\log\left(|\lambda_1\mu_1|+|\lambda_1\mu_2|+|\lambda_2\mu_1|+\cdots+|\lambda_n\mu_m|\right)=K\log(|\lambda_1|+\cdots+|\lambda_n|)+K\log(|\mu_1|+\cdots|\mu_m|). \tag{4.70}$$

Since,

$$K\log\left(nm\right)=K\log\left(n\right)+K\log\left(m\right), \tag{4.71}$$

Subtracting (4.70) from (4.71) each side, we have

$$K\log\left(\frac{nm}{|\lambda_1\mu_1|^\alpha+|\lambda_1\mu_2|^\alpha+|\lambda_2\mu_1|^\alpha+\cdots+|\lambda_n\mu_m|^\alpha}\right)=K\log\left(\frac{n}{|\lambda_1|^\alpha+\cdots+|\lambda_n|^\alpha}\right)+K\log\left(\frac{m}{|\mu_1|^\alpha+\cdots|\mu_m|^\alpha}\right), \tag{4.72}$$

and

$$K\log\left(\frac{n}{|\lambda_1|^\alpha+\cdots+|\lambda_n|^\alpha}\right) \tag{4.73}$$

as an index.

[15] See, for example, Marcus and Minc (1964, p. 133, 5.13.3) and Gantmacher (1959, p. 100).

When the transition matrix is diagonal or antidiagonal, or all absolute values of eigenvalues are 1, we have $\log 1 = 0$, Meanwhile, when all elements of the transition matrix are the same, or the only one eigenvalue is 1 and the other eigenvalues are 0, we have $\log(n/1) > 0$. Therefore, $K > 0$. □

PROOF OF THEOREM 4.6 (NECESSITY). We show that the index (4.74) satisfies each axiom.

(i) It follows from (4.72).

(ii) The same argument as Theorem 4.2 (ii) applies.

(iii) The same argument as Theorem 4.2 (iii) applies.

(iv) By Theorem 4.2 (iv) eigenvalues of a matrix is continuous on the elements of the matrix. Since n is constant, λ_i/n is also continuous. The continuity of ϕ follows from the continuity of logarithmic functions.

(v) When all elements of the transition matrix are the same, then the only one eigenvalue is 1 and the other eigenvalues are 0, we have $K\log(n/(|\lambda_1|^\alpha + \cdots + |\lambda_n|^\alpha)) = K\log(n/1)$. In other cases, $|\lambda_1|^\alpha + \cdots + |\lambda_n|^\alpha > 1$; therefore, we have maximum value under the given condition.

(vi) Since $|\lambda_i| \leq 1$, $K\log(|\lambda_1|^\alpha + \cdots + |\lambda_n|^\alpha/n) = K\log(n/n) = K\log 1 = 0$ is the minimum value which is constant regardless of n. □

DEFINITION 4.2 (index II).

$$K\log\left(\frac{n}{|\lambda_1|^\alpha + \cdots + |\lambda_n|^\alpha}\right), \tag{4.74}$$

where $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $K \in \mathbb{R}$, and $K > 0$.

4.3.4. Entropic mobility indices. We demonstrate that we can derive the entropic mobility index introduced in Chapter 3 by adding Axiom 8 (Bock consistency) as a property of mobility indices to the previous argument and, as a result, provide its axiomatic characterization. Because of the incompatibility between *Maximum invariance to dimensions* and *Minimum invariance to dimensions* which is shown in Theorem 4.5, we provide an index that is a counterpart of the entropic mobility index given in Chapter 3 and its intuitive illustration.

THEOREM 4.7. *If, and only if, the mobility index satisfies (i) Decomposability of independent factors, (ii) Permutation equivalence, (iii) Symmetry, (iv) Continuity, (v) Maximum invariance to dimensions, and (vi) Block consistency, then it is represented by a function* $\phi : \mathscr{A} \to \mathbb{R}$ *such that*

$$s \sum_{i=1}^{n} f_i \log f_i, \tag{4.75}$$

where

$$f_i = \frac{|\lambda_i|^{\alpha}}{\sum_{i=1}^{n} |\lambda_i|^{\alpha}} \tag{4.76}$$

and $\lambda_i \in \sigma(\mathbf{A})$, $\mathbf{A} \in \mathscr{A}$, $\alpha \in \mathbb{R}$, $s \in \mathbb{R}_{++}$, *with the convention*

$$0 \log 0 = 0. \tag{4.77}$$

PROOF OF THEOREM 4.7 (SUFFICIENCY). By Theorem 4.4, *Decomposability of independent factors*, *Permutation equivalence*, *Symmetry*, *Continuity*, and *Maximum invariance to dimensions* imply Index I (−):

$$K \log(|\lambda_1|^{\alpha} + \cdots + |\lambda_n|^{\alpha}), \tag{4.78}$$

where $K < 0$, which satisfies *Equalization of life chances*.

By *Block consistency*, the index values are constant even if elements of matrices become smaller at a certain rate. For a $n \times n$ transition matrix $\mathbf{B}$, the characteristic polynomial is $\det[\mathbf{B} - \lambda I]$, while the characteristic polynomial of $a_{i,j} \cdot \mathbf{B}$ is $\det[a_{i,j} \cdot \mathbf{B} - \mu I] = (a_{i,j})^n \det[\mathbf{B} - (\mu/a_{i,j})I]$. Thus, the eigenvalues of $a_{i,j} \cdot \mathbf{B}$ are $a_{i,j}$ times the eigenvalues of $\mathbf{B}$.

Thus, we consider a proportion of the absolute value of eigenvalues to the sum of them, that is, we set

$$f_i = \frac{|\lambda_i|^\alpha}{\sum_{i=1}^n |\lambda_i|^\alpha}. \tag{4.79}$$

The index using f_i can be deduced by the following operation. By *Decomposability of independent factors*,

$$\begin{aligned} K\log\left(|\lambda_1\mu_1|^\alpha + |\lambda_1\mu_2|^\alpha + |\lambda_2\mu_1|^\alpha + \cdots + |\lambda_n\mu_m|^\alpha\right) =& K\log(|\lambda_1|^\alpha + \cdots + |\lambda_n|^\alpha) \\ &+ K\log(|\mu_1|^\alpha + \cdots |\mu_m|^\alpha). \end{aligned} \tag{4.80}$$

We consider a single variable case of (4.80):

$$K\log\left(|\lambda_i\mu_j|^\alpha\right) = K\log(|\lambda_i|^\alpha) + K\log(|\mu_j|^\alpha). \tag{4.81}$$

Subtracting (4.80) from (4.81) each side, the following equation holds.

$$\begin{aligned} &K\log\left(|\lambda_i\mu_j|^\alpha\right) - K\log\left(|\lambda_1\mu_1|^\alpha + |\lambda_1\mu_2|^\alpha + |\lambda_2\mu_1|^\alpha + \cdots + |\lambda_n\mu_m|^\alpha\right) \\ &= K\log(|\lambda_i|^\alpha) - K\log(|\lambda_1|^\alpha + \cdots + |\lambda_n|^\alpha) + K\log(|\mu_j|^\alpha) - K\log(|\mu_1|^\alpha + \cdots |\mu_m|^\alpha). \end{aligned} \tag{4.82}$$

For each λ_i, μ_j, we have

$$K\log\left(\frac{|\lambda_i\mu_j|^\alpha}{|\lambda_1\mu_1|^\alpha+|\lambda_1\mu_2|^\alpha+|\lambda_2\mu_1|^\alpha+\cdots+|\lambda_n\mu_m|^\alpha}\right)$$

$$= K\log\left(\frac{|\lambda_i|^\alpha}{|\lambda_1|^\alpha+\cdots+|\lambda_n|^\alpha}\right)+K\log\left(\frac{|\mu_j|^\alpha}{|\mu_1|^\alpha+\cdots|\mu_m|^\alpha}\right). \tag{4.83}$$

For each f_i, g_j,

$$K\log\left(f_i g_j\right) = K\log f_i + K\log g_j. \tag{4.84}$$

Since

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\log\left(f_i g_j\right) \neq \sum_{i=1}^{n}\log f_i + \sum_{j=1}^{m}\log g_j \tag{4.85}$$

in general, we find an appropriate constant K and derive a function ψ : $(f_1, f_2, ..., f_n) \to \mathbb{R}$. For arbitrary constants K and L,

$$\sum_{i=1}^{n}K\log f_i + \sum_{j=1}^{m}L\log g_j = \sum_{j=1}^{m}g_j\sum_{i=1}^{n}K\log f_i + \sum_{i=1}^{n}f_i L\sum_{j=1}^{m}\log g_j \tag{4.86}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}g_j K\log f_i + \sum_{i=1}^{n}\sum_{j=1}^{m}f_i L\log g_j \tag{4.87}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}\left(g_j K\log f_i + f_i L\log g_j\right). \tag{4.88}$$

Setting $K = sf_i$ and $L = sg_j$, where s is an arbitrary constant,

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\left(g_j s f_i\log f_i + f_i s g_j\log g_j\right) = s\sum_{i=1}^{n}\sum_{j=1}^{m}f_i g_j\left(\log f_i + \log g_j\right) \tag{4.89}$$

$$= s\sum_{i=1}^{n}\sum_{j=1}^{m}f_i g_j\log\left(f_i g_j\right). \tag{4.90}$$

Therefore, we have

$$\psi(f_1, f_2, ..., f_n) = s\sum_{i=1}^{n} f_i \log f_i, \tag{4.91}$$

where $s \in \mathbb{R}_{++}$ is an arbitrary constant.

Since $f_i \log f_i < 0$ for $f_i \in \mathbb{R}^*$,

$$s\sum_{i=1}^{n} f_i \log f_i < 0. \tag{4.92}$$

Continuity requres the convention such that,

$$0 \log 0 = 0, \tag{4.93}$$

which completes the proof. □

PROOF OF THEOREM 4.7 (NECESSITY). We show that the index (4.76) satisfies each axiom.

(i) Using the fact that Kronecker product of **A** and **B** is $\lambda_i \mu_j$,

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{|\lambda_i\mu_j|^\alpha}{\sum_{i=1}^{n}\sum_{j=1}^{m}|\lambda_i\mu_j|^\alpha}\log\frac{|\lambda_i\mu_j|^\alpha}{\sum_{i=1}^{n}\sum_{j=1}^{m}|\lambda_i\mu_j|^\alpha}\right)\\
&=\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{|\lambda_i|^\alpha}{\sum_{i=1}^{n}|\lambda_i|^\alpha}\frac{|\mu_j|^\alpha}{\sum_{j=1}^{m}|\mu_j|^\alpha}\log\frac{|\lambda_i|^\alpha}{\sum_{i=1}^{n}|\lambda_i|^\alpha}\frac{|\mu_j|^\alpha}{\sum_{j=1}^{m}|\mu_j|^\alpha}\right) \qquad (4.94)\\
&=\sum_{i=1}^{n}\sum_{j=1}^{m}\left(f_i g_j \log f_i g_j\right). \qquad (4.95)
\end{aligned}$$

We need to show that

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\left(f_i g_j \log f_i g_j\right) = \sum_{i=1}^{n}\left(f_i \log f_i\right) + \sum_{i=1}^{n}\left(g_j \log g_j\right), \tag{4.96}$$

and it follows from the additivity of entropy (Aczél and Dároczy, 1975, pp. 30–31).

(ii) Since $\mathbf{B}$ is permutation equivalent to $\mathbf{A}$, we have $\sigma(\mathbf{A}) = \sigma(\mathbf{P^T B P})$. Thus, $\phi(\mathbf{A}) = \phi(\mathbf{P^T B P})$.

(iii) Since f_i is continuous on $p_{i,j}$, the continuity of ϕ follows from the continuity of logarithmic functions and the sum of continuous functions.

(iv) The same argument as Theorem 4.2 (iii) applies.

(v) Since *Equalization of life chances* holds by Theorem 3.1, the index value is maximized when all elements of the matrix are the same. Then, by Lemma 3.1 (iii), the only one eigenvalue is 1 and the other eigenvalues are all 0 for any dimensions; thus, *Maximum invariance to dimensions* holds.

(vi) As we have shown, the eigenvalues of $a_{i,j} \cdot \mathbf{A}$ are $a_{i,j}$ times the eigenvalues of $\mathbf{A}$. The value of f_i is the same for $\mathbf{A}$ and $a_{i,j} \cdot \mathbf{A}$; therefore, *Block consistency* holds. □

As a result, we have the index III as presented in Chapter 3. For two dimensional case, see Figure 3.1.

DEFINITION 4.3 (index III). Let f_i be be defined as

$$f_i = \frac{|\lambda_i|^\alpha}{\sum_{i=1}^n |\lambda_i|^\alpha}, \tag{4.97}$$

where α is a nonzero real number, which can be interpreted as a parameter of sensitivity to the deviation from the equal opportunity situation. The mobility index is given by a function ϕ defined on a set of nonnegative square matrices $\mathscr{A}$:

$$\phi(\mathbf{A}) = s \sum_{i=1}^n f_i \log f_i, \tag{4.98}$$

where $s \in \mathbb{R}$ is a constant scaling factor, with the convention

$$0 \ln 0 = 0. \tag{4.99}$$

COROLLARY 4.2. *The minimum value of index I and index III are the same.*

PROOF OF COROLLARY 4.2. Since all eigenvalues are 1, we have

$$\sum_{i=1}^{n} \left(\frac{1}{\sum_{i=1}^{n} 1} \log \frac{1}{\sum_{i=1}^{n} 1} \right)$$
$$= n \left(\frac{1}{n} \log \frac{1}{n} \right) = \log \frac{1}{n} = \log n^{-1} = -\log n = -\log \left(\sum_{i=1}^{n} 1 \right). \tag{4.100}$$

□

THEOREM 4.8. *If, and only if, the mobility index satisfies (i) Decomposability of independent factors, (ii) Permutation equivalence, (iii) Symmetry, (iv) Continuity, (v) Equalization of life chances, (vi) Minimum invariance to dimensions, and (vii) Block consistency, then it is represented by a function* $\phi : \mathscr{A} \to \mathbb{R}$ *such that*

$$\phi(\mathbf{A}) = s \sum_{i=1}^{n} h_i \log h_i, \tag{4.101}$$

where

$$h_i = \frac{n|\lambda_i|^{\alpha}}{\sum_{i=1}^{n} |\lambda_i|^{\alpha}} \tag{4.102}$$

and $\lambda_i \in \sigma(\mathbf{A})$, $\mathbf{A} \in \mathscr{A}$, $\alpha \in \mathbb{R}$, $s \in \mathbb{R}_{++}$, *with the convention*

$$0 \ln 0 = 0. \tag{4.103}$$

PROOF OF THEOREM 4.8 (SUFFICIENCY). By Theorem 4.6, *Decomposability of independent factors*, *Permutation equivalence*, *Symmetry*, *Continuity*, *Equalization of life chances*, and *Minimum invariance to dimensions* imply Index II:

$$K \log \left(\frac{n}{|\lambda_1|^\alpha + \cdots + |\lambda_n|^\alpha} \right), \tag{4.104}$$

where $\lambda_i \in \sigma(\mathbf{A})$, $\alpha \in \mathbb{R}$, and $K > 0$.

By *Decomposability of independent factors*,

$$K \log \left(\frac{nm}{|\lambda_1 \mu_1|^\alpha + |\lambda_1 \mu_2|^\alpha + |\lambda_2 \mu_1|^\alpha + \cdots + |\lambda_n \mu_m|^\alpha} \right) = K \log \left(\frac{n}{|\lambda_1|^\alpha + \cdots + |\lambda_n|^\alpha} \right) + K \log \left(\frac{m}{|\mu_1|^\alpha + \cdots |\mu_m|^\alpha} \right). \tag{4.105}$$

We consider a single variable case of (4.105):

$$K \log \left(\frac{1}{|\lambda_i \mu_j|^\alpha} \right) = K \log \left(\frac{1}{|\lambda_i|^\alpha} \right) + K \log \left(\frac{1}{|\mu_j|^\alpha} \right). \tag{4.106}$$

Subtracting (4.105) from (4.106) each side, the following equation holds.

$$K\log\left(\frac{1}{|\lambda_i\mu_j|^\alpha}\right)-K\log\left(\frac{nm}{|\lambda_1\mu_1|^\alpha+|\lambda_1\mu_2|^\alpha+|\lambda_2\mu_1|^\alpha+\cdots+|\lambda_n\mu_m|^\alpha}\right)$$

$$=K\log\left(\frac{1}{|\lambda_i|^\alpha}\right)-K\log\left(\frac{n}{|\lambda_1|^\alpha+\cdots+|\lambda_n|^\alpha}\right) \tag{4.107}$$

$$+K\log\left(\frac{1}{|\mu_j|^\alpha}\right)-K\log\left(\frac{m}{|\mu_1|^\alpha+\cdots|\mu_m|^\alpha}\right). \tag{4.108}$$

For each λ_i, μ_j, we have

$$K\log\left(\frac{nm|\lambda_i\mu_j|^\alpha}{|\lambda_1\mu_1|^\alpha+|\lambda_1\mu_2|^\alpha+|\lambda_2\mu_1|^\alpha+\cdots+|\lambda_n\mu_m|^\alpha}\right)$$

$$=K\log\left(\frac{n|\lambda_i|^\alpha}{|\lambda_1|^\alpha+\cdots+|\lambda_n|^\alpha}\right)+K\log\left(\frac{m|\mu_j|^\alpha}{|\mu_1|^\alpha+\cdots|\mu_m|^\alpha}\right). \tag{4.109}$$

In the same ways as in the Proof of Theorem 4.7 (Sufficiency), we have

$$s\sum_{i=1}^{n}h_i\log h_i. \tag{4.110}$$

where $s\in\mathbb{R}_{++}$ is an arbitrary constant., and *Continuity* requires the convention such that,

$$0\log 0=0, \tag{4.111}$$

which completes the proof. □

PROOF OF THEOREM 4.8 (NECESSITY). We show that the index (4.102) satisfies each axiom.

(i) Using the fact that Kronecker product of **A** and **B** is $\lambda_i\mu_j$,

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{nm|\lambda_i\mu_j|^\alpha}{\sum_{i=1}^{n}\sum_{j=1}^{m}|\lambda_i\mu_j|^\alpha}\log\frac{nm|\lambda_i\mu_j|^\alpha}{\sum_{i=1}^{n}\sum_{j=1}^{m}|\lambda_i\mu_j|^\alpha}\right)$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{n|\lambda_i|^\alpha}{\sum_{i=1}^{n}|\lambda_i|^\alpha}\frac{m|\mu_j|^\alpha}{\sum_{j=1}^{m}|\mu_j|^\alpha}\log\frac{n|\lambda_i|^\alpha}{\sum_{i=1}^{n}|\lambda_i|^\alpha}\frac{m|\mu_j|^\alpha}{\sum_{j=1}^{m}|\mu_j|^\alpha}\right) \tag{4.112}$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{m}\left(h_ik_j\log h_ik_j\right). \tag{4.113}$$

We need to show that

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\left(h_ik_j\log h_ik_j\right)=\sum_{i=1}^{n}\left(h_i\log h_i\right)+\sum_{i=1}^{n}\left(k_j\log k_j\right), \tag{4.114}$$

and it follows from the additivity of entropy the same as the proof of Theorem 4.7 (i).

(ii) The same argument as Theorem 4.2 (ii) applies.

(iii) The same argument as Theorem 4.7 (iii) applies.

(iv) The same argument as Theorem 4.2 (iii) applies.

(v) We apply the proof of the Theorem 3.2. We consider the following constrained maximization problem:

$$\max \sum_{i=1}^{n} h_i\log h_i \tag{4.115}$$

$$\text{s.t.} \sum_{i=1}^{n} h_i = 1. \tag{4.116}$$

Using the method of Lagrange multipliers,

$$L=\sum_{i=1}^{n} h_i\log h_i-\lambda\left(\sum_{i=1}^{n} h_i-1\right). \tag{4.117}$$

We obtain the first-order condition (FOC):

$$\frac{\partial L}{\partial h_i} = \log h_i + 1 - \lambda = 0 \tag{4.118}$$

$$f_i = e^{\lambda - 1}. \tag{4.119}$$

The FOC (4.119) indicates that each h_i is the same constant. Since, by the definition, $\sum_{i=1}^{n} h_i = n$, we have

$$h_i = 1. \tag{4.120}$$

By Lemma 3.1 (i) and (ii), if **A** is diagonal or antidiagonal, then all eigenvalues are 1. Hence,

$$h_i = \frac{n|\lambda_i|^\alpha}{\sum_{i=1}^{n} |\lambda_i|^\alpha} = \frac{n1^\alpha}{1^\alpha + 1^\alpha + \cdots + 1^\alpha} = 1. \tag{4.121}$$

Therefore, if **A** is diagonal or antidiagonal, then the first-order condition of the maximization problem is satisfied.

(vi) Since *Equalization of life chances* holds, the index value is maximized when all elements of the matrix are the same. Then, by Lemma 3.1 (iii), the only one eigenvalue is 1 and the other eigenvalues are all 0 for any dimensions; thus, *Maximum invariance to dimensions* holds.

(vii) The same argument as Theorem 4.7 (vii) applies. □

DEFINITION 4.4 (index IV). Let h_i be be defined as

$$h_i = \frac{n\lambda_i}{\sum_{i=1}^{n} \lambda_i}, \tag{4.122}$$

where α is a nonzero real number, which can be interpreted as a parameter of sensitivity to the deviation from the equal opportunity situation. The mobility index is given by a function ϕ defined on a set of nonnegative square matrices $\mathscr{A}$:

$$\phi(\mathbf{A}) = s\sum_{i=1}^{n} h_i \log h_i, \tag{4.123}$$

where $s \in \mathbb{R}$ is a constant scaling factor, with the convention

$$0 \ln 0 = 0. \tag{4.124}$$

EXAMPLE 4.5. We consider the 2×2 transition matrix:

$$\mathbf{A} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}, \tag{4.125}$$

where p is real value, and $0 \leq p \leq 1$. The values of the mobility index $\phi(\mathbf{A})$ are plotted in Figure 4.1.

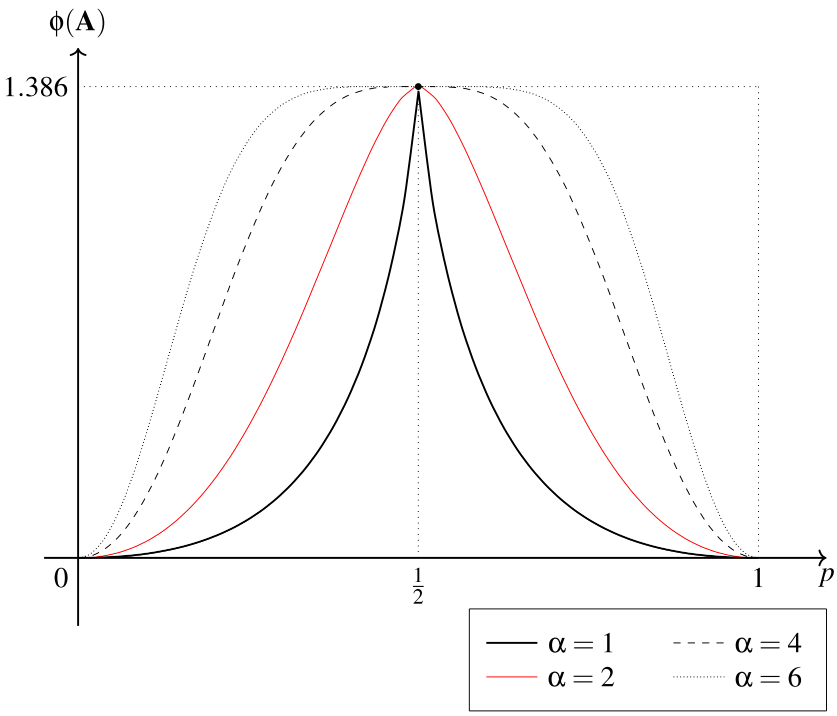

FIGURE 4.1. The values of the mobility index IV in the two-dimensional case.

The logical relationship between axioms and indices we provided is summarized in Figure 4.2.

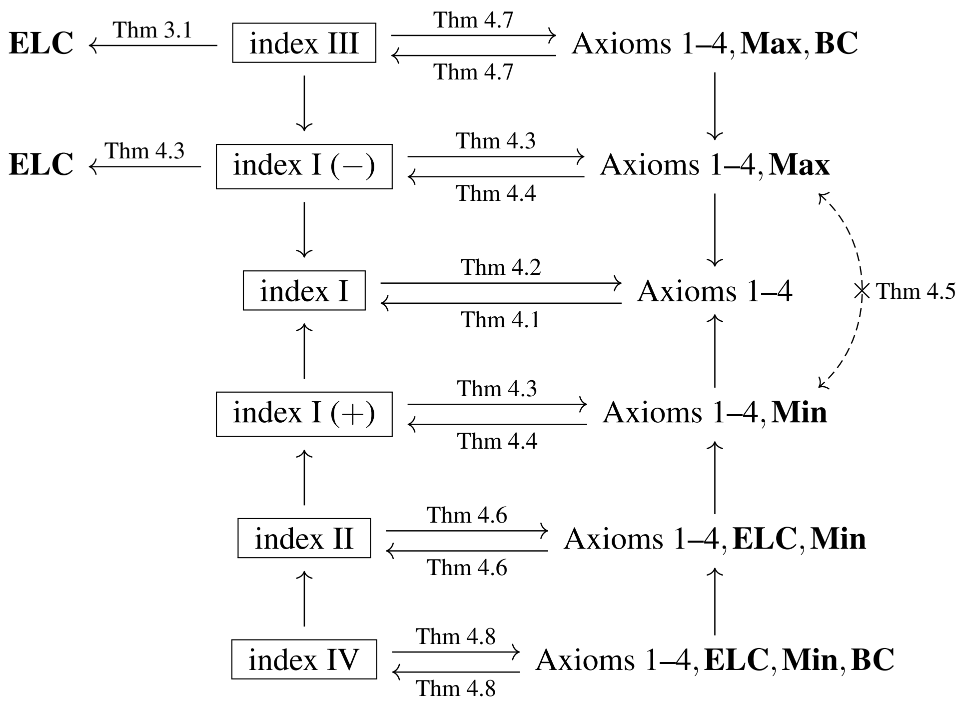

FIGURE 4.2. Logical implications between indices and axioms.
Axiom 1: Decomposability of independent factors
Axiom 2: Permutation equivalence
Axiom 3: Symmetry
Axiom 4: Continuity
Axiom 5: Equalization of life chances (**ELC**)
Axiom 6: Maximum invariance to dimensions (**Max**)
Axiom 7: Minimum invariance to dimensions (**Min**)
Axiom 8: Block consistency (**BC**)

REMARK 4.1. We consider deriving mobility indices, each of which is a function $\phi : \mathscr{A} \to \mathbb{R}$, where $\mathscr{A}$ is a set of $n \times n$ nonnegative square matrices. However, as we describe mobility by the doubly stochastic matrix, the domain of the function ϕ should naturally be doubly stochastic. The reason that we suppose nonnegative square matrix as a domain is to apply Axiom 8 (Block consistency); that is, each block is never doubly stochastic.

The question is, what is the information that such a general matrix embodies, and what does the function represent as a mechanism to evaluate? One of the plausible solutions is to redefine the domain of the function to be the doubly stochastic matrix and to replace Axiom 8 to another one to characterize Index III and IV. This is a remaining issue.

4.4. Concluding remarks

We provided a fundamental framework to evaluate social mobility for the measurement of equality of opportunity with emphasis on the additive decomposability property. We discuss significance and the position of our study in the literature on equality of opportunity.

Firstly, *Decomposability of independent factors* can be applied to factors that are independent and decomposable, which is a strong requirement. we should question the appropriateness of setting up the model in such a way that non-responsible and responsible factors can be decomposable and independent. Roemer (1986) argues that in the context of economic resource allocation, axiomatic systems that embody resource egalitarianism consequently derive only equality of welfare solutions. It is now known that the result is due to the fact that the proposed system is not responsibility-sensitive egalitarian, but rather outcome egalitarian.[16] Thus, in the context of economic resource allocation by Roemer (1986) where responsibility and non-responsibility factors can be explicitly distinguished, our study can be positioned as the characterization of axiomatic systems for deriving a outcome egalitarian mobility measures, which are not responsibility-sensitive egalitarian.

[16] Yoshihara (2003) proposes axiomatic systems different from Roemer (1986) and argues that they are persuasive from the view point of responsibility-sensitive egalitarianism. It is also shown that the solutions derived from these are different from equality of welfare.

Secondly, however, our research can be considered as a starting point of the analysis of responsibility-sensitive egalitarian mobility measures. It is, of course, desirable to provide an index that distinguishes between non-responsible and responsible factors to evaluate social mobility when the factors *cannot* be treated as decomposable and independent each other. We provided indices assuming a simplified case (i.e., decomposable and independent), and then we consider relaxing this assumption in the future research. Furthermore, it is significant that the theorems presented in this chapter identify the characteristics of axiomatic systems in which mobility indices are outcome egalitarian *even* under the assumption that they are decomposable and independent of each other.

For future studies, we should examine deriving responsibility-sensitive egalitarian indices by weakening the axioms, particularly *Decomposability of independent factors*. It may be helpful if the principles of responsibility-sensitive egalitarianism are directly incorporated in the axioms. Moreover, there is the issue of difficulty in applying the results to empirical analysis. As sources, or roots, of inequality have been estimated in recent literature (e.g., Brunori et al., 2018), sources of mobility are also expected to be studied empirically.

CHAPTER 5

Conclusion

5.1. Summary

We attempted to contribute to a strand of research that overcomes the negative observation on normative economics by Graaff (1957): "I do feel very strongly that the greatest contribution economics is likely to make to human welfare, broadly conceived, is through *positive* studies ... rather than normative welfare theory itself" (p. 170).

Even today, many economists hesitate to deal with normative issues:

> We economists must recognize ... the limits on the ability of our discipline to prescribe policy responses. Economists who discuss policy responses to increasing inequality are often playing the role of amateur political philosopher (and, admittedly, I will do so in this essay). Given the topic, that is perhaps inevitable. But it is useful to keep in mind when we are writing as economists and when we are venturing beyond the boundaries of our professional expertise" (Mankiw, 2013, p. 22).

Furthermore:

> Economists are put in an awkward position when asked to calculate the welfare consequences of changes to economic policy or of shocks to the economy: we are asked to act as moral philosophers (Lockwood and Weinzierl, 2016, p. 30).

Our task was to surmount "the disappearance from economics of discussion of the principles underlying normative statements" (Atkinson, 2001, p. 193) and to contribute to normative economics particularly from the perspective of opportunity equality.

We have paved the ways for the pursuit for equality of opportunity. In Chapter 1, we demonstrated an escape from the incompatibility between the *ex ante* and the *ex post* perspectives of compensation, which is recognized as a deeper root of conflicts between principles reward and compensation. It opened up a possibility that responsibility-sensitive egalitarian policies are implementable in a reasonable environment. In Chapter 2, we proposed a mobility index for the measure of equality of opportunity. In Chapter 3, we introduced principles of responsibility-sensitive egalitarianism to mobility measurement. Several possible measures were provided, including entropic mobility index, and the assessment of mobility incorporated fundamental normative principles that should be taken into account as a measure of (in)equality of opportunity.

They are, however, first steps for addressing the neglected issues. We need further exploration to accommodate various ethical values regarding equality of opportunity. Moreover, positive and empirical studies should also be supplemented for developing taxation and redistribution policies. Notwithstanding, "the social choice theory may be construed as a formal framework for examining the general workability of a moral principle in the resolution of social conflicts" (Suzumura and Suga, 1987, p. 268). To use Pigou's ([1920] 1932, p. vii) phrase, the complicated analyses which we endeavoured to carry through are not mere gymnastic. They are instruments for the bettering of human life.

5.2. Welfarism, consequentialism, and beyond

To conclude this dissertation, we evaluate our contributions in view of "enriching information for the possibility of social choice" (Sen, 2011, p. 40). We employ the *informational tree of normative social evaluations* which is introduced and utilized in Suzumura (2011, 2016a, 2020).[1]

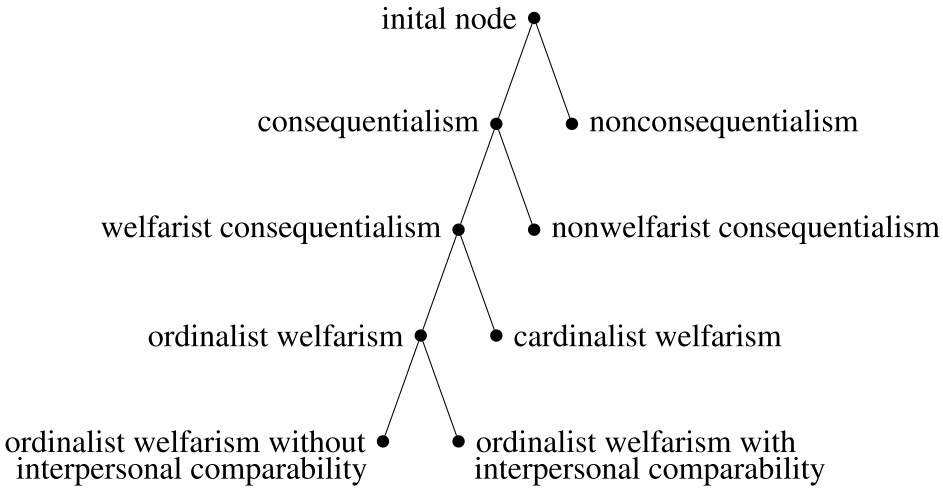

FIGURE 5.1. Informational tree of normative social evaluations (reproduced from Suzumura (2016a, p. 767, Figure 28.1) and Suzumura (2020, p. 108, Figure 6.1)).

At the initial node of this tree, we are in the position of evaluating the informational requirement whether or not to judge economic systems and/or economic policies according to their consequential outcomes. The stance of requiring solely consequential outcomes as information is called *consequentialism*; the stance of requiring information beyond consequential outcomes is called *nonconsequentialism*.[2]

[1]This analytical device is hinted in Suzumura (1999, 2000b), and virtually appeared first in Suzumura (2000a). Refer to these references for the more detail descriptions of the tree. The origin of the tree diagram can be traced back to Shionoya (1984), which is attributed to Sen (1974), Strasnick (1976), Sen (1977), Sen (1979a), and Sen (1979b).

[2]Note that the *nonconsequentialist* does not necessarily neglect consequential information; it considers *nonconsequential* information in addition to *consequential* information. A special class of the *nonconsequentialism* is the *deontological* stance, which focuses solely on *nonconsequentialist* information.

While we aim to clarify the moral stance of our studies according to this diagram, it is necessary to point out that it has limitation in classifying our coherent topic: *equality of opportunity*. Indeed, since we consider opportunities, we never restrict our attention to outcomes, and it implies that we are *nonconsequentialist*.[3] Meanwhile, to resolve the opportunity paradox, we restricted to pay attention to interpersonally comparable utility information. In this sense, we are *welfaristic*; thus, we are also in the stance of *ordinalist welfarist with interpersonal comparability*. It is due to the fact that we use the indirect approach and it is *consequentialistic* in nature. Moreover, social mobility is also not outcome but process. For the same reason, we are *nonconsequentialist* when we consider social mobility.

There are two issues. First, inconsistencies arise when we rely on the indirect approach. Second, as long as we are *nonconsequentialist*, the tree offers no more meaningful facts about the informational basis of normative social evaluations. In any case, we should explore the path beyond the *nonconsequentialism* node. For example, we can include uncertainty to consider the *ex ante* perspective of compensation, and we can introduce dynamic concept to take note of social mobility. The bottom line is that we should extensively redesign this tree to fully comprehend our contributions in the light of informational basis.

Theories on equality of opportunity is part and parcel of the cutting-edge normative economics that pursues the betterment of human life by enriching the informational basis, embracing political philosophy. To deepen our insights, we also need to continue developing analytical device to overview our studies.

[3]According to Suzumura (2020, 2021), Pigou ([1920] 1932) and Hicks (1969) are the precursors of *nonconsequentialism* as well as the theory of justice by means of social primary goods by Rawls ([1971] 1999) and the capability approach by Sen (1980).

Epilogue

On April 23, 2018, I asked Professor Amartya K. Sen what he expects to see in research on inequality and poverty, particularly in the fields of welfare economics and social choice theory, and he gave me the following suggestions.

I think three things to say there. There are a lot of interesting problems of inequality and poverty, which have not been dealt with. Certainly, the idea of relative poverty is not widely understood. So, there are some important issues which have not been as much addressed as they should have been. And first of all, I would like people to think about whether they want to work in this area and seriously work.

Secondly, there's an issue about why are these important problems and also why is it that they have not been addressed? Could it be because it's difficult to address them or could it be that people are lazy or could it be that there are some intrinsic philosophical issues that need to be addressed? And you have to think about that. So, the second thing is, aside from thinking about working in an area which is important but neglected, you have to ask also why is it important and why has it been neglected.

And the third question is that you're not the only one who is working in the world. And some of them, other people are addressing, and some other people are neglecting altogether. And you might miss to choose the problem, which others are neglecting for one reason or another, this will *come out from the second question. So, the first question, think about doing something which is important and neglected. Second question, ask why is it important, why is it neglected. And third is, if it is neglected for some reason, which makes it unlikely that others will address them, then you have a stronger case for working on that area, compared with those which are neglected now, but you could see lots of people are getting ready to address them. So that's how you address these things.*

And then ultimately, I believe that this is an appendix, that you should never choose a problem only because it's important. You have to choose a problem because you are engaged in it and it's fun to work on it. Ultimately, if it's not fun to work on it, you won't do work very well. You have to see whether it's something that you could enjoy doing, and don't neglect the fact, and it's not hedonism only, it is effectiveness. I don't think people have produced great work in which they were not involved and excited in working in that area. That's a much bigger issue.

Professor Mamoru Kaneko asked a follow-up question:

> Neglecting and not knowing are different? So, neglecting is rather really "active action," but not knowing is not really "active action," so those two are really different. But your third comment is about neglecting, so be conscious about neglecting is very important. But not knowing is also very important.

I totally agree, and I would say that among the reasons for neglecting, there are two types, ultimately, I say three types but two types to start with. One is, you know the problem, you know what is to be done, but it's very *complicated and then you say, "Look, I don't want to do it." And the other is to say, "Well, I think we have done all the work we have to do," but you don't know that there is something really important there that you have not thought about. And when a problem is neglected it could people haven't seen, they don't know that there is a neglect, that's a much harder thing. Now, sometimes, and that's the third point, that you have not absolutely ultimate distinction between being not clear whether something is being not known or unknown. That is, you may say, "Well, we neglect it, maybe there is some bigger issue there but I don't know." Another case you say, "There is a bigger issue here, but we don't know it because we have not done the work about how, for example, the sea level rise will affect lives as a result of environmental change in the future." But there is some, take the environment, they neglect, they're not knowing and they're not knowing in the sense that you don't quite know how to do it; you know how to study it. But there are others which you don't know anything much about at all and you don't even know how to study it. So, all these things are very important.*

Please allow me to write a response to the invaluable advice. I think that a fair amount your marvelous contributions focuses on the "utilitarian neglect of distributional issues" (Sen, 1999, p. 352) and the "informational neglect" (Sen, 2000, p. 65) of welfarism. Moreover, I have been fascinated by your position "of being supportive of the underdogs—those who are neglected by society" (Sen et al., 2020, p. 14). Hence, it seems natural to me that you emphasized the importance of neglected aspects of social problems, and I think your advice is so Amartya Sen. I believe that equality of opportunity—the coherent theme of this dissertation—belongs to one of them.

As Atkinson (1983a) observed that "many writers treat [equality of opportunity] as self-evident merit" (p. 77), few normative analyses have been done particularly in the empirical social mobility literature. Further, as you wrote in the praise of Fleurbaey (2008),[1] fairness that can be realized by the responsibility cut has not been seriously considered until recently.

I tried figuring out issues regarding on equality of opportunity that are neglected yet. I discovered an escape route from the "opportunity paradox," which is a question sprouted out from the contributions by Fleurbaey and others in Chapter 2. Then, I examined normative aspects of social mobility. I proposed a new mobility index for the measurement of (in)equality of opportunity in Chapter 3. I introduced principles of responsibility-sensitive egalitarianism to the measurement of social mobility and provided axiomatic characterizations in Chapter 4. All of them have been neglected in the literature. In fact, I cannot find exactly why they are neglected. I believe that they are all important and should be addressed, and I conjecture that they would be classified in one of the "not knowing" categories. I would like to think about it as well as remaining issues in future studies.

After all, I enjoyed writing this dissertation. I will continue working on this important and exciting area. Thank you, if I may, Amartya.

[1] "The role of personal responsibility is important for the foundations of justice, but the subject has not received the critical investigation it deserves. In this book, Marc Fleurbaey has gone a long distance in meeting this gap. This is a deeply illuminating contribution on a neglected aspect of welfare economics."

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