A study on the qualitative theory of solutions for some parabolic equations with nonlinear boundary conditions

非線形境界条件を伴う放物型方程式の解の定性理論の研究

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Part I

Second Order Parabolic Equations with Nonlinear Boundary Conditions

Introduction

In this part, we consider the following initial boundary value problem for the nonlinear heat equation with nonlinear boundary conditions:

$$
\text{(P)} \qquad \begin{cases} \partial_t u - \Delta u = |u|^{p-2}u, & t > 0,\ x \in \Omega, \\ -\partial_\nu u \in \beta(u), & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases}
$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$; $\nu = \nu(x)$ is the unit outward normal vector at $x \in \partial\Omega$; $p \in (2, \infty)$ is a given number; $u : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is a real-valued unknown function. As for the boundary condition, β is a maximal monotone graph on $\mathbb{R} \times \mathbb{R}$. More precisely, for some proper convex lower semicontinuous function $j : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, β is given by the subdifferential of j, that is, $\beta = \partial j$. The typical example of β is a singleton power type nonlinear term such as

$$
\beta_q(r) = |r|^{q-2}r, \qquad q \in (1, +\infty).
$$

This problem (P) is a prototype of nonlinear heat equations with nonlinear boundary conditions of radiation type.

When one tries to set up mathematical models for describing actual nonlinear phenomena, it is crucial to determine right ruling nonlinear structures in domains where the phenomena occur, but it is also very important to pay careful attention to the choice of the boundary conditions. For instance, when we are concerned with the heat diffusion, it should be noted that the standard boundary condition such as Dirichlet or Neumann boundary condition can be realized only when some artificial control of the heat flux is given on the boundary. For a large scale system, however, it is impossible to give such a control on the boundary. If there is no control of heat flux on the boundary, there is a prototype model in physics well known as Stefan-Boltzmann's law, which says that the heat energy radiation from the surface of the body is proportional to the fourth power of the difference of temperatures between the inside and outside of the body in $\mathbb{R}^3$.

In this sense, from a physical point of view, it could be more natural to consider nonlinear boundary conditions rather than the linear boundary conditions such as the homogeneous Dirichlet or Neumann boundary condition.

In spite of its importance, however, there are few studies on parabolic equations with nonlinear boundary conditions of radiation type. The first treatment for the dissipative parabolic systems in this direction was given by H. Brézis in [10], where he dealt with the following parabolic equation:

$$\text{(I.1)} \qquad \begin{cases} \partial_t u - \Delta u = f(t,x), & t > 0,\ x \in \Omega, \\ -\partial_\nu u \in \beta(u), & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases}$$

Here $f \in L^2(0,T;L^2(\Omega))$ is a given forcing term, $u_0 \in L^2(\Omega)$, and the other settings are the same as for (P). He proved the well-posedness of (I.1) by establishing a new class of maximal monotone operators within Kōmura's nonlinear semi-group theory, i.e., a class of subdifferential operators, which characterizes the parabolicity in the theory of evolution equation. This result gave a breakthrough in the study of parabolic equations with nonlinear boundary conditions. Nevertheless, the research in this framework is not fully pursued for the non-dissipative system which may admit blow-up solutions such as (P).

In Part I, we begin with the most fundamental problem in the study of partial differential equations, the local well-posedness, and then consider the qualitative properties of solutions of (P). The plan of Part I is as follows. In Chapter 1, we briefly summarize some notations and fundamental mathematical tools to be used in the following chapters.

In Chapter 2, we consider (P) with $\beta = \beta_q$ (denoted by $(\mathrm{P})_q$):

$$(\mathrm{P})_q \qquad \begin{cases} \partial_t u - \Delta u = |u|^{p-2}u, & t > 0,\ x \in \Omega, \\ -\partial_\nu u = |u|^{q-2}u, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

where $q \in (1,+\infty)$. We here show the existence and the uniqueness of local solutions of $(\mathrm{P})_q$. For semilinear equations such as (P) with the homogeneous Dirichlet boundary condition, the standard way to derive their local well-posedness is to rely on Duhamel's principle and apply the fixed point theorem to their transformed integral equations. Because of the presence of the nonlinear term on the boundary, however, it is not possible to follow the same strategy for our problem $(\mathrm{P})_q$. To cope with this difficulty, we reduce $(\mathrm{P})_q$ to an abstract evolution equation in $L^2(\Omega)$ (see [9]) and apply the theory of non-monotone perturbations for nonlinear parabolic equations associated with subdifferential operators developed by Ôtani [45, 47]. Moreover, for $q < p$, following the argument in [47], we show the existence of global solutions to $(\mathrm{P})_q$ for small initial data.

In Chapter 3, we are concerned with the asymptotic behavior of global solutions to $(\mathrm{P})_q$, more precisely, the question whether any global solution of $(\mathrm{P})_q$ is uniformly bounded in time. There are large amounts of works concerning the asymptotic behavior of solutions of the following nonlinear heat equation with the homogeneous Dirichlet boundary condition:

$$\text{(I.2)} \qquad \begin{cases} \partial_t u - \Delta u = |u|^{p-2}u & t > 0,\ x \in \Omega, \\ u = 0 & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x) & x \in \Omega. \end{cases}$$

Uniform bounds of global solutions of (I.2) was first studied by [46] in an abstract setting, where it is shown that every global solution of (I.2) is uniformly bounded in $H^1_0(\Omega)$ with respect to time for $p \in (2, p_S)$. Here p_S is the Sobolev critical exponent defined by

$p_S = \infty$ for $N = 1, 2$; $p_S = \frac{2N}{N-2}$ for $N \geq 3$. Cazenave-Lions [13] showed that every global solution (allowing sing-changing) is bounded in $L^\infty(\Omega)$ uniformly in time provided that $p \in (2, p_{CL})$, where $p_{CL} = \infty$ when $N = 1$; $p_{CL} = 2 + \frac{12}{3N-4}$ when $N \geq 2$. (Note that $p_{CL} \leq p_S$ for any $N \in \mathbb{N}$). Giga [23] removed this restriction on p for positive global solutions. Namely the uniform boundedness of every positive global solution of (I.2) in $L^\infty(\Omega)$ was shown for any $p \in (2, p_S)$. Quittner [54] extended this result for sign-changing solutions. The main tool in [23] is the rescaling argument and [54] relies on the bootstrap argument based on the interpolation and the maximal regularity theory. However it seems to be difficult to apply these devices for our problem $(\mathrm{P})_q$ because of the presence of the nonlinear boundary condition. The main purpose of this chapter is to derive the uniform boundedness in $H^1(\Omega)$ and $L^\infty(\Omega)$ for every global solution of $(\mathrm{P})_q$ by following the same strategy as that in [46]. However, we can not directly apply arguments in [46], since the functional associated with the Laplacian with nonlinear boundary conditions is not homogeneous, which is one of basic tools used in [46]. Nevertheless by introducing a new substitutive argument to avoid the use of the homogeneity of functionals, we are able to derive uniform bounds for global solutions in $H^1(\Omega)$. Moreover with the aid of Moser's iteration scheme, the uniform bound in $L^\infty(\Omega)$ is also obtained.

In Chapter 4, we set up a new type of comparison theorem which can cover both linear and nonlinear boundary conditions for a system of equations which have more general forms than that of (P).

Mathematical models for various types of phenomena arising from physics, chemistry, biology and so on are often described as reaction diffusion equations which give typical examples of second order nonlinear parabolic equations. It is widely known that comparison theorems yield very powerful tools for analyzing the second order parabolic equations, e.g., for constructing super-solutions or sub-solutions; and for examining the asymptotic behavior of solutions. However, most of the existing results on comparison theorems for nonlinear diffusion equations are concerned with the standard linear boundary conditions such as Dirichlet or Neumann boundary conditions (see [55]). Furthermore, these comparison theorems are only applicable to problems whose imposed boundary conditions are of the same form. There is a result on comparison theorems covering nonlinear boundary conditions by Bénilan and Díaz [8], which also compares two solutions satisfying nonlinear boundary conditions of the same form. Our comparison theorem has an advantage that it allows us to compare two solutions satisfying different types of nonlinear (including linear) boundary conditions. Moreover as applications of this comparison theorem, we can show the existence of blow-up solutions satisfying nonlinear boundary conditions for some parabolic equations in §4.2 and for some parabolic systems in §9.2.

In Chapter 5, we consider the existence and nonexistence of global solutions of (P). Concerning the blow-up phenomena in the whole domain, it is well known that there exists the critical Fujita exponent $p_c = 2 + 2/N$ which gives the threshold of p for the existence of global solutions. Namely if $p \in (2, p_c)$, then every positive solution blows up in finite time and for p grater than p_c, there exists (small) global solutions. As for the bounded domain, it is also well known that for the homogeneous Neumann boundary condition, every positive solution blows up in finite time and for the homogeneous Dirichlet boundary condition there exist (small) global solutions. As an analogy of the existence of the Fujita exponent for (P) in $\mathbb{R}^n$ (with no boundary condition), we give a special family

$(\beta^\alpha(\cdot))_{\alpha>0}$ of $\beta(\cdot)$ such that there exists a critical value $\alpha_c > 0$ which plays the same role as the Fujita exponent. More precisely, $(\beta^\alpha)_{\alpha>0}(\cdot)$ connects the homogeneous Neumann boundary condition (when $\alpha = \infty$) and the homogeneous Dirichlet boundary condition (when $\alpha = 0$) such that if α is grater than α_c, then every positive solution blows up in finite time and if α is smaller than α_c, there exists (small) global solutions.

In the last chapter of Part I, we are concerned with the structural stability of solutions to $(\mathrm{P})_q$, that is, the continuous dependence of solutions of $(\mathrm{P})_q$ with respect to the nonlinearity parameter q. Here we treat not only the case $q \in (1, \infty)$ but also the limiting cases where $q = 1$ and $q = +\infty$ given by

$$\beta_1(r) := \begin{cases} 1 & r > 0, \\ [-1, 1] & r = 0, \\ -1 & r < 0, \end{cases} \qquad \text{and} \qquad \beta_\infty(r) := \begin{cases} [0, \infty) & r = 1, \\ 0 & r \in (-1, 1), \\ (-\infty, 0] & r = -1, \\ \emptyset & |r| > 1, \end{cases}$$

respectively. To carry out this aim, we work in the abstract setting given in [9]. In this setting, Attouch [4] studied the convergence of solutions of evolution equations governed by subdifferential operators $\partial\varphi_n$ and showed that if convex functions φ_n converges to φ as $n \to \infty$ in the sense of Mosco, then the solutions of equations governed by $\partial\varphi_n$ converge to the solution of the equation governed by $\partial\varphi$ in an appropriate sense. So we here extend this result to the case where equations contain a Lipschitz perturbation term, and apply this abstract result to $(\mathrm{P})_q$ by showing the Mosco convergence of the associated convex functions. (Since we are here concerned with solutions belonging to $L^\infty(0, T; L^\infty(\Omega))$, the perturbation term $|u|^{p-2}u$ can be regarded as a Lipschitz perturbation.)

Chapter 1

Preliminaries

Here we collect some basic facts of functional analysis to be used later on.

1.1 Function Spaces

In this section, we fix some notations and basic facts on function spaces where we work. For the details of them, we refer the reader to [7, 12]. We first state some basic properties of the Lebesgue space. Let $N \in \mathbb{N}$ and let Ω be a open set of $\mathbb{R}^N$. For $1 \leq p \leq \infty$, we define Lebesgue space $L^p(\Omega)$ by

$$L^p(\Omega) := \{u : \Omega \to \mathbb{R}\,;\, \text{mesurable}, \|u\|_p < \infty\},$$

where

$$\|u\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |u(x)|^p dx\right)^{1/p}, & \text{if} \quad 1 \leq p < \infty, \\ \inf\{C \in \mathbb{R}\,;\, |u(x)| \leq C,\ a.e. \text{ on } \Omega\}, & \text{if} \quad p = \infty. \end{cases}$$

For simplicity, we may denote this L^p norm $\|\cdot\|_{L^p(\Omega)}$ as $\|\cdot\|_p$ or $\|\cdot\|_{p,\Omega}$. We also denote by $L^p_{loc}(\Omega)$ the set of all functions $u : \Omega \to \mathbb{R}$ which are measurable and belong $L^p(\omega)$ for any compact set $\omega \subset \Omega$. It is well known that $L^p(\Omega)$ is a Banach space and that $L^p(\Omega)$ is reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$. We can easily see that $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v) := \int_\Omega u(x)v(x)dx \qquad \forall u, v \in L^2(\Omega).$$

Lemma 1.1.1 (Young's inequality)**.** *Let a, $b \geq 0$ and $1 < p < \infty$. Then, the following inequality holds:*

$$ab \leq \frac{1}{p}a^p + \frac{1}{p'}b^{p'},$$

where p' satisfies $p^{-1} + (p')^{-1} = 1$.

Corollary 1.1.2 (Young's inequality with ε)**.** *Let a, $b \geq 0$ and $1 < p < \infty$. Then, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$ab \leq \varepsilon a^p + C_\varepsilon b^{p'}.$$

Lemma 1.1.3 (Hölder's inequality)**.** *Let $1 \leq p \leq \infty$. If $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, then $fg \in L^1(\Omega)$ and the following inequality holds:*

$$\|fg\|_1 \leq \|f\|_p \|g\|_{p'}.$$

Lemma 1.1.4 (Generalized Hölder's inequality)**.** *Let $1 \leq p_i \leq \infty$, $i = 1, 2, \cdots, n$ $(n \in \mathbb{N})$ and $r^{-1} = \Sigma_{i=1}^n p_i^{-1}$ with $r \geq 1$. If $f_i \in L^{p_i}(\Omega)$, $i = 1, 2, \cdots, n$, then $\Pi_{i=1}^n f_i \in L^r(\Omega)$ and*

$$\left\| \prod_{i=1}^n f_i \right\|_r \leq \prod_{i=1}^n \|f_i\|_{p_i}.$$

Lemma 1.1.5. *Let $1 \leq p \leq r \leq q \leq \infty$. If $f \in L^p(\Omega) \cap L^q(\Omega)$, then $f \in L^r(\Omega)$ with the estimate*

$$\|f\|_r \leq \|f\|_p^\theta \|f\|_q^{1-\theta},$$

for some $\theta \in [0, 1]$ satisfying $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$.

Let $\delta \in (0, 1]$. We say that a function $u : \overline{\Omega} \to \mathbb{R}$ is Hölder continuous with exponent δ if

$$|u(x) - u(y)| \leq C|x - y|^\alpha \qquad \forall x, y \in \overline{\Omega},$$

for some $C \geq 0$, which is called Hölder constant. We define $C^{0,\delta}(\overline{\Omega})$ as a set of all Hölder continuous functions on $\overline{\Omega}$ with exponent δ. It is well known that $C^{0,\delta}(\overline{\Omega})$ is a Banach space with the norm

$$\|u\|_{C^{0,\delta}(\overline{\Omega})} := \|u\|_\infty + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\delta}.$$

Next we briefly touch on the Sobolev spaces. For $m \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$, we set

$$W^{m,p}(\Omega) := \left\{ u \in L^p(\Omega) \,;\, D^\alpha u \in L^p(\Omega),\ \forall \alpha \in (\mathbb{N} \cup \{0\})^N,\ |\alpha| \leq m \right\},$$

and we also set

$$\|u\|_{W^{m,p}(\Omega)} := \begin{cases} \left(\displaystyle\sum_{|\alpha| \leq m} \|D^\alpha u\|_p^p \right)^{1/p} & \text{if} \quad 1 \leq p < \infty, \\ \displaystyle\max_{|\alpha| \leq m} \|D^\alpha u\|_\infty & \text{if} \quad p = \infty. \end{cases}$$

It is well known that this is a norm on $W^{m,p}(\Omega)$ and $(W^{m,p}(\Omega), \|\cdot\|_{W^{m,p}(\Omega)})$ is called the Sobolev space. We can see that Sobolev space $W^{m,p}(\Omega)$ is a Banach space and if $p = 2$ then $W^{m,2}(\Omega) =: H^m(\Omega)$ is a Hilbert space with the following scalar product

$$(u, v)_{H^m(\Omega)} := \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u D^\alpha v dx \qquad \forall u, v \in H^m(\Omega).$$

Moreover, we set

$$W_0^{m,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{m,p}(\Omega)}}, \qquad H_0^m(\Omega) := W_0^{m,2}(\Omega),$$

where $C_c^\infty(\Omega)$ is the space of C^∞ functions with compact support in Ω.

We also prepare the theory of the trace on $\partial\Omega$ of a function $u \in W^{1,p}(\Omega)$. If Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, then it is well known that every $u \in C(\overline{\Omega})$ is well defined on $\partial\Omega$. However, in general, a similar approach to the above makes it difficult to give a direct meaning to the values on the boundary of u. Note that since $\partial\Omega$ has N-dimensional Lebesgue measure zero, $u|_{\partial\Omega}$ is no longer well defined for $u \in L^p(\Omega)$. To cope with this difficulty, we introduce the notion of a trace operator.

Lemma 1.1.6. *Let Ω be a domain with compact smooth boundary $\partial\Omega$, and let $p \in [1,\infty]$. Then there exists a unique bounded linear operator $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that the followings hold.*

(i) $\gamma_0(u) = u|_{\partial\Omega}$ *for all* $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$.

(ii) *There exists C depending only on p and Ω such that*

$$\|\gamma_0(u)\|_{L^p(\partial\Omega)} \le C\|u\|_{W^{1,p}(\Omega)} \qquad \forall u \in W^{1,p}(\Omega).$$

By using this notation, $\gamma_0(u)$ is termed the trace of u on $\partial\Omega$.

Lemma 1.1.7. *Let Ω be a bounded domain of $\mathbb{R}^N$ and $\partial\Omega$ be a class of C^1, and assume that $u \in H^1(\Omega)$. Then, the following two properties are equivalent:*

(i) $u \in H_0^1(\Omega)$;

(ii) $\gamma_0(u) = 0$.

Let us recall the following results.

Lemma 1.1.8 (Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Then there exists a constant $C = C(\Omega) > 0$ such that*

$$\|u\|_2^2 \le C\|\nabla u\|_2^2 \qquad \forall u \in H_0^1(\Omega).$$

As a consequence, $\|\nabla u\|_2$ is a equivalent norm in $H_0^1(\Omega)$.

Next result is crucial for our study on parabolic and elliptic equations with Robin and nonlinear boundary conditions.

Lemma 1.1.9 (Poincaré - Friedrichs inequality)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Then there exists a constant $C > 0$ such that*

$$\|u\|_2^2 \le C\left(\|\nabla u\|_2^2 + \int_{\partial\Omega} u^2 d\sigma\right) \qquad \forall u \in H^1(\Omega).$$

As a consequence, $(\|\nabla u\|_2^2 + \|u\|_{2,\partial\Omega}^2)^{1/2}$ is a equivalent norm in $H^1(\Omega)$.

Lemma 1.1.10 ([56])**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. For $p \in [1, \infty)$ there exists a constant $C = C(\Omega, p) > 0$ such that*

$$\left\| u - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} u d\sigma \right\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \qquad \forall u \in W^{1,p}(\Omega).$$

We note that the embedding from X into Y is denoted by $X \hookrightarrow Y$, that is, $X \hookrightarrow Y$ means $X \subset Y$ and the injection $\iota : X \to Y$ is continuous. More precisely, if $X \hookrightarrow Y$, then there exists $C > 0$ such that

$$\|u\|_Y \leq C \|u\|_X \qquad \forall u \in X.$$

Moreover, if $\iota : X \to Y$ is compact then we say the embedding $X \hookrightarrow Y$ is compact.

Lemma 1.1.11 (Sobolev's embedding theorem)**.** *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, and let $m \in \mathbb{N}$ and $p \in [1, \infty]$. Then the followings hold.*

(i) *If $mp < N$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [1, \frac{Np}{N-mp}]$.*

(ii) *If $mp = N$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [1, \infty)$.*

(iii) *If $mp > N$, then $W^{m,p}(\Omega) \hookrightarrow C^{0,\delta}(\overline{\Omega})$, where*

$$\delta \begin{cases} = m - \frac{N}{p} & \textit{if} \quad m - \frac{N}{p} < 1, \\ \in [0, 1) \quad (\textit{arbitrary}) & \textit{if} \quad m - \frac{N}{p} = 1, p > 1, \\ = 1 & \textit{if} \quad m - \frac{N}{p} > 1. \end{cases}$$

Lemma 1.1.12 (Rellich - Kondrachov theorem)**.** *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, and let $m \in \mathbb{N}$ and $p \in [1, \infty]$. Then the followings hold.*

(i) *If $mp < N$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for all $q \in [1, \frac{Np}{N-mp})$.*

(ii) *If $mp = N$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for all $q \in [1, \infty)$.*

(iii) *If $mp > N$, then $W^{m,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact.*

Lemma 1.1.13 (Gagliardo - Nirenberg inequality)**.** *Assume $1 \leq p, q, r \leq \infty$ and let j, m be two integers satisfying $0 \leq j < m$. If*

$$\frac{1}{p} - \frac{j}{N} = a \left(\frac{1}{r} - \frac{m}{N} \right) + (1 - a) \frac{1}{q}$$

for some $a \in [\frac{j}{m}, 1]$ (or $a < 1$ if $r > 1$ and $m - j - \frac{N}{r}$), then there exists $C = C(m, j, p, q, r, N) > 0$ such that

$$\sum_{|\alpha|=j} \|D^\alpha u\|_p \leq C \left(\sum_{|\alpha|=m} \|D^\alpha u\|_r \right)^a \|u\|_q^{1-a} \qquad \forall u \in \mathcal{D}(\mathbb{R}^N).$$

We often use the positive part u^+ and the negative part u^- defined by

$$u^+ := \max(u, 0), \qquad u^- := \max(-u, 0)$$

in this thesis, for example, to prove the nonnegativity of solutions and a comparison principle. The next results are essential to deal with the positive and negative part.

Lemma 1.1.14. *Let $F : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function such that $F(0) = 0$, and let $p \in [1, \infty]$. If $u \in W^{1,p}(\Omega)$, then $F(u) \in W^{1,p}(\Omega)$ and $\nabla F(u) = F'(u)\nabla u$ a.e. on Ω. Moreover, if $p < \infty$, then the mapping $u \mapsto F(u)$ is continuous from $W^{1,p}(\Omega)$ into itself.*

Corollary 1.1.15. *Let $p \in [1, \infty]$.*

(i) *Given $u \in W^{1,p}(\Omega)$, it follows that u^+, u^-, $|u| = u^+ + u^-$ are belong to $W^{1,p}(\Omega)$. Moreover,*

$$\nabla u^+ = \begin{cases} \nabla u & \text{if } u > 0, \\ 0 & \text{if } u \leq 0, \end{cases} \qquad \nabla u^- = \begin{cases} -\nabla u & \text{if } u < 0, \\ 0 & \text{if } u \geq 0, \end{cases}$$

and

$$\nabla |u| = \begin{cases} \nabla u & \text{if } u > 0, \\ 0 & \text{if } u = 0, \\ -\nabla u & \text{if } u < 0, \end{cases}$$

a.e. on Ω. In particular, $|\nabla |u|| = |\nabla u|$ a.e. on Ω. If $p < \infty$, then the mappings $u \mapsto u^+$, $u \mapsto u^-$ and $u \mapsto |u|$ are continuous from $W^{1,p}(\Omega)$ into itself.

(ii) *If $u, v \in W^{1,p}(\Omega)$, then $\max(u, v) \in W^{1,p}(\Omega)$ and $\min(u, v) \in W^{1,p}(\Omega)$.*

(iii) *Assume $M \in W^{1,p}_{loc}(\Omega)$ such that $\nabla M \in L^p(\Omega)$. If $M^- \in L^p(\Omega)$, then $(u - M)^+ \in W^{1,p}(\Omega)$ for every $u \in W^{1,p}(\Omega)$, and*

$$\nabla (u - M)^+ = \begin{cases} \nabla u - \nabla M & \text{if } u > M, \\ 0 & \text{if } u \leq M \end{cases} \qquad \text{a.e. on } \Omega.$$

Moreover, if $p < \infty$, then the mapping $u \mapsto (u - M)^+$ is continuous from $W^{1,p}(\Omega)$ into itself. In particular, these results apply to the case where M is a nonnegative constant.

We now consider an open interval $I \subset \mathbb{R}$ (bounded or not) and state a few results concerning functions on I with values in Sobolev sapces and its function spaces, which play an important role of the theory of evolution equations. For the rest of this section, let X be a Banach space with the norm $\|\cdot\|$.

Definition 1.1.16 (Measurable functions)**.** *A function $f : I \to X$ is measurable if there exists a set $E \subset I$ with $|E| = 0$ and a sequence $(f_n)_{n=1}^{\infty} \subset C_c(I; X)$ such that*

$$f_n(t) \to f(t) \qquad \text{as} \quad n \to \infty,$$

for all $t \in I \setminus E$.

Proposition 1.1.17 (Pettis' theorem). *Let $f : I \to X$. Then f is measurable if and only if the following two conditions are satisfied:*

(i) *for every $x^* \in X^*$, the function $I \ni t \mapsto \langle x^*, f(t)\rangle \in \mathbb{R}$ is measurable in the sense of real-valued functions;*

(ii) *there exists a set $N \subset I$ with $|N| = 0$ such that $f(I \setminus N)$ is separable.*

Definition 1.1.18 (Integrable functions). *A measurable function $f : I \to X$ is integrable if there exists a sequence $(f_n)_{n=1}^{\infty} \subset C_c(I; X)$ such that*

$$\int_I \|f_n(t) - f(t)\|dt \to 0 \qquad as \quad n \to \infty.$$

Proposition 1.1.19 (Bochner's theorem). *Let $f : I \to X$ be a measurable function. Then f is integrable if and only if $\|f\|$ is integrable in the sense of $\mathbb{R}$. Moreover, the following holds:*

$$\left\| \int_I f(t)dt \right\| \leq \int_I \|f(t)\|dt.$$

Now, we define Bochner space $L^p(I; X)$. For $1 \leq p \leq \infty$, $L^p(I; X)$ denotes the set of equivalence class of measurable functions $f : I \to X$ such that $I \ni t \to \|f(t)\| \in \mathbb{R}$ belongs to $L^p(I)$. For $f \in L^p(I; X)$, we define its norm

$$\|f\|_{L^p(I;X)} = \begin{cases} \left(\int_I \|f(t)\|^p dt \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ ess\,\sup_{t \in I} \|f(t)\| & \text{if } p = \infty. \end{cases}$$

It is well known that $(L^p(I; X), \|\cdot\|_{L^p(I;X)})$ becomes a Banach space, and if X is reflexive, then $L^p(I; X)$ is also reflexive for $1 < p < \infty$. Moreover, for $1 \leq p < \infty$, if X is reflexive or X^* is separable, then $(L^p(I; X))^* \cong L^{p'}(I; X^*)$. We also define $L^p_{loc}(I; X)$ as the set of measurable functions $f : I \to X$ such that for any compact interval $J \subset I$, $f \in L^p(J; X)$. It is obvious that if I is bounded, then $L^p(I; X) \hookrightarrow L^q(I; X)$ for $1 \leq q \leq p \leq \infty$.

We denote by $W^{1,p}(I; X)$ the space of equivalence class of functions $f \in L^p(I; X)$ such that $f' \in L^p(I; X)$, where the derivative is the sense of $\mathcal{D}'(I; X)$, for $1 \leq p \leq \infty$. For $f \in W^{1,p}(I; X)$, we define

$$\|f\|_{W^{1,p}(I;X)} = \|f\|_{L^p(I;X)} + \|f'\|_{L^p(I;X)}.$$

It is well known that $(W^{1,p}(I; X), \|\cdot\|_{W^{1,p}(I;X)})$ is a Banaca space. We can define $W^{1,p}_{loc}(I; X)$ as well as $L^p_{loc}(I; X)$.

Lemma 1.1.20. *Let $p \in [1, \infty]$ and assume that $f \in L^p(I; X)$. Then the following five properties are equivalent:*

(i) *$f \in W^{1,p}(I; X)$;*

(ii) *there exists $g \in L^p(I; X)$ such that we have*

$$f(t) = f(t_0) + \int_{t_0}^{t} g(s)ds$$

for almost every t_0, $t \in I$;

(iii) *there exists* $g \in L^p(I;X)$, $x_0 \in X$ *and* $t_0 \in I$ *such that we have*

$$f(t) = x_0 + \int_{t_0}^{t} g(s)ds$$

for almost all $t \in I$*;*

(iv) f *is absolutely continuous, differentiable almost everywhere, and* $f' \in L^p(I;X)$*;*

(v) f *is weakly absolutely continuous, weakly differentiable almost everywhere and* $f' \in L^p(I;X)$*.*

1.2 Some Fundamental Lemmas

In this section, we are going to summarize frequently used inequalities and fundamental lemmas.

Lemma 1.2.1 (cf. Showalter [59], Lemma IV.4.1)**.** *Let* a, $b \in L^1(0,T)$ *with* $b \geq 0$*, and let* $y : [0,T] \to \mathbb{R}^+$ *be an absolutely continuous function satisfying*

$$(1-\alpha)y'(t) \leq a(t)y(t) + b(t)y^{\alpha}(t) \qquad a.e.\ t \in [0,T]$$

for some $\alpha \in [0,1)$*. Then the following inequality holds:*

$$y^{1-\alpha}(t) \leq y(0)^{1-\alpha} e^{\int_0^t a(s)ds} + \int_0^t e^{\int_s^t a(\tau)d\tau} b(s)ds, \qquad t \in [0,T].$$

The following two lemmas are necessary to apply L^∞-energy method developed by Ôtani in [45]. This is very useful method to show the local well-posedeness of nonlinear parabolic problems (see the following chapters). It is well known that the choice of the function spaces is crucial role to study of nonlinear partial differential equations and L^2 space or L^p space ($1 \leq p < \infty$) is frequently used to analyze them. Since L^∞ space is not reflexive and separable, L^∞ space is not often directly used and hence L^∞ bounds may be derived from $W^{m,p}$-estimate via Sobolev's imbedding theorem. Therefore a priori bounds of the solutions in L^∞ space are usually subject to the restriction of the space dimension. We can deal with this difficulty by using L^∞-energy method.

Lemma 1.2.2. ([45]) *Let* Ω *be any domain in* $\mathbb{R}^N$ *and assume that exists a number* $r_0 \geq 1$ *and a constant* C *independent of* $r \in [r_0,\infty)$ *such that*

$$\|u\|_{L^r(\Omega)} \leq C \qquad \forall r \in [r_0,\infty),$$

then u *belongs to* $L^\infty(\Omega)$ *and the following property holds.*

$$\lim_{r\to\infty} \|u\|_{L^r(\Omega)} = \|u\|_{L^\infty(\Omega)}. \tag{1.2.1}$$

Conversely, assume that $u \in L^{r_0}(\Omega) \cap L^\infty(\Omega)$ *for some* $r_0 \in [1,\infty)$*, then* u *satisfies* (1.2.1)*.*

Lemma 1.2.3 ([45]). *Let $y(t)$ be a bounded measurable non-negative function on $[0,T]$ and suppose that there exists $y_0 \geq 0$ and a monotone non-decreasing function $m(\cdot) : [0,+\infty) \to [0,+\infty)$ such that*

$$y(t) \leq y_0 + \int_0^t m(y(s))ds \qquad a.e.\ t \in (0,T).$$

Then there exists a number $T_0 = T_0(y_0, m(\cdot)) \in (0,T]$ such that

$$y(t) \leq y_0 + 1 \qquad a.e.\ t \in [0,T_0].$$

Lemma 1.2.4. *Let $p \in (2, 2^*)$, then there exists a constant $\lambda = \lambda(N,p) \in (0,2]$ such that*

$$\|u\|_{2(p-1)}^{2(p-1)} \leq C\|u\|_{H^2(\Omega)}^{2-\lambda}\|u\|_{H^1(\Omega)}^{2p-4+\lambda} \qquad \forall u \in H^2(\Omega) \tag{1.2.2}$$

for some $C > 0$.

Proof. First of all, if $N = 1,\ 2$; or $N \geq 3$ and $p \leq \frac{2(N-1)}{N-2}$, then we can take $\lambda = 2$ by Sobolev's embedding $H^1(\Omega) \subset L^{2(p-1)}(\Omega)$. For the case of $N \geq 3$ and $p > \frac{2(N-1)}{N-2}$, we note that the following Gagliardo-Nirenberg inequality holds:

$$\|v\|_{2(p-1)} \leq C\|v\|_{H^2(\Omega)}^{\theta}\|v\|_{\frac{2N}{N-2}}^{1-\theta} \qquad \forall v \in H^2(\Omega), \tag{1.2.3}$$

where $\theta \in (0,1)$ satisfies

$$\tfrac{1}{2(p-1)} = \theta\left(\tfrac{1}{2} - \tfrac{2}{N}\right) + (1-\theta)\tfrac{N-2}{2N}.$$

Then we see that $\frac{2(N-1)}{N-2} < p < \frac{2N}{N-2}$ implies $0 < \theta = \frac{(N-2)p-2N+2}{2(p-1)} < 1$ and $0 < 2(p-1)\theta = (N-2)p - 2N + 2 < 2$. Since $H^1(\Omega)$ is continuously embedded in $L^{\frac{2N}{N-2}}(\Omega)$, it follows from (1.2.3) that (1.2.2) holds with $\lambda = 2N - (N-2)p \in (0,2)$. □

Lemma 1.2.5. *Let $(a_n^1)_{n\in\mathbb{N}}$, $\cdots$, $(a_n^l)_{n\in\mathbb{N}}$ be a finite family of real-valued sequences for some $l \geq 2$ satisfying*

$$\sum_{k=1}^{l} a_n^k = 0, \qquad \forall\ n \in \mathbb{N}.$$

Suppose that there exist $a^k \in \mathbb{R}$ $(k = 1, \cdots, l)$ such that

$$a^k \leq \liminf_{n\to\infty} a_n^k, \qquad \forall\ k = 1, \cdots, l,$$

$$\sum_{k=1}^{l} a^k \geq 0.$$

Then

$$\lim_{n\to\infty} a_n^k = a^k, \qquad \forall\ k = 1, \cdots, l.$$

Proof. Since it follows from the assumption that

$$a_n^1 = -\sum_{k\neq 1} a_n^k, \qquad \forall\, n \in \mathbb{N},$$

we have

$$\begin{aligned}
\limsup_{n\to\infty} a_n^1 &= \limsup_{n\to\infty} \left(-\sum_{k\neq 1} a_n^k \right) \\
&= -\liminf_{n\to\infty} \left(\sum_{k\neq 1} a_n^k \right) \\
&\leq -\sum_{k\neq 1} \left(\liminf_{n\to\infty} a_n^k \right) \\
&\leq -\sum_{k\neq 1} a^k \leq a^1.
\end{aligned}$$

Therefore, combining the above inequality and the assumption $a^1 \leq \liminf_{n\to\infty} a_n^1$, we can deduce that

$$\lim_{n\to\infty} a_n^1 = a^1.$$

By the similar argument as above for $a^2, \cdots, a^l$, we conclude that

$$\lim_{n\to\infty} a_n^k = a^k, \qquad \forall\, k = 1, \cdots, l.$$

□

1.3 Maximal Monotone Operators

In this section, we define the maximal monotone operators on H, and we also mention some properties of them. In particular, the latter part of this section is devoted to the subdifferential operators, which play an important role throughout this thesis. The definitions and propositions for the maximal monotone operators and the subdifferential operators given in this section are explained in detail in [9, 7].

1.3.1 Maximal Monotone Operators

First let us introduce the notion of (nonlinear) maximal monotone operators. Let H be a real Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $|\cdot|_H$, and let $A : H \to 2^H$ be a possibly multivalued operator with domain $D(A) := \{u \in H\,;\, Au \neq \emptyset\}$. In particular, if an operator A is multivalued, we may identify it with its graph in $H \times H$. For A, $B : H \to 2^H$, $A \subset B$ means that $D(A) \subset D(B)$ and $Ax = Bx$ for all $x \in D(A)$. Moreover $A = B$ if and only if $A \subset B$ and $B \subset A$.

Definition 1.3.1. *An operator $A : H \to 2^H$ is monotone if the following holds:*

$$(y_1 - y_2, x_1 - x_2)_H \geq 0 \qquad \forall x_1, x_2 \in D(A),\ \forall y_1 \in Ax_1,\ \forall y_2 \in Ax_2.$$

In particular, a monotone operator A is maximal if there exists no monotone extension of A, i.e., if $B : H \to 2^H$ is monotone and satisfy $A \subset B$, then $A = B$.

The necessary and sufficient condition for the maximality of a monotone operator is given by the following proposition. In fact, it is often more convenient to show the maximality of a monotone operator by the following equivalent proposition instead of the definition.

Proposition 1.3.2. *Let $A : H \to 2^H$ be a monotone operator. Then the followings are equivalent.*

(i) *A is maximal.*

(ii) *$R(I + A) = H$.*

(iii) *$R(I + \lambda A) = H \qquad \forall\, \lambda > 0$.*

By virtue of this proposition, we know that to show the maximality of some monotone operator, we only need to show the range condition (ii), and it is often equivalent to the existence and the regularity of solutions of the associated nonlinear elliptic equations.

The following notion of closedness of an operator is important when we show the existence of solution to nonlinear evolution equations.

Definition 1.3.3. *An operator $A : H \to 2^H$ is demiclosed if and only if the following condition holds: if $y_n \in Ax_n$ satisfies $x_n \to x$ strongly in H and $y_n \rightharpoonup y$ weakly in H, than $x \in D(A)$ and $y \in Ax$.*

Proposition 1.3.4. *Let $A : H \to 2^H$ be a maximal monotone operator. Then A is demiclosed.*

Proposition 1.3.5. *Let $A : H \to 2^H$ be a maximal monotone operator. Then a set Ax is closed and convex for every $x \in D(A)$.*

By Proposition 1.3.5, we can see that for every $x \in D(A)$, there exists a unique $y_0 \in Ax$ such that $|y_0|_H = \inf\{|y|_H\,;\, y \in Ax\}$. Thus we define a single-valued mapping $A^0 : H \to H$ by

$$A^0 x := \{y_0\,;\, y_0 \in Ax,\ |y_0|_H = \inf_{y \in Ax} |y|_H\}, \qquad x \in D(A^0) := D(A).$$

This operator A^0 is called the canonical restriction (or the minimal section) of A.

Let $A : H \to 2^H$ be a maximal monotone operator. Then by Proposition 1.3.2 we know $R(I + \lambda A) = H$ for all $\lambda > 0$. We define $J_\lambda := (I + \lambda A)^{-1} : H \to D(A)$ as the resolvent of A for $\lambda > 0$, which is a single-valued nonexpansive mapping on H. Now let us introduce the Yosida approximation.

Definition 1.3.6. *Let $A : H \to 2^H$ be a maximal monotone operator. Then the Yosida approximation $A_\lambda : H \to H$ of A is given by*

$$A_\lambda x := \frac{1}{\lambda}(x - J_\lambda x). \tag{1.3.1}$$

Proposition 1.3.7. *Let $A : H \to 2^H$ be a maximal monotone operator. Then the following properties hold.*

(i) $A_\lambda x \in A(J_\lambda x), \qquad \forall\, x \in H,\ \forall\, \lambda > 0.$

(ii) *A_λ is maximal monotone and Lipschitz continuous on H with Lipschitz constant λ^{-1}.*

(iii) $(A_\lambda)_\mu = A_{\lambda+\mu}, \qquad \forall\, \lambda,\, \mu > 0.$

(iv) $|A_\lambda x|_H \le |A^0 x|_H, \qquad \forall\, \lambda > 0,\ \forall\, x \in D(A),$

$\lim_{\lambda\to 0} A_\lambda x = A^0 x, \qquad \forall\, x \in D(A).$

1.3.2 Subdifferential Operators

Let $\phi : H \to (-\infty, +\infty]$ be a functional with the effective domain $\phi := \{u \in H\,;\, \phi(u) < +\infty\}$. Functional ϕ is said to be *proper* if its effective domain $D(\phi)$ is not empty, i.e., $\phi(u) \not\equiv +\infty$ for $u \in H$. Moreover ϕ is said to be *convex* if the following inequality holds:

$$\phi((1-\theta)u + \theta v) \le (1-\theta)\phi(u) + \theta\phi(v), \qquad \forall u, v \in H,\ \forall \theta \in [0,1];$$

and ϕ is said to be *lower semicontinuous* if for $(u_n)_{n\in\mathbb{N}} \subset H$ and $u \in H$ satisfying $u_n \to u$ strongly in H, the following inequality holds:

$$\liminf_{n\to\infty} \phi(u_n) \ge \phi(u).$$

From now on, $\Phi(H)$ denotes the set of all proper convex lower semicontinuous functionals $\phi : H \to (-\infty, +\infty]$.

Let us briefly touch on the notion of subdifferential operators. Let $\phi \in \Phi(H)$. For each $u \in D(\phi)$, the subdifferential $\partial\phi(u)$ of ϕ at u (in H) is defined by

$$\partial\phi(u) := \{f \in H\,;\, \phi(v) - \phi(u) \ge (f, v-u)_H, \quad \forall v \in D(\phi)\}.$$

Then $\partial\phi : H \to 2^H$ becomes a possibly multivalued maximal monotone operator with domain $D(\partial\phi) := \{u \in D(\phi)\,;\, \partial\phi(u) \neq \emptyset\}$, which is called by subdifferential operator.

Proposition 1.3.8. *Let $\phi \in \Phi(H)$. Then the following statements hold.*

(i) *$\partial\phi$ is maximal monotone on H.*

(ii) $D(\partial\phi) \subset D(\phi) \subset \overline{D(\phi)} = \overline{D(\partial\phi)}.$

(iii) *If ϕ is Gâteaux differentiable at u, then $\partial\phi(u) = D_G\phi(u)$.*

For every $\lambda > 0$, we define the functional ϕ_λ by

$$\phi_\lambda(u) := \inf_{v\in H}\left(\frac{1}{2\lambda}|u-v|_H^2 + \phi(v)\right),$$

where $\phi \in \Phi(H)$. The functional ϕ_λ is called the Moreau-Yosida regularization of ϕ.

Proposition 1.3.9. *Let $\phi \in \Phi(H)$. Then the infimum in the definition of ϕ_λ is attained at $J_\lambda x$. More precisely, it follows that*

$$\phi_\lambda(u) = \frac{1}{2\lambda}|J_\lambda u - u|_H^2 + \phi(J_\lambda u) = \frac{\lambda}{2}|(\partial\phi)_\lambda u|_H^2 + \phi(J_\lambda u).$$

Furthermore, the following properties hold.

(i) *ϕ_λ is convex and Fréchet differentiable, in particular, $(\phi_\lambda)' = \partial\phi_\lambda = (\partial\phi)_\lambda$.*

(ii) $\phi(J_\lambda u) \le \phi_\lambda(u) \le \phi(u), \qquad \forall\, u \in H, \forall\, \lambda > 0.$

(iii) $\lim_{\lambda\to 0} \phi_\lambda(u) = \phi(u), \qquad \forall\, u \in H.$

The following two propositions are useful in showing that the sum of subdifferential operators is again maximal.

Proposition 1.3.10. *Let A be a maximal monotone operator on H, and let $\phi \in \Phi(H)$. Assume that there exists $C > 0$ such that*

$$\phi((1+\lambda A)^{-1}u) \le \phi(u) + C\lambda \qquad \forall u \in H, \forall \lambda > 0.$$

Then $A + \partial\phi$ is maximal monotone and the following inequality holds:

$$|A^0 u|_H \le |(A+\partial\phi)^0 u|_H + \sqrt{C} \qquad \forall u \in D(A) \cap D(\partial\phi).$$

Moreover, $\overline{D(A+\partial\phi)} = \overline{D(A)\cap D(\partial\phi)} = \overline{D(A)} \cap \overline{D(\partial\phi)}$.

Proposition 1.3.11. *Let A be a maximal monotone operator on H, and let $\phi \in \Phi(H)$ satisfying*

$$\phi(\mathrm{Proj}_{\overline{D(A)}} u) \le \phi(u) \qquad \forall u \in H.$$

Then the followings are equivalent.

(i) *$\phi((1+\lambda A)^{-1}u) \le \phi(u)$ for all $u \in H$ and $\lambda > 0$.*

(ii) *$(A_\lambda u, v)_H \ge 0$ for all $(u,v) \in \partial\phi$ and $\lambda > 0$.*

Finally we recall the chain rule for $\phi \in \Phi(H)$. The following result plays an important role to derive a priori estimates of solutions to nonlinear evolution equations associated with subdifferential operator in the next section and Chapter 6.

Proposition 1.3.12. *Let $\phi \in \Phi(H)$ and suppose that $u \in W^{1,2}(0,T;H)$ with $u(t) \in D(\partial\phi)$ for a.e. $t \in (0,T)$. If there exists $g \in L^2(0,T;H)$ with $g(t) \in \partial\phi(u(t))$ for a.e. $t \in (0,T)$, then the function $t \mapsto \phi(u(t))$ is absolutely continuous on $[0,T]$ and*

$$\frac{d}{dt}\phi(u(t)) = \left(h(t), \frac{du}{dt}(t)\right)_H \qquad a.e.\ t \in [0,T]$$

for any $h \in \partial\phi(u)$ a.e. on $[0,T]$.

1.3.3 Formulation of (P) as Evolution Equation

Here we are going to formulate (P) as a nonlinear evolution equation associated with a subdifferential operator in $L^2(\Omega)$. Recall our problem:

$$\text{(P)} \qquad \begin{cases} \partial_t u - \Delta u = |u|^{p-2}u, & t > 0,\ x \in \Omega, \\ -\partial_\nu u \in \beta(u), & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

where Ω is bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p > 2$ is a given constant, ν denotes the unit outward normal vector on $\partial\Omega$. The multivalued map β is a maximal monotone operator on $\mathbb{R}$. More precisely, $\beta = \partial j$, where $j : \mathbb{R} \to (-\infty, +\infty]$ is some proper convex lower semicontinuous function.

Let $\varphi : L^2(\Omega) \to (-\infty, +\infty]$ be the functional defined by

$$D(\varphi) := \{u \in H^1(\Omega)\,;\, j(u) \in L^1(\partial\Omega)\},$$

$$\text{(1.3.2)} \qquad \varphi(u) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} j(u) d\sigma, & u \in D(\varphi), \\ +\infty, & u \in L^2(\Omega) \setminus D(\varphi). \end{cases}$$

In [10], Brézis showed the following result.

Proposition 1.3.13. *Let $\varphi : L^2(\Omega) \to (-\infty, +\infty]$ be as above. Then $\varphi \in \Phi(L^2(\Omega))$ and its subdifferential coincides with the Laplacian under nonlinear boundary conditions, namely,*

$$D(\partial\varphi) = \{u \in H^2(\Omega)\,;\, -\partial_\nu u \in \beta(u), \quad \text{a.e. on } \partial\Omega\},$$
$$\partial\varphi(u) = -\Delta u, \qquad u \in D(\partial\varphi).$$

Furthermore the followings hold.

(i) $\overline{D(\partial\varphi)} = L^2(\Omega)$.

(ii) (*Elliptic estimate*) *There exist c_1, $c_2 > 0$ such that*

$$\text{(1.3.3)} \qquad \|u\|_{H^2(\Omega)} \le c_1\| - \Delta u + u\|_2 + c_2, \qquad u \in D(\partial\varphi).$$

Using the above notations, we can rewrite (P) into the following evolution equation in $L^2(\Omega)$:

$$\text{(AC)} \qquad \begin{cases} \dfrac{d}{dt}u(t) + \partial\varphi(u(t)) + B_p(u(t)) = 0, & t > 0, \\ u(0) = u_0, \end{cases}$$

where $B_p : L^2(\Omega) \to L^2(\Omega)$ is defined by $B_p(u) = -|u|^{p-2}u$. In Chapter 3, the local solvability of (P) will be shown based on the abstract theory of evolution equations described in the next section using this formulation (AC).

Note that (AC) has another description. Indeed, we define a functional $\psi_p : L^2(\Omega) \to (-\infty, +\infty]$ by

$$\psi_p(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |u|^p dx, & u \in D(\psi_p) := L^p(\Omega), \\ +\infty, & u \in L^2(\Omega) \setminus L^p(\Omega). \end{cases} \tag{1.3.4}$$

Since we can see that $\psi_p \in \Phi(L^2(\Omega))$, we can define its subdifferential, which coincides with the (single-valued) power type nonlinear operator, i.e.,

$$\partial\psi_p(u) = |u|^{p-2}u, \qquad u \in D(\partial\psi_p) = \{u \in L^2(\Omega)\,;\, |u|^{p-2}u \in L^2(\Omega)\}.$$

By using this notation, (AC) is also represented as

$$\text{(AC)}^* \qquad \begin{cases} \dfrac{d}{dt}u(t) + \partial\varphi(u(t)) - \partial\psi_p(u(t)) = 0, & t > 0, \\ u(0) = u_0, \end{cases}$$

The form of evolution equations with the difference term of two subdifferentials make it easy to understand the energy structure. The asymptotic behavior of solutions to this type equation have been studied by Ôtani [46, 48] and Ishii [29]. In Chapter 3, we are going to clarify the large time behavior of global solutions to (P) via an analysis for $(\mathrm{AC})^*$.

Finally, let $(\mathrm{AC})_q$ or $(\mathrm{AC})_q^*$ denote (AC) with $\beta = \beta_q$ (i.e., $j(r) = \frac{1}{r}|r|^q$) for $q \in (1, +\infty)$.

1.4 Evolution Equations

This section is devoted to summarizing the theory of abstract evolution equations, which will be used to prove the existence of solution in the latter chapters.

We consider the following evolution equation governed by subdifferential operators in a real Hilbert space H:

$$\text{(E)} \qquad \begin{cases} \dfrac{du}{dt}(t) + \partial\phi(u(t)) + B(u(t)) \ni f(t), & t > 0, \\ u(0) = u_0, \end{cases}$$

where $\phi \in \Phi(H)$, $f \in L^2(0, T; H)$ and $u_0 \in \overline{D(\partial\phi)}$. We first assume that B is a Lipschitz continuous on H, that is, there exists $L > 0$ such that

$$|B(u) - B(v)|_H \le L|u - v|_H, \qquad u, v \in H. \tag{1.4.1}$$

Proposition 1.4.1. *Let $\phi \in \Phi(H)$ and let $B : H \to H$ with* (1.4.1)*. Then for every $f \in L^2(0, T; H)$ and $u_0 \in \overline{D(\partial\phi)}$, there exists a unique solution $u \in C([0, T]; H)$ of* (E) *satisfying the following properties.*

(i) *$u(t) \in D(\partial\phi)$ for a.e. $t \in [0, T]$, and u satisfies* (E) *for a.e. on $[0, T]$.*

(ii) *$\sqrt{t}\dfrac{du}{dt}$, $\sqrt{t}g \in L^2(0, T; H)$, where $g(t) \in \partial\phi(u(t))$ for a.e. $t \in [0, T]$.*

(iii) $\phi(u(t)) \in L^1(0,T)$, $t\phi(u(t)) \in L^\infty(0,T)$.

In addition, if $u_0 \in D(\phi)$, *then* $\dfrac{du}{dt}$, $g \in L^2(0,T;H)$.

Remark 1.4.2. *Note that, in Proposition 1.4.1, the function* $t \mapsto \phi(u(t))$ *is absolutely continuous on* $(0,T]$ (resp. $[0,T]$) *for* $u_0 \in \overline{D(\phi)}$ (resp. $u_0 \in D(\phi)$) *by virtue of Proposition 1.3.12.*

This result is well known (e.g., see Proposition 3.12 in Brezis [9]), but we here give a brief proof of the existence of solutions for later use. The argument of a priori estimates in this proof will also be used in Chapter 6.

Proof. For the rest of this proof, let $A = \partial\phi$. We first show the existence of solutions for the case where $u_0 \in D(\phi)$. To do this, we consider the following approximate equation of (E):

$$
(\mathrm{E})_\lambda \qquad \begin{cases} \dfrac{du}{dt}(t) + A_\lambda u(t) + B(u(t)) \ni f(t), & t > 0, \\ u(0) = u_0. \end{cases}
$$

Here A_λ is the Yosida approximation of A, and note that $A_\lambda = (\partial\phi)_\lambda = \partial\phi_\lambda$. It is well known that the corresponding integral equation of $(\mathrm{E})_\lambda$ possesses a unique solution $u_\lambda \in C([0,T];H)$. We are going to derive a priori estimates of solution to $(\mathrm{E})_\lambda$.

For fixed $v \in D(A)$, we set

$$
\tilde{\phi}_\lambda(u) := \phi_\lambda(u) - \phi_\lambda(v) - (A_\lambda v, u - v)_H.
$$

Then we easily verify

$$
\tilde{\phi}_\lambda(u) \geq 0, \qquad u \in D(\phi), \tag{1.4.2}
$$

$$
\tilde{\phi}_\lambda(v) = 0. \tag{1.4.3}
$$

Moreover put $\tilde{B}(u) := B(u) - B(0)$. Then $\tilde{B}$ is Lipschitz continuous on H which satisfy $\tilde{B}(0) = 0$. By using $\tilde{\phi}$ and $\tilde{B}$, the equation $(\mathrm{E})_\lambda$ can be rewritten in the following equation.

$$
\frac{du_\lambda}{dt} + \partial\tilde{\phi}_\lambda(u_\lambda(t)) + \tilde{B}(u_\lambda(t)) \ni f(t) - A_\lambda v + B(0) =: \tilde{f}(t).
$$

We here note that the estimates of $\tilde{f}$ is independent of λ. Indeed, by using Proposition 1.3.7,

$$
\begin{aligned}
&|\tilde{f}(t)|_H \leq |f(t)|_H + |A^0 v|_H + |B(0)|_H, \\
&\|\tilde{f}\|^2_{L^2(0,T;H)} \leq 3\left(\|f\|^2_{L^2(0,T;H)} + T|A^0 v|^2_H + T|B(0)|^2_H\right).
\end{aligned}
$$

Thus, without loss of generality, we can assume that (1.4.2), (1.4.3) and $B(0) = 0$ hold.

Multiplying $(\mathrm{E})_\lambda$ by $u_\lambda - v$, we have

$$\frac{1}{2}\frac{d}{dt}|u_\lambda(t)-v|_H^2 + (A_\lambda u_\lambda, u_\lambda - v)_H + (Bu_\lambda, u_\lambda - v)_H \le |f(t)|_H|u_\lambda(t)-v|_H.$$

By the definition of subdifferential of ϕ_λ, (1.4.2) and (1.4.3), we can see that

$$(A_\lambda u_\lambda, u_\lambda - v)_H \ge \phi_\lambda(u_\lambda) - \phi_\lambda(v) \ge 0.$$

Therefore we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_\lambda(t)-v|_H^2 &\le |f(t)|_H|u_\lambda(t)-v|_H + |(Bu_\lambda - Bv, u_\lambda - v)_H| + |(Bv, u_\lambda - v)_H| \\ &\le (|f(t)|_H + |Bv|_H)\,|u_\lambda(t)-v|_H + L|u_\lambda(t)-v|_H^2\end{aligned}$$

Applying Lemma 1.2.1 with $\alpha = 1/2$, $y(t) = |u_\lambda(t)-v|_H^2$, $a(t) = L$ and $b(t) = |f(t)|_H + |Bv|_H$, we obtain

$$|u_\lambda(t)-v|_H \le |u_0 - v|_H e^{LT} + e^{LT}\int_0^T (|f(t)|_H + |Bv|_H)\,dt, \qquad t\in[0,T],$$

which implies

$$\sup_{t\in[0,T]}|u_\lambda(t)|_H \le |v|_H + e^{LT}\left(|u_0 - v|_H + \|f\|_{L^1(0,T;H)} + T|Bv|_H\right) =: C_1. \tag{1.4.4}$$

Multiplying $(\mathrm{E})_\lambda$ by $\dfrac{du_\lambda}{dt}$, we have

$$\left|\frac{du_\lambda}{dt}\right|_H^2 + \left(A_\lambda u_\lambda, \frac{du_\lambda}{dt}\right)_H + \left(B(u_\lambda), \frac{du_\lambda}{dt}\right)_H = \left(f(t), \frac{du_\lambda}{dt}\right)_H.$$

Note that, by virtue of Proposition 1.3.9, since ϕ_λ is Fréchet differentiable and $(\phi_\lambda)' = A_\lambda$, we see that

$$\left(A_\lambda u_\lambda, \frac{du_\lambda}{dt}\right)_H = \frac{d}{dt}\phi_\lambda(u_\lambda(t)).$$

Hence, applying Schwarz inequality and Young inequality, we get

$$\begin{aligned}\left|\frac{du_\lambda}{dt}\right|_H^2 + \frac{d}{dt}\phi_\lambda(u_\lambda(t)) &\le \left|\left(B(u_\lambda), \frac{du_\lambda}{dt}\right)_H\right| + \left(f(t), \frac{du_\lambda}{dt}\right)_H \\ &\le |Bu_\lambda|_H^2 + \frac{1}{4}\left|\frac{du_\lambda}{dt}\right|_H^2 + |f(t)|_H^2 + \frac{1}{4}\left|\frac{du_\lambda}{dt}\right|_H^2.\end{aligned}$$

By (1.4.1) and (1.4.4), we obtain

$$\frac{1}{2}\left|\frac{du_\lambda}{dt}\right|_H^2 + \frac{d}{dt}\phi_\lambda(u_\lambda(t)) \le L^2C_1^2 + |f(t)|_H^2.$$

Noting that $\phi_\lambda(u) \le \phi(u)$, we have

$$\sup_{t\in[0,T]} \phi_\lambda(u_\lambda(t)) + \frac{1}{2}\int_0^T \left|\frac{du_\lambda}{dt}\right|_H^2 dt \le \phi(u_0) + TL^2C_1^2 + \|f\|^2_{L^2(0,T;H)} =: C_2. \tag{1.4.5}$$

Furthermore since $A_\lambda u_\lambda = f - Bu_\lambda - \dfrac{du_\lambda}{dt}$, we can also derive

$$\|A_\lambda u_\lambda\|^2_{L^2(0,T;H)} \le 3\left(\|f\|^2_{L^2(0,T;H)} + TL^2C_1^2 + 2C_2\right) =: C_3. \tag{1.4.6}$$

We remark that C_1, C_2 and C_3 are independent of λ.

Put $w_{\lambda,\mu}(t) = u_\lambda(t) - u_\mu(t)$. Then

$$\frac{d}{dt}w_{\lambda,\mu}(t) + A_\lambda u_\lambda - A_\mu u_\mu + Bu_\lambda - Bu_\mu = 0. \tag{1.4.7}$$

By using the monotonicity of A and $A_\lambda u \in A(J_\lambda u)$, we see that

$$\begin{aligned}(A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu)_H &= (A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu)_H \\ &\qquad + (A_\lambda u_\lambda - A_\mu u_\mu, J_\lambda u_\lambda - J_\mu u_\mu)_H \\ &\ge \lambda|A_\lambda u_\lambda|_H^2 + \mu|A_\mu u_\mu|_H^2 - (\lambda+\mu)(A_\lambda u_\lambda, A_\mu u_\mu)_H \\ &\ge \lambda|A_\lambda u_\lambda|_H^2 + \mu|A_\mu u_\mu|_H^2 - (\lambda+\mu)|A_\lambda u_\lambda|_H|A_\mu u_\mu|_H \\ &\ge -\frac{\lambda}{4}|A_\mu u_\mu|_H^2 - \frac{\mu}{4}|A_\lambda u_\lambda|_H^2.\end{aligned}$$

Multiplying (1.4.7) by $w_{\lambda,\mu}$ and using the above inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w_{\lambda,\mu}(t)|_H^2 &= -(A_\lambda u_\lambda - A_\mu u_\mu, w_{\lambda,\mu})_H - (Bu_\lambda - Bu_\mu, w_{\lambda,\mu})_H \\ &\le \frac{\lambda}{4}|A_\mu u_\mu(t)|_H^2 + \frac{\mu}{4}|A_\lambda u_\lambda(t)|_H^2 + L|w_{\lambda,\mu}(t)|_H^2.\end{aligned}$$

By virtue of (1.4.6), integrating this inequality over $[0,t]$, we obtain

$$|w_{\lambda,\mu}(t)|_H^2 \le \frac{\lambda+\mu}{4}C_3 + L\int_0^t |w_{\lambda,\mu}(s)|_H^2 ds.$$

By Gronwall's inequality we can deduce that

$$\sup_{t\in[0,T]} |w_{\lambda,\mu}(t)|_H^2 \le e^{LT}C_3\frac{\lambda+\mu}{4},$$

which implies that $(u_{\lambda_n})_{n\in\mathbb{N}}$ is Cauchy sequence in $C([0,T];H)$, where (λ_n) is a sequence satisfying $\lambda_n \to 0$ as $n \to +\infty$. Therefore there exists $u \in C([0,T];H)$ such that

$$u_{\lambda_n} \to u \qquad \text{strongly in} \quad C([0,T];H). \tag{1.4.8}$$

Moreover we can get

$$J_{\lambda_n} u_{\lambda_n} \to u \qquad \text{strongly in} \quad L^2(0,T;H). \tag{1.4.9}$$

Indeed, by (1.4.6),

$$\begin{aligned}
\|J_\lambda u_\lambda - u\|_{L^2(0,T;H)} &\le \|J_\lambda u_\lambda - u_\lambda\|_{L^2(0,T;H)} + \|u_\lambda - u\|_{L^2(0,T;H)} \\
&= \lambda \|A_\lambda u_\lambda\|_{L^2(0,T;H)} + \|u_\lambda - u\|_{L^2(0,T;H)} \\
&\le \lambda\sqrt{C_3} + \sqrt{T} \sup_{t\in[0,T]} |u_\lambda(t) - u(t)|_H \\
&\to 0 \qquad\qquad \text{as } \lambda \to 0.
\end{aligned}$$

On the other hand, since $A_\lambda u_\lambda$ and $\dfrac{d}{dt}u_\lambda$ are bounded in $L^2(0,T;H)$ by (1.4.5) and (1.4.6), there exists a subsequence of (n) denoted by (n) again such that

$$A_{\lambda_n} u_{\lambda_n} \rightharpoonup g \qquad \text{weakly in} \quad L^2(0,T;H), \tag{1.4.10}$$

$$\frac{d}{dt} u_{\lambda_n} \rightharpoonup \chi \qquad \text{weakly in} \quad L^2(0,T;H), \tag{1.4.11}$$

for some g and $\chi \in L^2(0,T;H)$. Let $\mathcal{A}$ be a realization of A on $L^2(0,T;H)$, that is,

$$(\mathcal{A}u)(t) := A(u(t)) \qquad \text{a.e. } t \in [0,T].$$

Since A is a maximal monotone operator, $\mathcal{A}$ is also maximal monotone. By (1.4.9), (1.4.10) and the demiclosedness of $\mathcal{A}$, we have

$$u \in D(\mathcal{A}), \quad g \in \mathcal{A}u,$$

which is equivalent to

$$u(t) \in D(A), \quad g(t) \in Au(t) = \partial\phi(u(t)), \qquad\qquad \text{a.e. } t \in [0,T].$$

Since $\dfrac{d}{dt}$ is a closed linear operator in $L^2(0,T;H)$, by (1.4.8) and (1.4.11), we get

$$\chi = \frac{d}{dt}u.$$

Moreover since B is Lipschitz continuous, it is clear that

$$B(u_{\lambda_n}) \to B(u) \qquad \text{strongly in} \quad C([0,T];H).$$

Then we conclude that

$$\frac{d}{dt}u(t) + g(t) + B(u(t)) = f(t) \qquad \text{in } H \quad \text{a.e. } t \in [0,T],$$

where $g(t) \in \partial\phi(u(t))$. Furthermore, by (1.4.8), we have $u(+0) = u_0$.

We next consider the case where $u_0 \in \overline{D(\phi)} = \overline{D(\partial\phi)}$. Let $(u_0^n) \subset D(\phi)$ be a sequence satisfying $u_0^n \to u_0$ strongly in H as $n \to \infty$, and let u^n be a solution to (E) with the initial data u_0^n for $n \in \mathbb{N}$. In the same way as the former part of the proof, we can deduce that there exists $C_4 > 0$ independent of n such that

$$\sup_{t\in[0,T]} |u^n(t)|_H \leq C_4. \tag{1.4.12}$$

Multiplying (E) by $t\dfrac{du^n}{dt}$, we have

$$t\left|\frac{du^n}{dt}\right|_H^2 + t\frac{d}{dt}\phi(u^n(t)) + t\left(B(u^n), \frac{du^n}{dt}\right)_H = t\left(f(t), \frac{du^n}{dt}\right)_H.$$

By integrating it over $[0,t]$ and using Schwarz's inequality and Young's inequality, we get

$$\begin{aligned}\frac{1}{2}\int_0^t s\left|\frac{du^n}{ds}\right|_H^2 ds + t\phi(u^n(t)) \\ \leq \int_0^t s\,|B(u^n(s))|_H^2\,ds + \int_0^t s\,|f(s)|_H^2\,ds + \int_0^t \phi(u^n(s))ds.\end{aligned} \tag{1.4.13}$$

On the other hand, for $g^n(t) \in \partial\phi(u^n(t))$, by the definition of the subdifferential operator, we see that

$$\begin{aligned}\phi(u^n(t)) &\leq (g^n, u^n - v)_H \\ &= \left(f(t) - \frac{du^n}{dt} - B(u^n), u^n - v\right)_H \\ &\leq |f(t)|_H|u^n(t) - v|_H - \frac{1}{2}\frac{d}{dt}|u^n(t) - v|_H^2 + L|u^n(t) - v|_H^2 + |Bv|_H|u^n(t) - v|_H,\end{aligned}$$

whence follows

$$\begin{aligned}\int_0^t \phi(u^n(s))ds \leq \sup_{t\in[0,T]} |u^n(t) - v|_H \left(\|f\|_{L^1(0,T;H)} + T|Bv|_H\right) \\ + \frac{1}{2}|u_0^n - v|_H^2 + LT\sup_{t\in[0,T]} |u^n(t) - v|_H^2.\end{aligned} \tag{1.4.14}$$

Combining (1.4.12), (1.4.13), (1.4.14) and the fact $u_0^n \to u_0$ strongly in H, we see that there exists $C_5 > 0$ independent of n such that

$$\int_0^T t\left|\frac{du^n}{dt}\right|_H^2 dt \leq C_5, \quad t\phi(u^n(t)) \leq C_5 \qquad \forall\, t \in [0,T]. \tag{1.4.15}$$

Moreover since $g^n = f(t) - \dfrac{du^n}{dt} - B(u^n)$, we also see that there exists $C_6 > 0$ independent of n such that

$$\int_0^T t\,|g^n(t)|_H^2 dt \leq C_6. \tag{1.4.16}$$

Put $w^{n,m}(t) = u^n(t) - u^m(t)$. Then

$$\frac{d}{dt}w^{n,m}(t) + \partial\phi(u^n(t)) - \partial\phi(u^m(t)) + B(u^n(t)) - B(u^m(t)) \ni 0.$$

By the monotonicity of $\partial\phi$, it follows from Gronwall's inequality

$$|w^{n,m}(t)|_H^2 \le e^{2LT}|u_0^n - u_0^m|_H^2.$$

Hence since (u^n) is a Cauchy sequence in $C([0,T];H)$, there exists $u \in C([0,T];H)$ such that

$$u^n \to u \qquad \text{strongly in} \quad C([0,T];H). \tag{1.4.17}$$

As before, by (1.4.15) and (1.4.16), taking a subsequence of (n) (which is denoted by (n) again), we conclude that

$$\sqrt{t}g^n \rightharpoonup \sqrt{t}g \qquad \text{weakly in} \quad L^2(0,T;H),$$
$$\sqrt{t}\frac{du^n}{dt} \rightharpoonup \sqrt{t}\frac{du}{dt} \qquad \text{weakly in} \quad L^2(0,T;H),$$

and u is a desired solution to (E). □

Remark 1.4.3. *In proposition 1.4.1, the assertions (i) - (iii) are still valid with the assumptions $f \in L^1(0,T;H)$ and $\sqrt{t}f \in L^2(0,T;H)$ instead of $f \in L^2(0,T;H)$ by considering slight modification in the above proof.*

We next introduce the abstract theory for some nonlinear evolution equations (E) associated with subdifferential operators with non-monotone perturbations B in a real Hilbert space H, which is developed by Ôtani [47]. In order to formulate a solvability result, we impose the following assumptions.

(A1) For any $L > 0$, the set $\{u \in H\,;\, \phi(u) + |u|_H^2 \le L\}$ is compact in H.

(A2) $B : H \to H$ satisfies the following ϕ-demiclosedness condition:
If $u_n \to u$ strongly in $C([0,T];H)$, $\partial\phi(u_n) \rightharpoonup \partial\phi(u)$ weakly in $L^2(0,T;H)$ and $B(u_n) \rightharpoonup b$ weakly in $L^2(0,T;H)$, then $b = B(u)$ holds a.e. in $t \in [0,T]$.

(A3) There exist a monotone increasing function $\ell(\cdot) : [0,\infty) \to [0,\infty)$ and $k \in [0,1)$ such that
$$|B(u)|_H^2 \le k|\partial\phi(u)|_H^2 + \ell(\phi(u) + |u|_H) \qquad \forall u \in D(\partial\phi).$$

The following result is a simplified version of the existence result which is founded in Ôtani [47], and note that more general setting is investigated in [47], e.g., B is a multivalued mapping and B, ϕ have a t-dependence.

Proposition 1.4.4. *[47] Let $\phi \in \Phi(H)$ and the assumptions* (A1) - (A3) *be satisfied. Then for any $u_0 \in D(\phi)$ and $f \in L^2(0,T;H)$, there exists a positive number $T_0 = T_0(|u_0|_H, \phi(u_0)) \in [0,T]$ such that the abstract Cauchy problem* (E) *in H possesses a strong solution $u \in C([0,T_0];H)$ such that*

$$\frac{d}{dt}u,\ \partial\phi(u),\ B(u) \in L^2(0,T_0;H). \tag{1.4.18}$$

1.5 Convergence of Functionals

In this section, we define Mosco convergence and mention some properties of it according to Attouch [4, 5] as the preparation for Chapter 6.

Mosco convergence is defined as follows.

Definition 1.5.1. *Let $(\phi^n) \subset \Phi(H)$ and $\phi \in \Phi(H)$. Then $\phi^n \to \phi$ in the sense of Mosco on H as $n \to \infty$ (denoted by $\phi^n \xrightarrow{M} \phi$), if the following two conditions (i) and (ii) hold:*

(i) *For every $u \in D(\phi)$, there exists a sequence $(u_n) \subset H$ such that $u_n \to u$ strongly in H and $\phi^n(u_n) \to \phi(u)$.*

(ii) *If $u_n \rightharpoonup u$ weakly in H, then $\phi(u) \leq \liminf_{n\to\infty} \phi^n(u_n)$.*

Remark 1.5.2. *In Definition 1.5.1, the condition $\phi^n(u_n) \to \phi(u)$ in the assumption (i) can be replaced by $\phi(u) \geq \limsup_{n\to\infty} \phi^n(u_n)$.*

We present some properties of Mosco convergence.

Proposition 1.5.3 (Theoreme 1.10 [4])**.** *For $(\phi^n) \subset \Phi(H)$ and $\phi \in \Phi(H)$, the following statements are equivalent.*

(i) $\phi^n \xrightarrow{M} \phi$,

(ii) (a) $(1 + \lambda\partial\phi^n)^{-1}u \to (1 + \lambda\partial\phi)^{-1}u$ *for all $\lambda > 0$ and $u \in H$,*

(b) *there exists $(u, v) \in \partial\phi$ and $(u_n, v_n) \in \partial\phi^n$ such that $u_n \to u$, $v_n \to v$ and $\phi^n(u_n) \to \phi(u)$.*

(iii) $\phi^n_\lambda(u) \to \phi_\lambda(u)$ *for all $\lambda > 0$ and $u \in H$, where ϕ^n_λ and ϕ_λ are the Moerau-Yosida regularizations of ϕ^n and ϕ respectively.*

Lemma 1.5.4 (Corollaire 1.17 [4])**.** *Let $(\phi^n) \subset \Phi(H)$ and $\phi \in \Phi(H)$ satisfying $\phi^n \xrightarrow{M} \phi$ on H. Assume that H is separable. Then,*

$$\int_0^T \phi^n dt \to \int_0^T \phi dt \qquad \text{in the sense of Mosco on } L^2(0,T;H).$$

For $\phi : H \to (-\infty, +\infty]$ with $D(\phi) \neq \emptyset$, we define the conjugate function (or the Fenchel-Legendre transform) $\phi^* : H \to (-\infty, +\infty]$ to be

$$\phi^*(f) := \sup_{x\in H} \{ (f\,,\, x)_H - \phi(x) \}.$$

It is known that ϕ^* is convex and lower semicontinuous on H, and if $\phi \in \Phi(H)$ then $\phi^* \in \Phi(H)$. Moreover, for $\phi \in \Phi(H)$, it holds that $x^* \in \partial\phi(x)$ if and only if

$$\phi(x) + \phi^*(x^*) = (x^*\,,\, x)_H. \tag{1.5.1}$$

In particular, for Mosco convergence of conjugate functions, the following holds.

Lemma 1.5.5 (Proposition 1.19 [4]). *Let* $(\phi^n) \subset \Phi(H)$ *and* $\phi \in \Phi(H)$. *Then* $\phi^n \xrightarrow{M} \phi$ *if and only if* $(\phi^n)^* \xrightarrow{M} \phi^*$.

Lemma 1.5.6. *Let* $(\phi^n)_{n\geq 1} \subset \Phi(H)$ *and* $\phi \in \Phi(H)$. *Suppose that if for any subsequence* $(\phi^{n_k})_{k\geq 1}$ *of* $(\phi^n)_{n\geq 1}$, *there exists a subsequence* $(\phi^{n_{k'}})_{k'\geq 1}$ *of* $(\phi^{n_k})_{k\geq 1}$ *such that* $(\phi^{n_{k'}})_{k'\geq 1}$ *converges to* ϕ *in the sense of Mosco on* H, *then* $\phi^n \xrightarrow{M} \phi$.

Proof. We are going to prove this lemma by contradiction. Let $(\phi^n) \subset \Phi(H)$ and $\phi \in \Phi(H)$, and suppose that ϕ^n does not converge to ϕ in the sense of Mosco on H. Then, by Proposition 1.5.3, we see that there exist $\lambda_0 > 0$, $u_0 \in H$ and $\epsilon_0 > 0$ such that for all $N \in \mathbb{N}$ there exists $n(N) \geq N$ such that

$$\left|\phi_{\lambda_0}^{n(N)}(u_0) - \phi_{\lambda_0}(u_0)\right| \geq \epsilon_0.$$

Put $n_1 := n(1) \geq 1$. Then

$$\left|\phi_{\lambda_0}^{n_1}(u_0) - \phi_{\lambda_0}(u_0)\right| \geq \epsilon_0.$$

Similarly, put $n_2 := n(n_1 + 1) \geq n_1 + 1 > n_1$. Then

$$\left|\phi_{\lambda_0}^{n_2}(u_0) - \phi_{\lambda_0}(u_0)\right| \geq \epsilon_0.$$

Define $(n_k)_{k\geq 1}$ by inductive as above. By the definition of (n_k), we see that $(\phi_{\lambda_0}^{n_k}(u_0))_{k\geq 1} \subset \mathbb{R}$ is a subsequence of $(\phi_{\lambda_0}^{n}(u_0))_{n\geq 1}$ and any subsequence of $(\phi_{\lambda_0}^{n_k}(u_0))_{k\geq 1}$ cannot converge to $\phi_{\lambda_0}(u_0)$ in $\mathbb{R}$. By Proposition 1.5.3, this lead a contradiction. □

Chapter 2

Local Well-posdness

In this chapter, we are concerned with the local well-posedness of $(\mathrm{P})_q$. For the first step, we are going to give a proof of the local well-posedness of $(\mathrm{P})_q$ for the case where the initial data belong to the domain of a functional associated with Laplacian under nonlinear boundary conditions. Moreover we also discuss the case where the initial data are bounded. Since it is difficult to use the Duhamel's principle and to apply the fixed point theorem for the integral equation, we mainly rely on the theory of nonlinear evolution equations developed by [47, 45].

2.1 Local Well-posdness for $D(\varphi)$-data

We first show the existence of time local solutions of $(\mathrm{P})_q$ for the initial values which belong to the effective domain $D(\varphi)$ of φ (note that $D(\varphi) \subset H^1(\Omega)$). We here emphasize that even though $\partial\varphi(u) = -\Delta u$ looks like a linear operator, this is not the case since $D(\partial\varphi)$ does not have the linear structure. Therefore, as mentioned above, we can not rely on the Duhamel principle (see also Introduction). Instead, we here rely on the abstract theory of nonlinear evolution equations associated with subdifferential operators given in Proposition 1.4.4. Our first main theorem can be stated as follows.

Theorem 2.1.1. *Let $p \in (2, 2^*)$ and $u_0 \in D(\varphi)$. Then there exists $T_0 = T_0(\varphi(u_0)) > 0$ such that $(\mathrm{P})_q$ possesses a unique solution u satisfying the following regularity*

$$\begin{aligned} &u \in C([0, T_0]; L^2(\Omega)), \\ &\partial_t u,\ \Delta u,\ |u|^{p-2}u \in L^2(0, T_0; L^2(\Omega)). \end{aligned} \tag{2.1.1}$$

Proof. (Existence) Recall that $(\mathrm{P})_q$ is reduced to $(\mathrm{AC})_q$. In order to show the existence of a solution of $(\mathrm{AC})_q$, we are going to apply Proposition 1.4.4. To do this, we have only to check three assumptions (A1), (A2) and (A3). It is clear that (A1) follows from the boundedness of the domain Ω and the Rellich-Kondrachov compactness theorem. Since $-B_p(u)$ is maximal monotone and the maximal monotone operator satisfies the demi-closedness property (in the standard sense), assumption (A2) is also satisfied. To verify (A3), by Lemma 1.2.4, we note that there exists $\lambda = \lambda(p, N) \in (0, 2]$ such that

$$\|u\|_{2(p-1)}^{2(p-1)} \le C\|u\|_{H^2(\Omega)}^{2-\lambda}\|u\|_{H^1(\Omega)}^{2p-4+\lambda} \qquad \forall u \in H^2(\Omega). \tag{2.1.2}$$

Then by virtue of (2.1.2), the elliptic estimate (1.3.3) and Young's inequality, we obtain

$$\begin{aligned}\|B(u)\|_2^2 &= \|u\|_{2(p-1)}^{2(p-1)} \\ &\le C\|u\|_{H^2(\Omega)}^{2-\lambda}\|u\|_{H^1(\Omega)}^{2p-4+\lambda} \\ &\le C\left(\|-\Delta u+u\|_2^{2-\lambda}+1\right)\|u\|_{H^1(\Omega)}^{2p-4+\lambda} \\ &\le C\left(\|-\Delta u\|_2^{2-\lambda}+\|u\|_2^{2-\lambda}+1\right)\|u\|_{H^1(\Omega)}^{2p-4+\lambda} \\ &\le k\|-\Delta u\|_2^2+C\|u\|_{H^1(\Omega)}^{\frac{2(2p-4+\lambda)}{\lambda}}+C\left(\|u\|_2^{2-\lambda}+1\right)\|u\|_{H^1(\Omega)}^{2p-4+\lambda},\end{aligned}$$

which ensures (A3). Thus, Proposition 1.4.4 implies that $(\mathrm{P})_q$ admits a local solution $u \in C([0,T_0];L^2(\Omega))$ satisfying (1.4.18).

(Uniqueness) Let u and v be two solutions of $(\mathrm{P})_q$ on $[0,T_0]$ with the initial data $u_0 \in D(\varphi)$ and $v_0 \in D(\varphi)$ respectively. Setting $w := u - v$, we have

$$(\mathrm{P}_w) \qquad \begin{cases}\partial_t w-\Delta w=|u|^{p-2}u-|v|^{p-2}v, & t>0,\ x\in\Omega,\\ \partial_\nu w+|u|^{q-2}u-|v|^{q-2}v=0, & t>0,\ x\in\partial\Omega,\\ w(0,x)=u_0(x)-v_0(x), & x\in\Omega.\end{cases}$$

Multiplying (P_w) by w and using integration by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2+\|\nabla w(t)\|_2^2+\int_{\partial\Omega}\left(|u|^{q-2}u-|v|^{q-2}v\right)w\,d\sigma \\ &\quad=\int_\Omega\left(|u|^{p-2}u-|v|^{p-2}v\right)w\,dx.\end{aligned}$$

Since $u \mapsto |u|^{q-2}u$ is monotone increasing, it is easy to see that

$$\int_{\partial\Omega}\left(|u|^{q-2}u-|v|^{q-2}v\right)w\,d\sigma \ge 0.$$

Moreover we note

$$\left||x|^{p-2}x-|y|^{p-2}y\right|=\left|\int_x^y(p-1)|s|^{p-2}ds\right|\le(p-1)\left(|x|^{p-2}+|y|^{p-2}\right)|x-y|$$

for all x, $y \in \mathbb{R}^1$. Hence, from Hölder's inequality and the above inequality, it follows that

$$\begin{aligned}\int_\Omega\left(|u|^{p-2}u-|v|^{p-2}v\right)w\,dx &\le (p-1)\int_\Omega\left(|u|^{p-2}+|v|^{p-2}\right)w^2dx \\ &\le (p-1)\left(\|u(t)\|_p^{p-2}+\|v(t)\|_p^{p-2}\right)\|w(t)\|_p^2.\end{aligned} \tag{2.1.3}$$

We here recall the following Gagliardo-Nirenberg interpolation inequality on a bounded domain (see [43])

$$\|u\|_p \le C\left(\|\nabla u\|_2^\eta\|u\|_2^{1-\eta}+\|u\|_2\right) \qquad \forall u \in H^1(\Omega),$$

where $\eta \in (0,1)$ is determined by $\frac{1}{p} = \eta\left(\frac{1}{2} - \frac{1}{N}\right) + (1-\eta)\frac{1}{2}$. Applying this inequality and Young's inequality to (2.1.3), we obtain

$$
\begin{aligned}
&\int_\Omega \left(|u|^{p-2}u - |v|^{p-2}v\right) w\, dx \\
&\le (p-1)\left(\|u(t)\|_p^{p-2} + \|v(t)\|_p^{p-2}\right)\|w(t)\|_p^2 \\
&\le C\left(\|u(t)\|_p^{p-2} + \|v(t)\|_p^{p-2}\right)\left(\|\nabla w(t)\|_2^{2\eta}\|w(t)\|_2^{2(1-\eta)} + \|w(t)\|_2^2\right) \\
&\le \frac{1}{2}\|\nabla w(t)\|_2^2 + C\left(\|u_1(t)\|_p^{p-2} + \|u_2(t)\|_p^{p-2}\right)^{\frac{1}{1-\eta}}\|w(t)\|_2^2 \\
&\qquad + C\left(\|u_1(t)\|_p^{p-2} + \|u_2(t)\|_p^{p-2}\right)\|w(t)\|_2^2.
\end{aligned}
$$

Since u and v satisfy the regularity (1.4.18) of Proposition 1.4.4, $\varphi(u)$ and $\varphi(v)$ are absolute continuous on $[0,T_0]$ (see [9]). Noting that $p \in (2,2^*)$ implies $\|u\|_p \le C(\varphi(u) + \|u\|_2^2)^{1/2}$, we deduce that $\|u\|_p$ and $\|v\|_p$ are bounded above by some constant $M > 0$ uniformly on $[0,T_0]$. Thus we get

$$
\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 + \frac{1}{2}\|\nabla w(t)\|_2^2 \le C\left((2M^{p-2})^{\frac{1}{1-\eta}} + 2M^{p-2}\right)\|w(t)\|_2^2.
$$

Then by Gronwall's inequality (see Lemma 1.2.1 with $\alpha = 0$), we derive

$$
\|u(t) - v(t)\|_2^2 \le \|u_0 - v_0\|_2^2\, e^{2C\left((2M^{p-2})^{\frac{1}{1-\eta}} + 2M^{p-2}\right)t} \qquad \forall t \in [0,T_0],
$$

whence follows the uniqueness. □

2.2 Local Well-posdness for $L^\infty(\Omega)$-data

In this section, we are going to show the local well-posedness of $(\mathrm{P})_q$ in $L^\infty(\Omega)$ without any restriction on the growth order p. Applying "L^∞-energy method" developed in [45], we can obtain the following result of the local well-posedness of $(\mathrm{P})_q$ in $L^\infty(\Omega)$.

Theorem 2.2.1. *Let $u_0 \in L^\infty(\Omega)$, then there exists $T_0 = T_0(\|u_0\|_\infty) > 0$ such that $(\mathrm{P})_q$ possesses a unique solution u satisfying the following regularity*

$$
\begin{aligned}
&u \in C([0,T_0]; L^2(\Omega)) \cap L^\infty(0,T_0; L^\infty(\Omega)), \\
&\sqrt{t}\,\partial_t u,\ \sqrt{t}\,\Delta u,\ \sqrt{t}\,|u|^{p-2}u \in L^2(0,T_0; L^2(\Omega)).
\end{aligned}
\tag{2.2.1}
$$

Proof. (Uniqueness) Let u and v be two solutions of $(\mathrm{P})_q$ with the same initial data $u_0 \in L^\infty(\Omega)$ satisfying the regularity (2.2.1). Then $w := u - v$ satisfies (P_w) with $w(0) = 0$. Multiplying (P_w) by w, we now get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 &\le \int_\Omega \left(|u|^{p-2}u - |v|^{p-2}v\right) w\, dx \\
&\le (p-1)\int_\Omega \left(|u|^{p-2} + |v|^{p-2}\right) w^2 dx \\
&\le (p-1)\left(\|u\|_{L^\infty(0,T;L^\infty(\Omega))}^{p-2} + \|v\|_{L^\infty(0,T;L^\infty(\Omega))}^{p-2}\right)\|w(t)\|_2^2 \\
&\le C\,\|w(t)\|_2^2,
\end{aligned}
$$

whence it follows from Gronwall's inequality

$$\|w(t)\|_2^2 \le \|w(0)\|_2^2 e^{2CT} = 0 \qquad \forall t \in (0,T).$$

Thus, for $u_0 \in L^\infty(\Omega)$, the solution to $(\mathrm{P})_q$ satisfying (2.2.1) is unique.

(Existence) We here consider the following auxiliary problem:

$$\begin{cases} \partial_t u - \Delta u = |[u]_M|^{p-2}\, u, & t>0,\ x\in\Omega, \\ \partial_\nu u + |u|^{q-2}u = 0, & t>0,\ x\in\partial\Omega, \\ u(0,x) = u_0(x), & x\in\Omega, \end{cases} \tag{2.2.2}$$

where $M > 0$ is a positive constant to be fixed later and $[u]_M$ is a cut-off function of u defined by

$$[u]_M = \begin{cases} M & u \ge M, \\ u & |u| \le M, \\ -M & u \le -M. \end{cases}$$

Set $B_M(u) = -|[u]_M|^{p-2}u$, then the auxiliary problem (2.2.2) can be reduced to the following evolution equation in $L^2(\Omega)$:

$$\frac{d}{dt}u(t) + \partial\varphi(u(t)) + B_M(u(t)) = 0, \qquad u(0) = u_0. \tag{2.2.3}$$

Note that $B_M : L^2(\Omega) \to L^2(\Omega)$ is Lipschitz continuous. Applying the abstract theory developed by H. Brézis (see Proposition 1.4.1), we know that (2.2.3) has a unique global solution $u \in C([0,T];L^2(\Omega))$ for $u_0 \in L^2(\Omega)$ satisfying the same regularity (except L^∞-estimate) of Proposition 2.2.1 with T_0 replaced by T.

Furthermore we can show that $u_0 \in L^\infty(\Omega)$ assures $u(t) \in L^\infty(\Omega)$ for all $t \ge 0$. To see this, put $v(t) := e^{-M^{p-2}t}u(t)$, then $v(t)$ satisfies

$$\partial_t v(t) - \Delta v(t) = \Big(|[u]_M|^{p-2} - M^{p-2}\Big)v(t), \quad v(0) = u_0. \tag{2.2.4}$$

Multiplying (2.2.4) by $[v(t)-M]^+ = \max(v(t)-M,0)$ and noting that $|[u]_M|^{p-2} - M^{p-2} \le 0$, we get

$$\frac{1}{2}\frac{d}{dt}\left\|[v(t)-M]^+\right\|_2^2 + \int_\Omega |\nabla[v(t)-M]^+|^2\,dx \le 0. \tag{2.2.5}$$

Here we used the fact that

$$\begin{aligned} -\int_\Omega \Delta v[v-M]^+dx &= \int_\Omega |\nabla[v-M]^+|^2dx - \int_{\partial\Omega} \partial_\nu v[v-M]^+d\sigma \\ &= \int_\Omega |\nabla[v-M]^+|^2dx + \int_{\partial\Omega} |u|^{q-2}v[v-M]^+d\sigma \\ &= \int_\Omega |\nabla[v-M]^+|^2dx + \int_{\partial\Omega} |u|^{q-2}M[v-M]^+d\sigma \\ &\ge \int_\Omega |\nabla[v-M]^+|^2dx. \end{aligned}$$

Hence $\|[v(t)-M]^+\|_2 \le \|[u_0-M]^+\|_2 = 0$ for a.e. $t\in[0,\infty)$. Thus we see that $v(t)\le M$, i.e., $u(t)\le Me^{M^{p-2}t}$.

Multiply again (2.2.4) by $[v(t)+M]^- = \max(-v(t)-M,0)$. Then in parallel with (2.2.5), we get

$$\frac{1}{2}\frac{d}{dt}\left\|[v(t)+M]^-\right\|_2^2 + \int_\Omega |\nabla[v(t)+M]^-|^2\,dx \le 0, \tag{2.2.6}$$

whence follows $u(t)\ge -Me^{M^{p-2}t}$. Thus we get $|u(t)|_{L^\infty}\le Me^{M^{p-2}t}$, which implies $u(t)\in L^\infty$ for a.e. $t\in[0,\infty)$. Hence noticing that $|u|^{r-2}u\in L^2(\Omega)$ and $||[u]_M|^{p-2}|\le |u|^{p-2}$, we multiply (2.2.2) by $|u|^{r-2}u$ to obtain

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\|u(t)\|_r^r + (r-1)\int_\Omega |\nabla u|^2|u|^{r-2}dx + \int_{\partial\Omega}|u|^{q+r-2}d\sigma &= \int_\Omega |[u]_M|^{p-2}|u|^r dx\\ &\le \int_\Omega |u|^{p+r-2}dx\\ &\le \|u(t)\|_\infty^{p-2}\|u(t)\|_r^r.\end{aligned}$$

Since the second term and third term of left hand side are nonnegative,

$$\|u(t)\|_r^{r-1}\frac{d}{dt}\|u(t)\|_r \le \|u(t)\|_\infty^{p-2}\|u(t)\|_r^r.$$

Divide both sides by $\|u(t)\|_r^{r-1}$ and integrate with respect to t on $[0,t]$, then we get

$$\|u(t)\|_r \le \|u_0\|_r + \int_0^t \|u(\tau)\|_\infty^{p-2}\|u(\tau)\|_r d\tau.$$

Note that even though $\|u(t)\|_r^{r-1}$ attains zero, we can justify this argument by Proposition 1 in [40]. Letting r tend to ∞, we derive

$$\|u(t)\|_\infty \le \|u_0\|_\infty + \int_0^t \|u(\tau)\|_\infty^{p-1}d\tau.$$

Hence applying lemma 1.2.3, we see that there exists T_0 such that

$$\|u(t)\|_\infty \le \|u_0\|_\infty + 1 \qquad \text{a.e. } t\in[0,T].$$

Therefore choosing $M > \|u_0\|_\infty + 1$, we can see that u gives a solution for $(\mathrm{P})_q$ on $[0,T]$ by the definition of cut-off function $[u]_M$. □

Remark 2.2.2. *If $y_0>0$ in Lemma 1.2.3, then we can derive*

$$y(t)\le 2y_0 \qquad \textit{a.e. } t\in[0,T_0],$$

and choose $T_0=\min\{\frac{y_0}{2m(y_0)},T\}$. From this observation we can deduce that the maximal existence time is sufficiently large for sufficiently small $\|u_0\|_\infty>0$.

2.2.1 More Generalized Equations

At the end of this chapter, we also consider the local well-posedness of the initial-boundary problem for the following parabolic equation, which is a more generalized version of $(\mathrm{P})_q$:

$$
(\mathrm{P})_F^\beta \qquad \begin{cases} \partial_t u - \Delta u - F(u) \ni 0, & t>0,\ x\in\Omega, \\ -\partial_\nu u \in \beta(u), & t>0,\ x\in\partial\Omega, \\ u(0,x) = u_0(x), & x\in\Omega. \end{cases}
$$

We impose the following assumptions on F and β.

(F) $F : \mathbb{R}^1 \to 2^{\mathbb{R}^1}$ is a (possibly multi-valued) operator satisfying the following (i) and (ii).

(2.2.7)
$$\text{(i) } 0 \in F(0), \quad \inf\{\, z\,;\, z\in F(u)\,\} \geq |u|^{p-2}u^+ \quad \forall u\in\mathbb{R}^1 \quad \text{with } p>2,$$

(2.2.8)
$$\text{(ii) } F(u) = F_s(u) + F_m^+(u) - F_m^-(u) \quad \forall u\in\mathbb{R}^1 \quad \text{and}$$
$F_s(\cdot)$ is singleton and locally Lipschitz continuous on $\mathbb{R}^1$,
$F_m^\pm(\cdot) : \mathbb{R}^1 \to 2^{\mathbb{R}^1}$ are maximal monotone operators such that $D(F_m^\pm) = \mathbb{R}^1$.

(β) $\beta : \mathbb{R}^1 \to 2^{\mathbb{R}^1}$ is a (possibly multi-valued) maximal monotone operator satisfying $0\in\beta(0)$.

Then, by using the L^∞-energy method, we can obtain the local well-posedness of $(\mathrm{P})_F^\beta$ for $u_0 \in L^\infty(\Omega)$. We apply here a different method from the previous proof, namely, we use the subdifferential of the indicator function.

Theorem 2.2.3. *Let $u_0 \in L^\infty(\Omega)$, then there exists $T_0 = T_0(\|u_0\|_{L^\infty}) > 0$ such that $(\mathrm{P})_F^\beta$ possesses a solution u satisfying the following regularity*

$$(2.2.9) \qquad u \in C([0,T_0]; L^2(\Omega)) \cap L^\infty(0,T_0; L^\infty(\Omega)), \quad \sqrt{t}\partial_t u, \sqrt{t}\Delta u \in L^2(0,T_0; L^2(\Omega)).$$

Moreover let $T_m = T_m(u)$ be the maximal existence time of u, then the following alternative holds:

- $T_m = +\infty$, *or*
- $T_m < +\infty$, $\displaystyle\lim_{t\to T_m} \|u(t)\|_{L^\infty} = +\infty.$

Proof. Since β is assumed to be maximal monotone, there exists a lower semi-continuous convex function $j : \mathbb{R}^1 \to (-\infty, +\infty]$ such that $j(r) \geq 0$, and $\partial j(u) = \beta(u)$ (see [9]).

For the rest of this proof, define the functional $\tilde{\varphi}$ on $L^2(\Omega)$ by

$$
\tilde{\varphi}(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 dx + \dfrac{1}{2}\int_\Omega |u|^2 dx + \int_{\partial\Omega} j(u) d\sigma, & u \in D(\tilde{\varphi}) := \{u\in H^1(\Omega); j(u)\in L^1(\partial\Omega)\}, \\ +\infty, & u \in L^2(\Omega)\setminus D(\tilde{\varphi}). \end{cases}
$$

Then we can see that $\tilde{\varphi}$ is a lower semi-continuous convex function on $L^2(\Omega)$ and the subdifferential operator $\partial\tilde{\varphi}$ associated with $\tilde{\varphi}$ is given as follows (see [7, 9, 10]):

$$\begin{cases} \partial\tilde{\varphi}(u) = -\Delta u + u, \\ D(\partial\tilde{\varphi}) = \{u \in H^2(\Omega)\,;\, -\partial_\nu u(x) \in \beta(u(x)) \quad \text{a.e. on } \partial\Omega\}. \end{cases}$$

Furthermore the following elliptic estimate for $\partial\tilde{\varphi}$ holds, i.e., there exist some constants c_1, $c_2 > 0$ such that

$$\|u\|_{H^2} \leq c_1\| - \Delta u + u\|_{L^2} + c_2 \quad \forall u \in D(\partial\tilde{\varphi}). \tag{2.2.10}$$

Then by putting $B(u) := -u - F(u)$, $(\mathrm{P})_F^\beta$ can be reduced to the following abstract evolution equation in $H = L^2(\Omega)$:

$$(\mathrm{CP}) \begin{cases} \dfrac{d}{dt}u(t) + \partial\tilde{\varphi}(u(t)) + B(u(t)) \ni 0, \quad t > 0, \\ u(0) = u_0. \end{cases}$$

In order to show the existence of time local solutions of $(\mathrm{P})_F^\beta$ belonging to $L^\infty(\Omega)$, we rely on "L^∞-Energy Method" developed in [45]. To this end, we introduce another maximal monotone graph $\beta_M(\cdot) = \partial\eta_M(\cdot)$ on $\mathbb{R}^1 \times \mathbb{R}^1$ by

$$\beta_M(r) = \begin{cases} \emptyset, & |r| > M, \\ (-\infty, 0], & r = -M, \\ 0, & |r| < M, \\ [0, +\infty), & r = M, \end{cases} \qquad \eta_M(r) = \begin{cases} 0, & |r| \leq M, \\ +\infty, & |r| > M, \end{cases}$$

where $M > 0$ is a positive constant which is determined later. The realizations of β_M and η_M in $H = L^2(\Omega)$ are given by

$$\beta_M(u) = \partial I_{K_M}(u) = \begin{cases} \emptyset, & |u(x)| > M, \\ (-\infty, 0], & u(x) = -M, \\ 0, & |u(x)| < M, \\ [0, +\infty), & u(x) = M, \end{cases}$$

$$I_{K_M}(u) := \begin{cases} 0, & u \in K_M := \{\, u \in L^2(\Omega)\,;\, |u(x)| \leq M \quad a.e.\ x \in \Omega\,\}, \\ +\infty, & u \in L^2(\Omega) \setminus K_M. \end{cases}$$

Here we put

$$\varphi_M(u) := \tilde{\varphi}(u) + I_{K_M}(u).$$

Then we can get

$$\partial\varphi_M(u) = \partial\tilde{\varphi}(u) + \beta_M(u) \qquad \forall u \in D(\partial\varphi_M) := D(\partial\tilde{\varphi}) \cap K_M. \tag{2.2.11}$$

In fact, since the Yosida approximation $(\beta_M)_\lambda(\cdot)$ of $\beta_M(\cdot)$ is given by

$$(\beta_M)_\lambda(u) = \begin{cases} \frac{u(x)+M}{\lambda}, & u(x) \le -M, \\ 0, & |u(x)| < M, \\ \frac{u(x)-M}{\lambda}, & u(x) \ge M, \end{cases}$$

we easily see

$$\begin{aligned} (\partial\tilde{\varphi}(u), (\beta_M)_\lambda(u))_{L^2} &= \int_\Omega (-\Delta u + u)(\beta_M)_\lambda(u)\,dx \\ &\ge \int_\Omega (\beta_M)'_\lambda(u)|\nabla u(x)|^2 dx + \int_{\partial\Omega} -\partial_\nu u(x)\,(\beta_M)_\lambda(u(x))\,d\sigma \ge 0. \end{aligned} \tag{2.2.12}$$

Here we used the fact that $u \cdot (\beta_M)_\lambda(u) \ge 0$, $(\beta_M)'_\lambda(u) \ge 0$, $-\partial_\nu u(x) \in \beta(u(x))$ and $0 \in \beta(0)$ implies that $\beta(u) \subset (-\infty, 0]$ if $u \le 0$ and $\beta(u) \subset [0, +\infty)$ if $u \ge 0$.

Consequently (2.2.12) together with Proposition 1.3.10 and Proposition 1.3.11 assures that $\partial\tilde{\varphi} + \partial I_M$ becomes maximal monotone. Hence since $\partial\tilde{\varphi}(u) + \partial I_M(u) \subset \partial\varphi_M(u)$ is obvious, we can conclude that (2.2.11) holds true.

Now consider the following auxiliary equation:

$$(\mathrm{CP})_M \quad \begin{cases} \dfrac{d}{dt}u(t) + \partial\varphi_M(u(t)) + B(u(t)) \ni 0, \quad t > 0, \\ u(0) = u_0, \end{cases}$$

where we choose $M > 0$ such that

$$M := \|u_0\|_{L^\infty} + 2. \tag{2.2.13}$$

Then we easily see that $u_0 \in \overline{D(\partial\varphi_M)}^{L^2} = K_M$.

Define a monotone increasing function $\ell(\cdot) : [0, \infty) \to [0, \infty)$ by

$$\ell(r) := r + \sup\{\,|z|\,;\, z \in F(\tau), \quad |\tau| \le r\,\}. \tag{2.2.14}$$

Here we note that $\ell(\cdot)$ takes a finite value for any finite r, which is assured by assumption $D(F) = D(F_m^+) = D(F_m^-) = \mathbb{R}^1$ and then we obtain

$$\sup\{\,|z|\,;\, z \in B(u(x))\,\} \le \ell(|u(x)|). \tag{2.2.15}$$

Hence we get

$$\begin{aligned} |||B(u)|||_{L^2} :=& \sup\{\|z\|_{L^2}; z \in B(u)\} \\ \le& \ell(\|u\|_{L^\infty})\,|\Omega|^{1/2} \le \ell(M)\,|\Omega|^{1/2} \qquad \forall u \in D(\partial\varphi_M), \end{aligned} \tag{2.2.16}$$

since $u \in D(\partial\varphi_M)$ implies $\|u\|_{L^\infty} \le M$. Now we are going to check some assumptions required in [47]. It is easy to see that (2.2.16) assures assumption (A5) of Theorem III and (A6) of Theorem IV in [47] by taking $H = L^2(\Omega)$. Furthermore the compactness assumption (A1), the set $\{u; \varphi_M(u) \le L\}$ is compact in $H := L^2(\Omega)$, is obviously satisfied, since Ω is bounded; and the demiclosedness assumption (A2) is also assured, since the maximal monotone parts $F_m^\pm$ are always demiclosed in $L^2(\Omega)$. Thus we can apply Theorem III and Corollary IV of [47] to conclude that $(\mathrm{P})_F^\beta$ admits a solution u on $[0,T]$ for any $T>0$ satisfying (2.2.9) with T_0 replaced by T.

Now we are going to show that there exists $T_0>0$ such that

$$\|u(t)\|_{L^\infty} \le M+1 \qquad \forall t\in[0,T_0], \tag{2.2.17}$$

whence follows $\beta_M(u(t))=0$ for all $t\in[0,T_0]$, which implies that u turns out to be the desired solution of the original equation $(\mathrm{P})_F^\beta$ on $[0,T_0]$.

To see this, multiplying $(\mathrm{CP})_M$ by $|u|^{r-2}u$, we get by (2.2.15)

$$\frac{1}{r}\frac{d}{dt}\|u(t)\|_{L^r}^r+(r-1)\int_\Omega |u|^{r-2}|\nabla u(t)|^2dx+\int_{\partial\Omega} b(t,x)\,|u|^{r-2}u(t)d\sigma$$
$$\le \ell(\|u(t)\|_{L^\infty})\|u(t)\|_{L^r}^{r-1}|\Omega|^{1/r},$$

where $b(t,x)\in\beta(u(t,x))$ and so $b(t,x)\,|u|^{r-2}u(t,x)\ge 0$. Hence

$$\frac{d}{dt}\|u(t)\|_{L^r}\le \ell(\|u(t)\|_{L^\infty})\,|\Omega|^{1/r}.$$

Letting $r\to\infty$, we obtain (see [45])

$$\|u(t)\|_{L^\infty}\le\|u_0\|_{L^\infty}+\int_0^t \ell(\|u(s)\|_{L^\infty})ds. \tag{2.2.18}$$

Then Lemma 1.2.3 assures that if we set

$$T_0:=\frac{1}{2\ell(\|u_0\|_{L^\infty}+1)}, \tag{2.2.19}$$

then (2.2.17) holds true.

In order to prove the alternative part, assume that $T_m<\infty$ and $\liminf_{t\to T_m}\|u(t)\|_{L^\infty}=: M_0<\infty$. Then there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that

$$t_n\to T_m \quad \text{as } n\to\infty \quad\text{and}\quad \|u(t_n)\|_{L^\infty}\le M_0+1 \qquad \forall n\in\mathbb{N}. \tag{2.2.20}$$

Hence in view of (2.2.19), the definition of T_0, regarding $u(t_n)$ as an initial data, we find that u can be continued up to $t_n+\frac{1}{2\ell(M_0+2)}$ which becomes strictly larger than T_m for sufficiently large n such that $T_m-t_n<\frac{1}{4\ell(M_0+2)}$. This leads to a contradiction. Thus the alternative assertion is verified. □

Remark 2.2.4. (1) *One can prove that under the same assumptions in Theorem 2.2.3, problem* $(\mathrm{P})_F^\beta$ *with the boundary condition replaced by the homogeneous Dirichlet (resp. Neumann) boundary condition, dented by* $(\mathrm{P})_F^D$ *(resp.* $(\mathrm{P})_F^N$*), admits a time local solution* u *satisfying* (2.2.9)*, which is denoted by* u_F^D *(resp.* u_F^N*). To do this, it suffices to repeat the same arguments as those in the proof of Theorem 2.2.3 with obvious modifications such as* $j(\cdot)\equiv 0, D(\varphi)=H_0^1(\Omega)$ *(resp.* $D(\varphi)=H^1(\Omega)$*).*

(2) *If assumption* (F) *is satisfied with* $F_m^-\equiv 0$*, then the solution of* $(\mathrm{P})_F^\beta$ *(or* $(\mathrm{P})_F^D$*,* $(\mathrm{P})_F^N$*) given in Theorem 2.2.3 is unique.*

Chapter 3

Asymptotic Behavior of Solutions

In this chapter, we discuss a uniform bound for global solutions of $(\mathrm{P})_q$. In order to investigate the uniform boundedness of global solutions of $(\mathrm{P})_q$, we make the most use of the variational structure of our problem $(\mathrm{AC})_q^*$. Note that the theory of the asymptotic behavior of global solutions to nonlinear evolution equations with the term of a difference of subdifferentials was established by [46] and [29] provided that functionals are homogeneous. However, in general, it is obvious that φ is not homogeneous in our setting.

3.1 Grow-up of Functionals

First of all, we recall the formulation of $(\mathrm{P})_q$ as a evolution equation on $L^2(\Omega)$. Set

(3.1.1) $$\psi_p(u) = \frac{1}{p}\|u\|_p^p.$$

Then we see that $(\mathrm{P})_q$ is equivalent to

$(\mathrm{AC})_q^*$ $$\begin{cases} \dfrac{d}{dt}u(t) + \partial\varphi(u(t)) - \partial\psi_p(u(t)) = 0, & t > 0, \\ u(0) = u_0, \end{cases}$$

Moreover we introduce the energy functional J and the Nehari functional j which are defined by

(3.1.2) $$J(u) = \varphi(u) - \psi_p(u),$$

(3.1.3) $$\begin{aligned} j(u) &= -(\partial\varphi(u) - \partial\psi_p(u), u)_2 \\ &= -\|\nabla u\|_2^2 - \int_{\partial\Omega} |u|^q d\sigma + \|u\|_p^p, \end{aligned}$$

respectively. Let u be a global solution of $(\mathrm{AC})_q^*$ satisfying (2.1.1). Then multiplying $(\mathrm{P})_q$ by u and $du(t)/dt$, we get the following equality:

(3.1.4) $$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 = j(u(t)) \qquad \forall\, t \in [0, \infty)$$

and

$$\frac{d}{dt}J(u(t)) + \left\| \frac{du}{dt}(t) \right\|_2^2 = 0 \qquad \text{for } a.e.\ t \in (0, \infty). \tag{3.1.5}$$

Hence, in particular, it follows from (3.1.5) that J is monotone non-increasing in $(0, \infty)$ and

$$J(u(t)) \le J_0 := J(u_0) \qquad \forall\, t \ge 0. \tag{3.1.6}$$

We now introduce some types of *growing up* (G.U.) of solutions to $(\mathrm{AC})_q^*$.

Definition 3.1.1. *For a global solution u of $(\mathrm{AC})_q^*$ satisfying (2.1.1), we define the following notions:*

(i) *u is said to be φ-G.U. if and only if* $\liminf_{t\to\infty} \varphi(u(t)) = +\infty$.

(ii) *u is said to be ψ_p-G.U. if and only if* $\liminf_{t\to\infty} \psi_p(u(t)) = +\infty$.

(iii) *u is said to be J-G.U. if and only if* $\liminf_{t\to\infty} J(u(t)) = -\infty$.

(iv) *u is said to be j-G.U. if and only if* $\liminf_{t\to\infty} j(u(t)) = +\infty$.

(v) *u is said to be H-G.U. if and only if* $\liminf_{t\to\infty} \|u(t)\|_2 = +\infty$.

From the definitions and simple properties of these functionals, we can see that each glow-up dose not occur.

Proposition 3.1.2. *Let u be a global solution to $(\mathrm{AC})_q^*$ with $u_0 \in D(\varphi)$ satisfying* (2.1.1) *and assume that $q \in (1, p)$ and $p \in (2, 2^*)$. Then each glowing up* (i) - (v) *in Definition 3.1.1 cannot occur.*

Proof. By the Hölder inequality and the Poincaré-Friedrichs inequality (Lemma 1.1.9), we can see that

$$\|u(t)\|_2 \le C \left\{\psi_p(u(t))\right\}^{\frac{1}{p}} \le C \left\{\varphi(u(t)) + 1\right\}^{\frac{1}{2}},$$

which implies that (v) $\Rightarrow$ (ii) $\Rightarrow$ (i). Moreover, (3.1.2) and (3.1.6) imply

$$\varphi(u(t)) \le J_0 + \psi_p(u(t)).$$

Hence (i) is equivalent to (ii). We can also verify that (iii) or (iv) implies (ii). Indeed, by the definition of J and j, we can easily show

$$J(u(t)) \ge -\psi_p(u(t)), \qquad j(u(t)) \le p\psi_p(u(t)),$$

respectively. These inequalities show (iii) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (ii).

On the other hand, if (ii) holds, then we can see that L^2 norm of $u(t)$ blows up in finite time (see Lemma 3.2.3 in the next section). Therefore (ii) does not occur. Especially, from the above observations, (i) - (v) cannot occur. □

3.2 Uniform Bounds for Global Solutions

In this section, we give more precise results on bounds of global solutions. Our main results can be stated as follows.

Theorem 3.2.1. *Assume that* $q \in (1,p)$, $p \in (2,2^*)$ *and* $u_0 \in D(\varphi)$. *Let* u *be a global strong solution of* $(\mathrm{P})_q$ *satisfying* (2.1.1). *Then we have*

$$\|u(t)\|_2 \le \Big[\frac{q_2\, p\, J_0\, |\Omega|^{\frac{p-2}{2}}}{p-q_2}\Big]^{1/p} \qquad \forall t \ge 0, \tag{3.2.1}$$

$$\sup_{t\ge 0} \varphi(u(t)) < \infty, \tag{3.2.2}$$

where $q_2 := \max(2,q)$.

Theorem 3.2.2. *Assume that* $q \in (1,p)$, $p \in (2,2^*)$ *and* $u_0 \in L^\infty(\Omega)$. *Let* u *be a global strong solution of* $(\mathrm{P})_q$ *satisfying* (2.2.1). *Then there exists* $C_\infty = C_\infty(p,q,|\Omega|)$ *such that*

$$\|u(t)\|_2 \le C_\infty\, \|u_0\|_\infty \qquad \forall t \ge 0, \tag{3.2.3}$$

$$\sup_{t\ge 0} \|u(t)\|_\infty < \infty. \tag{3.2.4}$$

In order to derive the estimate of φ, we here mainly rely on the variational structure of $(\mathrm{AC})_q^*$. We first show that negative energy causes a finite-time blowing up of solutions. Moreover, for a global solution to $(\mathrm{AC})_q^*$, we also derive a uniform bound of L^2-norm.

Lemma 3.2.3. *Let* $q_2 < p$ *and let* u *be a global solution of* $(\mathrm{AC})_q^*$ *satisfying* (2.1.1). *Then we have*

$$0 \le J(u(t)) \le J_0 \qquad \forall t \ge 0, \tag{3.2.5}$$

$$\|u(t)\|_2 \le B_{L^2} := \Big[\frac{q_2\, p\, J_0\, |\Omega|^{\frac{p-2}{2}}}{p-q_2}\Big]^{1/p} \qquad \forall t \ge 0. \tag{3.2.6}$$

Furthermore there exists a constant C_0 *depending only on* p, q, J_0 *and* $|\Omega|$ *such that*

$$\sup_{t\ge 0} \int_t^{t+1} \big(\psi_p(u(s))^2 + \varphi(u(s))^2\big)\, ds \le C_0. \tag{3.2.7}$$

Proof. From (3.1.4), (3.1.3) and (3.1.6), it follows that

$$\frac{d}{dt}\|u(t)\|_2^2 = -2\left(\|\nabla u(t)\|_2^2 + \int_{\partial\Omega} |u(t)|^q d\sigma - \|u(t)\|_p^p\right)$$

$$\geq -2\left(\frac{q_2}{2}\|\nabla u(t)\|_2^2 + \frac{q_2}{q}\int_{\partial\Omega} |u(t)|^q d\sigma - \frac{q_2}{p}\|u(t)\|_p^p\right) + \frac{2(p-q_2)}{p}\|u(t)\|_p^p$$

$$\geq -2\, q_2\, J(u(t)) + \frac{2(p-q_2)}{p}\|u(t)\|_p^p \tag{3.2.8}$$

$$\geq -2\, q_2\, J(u(t)) + \frac{2(p-q_2)}{p}\,|\Omega|^{\frac{2-p}{2}}\,\|u(t)\|_2^p \tag{3.2.9}$$

$$\geq -2\, q_2\, J_0 + \frac{2(p-q_2)}{p}\,|\Omega|^{\frac{2-p}{2}}\,\|u(t)\|_2^p \qquad \forall t \in [0,\infty). \tag{3.2.10}$$

Since $J(u(t)) \leq J_0$ for all $t \geq 0$ holds, in order to show (3.2.5), it suffices to verify that the energy functional cannot take any negative value for a global solution. Suppose that $J(u(t_1)) < 0$ for some $t_1 \in [0,\infty)$, then from (3.1.5) it follows that $J(u(t)) < 0$ for all $t \in [t_1,\infty)$, which together with (3.2.9) yields

$$\frac{d}{dt}\|u(t)\|_2^2 \geq \frac{2(p-q_2)}{p}\,|\Omega|^{\frac{2-p}{2}}\,\|u(t)\|_2^p \qquad \forall t \in [t_1,\infty). \tag{3.2.11}$$

Since $p > q_2 \geq 2$ and $J(u(t_1)) < 0$ implies $\|u(t_1)\|_2 > 0$, by (3.2.11) we can see that $\|u(t)\|_2$ blows up in finite time, which leads to a contradiction. Thus (3.2.5) is derived.

Suppose now that $\|u(t_2)\|_2 > B_{L^2}$ for some $t_2 \in [0,\infty)$, then (3.2.10) implies $\frac{d}{dt}\|u(t_2)\|_2^2 > 0$. Thus we see that $\|u(t)\|_2$ is monotone increasing in the neighborhood of $t = t_2$. Therefore, by (3.2.10), we can easily see that

$$\frac{d}{dt}\|u(t)\|_2^2 \geq \delta := -2J_0 + \frac{2(p-q_2)}{p}\,|\Omega|^{\frac{2-p}{2}}\,\|u(t_2)\|_2^p > 0 \qquad \forall t \in [t_2,\infty),$$

which implies that $\|u(t)\|_2$ is strictly monotone increasing and tends to ∞ as $t \to \infty$. Hence there exists $t_3 > t_2$ such that

$$\frac{d}{dt}\|u(t)\|_2^2 \geq \frac{(p-q_2)}{p}\,|\Omega|^{\frac{2-p}{2}}\,\|u(t)\|_2^p \qquad \forall t \in [t_3,\infty).$$

This leads to a contradiction as before. Thus (3.2.6) is verified.

Furthermore, since

$$\frac{d}{dt}\|u(t)\|_2^2 = 2\left(u(t), \frac{du(t)}{dt}\right)_{L^2} \leq 2\,\|u(t)\|_2 \left\|\frac{du(t)}{dt}\right\|_2$$

holds, (3.1.5), (3.1.6) and (3.2.6) assure that there exists $C > 0$ such that

$$\int_t^{t+1} \left|\frac{d\|u(s)\|_2^2}{ds}\right|^2 ds \leq C.$$

Hence, in view of (3.1.2) and (3.1.6), we can derive (3.2.7) from (3.2.8). □

As a consequence of Lemma 3.2.3 and monotonicity of $J(u(t))$, we can conclude that

$$\lim_{t\to\infty} J(u(t)) =: J_\infty \geq 0. \tag{3.2.12}$$

Remark 3.2.4. *Estimate* (3.2.6) *implies that if* $J_0 = 0$, *then there is no global solution of* $(\mathrm{P})_q$ *except the trivial solution* $u(t) \equiv 0$.

Lemma 3.2.5. *Let* $q_2 < p$ *and let* u *be a global solution of* $(\mathrm{P})_q$ *satisfying* (2.1.1). *Then we have*

$$\liminf_{t\to\infty} \varphi(u(t)) \leq \frac{pJ_0 + 1}{p - q_2}. \tag{3.2.13}$$

Proof. Suppose that

$$\liminf_{t\to\infty} \varphi(u(t)) > \frac{pJ_0 + 1}{p - q_2}.$$

Then we can see that there exists $t_0 > 0$ such that

$$\varphi(u(t)) \geq \frac{pJ_0 + 1}{p - q_2} \qquad \forall t \geq t_0. \tag{3.2.14}$$

From (3.1.4) and (3.2.14), it follows that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 &= j(u(t)) \\
&= -\|\nabla u(t)\|_2^2 - \int_{\partial\Omega} |u(t)|^q d\sigma + \|u(t)\|_p^p \\
&\geq -\frac{q_2}{2}\|\nabla u(t)\|_2^2 - \frac{q_2}{q}\int_{\partial\Omega} |u(t)|^q d\sigma + \|u(t)\|_p^p \\
&= -q_2\varphi(u(t)) + p\psi_p(u(t)) \\
&= -pJ(u(t)) + (p - q_2)\varphi(u(t)) \\
&\geq -pJ_0 + (p - q_2)\varphi(u(t)) \geq 1 \qquad \forall t \geq t_0.
\end{aligned} \tag{3.2.15}$$

Hence we get

$$\|u(t)\|_2^2 \geq \|u(t_0)\|_2^2 + 2(t - t_0) \qquad \forall t \geq t_0,$$

whence it follows that $\|u(t)\|_2 \to \infty$ as $t \to \infty$, which contradicts (3.2.6). □

Lemma 3.2.6. *Let* $p \in (2, 2^*)$ *and* u *be a global solution of* $(\mathrm{P})_q$. *Then there exists a monotone decreasing function* $T_0(\cdot) : [0, \infty) \to (0, \infty)$ *such that for every* $t_0 > 0$

$$\varphi(u(t)) \leq \varphi(u(t_0)) + 1 \qquad \forall t \in [t_0, t_0 + T_0(\varphi(u(t_0)))].$$

Proof. By multiplying $(\mathrm{P})_q$ by $-\Delta u = \partial\varphi(u(t))$, it follows from (1.2.2) that

$$\begin{aligned}
\frac{d}{dt}\varphi(u(t)) + \|\Delta u(t)\|_2^2 &\leq \int_\Omega |\Delta u||u|^{p-1} dx \\
&\leq \frac{1}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}\|u(t)\|_{2(p-1)}^{2(p-1)} \\
&\leq \frac{1}{2}\|\Delta u(t)\|_2^2 + C\|u(t)\|_{H^2(\Omega)}^{2-\lambda}\|u(t)\|_{H^1(\Omega)}^{2p-4+\lambda}.
\end{aligned}$$

Using (2.2.10) and Young's inequality, we can see that, for any $\eta > 0$, there exists C_η such that

$$\begin{aligned}\|u\|_{H^2(\Omega)}^{2-\lambda}\|u\|_{H^1(\Omega)}^{2p-4+\lambda} &\leq \eta\,\|u\|_{H^2(\Omega)}^2 + C_\eta\|u\|_{H^1(\Omega)}^{\frac{2(2p-4+\lambda)}{\lambda}}\\ &\leq \eta\, C\left(\|\Delta u\|_2^2 + \|u\|_2^2 + 1\right) + C_\eta\|u\|_{H^1(\Omega)}^{\frac{2(2p-4+\lambda)}{\lambda}}\\ &\leq \eta\, C\|\Delta u\|_2^2 + M_\eta(\varphi(u)),\end{aligned}$$

where $M_\eta(\cdot)$ is a monotone increasing function on $\mathbb{R}^+$ of the form

$$M_\eta(s) = C_\eta(s+1)^{\frac{2p-4+\lambda}{\lambda}} + \eta\, C(s+1),$$

and we used the fact that $\|u\|_{H^1(\Omega)}^2 \leq C(\varphi(u)+1)$, which is verified by the Poincaré-Friedrichs inequality, that is, $\|u\|_2^2 \leq C(\|\nabla u\|_2^2 + \int_{\partial\Omega}|u|^q d\sigma + 1)$ holds for any $q \in (1,\infty)$. Thus, taking $\eta > 0$ sufficiently small, we obtain

$$\frac{d}{dt}\varphi(u(t)) \leq M_\eta(\varphi(u(t))).$$

Hence by applying Lemma 1.2.3, we can conclude the claim of this lemma (cf. [45]). □

Lemma 3.2.7. *Let $q_2 < p$ and let u be a global solution of* $(\mathrm{P})_q$ *satisfying* (2.1.1). *Then we have*

$$\limsup_{t\to\infty}\varphi(u(t)) \leq \frac{pJ_0+1}{p-q_2} + 3. \tag{3.2.16}$$

Proof. Suppose that

$$\limsup_{t\to\infty}\varphi(u(t)) > \frac{pJ_0+1}{p-q_2} + 3.$$

Then, by (3.2.13) of Lemma 3.2.5, there exists a couple of sequences $\{t_n^i\}_{n=1}^\infty$ and $\{t_n^s\}_{n=1}^\infty$ such that

$$t_n^i < t_n^s < t_{n+1}^i, \qquad t_n^i \to \infty \quad \text{as } n\to\infty, \tag{3.2.17}$$

$$\varphi(u(t_n^i)) = \frac{pJ_0+1}{p-q_2} + 1, \qquad \varphi(u(t_n^s)) = \frac{pJ_0+1}{p-q_2} + 3, \tag{3.2.18}$$

$$\varphi(u(t)) \geq \frac{pJ_0+1}{p-q_2} + 1 \qquad \forall t \in [t_n^i, t_n^s]. \tag{3.2.19}$$

Integrating (3.1.5) over $[0,t]$, we obtain

$$\int_0^t \left\|\frac{du}{d\tau}(\tau)\right\|_2^2 = J_0 - J(u(t)) \leq J_0 - J_\infty.$$

Therefore $\frac{du}{dt} \in L^2(0,\infty;L^2(\Omega))$ holds and we get

$$\varepsilon(t) := \left\|\frac{du}{d\tau}\right\|_{L^2(t,\infty;L^2(\Omega))} \to 0 \qquad \text{as } t\to\infty. \tag{3.2.20}$$

In view of (3.2.14) and (3.2.19), by the same argument as for (3.2.15), we have

$$1 < \frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 \leq \|u(t)\|_2 \Big\| \frac{du}{dt}(t) \Big\|_2 \qquad \forall t \in [t_n^i, t_n^s]. \tag{3.2.21}$$

Hence $\|u(t)\|_2^2$ is monotone increasing in $t \in [t_n^i, t_n^s]$, so we get

$$\|u(t)\|_2^2 \leq \|u(t_n^s)\|_2^2 \leq C\,(\varphi(u(t_n^s)) + 1) \qquad \forall t \in [t_n^i, t_n^s]. \tag{3.2.22}$$

Integrating (3.2.21) over $[t_n^i, t_n^s]$ and making use of (3.2.22), we get

$$\begin{aligned} t_n^s - t_n^i &< \int_{t_n^i}^{t_n^s} \|u(\tau)\|_2 \Big\| \frac{du}{d\tau}(\tau) \Big\|_2 d\tau \\ &\leq C\,(\varphi(u(t_n^s)) + 1) \int_{t_n^i}^{t_n^s} \Big\| \frac{du}{d\tau}(\tau) \Big\|_2 d\tau \\ &\leq C\,(\varphi(u(t_n^s)) + 1) \left(\int_{t_n^i}^{t_n^s} \Big\| \frac{du}{d\tau}(\tau) \Big\|_2^2 d\tau \right)^{\frac{1}{2}} \left(\int_{t_n^i}^{t_n^s} d\tau \right)^{\frac{1}{2}} \\ &\leq C\Big(\frac{pJ_0 + 1}{p - q_2} + 4 \Big) \sqrt{t_n^s - t_n^i}\; \varepsilon(t_n^i). \end{aligned}$$

Therefore from (3.2.20), we can derive that $t_n^s - t_n^i \to 0$ as $n \to \infty$, which contradicts Lemma 3.2.6 and (3.2.18) with a sufficiently large n. □

Now we are ready to give a proof of Theorem 3.2.1.

Proof of Theorem 3.2.1. The assertion (3.2.1) is nothing but (3.2.6) given in Lemma 3.2.3. By virtue of (3.2.16) in Lemma 3.2.7, there exists some $T_1 > 0$ such that

$$\sup_{t \geq T_1} \varphi(u(t)) \leq \frac{p\,J_0 + 1}{p - q_2} + 4.$$

Since $\varphi(u(t))$ is continuous on $[0, \infty)$, we have

$$\sup_{0 \leq t \leq T_1} \varphi(u(t)) < \infty.$$

Therefore (3.2.2) is verified. □

Remark 3.2.8. *According to the proof of Proposition 3.1.2, under the same assumptions in Theorem 3.2.1, we can see that*

$$\sup_{t \geq 0} \psi_p(u(t)) < +\infty, \qquad \sup_{t \geq 0} j(u(t)) < +\infty.$$

In order to discuss a uniform bound of solutions in $L^\infty(\Omega)$, we prepare the following device, which is a variant of results by Alikakos [1] and Nakao [41]. Its proof can be done along essentially the same lines in the proof of Lemma 3.1 in [41]. To make this thesis self-contained, we shall give its proof.

Lemma 3.2.9. *Let $w \in W^{1,2}_{loc}([0,\infty); L^2(\Omega)) \cap L^\infty_{loc}([0,\infty); L^\infty(\Omega) \cap H^1(\Omega))$ and assume that w satisfies*

$$\frac{d}{dt}\|w(t)\|_r^r + c_0 r^{-\theta_0} \| |w(t)|^{\frac{r}{2}} \|^2_{H^1(\Omega)} \le c_1 r^{\theta_1} \|w(t)\|_r^r \qquad a.e.\ t \in (0,\infty) \tag{3.2.23}$$

for all $r \in [2,\infty)$, where $c_0 > 0$ and $c_1,\ \theta_0,\ \theta_1 \ge 0$. Then there exist some positive constants a, b, c such that

$$\sup_{t\ge 0} \|w(t)\|_\infty \le a^{\frac{1}{2}} 2^{\theta_1 + (\theta_0+\theta_1)b} M_0,$$

where $M_0 = \max\ (1,\ c\,\|w(0)\|_\infty,\ \sup_{t\ge0}\|w(t)\|_2)$.

Proof. For each $k \in \mathbb{N}$, setting

$$r_k = 2^{k+1}, \quad \alpha_k = c_1 r_k^{\theta_1}, \quad \nu_k = c_0 r_k^{-\theta_0}, \quad v = w^{2^k}$$

by (3.2.23), we get the following inequality

$$\frac{d}{dt}\|v(t)\|_2^2 \le -\nu_k \|v(t)\|^2_{H^1(\Omega)} + \alpha_k \|v(t)\|_2^2. \tag{3.2.24}$$

We here note that the following Gagliardo-Nirenberg interpolation inequality

$$\|v\|_2^2 \le C\|v\|^{2\theta}_{H^1(\Omega)} \|v\|_1^{2(1-\theta)} \le \epsilon_k \|v\|^2_{H^1(\Omega)} + C_{\epsilon_k}\|v\|_1^2$$

holds with $\theta = \frac{N}{N+2}$. Here set $C_{\epsilon_k} = C^{\frac{1}{1-\theta}} \epsilon_k^{-\frac{\theta}{1-\theta}}$, and we take $\epsilon_k > 0$ sufficiently small so that $\epsilon_k\alpha_k + \epsilon_k^2 \le \nu_k$ and $C_{\epsilon_k} \ge 1$. Then we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |w|^{r_k} dx &\le -\epsilon_k^2 \|v(t)\|^2_{H^1(\Omega)} + \alpha_k C_{\epsilon_k} \|v(t)\|_1^2 \\
&\le -\epsilon_k \|v(t)\|_2^2 + (\epsilon_k + \alpha_k)\, C_{\epsilon_k} \|v(t)\|_1^2 \\
&\le -\epsilon_k \int_\Omega |w|^{r_k} dx + (\epsilon_k + \alpha_k)\, C_{\epsilon_k} \Big(\int_\Omega |w|^{r_{k-1}} dx \Big)^2 \\
&\le -\epsilon_k \int_\Omega |w|^{r_k} dx + (\epsilon_k + \alpha_k)\, C_{\epsilon_k} \Big(\sup_{t\ge0} \int_\Omega |w|^{r_{k-1}} dx \Big)^2,
\end{aligned}$$

whence follows

$$\sup_{t\ge0} \int_\Omega |w(t)|^{r_k} dx \le \max\ \Big\{ \delta_k \Big(\sup_{t\ge0} \int_\Omega |w(t)|^{r_{k-1}} dx \Big)^2, \int_\Omega |w(0)|^{r_k} dx \Big\}, \tag{3.2.25}$$

where $\delta_k = \frac{(\epsilon_k+\alpha_k)C_{\epsilon_k}}{\epsilon_k} \ge 1$. Indeed, it is not difficult to show that $y'(t) \le -\epsilon y(t) + C$ yields

$$\sup_{t\ge0} y(t) \le \max\{\frac{C}{\epsilon}, y(0)\}.$$

Then the iterative use of (3.2.25) gives

$$\begin{aligned}&\int_\Omega |w|^{r_k} dx \le \delta_k\, \delta_{k-1}^2 \cdots \delta_1^{2^{(k-1)}}\, M_0^{r_k},\\ &M_0 := \max\,(1, c\,\|w(0)\|_\infty,\ \sup_{t\ge 0}\|w(t)\|_2) \quad \text{with} \quad c = \max\,(1, |\Omega|).\end{aligned} \tag{3.2.26}$$

Set $\epsilon_k = \eta\, 2^{-(\theta_0+\theta_1)k}$ and choose $\eta > 0$ sufficiently small so that $\epsilon_k\alpha_k + \epsilon_k^2 \le \nu_k$ and $C_{\epsilon_k} \ge 1$ are satisfied, then rewriting $C_{\epsilon_k} = C\epsilon_k^{-\gamma}$ with $\gamma = \frac{\theta}{1-\theta} > 0$, we have

$$\begin{aligned}\delta_k = \frac{(\epsilon_k + \alpha_k)C_{\epsilon_k}}{\epsilon_k} &= C(\epsilon_k + \alpha_k)\epsilon_k^{-\gamma-1}\\ &\le C\,\nu_k\,\epsilon_k^{-\gamma-2}\\ &\le Cc_0\, 2^{-\theta_0(k+1)}\eta^{-(\gamma+2)}\, 2^{(\theta_0+\theta_1)(\gamma+2)k}\\ &= C\,2^{-\theta_0}c_0\,\eta^{-(\gamma+2)}\, 2^{\{\theta_1+(\theta_0+\theta_1)(\gamma+1)\}k}\\ &=: a\, 2^{\{\theta_1+(\theta_0+\theta_1)b\}k},\end{aligned}$$

where we put $a = C2^{-\theta_0}c_0\eta^{-(\gamma+2)}$ and $b = \gamma+1$. Then by virtue of (3.2.26) with inductive reasoning, we easily obtain

$$\|w(t)\|_{r_k} \le a^{p_k} 2^{q_k} M_0, \tag{3.2.27}$$

where

$$p_k = \frac{2^k - 1}{2^{k+1}}, \qquad q_k = \frac{(2^{k+1} - k - 2)\{\theta_1 + (\theta_0 + \theta_1)b\}}{2^{k+1}}.$$

Since

$$p_k \uparrow \frac{1}{2}, \qquad q_k \uparrow \theta_1 + (\theta_0 + \theta_1)b \qquad \text{as} \quad k \uparrow \infty,$$

from (3.2.27) we can derive (see [45])

$$\|w(t)\|_\infty \le a^{\frac{1}{2}} 2^{\{\theta_1+(\theta_0+\theta_1)b\}} M_0 \qquad a.e.\ t \in [0,\infty).$$

□

Finally, we are going to give a proof for Theorem 3.2.2 by applying Lemma 3.2.9.

Proof of Theorem 3.2.2. If $\|u_0\|_\infty = 0$, then the unique solution of (P) is the trivial solution $u(t) \equiv 0$, so (3.2.4) is obvious. Let $\|u_0\|_\infty > 0$, then as is stated in Remark 2.2.2, we have

$$\|u(t)\|_\infty \le 2\,\|u_0\|_\infty \qquad a.e.\ t \in [0, T_0] \quad \text{with } T_0 = \frac{1}{2^p\|u_0\|_\infty^{p-2}}. \tag{3.2.28}$$

In order to apply results prepared for the proof of Theorem 3.2.1, we are going to derive a priori bounds for $\varphi(u(t))$. Multiplying $(\mathrm{AC})_q^*$ by u, we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \varphi(u(t)) \le \|u(t)\|_p^p \le \|u(t)\|_\infty^p\,|\Omega|,$$

where we used the fact that $\varphi(0) = 0$ and the definition of subdifferential yield $\varphi(u) \le (\partial\varphi(u), u)_{L^2}$. Integrating this over $(0, T_0)$ and using (3.2.28), we obtain

$$\int_0^{T_0} \varphi(u(t))\, dt \le 2^p \|u_0\|_\infty^p |\Omega| \frac{1}{2^p \|u_0\|_\infty^{p-2}} + \frac{1}{2}\|u_0\|_\infty^2 = \left(|\Omega| + \frac{1}{2}\right) \|u_0\|_\infty^2. \tag{3.2.29}$$

We now multiply $(\mathrm{AC})_q^*$ by $t\frac{du(t)}{dt}$ to get

$$t \left\| \frac{du}{dt}(t) \right\|_2^2 + t\, \frac{d}{dt}\varphi(u(t)) \le \frac{t}{2} \left\| \frac{du}{dt}(t) \right\|_2^2 + \frac{t}{2} \|u(t)\|_{2(p-1)}^{2(p-1)}.$$

By integrating this over $(0, T_0)$, we have

$$T_0\, \varphi(u(T_0)) \le \int_0^{T_0} \varphi(u(t))\, dt + \frac{T_0^2}{4} \sup_{0 \le t \le T_0} \|u(t)\|_\infty^{2(p-1)} |\Omega|.$$

Hence in view of (3.2.28) and (3.2.29), we can see that

$$\begin{aligned} \varphi(u(T_0)) &\le 2^p \|u_0\|_\infty^{p-2} \left(|\Omega| + \frac{1}{2}\right) \|u_0\|_\infty^2 + 2^{p-4} \|u_0\|_\infty^p |\Omega| \\ &\le 2^{p+1} \left(|\Omega| + \frac{1}{2}\right) \|u_0\|_\infty^p. \end{aligned} \tag{3.2.30}$$

Consequently, from (3.2.30) and (3.2.6) of Lemma 3.2.3, it follows that

$$\sup_{T_0 \le t < \infty} \|u(t)\|_2 \le \left[\frac{q_2\, p\, |\Omega|^{\frac{p-2}{2}}\, 2^{p+1} \left(|\Omega| + \frac{1}{2}\right)}{p - q_2} \right]^{1/p} \|u_0\|_\infty. \tag{3.2.31}$$

Hence since $\|u(t)\|_2 \le \|u(t)\|_\infty |\Omega|^{1/2} \le 2\, \|u_0\|_\infty |\Omega|^{1/2}$ for all $t \in [0, T_0]$, (3.2.3) is derived. In order to derive the uniform bound of solutions in $L^\infty(\Omega)$ on $[T_0, \infty)$, we rely on Lemma 3.2.9.

To do this, we rewrite $(\mathrm{P})_q$ in the following way:

$$\partial_t u - \Delta u + u = |u|^{p-2}u + u. \tag{3.2.32}$$

Multiplying (3.2.32) by $|u|^{r-2}u$ $(r \ge 2)$, we obtain

$$\frac{1}{r}\frac{d}{dt}\|u(t)\|_r^r - \int_\Omega |u|^{r-2}u\Delta u dx + \|u(t)\|_r^r = \int_\Omega |u|^{p+r-2}dx + \|u(t)\|_r^r. \tag{3.2.33}$$

By transforming (3.2.33), we are going to derive an inequality of the form (3.2.23) in Lemma 3.2.9. Note that the left-hand side of (3.2.33), denoted by (LHS), can be estimated from below as follows:

$$\begin{aligned} (\mathrm{LHS}) &= \frac{1}{r}\frac{d}{dt}\|u(t)\|_r^r + (r-1)\int_\Omega |\nabla u|^2 |u|^{r-2} dx + \int_{\partial\Omega} |u|^{q+r-2} d\sigma + \|u(t)\|_r^r \\ &\ge \frac{1}{r}\frac{d}{dt}\|u(t)\|_r^r + \frac{4(r-1)}{r^2}\int_\Omega \left|\nabla |u|^{\frac{r}{2}}\right|^2 dx + \left\| |u(t)|^{\frac{r}{2}} \right\|_2^2 \\ &\ge \frac{1}{r}\frac{d}{dt}\|u(t)\|_r^r + \frac{4(r-1)}{r^2} \left\| |u(t)|^{\frac{r}{2}} \right\|_{H^1(\Omega)}^2. \end{aligned}$$

On the other hand, in order to give an estimate for the right-hand side of (3.2.33), denoted by (RHS), we first consider the first term of (RHS). We apply generalized Hölder's inequality (see Lemma 1.1.4) of the following form:

$$\int_\Omega |u|^{p+r-2}dx \le \|u\|_r^{r(1-\alpha)}\,\|u\|_p^{p-2}\,\|u\|_{\frac{sr}{2}}^{\alpha r} \quad \text{with} \quad \alpha = \frac{(p-2)s}{p(s-2)}. \tag{3.2.34}$$

This is valid for all $\alpha \in (0,1)$, which holds if and only if $p < s$. So we take $s = 2^*$ for $N \ge 3$ and $s = 2p$ for $N = 2$ to get

$$\|u\|_{\frac{sr}{2}}^{\alpha r} = \left\| |u|^{\frac{r}{2}} \right\|_s^{2\alpha} \le C \left\| |u|^{\frac{r}{2}} \right\|_{H^1(\Omega)}^{2\alpha}. \tag{3.2.35}$$

Then, recalling that $\|u\|_p \le C(\varphi(u)+1)^{1/2}$ which is uniformly bounded by (3.2.2), we obtain by (3.2.34) and (3.2.35)

$$\begin{aligned}
\text{(RHS)} &\le \|u(t)\|_r^{r(1-\alpha)}\|u(t)\|_p^{p-2}\|u(t)\|_{\frac{sr}{2}}^{\alpha r} + \|u(t)\|_r^r \\
&\le C\|u(t)\|_r^{r(1-\alpha)}\Big(\sup_{t\ge T_0}\varphi(u(t))+1\Big)^{\frac{p-2}{2}} \left\| |u(t)|^{\frac{r}{2}} \right\|_s^{2\alpha} + \|u(t)\|_r^r \\
&\le \frac{2(r-1)}{r^2}\left\| |u(t)|^{\frac{r}{2}} \right\|_{H^1(\Omega)}^2 + C\left(\frac{2(r-1)}{r^2}\right)^{-\frac{\alpha}{1-\alpha}} \|u(t)\|_r^r + \|u(t)\|_r^r,
\end{aligned}$$

Thus since $\frac{r^2}{2(r-1)} \le r$ and $\frac{2(r-1)}{r} \ge 1$ for all $r \ge 2$, from (3.2.33) we deduce

$$\frac{d}{dt}\|u(t)\|_r^r + \left\| \, |u(t)|^{\frac{r}{2}} \right\|_{H^1(\Omega)}^2 \le Cr^{\frac{1}{1-\alpha}}\|u(t)\|_r^r \qquad \forall t \in [T_0,\infty). \tag{3.2.36}$$

Then (3.2.36) implies that u satisfies (3.2.23) with $c_0 = 1$, $c_1 = C$, $\theta_0 = 0$ and $\theta_1 = \frac{1}{1-\alpha}$. Thus the desired bound of u in $L^\infty([T_0,\infty);L^\infty(\Omega))$ is derived from Lemma 3.2.9 and (3.2.3). □

Remark 3.2.10. *It is possible to show that the global bounds of $\varphi(u(t))$ and $\|u(t)\|_\infty$ depend only on initial data $\varphi(u_0)$ and $\|u_0\|_\infty$ (as well as on $p, q, |\Omega|$) respectively, if p satisfies the following more restrictive condition: $2 < p < 2_*$, where $2_* = \infty$ for $N = 1$ and $2_* = 2 + \frac{12}{3N-4}$ for $N \ge 2$ ($2_* < 2^*$ for $N \ge 2$, see* [13]*).*

Chapter 4

Comparison Theorem

This chapter is devoted to studying comparison theorem for an initial-boundary problem for a system of nonlinear parabolic equations with nonlinear boundary conditions. The advantage of our comparison theorem over the classical ones lies in the fact that it enables us to compare two solutions satisfying different types of boundary conditions. We first prove our main result on a general domain. In the latter half, we give an application of this theorem to nonlinear heat equations with nonlinear boundary conditions. More precisely, we consider the generalized version of nonlinear heat equations $(\mathrm{P})^{\beta}_{F}$ (see Chapter 2).

4.1 Main Statement and Its Proof

The main purpose of this chapter is to give a comparison theorem for a rather wide class of nonlinear systems of reaction diffusion equations with nonlinear boundary conditions, i.e., the following system of equations for $U = (u^1, u^2, \cdots, u^m)$ given by

$$
(\mathrm{GP}) \quad \begin{cases} \dfrac{\partial u^k}{\partial t} - \displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a^k_{ij}(t,x)\frac{\partial u^k}{\partial x_i}\right) + \gamma^k(t,x,u^k) - F^k(t,x,U) \ni 0, & (t,x) \in Q_T, \\ -\displaystyle\sum_{i,j=1}^{N} a^k_{ij}(t,x)\,\nu_j \frac{\partial u^k}{\partial x_i} \in \beta^k(t,x,u^k), & (t,x) \in \Gamma_T, \\ u^k(0,x) = a^k(x), & x \in \Omega, \end{cases}
$$

where Ω is a general domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $Q_T := (0,T)\times\Omega$, $\Gamma_T := (0,T)\times\partial\Omega$, $\nu = \nu(x) = (\nu_1, \cdots, \nu_N)$ is the unit outward vector at $x \in \partial\Omega$, $u^k : Q_T \to \mathbb{R}$ $(k = 1, 2, \cdots, m)$ are the unknown functions.

As for the coefficients a^k_{ij} $(k = 1, 2, \cdots, m)$, we assume

$$
(4.1.1) \quad \exists \lambda^k \geq 0 \quad \text{such that} \quad \lambda^k|\xi|^2 \leq \sum_{i,j=1}^{N} a^k_{ij}(t,x)\,\xi_i\xi_j \qquad \forall \xi \in \mathbb{R}^N, \quad \text{a.e. } (t,x) \in Q_T,
$$

$$
(4.1.2) \qquad a^k_{i,j} \in L^\infty(Q_T), \quad a^k_{i,j}|_{\Gamma_T} \in L^\infty(\Gamma_T).
$$

We also assume that $F^k : Q_T \times \mathbb{R}^m \to 2^{\mathbb{R}^1}$ $(k = 1, 2, \cdots, m)$ are (possibly multi-valued) nonlinear mappings; $\gamma^k(t,x,\cdot)$ and $\beta^k(t,x,\cdot)$ $(k = 1, 2, \cdots, m)$ are maximal monotone graphs on $\mathbb{R}^1 \times \mathbb{R}^1$ for *a.e.* (t, x). More precisely, there exist lower semi-continuous convex functions $j^k(t, x, r) : \Gamma_T \times \mathbb{R} \to (-\infty, +\infty]$ and $\eta^k(t, x, r) : Q_T \times \mathbb{R} \to (-\infty, +\infty]$ such that $\beta^k = \partial j^k$ and $\gamma^k = \partial \eta^k$, respectively. Here ∂j^k and $\partial \eta^k$ denote subdifferentials of j^k and η^k with respect to $r \in \mathbb{R}$, respectively.

The problem with this type of boundary conditions appears in models describing diffusion phenomena taking into consideration some nonlinear radiation law on the boundary (see Brézis[10] and Barbu [7]) and the solvability for (GP) is examined in detail under various settings (see [10, 7, 45]).

In what follows we work with solutions of (GP) in the following sense.

Definition 4.1.1. *A function $U = (u^1, u^2, \cdots, u^m) : Q_T \to \mathbb{R}^m$ is called a super-solution (resp. sub-solution) of* (GP) *on $[0, T]$ if and only if for all $k \in \{1, 2, \cdots, m\}$,*

(4.1.3)
$$u^k \in C([0,T]; L^2(\Omega)) \cap L^\infty([0,T]; L^\infty(\Omega)) \cap W^{1,2}_{loc}((0,T]; L^2(\Omega)) \cap L^2_{loc}((0,T]; H^2(\Omega)),$$

and there exist sections $f^k, g^k \in L^2_{loc}((0,T]; L^2(\Omega))$ of $F^k(t, x, U(t, x))$, $\gamma^k(t, x, u^k(t, x))$ and a section $b^k \in L^2_{loc}((0,T]; L^2(\partial\Omega))$ of $\beta^k(t, x, u^k(t, x))$ satisfying (GP)*, i.e.,*

$$\begin{cases} \dfrac{\partial u^k}{\partial t} - \displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a^k_{ij}(t,x) \frac{\partial u^k}{\partial x_i} \right) + g^k(t,x) - f^k(t,x) \geq 0 \ (\textit{resp.} \ \leq 0), \\ \qquad f^k(t,x,U) \in F^k(t,x,U(t,x)), \quad g^k(t,x) \in \gamma^k(t,x,u^k(t,x)), \quad \textit{a.e.}\ (t,x) \in Q_T, \\ -\displaystyle\sum_{i,j=1}^{N} a^k_{ij}(t,x)\, \nu_j \frac{\partial u^k}{\partial x_i} \leq b^k(t,x) \ (\textit{resp.} \ \geq), \\ \qquad b^k(t,x) \in \beta^k(t,x,u^k(t,x)) \quad \textit{a.e.}\ (t,x) \in \Gamma_T, \\ u^k(0,x) = a^k(x), \quad \textit{a.e.}\ x \in \Omega. \end{cases}$$

If U is a super- and sub-solution of (GP) *on $[0,T]$ with the same sections f^k, b^k, g^k, then U is called a solution of* (GP) *on $[0,T]$.*

We also define the maximal existence time $T_m = T_m(U)$ of a solution U by

$$T_m(U) := \sup\{\, T > 0\, ;\ U \textit{ is extended to } [0,T] \textit{ as a solution of } \text{(GP)} \textit{ in the sense above.}\}$$

Remark 4.1.2. *When the existence of solution is concerned, the assumption $D(\beta^k) \cap D(\gamma^k) \neq \emptyset$ is usually required for each k (see* [10, 7]*). However we do not apparently need this assumption to derive our comparison theorem, since the existence of solutions satisfying* (4.1.3) *is always assumed in our setting.*

We now state our comparison theorem for (GP) and give a proof of it. The idea of proof is standard and elementary, however, this type comparison theorem can cover various types of nonlinear parabolic equations including those with classical linear boundary conditions. The applicability of this comparison theorem will be exemplified in the next section.

Consider the following two systems of equations:

$$
(\mathrm{GP})_1 \quad \begin{cases} \dfrac{\partial u^k}{\partial t} - \displaystyle\sum_{i,j=1}^{N} \dfrac{\partial}{\partial x_j}\left(a_{ij}^k(t,x)\dfrac{\partial u^k}{\partial x_i}\right) + \gamma_1^k(t,x,u^k) - F_1^k(t,x,U) \ni 0, & (t,x) \in Q_T, \\ -\displaystyle\sum_{i,j=1}^{N} a_{ij}^k(t,x)\nu_j \dfrac{\partial u^k}{\partial x_i} \in \beta_1^k(t,x,u^k), & (t,x) \in \Gamma_T, \\ u^k(0,x) = a_1^k(x), & x \in \Omega, \end{cases}
$$

and

$$
(\mathrm{GP})_2 \quad \begin{cases} \dfrac{\partial u^k}{\partial t} - \displaystyle\sum_{i,j=1}^{N} \dfrac{\partial}{\partial x_j}\left(a_{ij}^k(t,x)\dfrac{\partial u^k}{\partial x_i}\right) + \gamma_2^k(t,x,u^k) - F_2^k(t,x,U) \ni 0, & (t,x) \in Q_T, \\ -\displaystyle\sum_{i,j=1}^{N} a_{ij}^k(x)\nu_j \dfrac{\partial u_k}{\partial x_i} \in \beta_2^k(t,x,u^k), & (t,x) \in \Gamma_T, \\ u^k(0,x) = a_2^k(x), & x \in \Omega, \end{cases}
$$

where for every $k \in \{1, 2, \cdots, m\}$, β_i^k, γ_i^k and F_i^k in $(\mathrm{GP})_i$ satisfy the same conditions as those for β^k, γ^k and F^k in (GP). Then our main theorem is stated as follows.

Theorem 4.1.3. *Let $U_1 = (u_1^1, u_1^2, \cdots, u_1^m)$ be a sub-solution of $(\mathrm{GP})_1$ on $[0,T]$ and $U_2 = (u_2^1, u_2^2, \cdots, u_2^m)$ be a super-solution of $(\mathrm{GP})_2$ on $[0,T]$, and let the following assumptions* (A1)-(A4) *be satisfied.*

(A1) $a_1^k(x) \le a_2^k(x)$ *a.e.* $x \in \Omega$ *for all* $k \in \{1, 2, \cdots, m\}$.

(A2) *For each* $k \in \{1, 2, \cdots, m\}$, *one of the following* (i)-(ii) *holds true.*

(i) $\gamma_1^k(t,x,\cdot) = \gamma_2^k(t,x,\cdot) = \gamma^k(t,x,\cdot)$ *a.e.* $(t,x) \in Q_T$.

(ii) $\sup\{\, g_2^k\,;\, g_2^k \in \gamma_2^k(t,x,r_2)\,\} \le \inf\{\, g_1^k\,;\, g_1^k \in \gamma_1^k(t,x,r_1)\,\}$
$\forall r_1 \in D(\gamma_1^k(t,x,\cdot))$, $\forall r_2 \in D(\gamma_2^k(t,x,\cdot))$ *with* $r_1 > r_2$ *a.e.* $(t,x) \in Q_T$.

(A3) *For each* $k \in \{1, 2, \cdots, m\}$, *one of the following* (i)-(iii) *holds true.*

(i) $\beta_1^k(t,x,\cdot) = \beta_2^k(t,x,\cdot) = \beta^k(t,x,\cdot)$ *a.e.* $(t,x) \in \Gamma_T$.

(ii) $\sup\{\, b_2^k\,;\, b_2^k \in \beta_2^k(t,x,r_2)\,\} \le \inf\{\, b_1^k\,;\, b_1^k \in \beta_1^k(t,x,r_1)\,\}$
$\forall r_1 \in D(\beta_1^k(t,x,\cdot))$, $\forall r_2 \in D(\beta_2^k(t,x,\cdot))$ *with* $r_1 > r_2$ *a.e.* $(t,x) \in \Gamma_T$.

(iii) $r_1^k \le r_2^k$ $\forall r_1^k \in D(\beta_1^k(t,x,\cdot))$, $\forall r_2^k \in D(\beta_2^k(t,x,\cdot))$ *a.e.* $(t,x) \in \Gamma_T$.

(A4) *For each* $k \in \{1, 2, \cdots, m\}$, *the following* (i) *and* (ii) *hold true.*

(i) $-\infty < \sup\{\, z; z \in F_1^k(t,x,U)\,\} \le \inf\{\, z; z \in F_2^k(t,x,U)\,\} < +\infty$ *a.e.* $(t,x,U) \in Q_T \times \mathbb{R}^m$.

(ii) $F_1^k(t,x,\cdot)$ *or* $F_2^k(t,x,\cdot)$ *is single-valued and satisfies the following structure condition* (SC) *with* F^k *replaced by* F_1^k *or* F_2^k*:*

(SC) $F^k(t,x,U)$ *is differentiable for almost all* $U \in \mathbb{R}^m$ *and satisfies*

$$\frac{\partial}{\partial u_j} F^k(t,x,U) \geq 0 \quad \text{for all } j \neq k \quad \text{for a.e. } (t,x,U) \in Q_T \times \mathbb{R}^m \tag{4.1.4}$$

and for any $M > 0$ *there exists* $L_M > 0$ *such that*

$$\sup \left\{ \left| \frac{\partial}{\partial u_j} F^k(t,x,U) \right| ; 1 \leq j \leq m, \quad (t,x,U) \in Q_T \times \{ U ; |U|_{\mathbb{R}^m} \leq M \} \right\} \leq L_M. \tag{4.1.5}$$

Then, we have

$$u_1^k(t,x) \leq u_2^k(t,x) \qquad \forall k \in \{1,2,\cdots,m\}, \quad \forall t \in [0,T], \quad a.e.\ x \in \Omega. \tag{4.1.6}$$

Proof. Let $f_i^k,\ b_i^k,\ g_i^k$ be the sections of $F_i^k(U_i),\ \beta^k(u_i^k),\ \gamma^k(u_i^k)$ appearing in $(\mathrm{GP})_i$, and let $w^k := u_1^k - u_2^k$. Since U_1 and U_2 are a sub-solution of $(\mathrm{GP})_1$ and a super-solution of $(\mathrm{GP})_2$ respectively, we can see that w^k satisfies

$$\begin{cases} \partial_t w^k - \displaystyle\sum_{i,j=1}^N \frac{\partial}{\partial x_j}\left(a_{ij}^k(t,x) \frac{\partial w^k}{\partial x_i} \right) + g_1^k - g_2^k \leq f_1^k(U_1) - f_2^k(U_2), & (t,x) \in Q_T, \\ -\displaystyle\sum_{i,j=1}^N a_{ij}^k(t,x) \nu_j \frac{\partial w^k}{\partial x_i} \geq b_1^k - b_2^k, & (t,x) \in Q_T, \\ w^k(0,x) = a_1^k(x) - a_2^k(x), & x \in \Omega. \end{cases} \tag{4.1.7}$$

Multiplying (4.1.7) by $(w^k)^+ := \max\,(w^k, 0)$, we have

$$\int_\Omega \partial_t w^k\, (w^k)^+ dx - \int_\Omega \sum_{i,j=1}^N \frac{\partial}{\partial x_j}\left(a_{ij}^k(t,x) \frac{\partial w^k}{\partial x_i} \right) (w^k)^+ dx + \int_\Omega (g_1^k - g_2^k)(w^k)^+ dx \leq \int_\Omega (f_1^k(U_1) - f_2^k(U_2))(w^k)^+ dx.$$

Here note that

$$\int_\Omega \partial_t w^k\, (w^k)^+ dx = \int_{\{w^k \geq 0\}} \partial_t w^k\, w^k dx = \frac{1}{2}\frac{d}{dt} \int_{\{w^k \geq 0\}} |w^k|^2 dx = \frac{1}{2}\frac{d}{dt} \int_\Omega |(w^k)^+|^2 dx,$$

and it follows from (4.1.1) that

$$
\begin{aligned}
-\int_\Omega \sum_{i,j=1}^N \frac{\partial}{\partial x_j}\left(a_{ij}^k(t,x)\frac{\partial w^k}{\partial x_i}\right)(w^k)^+dx \\
&= \int_\Omega \sum_{i,j=1}^N a_{ij}^k(t,x)\frac{\partial w^k}{\partial x_i}\frac{\partial (w^k)^+}{\partial x_j}dx - \int_{\partial\Omega}\sum_{i,j=1}^N a_{ij}^k(t,x)\nu_j\frac{\partial w^k}{\partial x_i}(w^k)^+d\sigma \\
&\geq \int_{\{w^k\geq 0\}}\sum_{i,j=1}^N a_{ij}^k(t,x)\frac{\partial w^k}{\partial x_i}\frac{\partial w^k}{\partial x_j}dx + \int_{\partial\Omega}(b_1^k-b_2^k)(w^k)^+d\sigma \\
&= \int_\Omega \sum_{i,j=1}^N a_{ij}^k(t,x)\frac{\partial (w^k)^+}{\partial x_i}\frac{\partial (w^k)^+}{\partial x_j}dx + \int_{\partial\Omega}(b_1^k-b_2^k)(w^k)^+d\sigma \\
&\geq \lambda^k\int_\Omega \sum_{j=1}^N\left|\frac{\partial (w^k)^+}{\partial x_j}\right|^2dx + \int_{\partial\Omega}(b_1^k-b_2^k)(w^k)^+d\sigma.
\end{aligned}
$$

Hence the following inequality holds:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|(w^k)^+(t)\|_{L^2}^2 + \int_{\partial\Omega}(b_1^k-b_2^k)(w^k)^+d\sigma + \int_\Omega(g_1^k-g_2^k)(w^k)^+dx \\
\leq \int_\Omega (f_1^k(U_1)-f_2^k(U_2))(w^k)^+dx.
\end{aligned}
\tag{4.1.8}
$$

Here we are going to show that

$$
I_{\partial\Omega} := \int_{\partial\Omega}(b_1^k-b_2^k)\,(w^k)^+d\sigma = \int_{\{u_1^k>u_2^k\}}(b_1^k-b_2^k)\,(u_1^k-u_2^k)d\sigma \geq 0.
\tag{4.1.9}
$$

In fact, if (i) of (A3) is satisfied, then (4.1.9) is derived from the monotonicity of β^k, and it is obvious that $I_{\partial\Omega}=0$ provided that (iii) of (A3), that is, $(w^k)^+|_{\partial\Omega}=0$. As for the case where (ii) of (A3) is satisfied, $u_1^k>u_2^k$ and $b_1^k\in\beta_1^k(u_1^k)$, $b_2^k\in\beta_2^k(u_2^k)$ imply that

$$
(b_1^k-b_2^k)\,(u_1^k-u_2^k)\geq 0,
$$

whence follows $I_{\partial\Omega}\geq 0$.

In the same way as above, it follows from (A2) that

$$
\int_\Omega (g_1^k-g_2^k)\,(w^k)^+dx\geq 0.
\tag{4.1.10}
$$

Here we consider the case where F_1^k is singleton and satisfies (SC) with F^k replaced by F_1^k. Then by (i) of (A4) we obtain

$$
\begin{aligned}
\text{(4.1.11)} \qquad \int_\Omega (f_1^k(U_1) - f_2^k(U_2))(w^k)^+ dx &= \int_\Omega (F_1^k(U_1) - f_2^k(U_2))(w^k)^+ dx \\
&= \int_\Omega (F_1^k(U_1) - F_1^k(U_2))(w^k)^+ dx \\
&\quad + \int_\Omega (F_1^k(U_2) - f_2^k(U_2))(w^k)^+ dx \\
&\le \int_\Omega (F_1^k(U_1) - F_1^k(U_2))(w^k)^+ dx.
\end{aligned}
$$

Furthermore by virtue of (SC), there exists some $\theta \in (0,1)$ such that

$$
\begin{aligned}
I_F^k &:= \int_\Omega (F_1^k(U_1) - F_1^k(U_2))(w^k)^+ dx \\
&= \int_\Omega \sum_{j=1}^m \frac{\partial}{\partial u_j} F_1^k(U_2 + \theta(U_1 - U_2))\, w^j\, (w^k)^+ dx \\
&= \int_\Omega \sum_{j=1}^m \frac{\partial}{\partial u_j} F_1^k(U_2 + \theta(U_1 - U_2))((w^j)^+ - (w^j)^-)(w^k)^+ dx \\
&\le \int_\Omega \sum_{j=1}^m \frac{\partial}{\partial u_j} F_1^k(U_2 + \theta(U_1 - U_2))\, (w^j)^+\, (w^k)^+ dx,
\end{aligned}
$$

where we used the fact that $w = w^+ - w^-$, $w^- := \max\,(-w, 0) \ge 0$ and $\frac{\partial}{\partial u_j} F_1^k(U_2 + \theta(U_1 - U_2))\,(w^j)^-(w^k)^+ \ge 0$ for $j \ne k$ and $(w^j)^-(w^k)^+ = 0$ for $j = k$.

Hence since $U_i \in L^\infty(0,T; L^\infty(\Omega))$ implies that there exists $M > 0$ such that

$$
\max_{i=1,2} \sup_{t \in (0,T)} \|U_i(t)\|_{\mathbb{L}^\infty} \le M,
$$

we obtain by (4.1.5)

$$
I_F^k \le L_M\, \|(w^k)^+\|_{L^2} \sum_{j=1}^m \|(w^j)^+\|_{L^2}. \tag{4.1.12}
$$

Thus in view of (4.1.8), (4.1.9), (4.1.10) and (4.1.12), we finally get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt} \sum_{k=1}^m \|(w^k)^+(t)\|_{L^2}^2 &\le L_M \Big(\sum_{k=1}^m \|(w^k)^+(t)\|_{L^2} \Big)^2 \\
&\le L_M\, m \sum_{k=1}^m \|(w^k)^+(t)\|_{L^2}^2 \qquad \forall t \in (0,T).
\end{aligned}
$$

Then integrating this over (s,t) with $0 < s < t \le T$, we obtain by Gronwall's inequality

$$\sum_{k=1}^{m} \|(w^k)^+(t)\|_{L^2}^2 \le \sum_{k=1}^{m} \|(w^k)^+(s)\|_{L^2}^2 \, e^{2mL_M(t-s)} \quad 0 < s \le t \le T.$$

Since $w^k \in C([0,T];L^2(\Omega))$, letting $s \to 0$, we obtain by (A1)

$$\sum_{k=1}^{m} \|(w^k)^+(t)\|_{L^2}^2 \le \sum_{k=1}^{m} \|(a_1^k - a_2^k)^+\|_{L^2}^2 \, e^{2mL_M T} = 0 \qquad \forall t \in [0,T],$$

whence follows (4.1.6).

As for the case where F_2^k is singleton and satisfies (SC) with F^k replaced by F_2^k, instead of (4.1.11) we can get

$$\int_\Omega (f_1^k(U_1) - f_2^k(U_2))(w^k)^+ dx \le \int_\Omega (F_2^k(U_1) - F_2^k(U_2))(w^k)^+ dx.$$

Then we can repeat the same argument as above with F_1^k replaced by F_2^k. □

Remark 4.1.4. (1) *If $f_1^k(U_1) \le f_2^k(U_2)$ is known a priori, we need not assume* (A4) *for F_1^k and F_2^k in Theorem 4.1.3.*

(2) *If $b_1^k(u_1^k) \le b_2^k(u_2^k)$ is known a priori, we need not assume* (A3) *for β_1^k and β_2^k in Theorem 4.1.3.*

(3) *If $m = 1$ in Theorem 4.1.3, then assumption* (4.1.4) *is not needed.*

(4) *When we discuss the existence of solutions for* $(\mathrm{GP})_i$ *($i = 1,2$), we need to assume that β_i^k and γ_i^k are maximal monotone graphs. In Theorem 4.1.3, however, we need only the monotonicity of β_i^k and γ_i^k, since the existence of solutions is always assumed in our setting.*

(5) *The following condition gives a sufficient condition for* (ii) *of* (A3).

$$(\mathrm{ii})' \begin{cases} D(\beta_1^k(t,x,\cdot)) \subset D(\beta_2^k(t,x,\cdot)) \quad \text{a.e. } (t,x) \in \Gamma_T, \quad \text{and} \\ \inf \{\, b_1^k \,;\, b_1^k \in \beta_1^k(t,x,r) \,\} \ge \sup \{\, b_2^k \,;\, b_2^k \in \beta_2^k(t,x,r) \,\} \quad \forall r \in D(\beta_1^k(t,x,\cdot)), \end{cases}$$

and the same assertion for (ii) *of* (A2) *as above holds true.*

4.2 Applications

In this section we give an example of the application of our comparison theorem to some nonlinear problem. Especially, we give a simple proof of the existence of blowing-up solutions for nonlinear diffusion equations with nonlinear boundary conditions. We here consider the following initial-boundary problem:

$$(\mathrm{P})_F^\beta \qquad \begin{cases} \partial_t u - \Delta u - F(u) \ni 0, & t > 0,\ x \in \Omega, \\ -\partial_\nu u \in \beta(u), & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x) \ge 0, & x \in \Omega. \end{cases}$$

We here assume that Ω is bounded in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ (Note that our comparison theorem holds without the assumption on boundedness of Ω). The existence of local solutions to this problem has been already discussed in Chapter 2 under some assumptions (F) and (β). Moreover, the condition which assures the uniqueness has been mentioned in Remark 2.2.4. For readers, we state the assumptions for F and β again.

(F) $F : \mathbb{R}^1 \to 2^{\mathbb{R}^1}$ is a (possibly multi-valued) operator satisfying the following (i) and (ii).

(i) $0 \in F(0), \quad \inf\{\, z\,;\, z \in F(u)\,\} \geq |u|^{p-2}u^+ \quad \forall u \in \mathbb{R}^1 \quad \text{with } p > 2,$

(ii) $F(u) = F_s(u) + F_m^+(u) - F_m^-(u) \quad \forall u \in \mathbb{R}^1$ and
$F_s(\cdot)$ is singleton and locally Lipschitz continuous on $\mathbb{R}^1$,
$F_m^\pm(\cdot) : \mathbb{R}^1 \to 2^{\mathbb{R}^1}$ are maximal monotone operators such that $D(F_m^\pm) = \mathbb{R}^1$.

(β) $\beta : \mathbb{R}^1 \to 2^{\mathbb{R}^1}$ is a (possibly multi-valued) maximal monotone operator satisfying $0 \in \beta(0)$.

In view of assumptions $0 \in F(0)$ and $0 \in \beta(0)$, we immediately see that $(\mathrm{P})_F^\beta$ possesses the trivial solution $v \equiv 0$ with sections $0 = f(v) \in F(v),\ 0 = b(v) \in \beta(v)$. Let u be any solution of $(\mathrm{P})_F^\beta$ with $u_0(x) \geq 0$ with sections $f(u) \in F(u),\ b(u) \in \beta(u)$ satisfying the regularity required in Definition 4.1.1, whose existence is assured in Theorem 2.2.3, then applying Theorem 4.1.3 with $m = 1;\ F_1 = F_2 = F;\ \gamma_1 = \gamma_2 = 0;\ \beta_1 = \beta_2 = \beta;\ a_1 = 0,\ a_2 = u_0;\ u_1 = v = 0,\ u_2 = u$, we conclude that $u \geq 0$ as far as u exists. Here we use the fact that $0 = f(u_1) \leq \min\{z; z \in F(u)\} \leq f(u_2)$ is assured a priori by (2.2.7) (see Remark 4.1.4).

Since we are here concerned only with non-negative solutions, the typical model of F and β is given by $F(u) = |u|^{p-2}u$ and $\beta(u) = |u|^{q-2}u$. For this special case, when $q < p$, i.e., the nonlinearity inside the region is stronger than that at the boundary, it might be straightforward to prove that there exist solutions of $(\mathrm{P})_F^\beta$ which blow up in finite time by applying the same strategy as that in [51]. Even though, it is difficult to apply such a method to $(\mathrm{P})_F^\beta$ for the case where $q \geq p$, and to derive the existence of blow-up solutions for this case by using the variational structure, one would need some complicated classifications on parameters (p, q) with heavy calculations (cf. [56]). We emphasize that our method for showing the existence of blow-up solutions relying on Theorem 4.1.3 provides us a much simpler device with wider applicability.

Our result on the existence of solutions of $(\mathrm{P})_F^\beta$ which blow up in finite time can be formulated in terms of the following eigenvalue problem:

$$
(4.2.1) \qquad \begin{cases} -\Delta\phi = \lambda\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases}
$$

Let $\lambda_1 > 0$ be the first eigenvalue of (4.2.1) and ϕ_1 be the associated positive eigenfunction normalized by $\int_\Omega \phi_1(x)dx = 1$.

We here consider the following fully studied problem:

$$
(\mathrm{P})_p^D \quad \begin{cases} \partial_t u - \Delta u = |u|^{p-2}u, & t > 0,\ x \in \Omega, \\ u = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x) \geq 0, & x \in \Omega, \end{cases}
$$

It is well known that $(\mathrm{P})_p^D$ admits the unique time local solution u_p^D for any $u_0 \in L^\infty(\Omega)$ and $T_m(u_p^D) < \infty$ if u_0 satisfies

$$
(4.2.2) \qquad u_0 \in L^\infty(\Omega), \quad 0 \leq u_0(x) \quad a.e.\ x \in \Omega, \quad \text{and} \quad \int_\Omega u_0(x)\,\phi_1(x)\,dx > \lambda_1^{\frac{1}{p-2}},
$$

which is proved by the so-called Kaplan's method (see [55]).

By comparing the solution u of $(\mathrm{P})_F^\beta$ with u_p^D, we obtain the following result.

Proposition 4.2.1. *Assume that u_0 satisfies (4.2.2) and let u_F^β be any solution of $(\mathrm{P})_F^\beta$, then $T_m(u_F^\beta) \leq T_m(u_p^D) < \infty$, i.e., u_F^β blows up in finite time.*

Proof. We apply Theorem 4.1.3 with $m = 1$, $a_{i,j} = \delta_{i,j}$ and $\gamma_1 = \gamma_2 = 0$, $a_1 = a_2 = u_0$. Then (A1) and (A2) are automatically satisfied. As for (A4), we take $F_1(t,x,u) = |u|^{p-2}u$ and $F_2(t,x,u) = F(u)$, then (2.2.7) assures (i) of (A4), and it is clear that F_1 satisfies (SC), since F_1 is of C^1-class with respect to u. As for the boundary conditions, we set

$$
(4.2.3) \qquad \beta_1(r) = \beta^D(r) := \begin{cases} \mathbb{R}^1 & \text{for } r = 0, \\ \emptyset & \text{for } r \neq 0, \end{cases}
$$

$$
(4.2.4) \qquad \beta_2(r) = \beta_e(r) := \begin{cases} \beta(r) & \text{for } r > 0, \\ (-\infty, 0] \cup \beta(0) & \text{for } r = 0, \\ \emptyset & \text{for } r < 0. \end{cases}
$$

Then we can easily see that β_2 is monotone, i.e., $(z_1 - z_2)(r_1 - r_2) \geq 0$ for all $[r_1, z_1], [r_2, z_2] \in \beta_2$. In fact, this is obvious when $r_i > 0$ or $r_i = 0$ $(i = 1, 2)$. Let $r_1 > 0$ and $r_2 = 0$, then $z_2 \in \beta(0)$ or $z_2 \in (-\infty, 0]$. If $z_2 \in \beta(0)$, the monotonicity of β assures the assertion; and if $z_2 \in (-\infty, 0]$, then since $0 \in \beta(0)$ implies $z_1 \geq 0$, we get $(z_1 - z_2)(r_1 - r_2) \geq z_1 r_1 \geq 0$.

Since $\beta(r) \subset \beta_2(r)$ for all $r \geq 0$ and $u_F^\beta(t,x) \geq 0$ *a.e.* $(t,x) \in \Gamma_T$, which is assured by $u_F^\beta(t,x) \geq 0$ *a.e.* $(t,x) \in Q_T$, $u_F^\beta(t,x)$ satisfies $-\partial_\nu u_F^\beta(t,x) \in \beta_2(u_F^\beta(t,x))$ *a.e.* $(t,x) \in \Gamma_T$.

On the other hand, $-\partial_\nu u_p^D(t,x) \in \beta_1(u_p^D)$ implies $u_p^D(t,x) \in D(\beta_1) = \{0\}$ and $-\partial_\nu u_p^D(t,x) \in \mathbb{R}^1$, i.e., $u_p^D(t,x)$ obeys the homogeneous Dirichlet boundary condition (see [9, 10, 7]).

Thus since $D(\beta_1) = \{0\}$ and $D(\beta_2) \subset [0, +\infty)$, (iii) of (A3) is satisfied. Consequently, applying Theorem 4.1.3, we find that

$$
(4.2.5) \qquad 0 \leq u_p^D(t,x) \leq u_F^\beta(t,x) \qquad \forall t \in [0,T) \ \ a.e.\ x \in \Omega,
$$

where $T = \min\,(T_m(u_F^\beta), T_m(u_p^D))$, whence follows

$$
(4.2.6) \qquad \|u_p^D(t)\|_{L^\infty} \leq \|u_F^\beta(t)\|_{L^\infty} \qquad \forall t \in [0,T).
$$

Here suppose that $T_m(u_p^D) < T_m(u_F^\beta)$, then it follows from (4.2.6) that

$$\lim_{t \to T_m(u_p^D)} \|u_F^\beta(t)\|_{L^\infty} = +\infty,$$

which contradicts the definition of $T_m(u_F^\beta)$. Hence we conclude that $T_m(u_F^\beta) \leq T_m(u_p^D) < +\infty$. $\square$

As the special case where $F(u) = |u|^{p-2}u$, we get the following (see (2) of Remark 2.2.4).

Corollary 4.2.2. *Assume that* u_0 *satisfies* (4.2.2) *and let* u_p^β *be the unique solution of* $(\mathrm{P})_F^\beta$ with $F(u) = |u|^{p-2}u$, denoted by $(\mathrm{P})_p^\beta$, then $T_m(u_p^\beta) \leq T_m(u_p^D) < \infty$, i.e., u_p^β blows up in finite time.

We next consider another typical classical boundary condition, namely, the following problem with the homogeneous Neumann boundary condition:

$$(\mathrm{P})_p^N \quad \begin{cases} \partial_t u - \Delta u = |u|^{p-2}u, & t > 0,\ x \in \Omega, \\ \partial_\nu u = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x)u = u_0(x) \geq 0, & x \in \Omega. \end{cases}$$

Then it is also well known that $(\mathrm{P})_p^N$ admits the unique positive local solution u_p^N for any $0 \leq u_0 \in L^\infty(\Omega)$ and $T_m(u_p^N) < \infty$ if u_0 is not identically zero in Ω.

Let u_F^N be any solution of $(\mathrm{P})_F^N$ (see Remark 2.2.4), and we apply Theorem 4.1.3 with $m = 1,\ a_{i,j} = \delta_{i,j}$ and $\gamma_1 = \gamma_2 = 0,\ \ \beta_1 = \beta_2 = \beta^N :\equiv 0,\ a_1 = a_2 = u_0$. Then (A1), (A2) and (A3) are automatically satisfied. As for (A4), we take $F_1(t,x,u) = |u|^{p-2}u$ and $F_2(t,x,u) = F(u)$, then (2.2.7) assures (i) of (A4), and it is clear that F_1 satisfies (SC). Then we get

$$\|u_p^N(t)\|_{L^\infty} \leq \|u_F^N(t)\|_{L^\infty} \quad \forall t \in [0,T) \quad \text{with } T = \min\,(T_m(u_p^N), T_m(u_F^N)), \tag{4.2.7}$$

whence follows

$$T_m(u_F^N) \leq T_m(u_p^N). \tag{4.2.8}$$

We now compare $(\mathrm{P})_p^N$ with $(\mathrm{P})_p^\beta$, i.e., $(\mathrm{P})_F^\beta$ with $F(u) = |u|^{p-2}u$. Let u_p^β be the unique non-negative solution of $(\mathrm{P})_p^\beta$ (cf. (2) of Remark 2.2.4). We apply Theorem 4.1.3 with $m = 1,\ a_{i,j} = \delta_{i,j}$ and $\gamma_1 = \gamma_2 = 0,\ a_1 = a_2 = u_0,\ F_1(u) = F_2(u) = |u|^{p-2}u$. Then (A1), (A2) and (A4) are satisfied. As for (A3), define $\beta_1(\cdot)$ and $\beta_2(\cdot)$ by

$$\beta_1(r) = \beta_e(r) := \begin{cases} \beta(r) & \text{for } r > 0, \\ (-\infty, 0] \cup \beta(0) & \text{for } r = 0, \\ \emptyset & \text{for } r < 0, \end{cases}$$

$$\beta_2(r) = \beta_e^N(r) := \begin{cases} 0 & \text{for } r > 0, \\ (-\infty, 0] & \text{for } r = 0, \\ \emptyset & \text{for } r < 0. \end{cases}$$

Then we can show that β_1, β_2 are monotone by the same reasoning as that for (4.2.4).

Moreover since $\beta(r) \subset \beta_1(r)$ and $0 \equiv \beta^N(r) \subset \beta_2(r)$ for $r \geq 0$, and $u_p^\beta(t,x), u_p^N(t,x) \geq 0$ *a.e.* $(t,x) \in \Gamma_T$ are assured by $u_p^\beta(t,x), u^N(t,x) \geq 0$ *a.e.* $(t,x) \in Q_T$, we get $-\partial_\nu u_p^\beta(t,x) \in \beta_1(u_p^\beta(t,x))$ and $-\partial_\nu u_p^N(t,x) \in \beta_2(u_p^N(t,x))$ for *a.e.* $(t,x) \in \Gamma_T$.

Furthermore for any $r_1 \in D(\beta_1)$, $r_2 \in D(\beta_2)$ with $r_2 < r_1$, since $D(\beta_2) = [0, +\infty)$ and $r_2 < r_1$ implies $0 < r_1$ and $0 \in \beta(0)$ is assumed, we have

$$\sup \{ b_2 ; b_2 \in \beta_2(r_2) \} \leq 0 \leq \inf \{ b_1 ; b_1 \in \beta_1(r_1) \}.$$

Hence (ii) of (A3) is satisfied. Consequently, applying Theorem 4.1.3, we find that

$$0 \leq u_p^\beta(t,x) \leq u_p^N(t,x) \qquad \forall t \in [0,T) \ \ a.e.\ x \in \Omega,$$

where $T = \min\ (T_m(u_p^\beta), T_m(u_p^N))$, whence follows

$$T_m(u_p^N) \leq T_m(u_p^\beta) \quad \text{and} \quad \|u_p^\beta(t)\|_{L^\infty} \leq \|u_p^N(t)\|_{L^\infty} \quad \forall t \in [0, T_m(u_p^N)). \tag{4.2.9}$$

Thus putting arguments above all together, we obtain the following observations.

Proposition 4.2.3. *Let* u_F^* *be any solution of* $(\mathrm{P})_F^*$ *and let* u_p^* *be the unique solution of* $(\mathrm{P})_p^*$ *($* = D, \beta, N$). Then the following hold.*

(i) $T_m(u_F^D) \leq T_m(u_p^D),\ T_m(u_F^\beta) \leq T_m(u_p^\beta),\ T_m(u_F^N) \leq T_m(u_p^N)$.

(ii) $T_m(u_p^N) \leq T_m(u_p^\beta) \leq T_m(u_p^D)$.

Remark 4.2.4. *By virtue of* (4.2.5)*, we can also derive some results on the strong maximum principle* (*see* [55]) *for* $(\mathrm{P})_F^\beta$.

Chapter 5

Existence and Nonexistence of Global Solutions

For nonlinear heat equations $\partial_t u - \Delta u = |u|^{p-2}u$ in the whole space $\mathbb{R}^N$, it is well known that there exists the critical Fujita exponent $p_c = 2 + \frac{2}{N}$ which gives the threshold of p that divides the existence and the non-existence of positive global solutions (see [22, 28, 55]). As for the same equation in bounded domains, there is no such a critical exponent of p. In this chapter, however, we show that the same threshold phenomenon can occur in bounded domains, which is controlled according to boundary conditions but not to the exponents p.

5.1 Main Result

We are concerned with the existence and the nonexistence of positive global solutions to (P):

$$\text{(P)} \qquad \begin{cases} \partial_t u - \Delta u = |u|^{p-2}u, & t > 0,\ x \in \Omega, \\ -\partial_\nu u \in \beta(u), & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases}$$

As in the previous chapters, we assume that Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ satisfying $\beta(0) \ni 0$. Moreover let $T_m = T_m(u_0)$ be the maximal existence time of the solution to (P). This type of boundary conditions imposed on (P) can cover classical linear boundary conditions such as the homogeneous Dirichlet or Neumann boundary condition. Indeed, set

$$\beta_D(r) = \begin{cases} \mathbb{R} & r = 0, \\ \emptyset & r \neq 0, \end{cases} \qquad \text{or} \qquad \beta_N(r) = 0 \quad \forall r \in \mathbb{R},$$

then the boundary condition of (P) with $\beta = \beta_D$ or $\beta = \beta_N$ becomes the homogeneous Dirichlet boundary condition or the homogeneous Neumann boundary condition respectively [10, 7]. To simplify the descriptions, we denote (P) with $\beta = \beta_D$ and $\beta = \beta_N$ by $(\text{P})_D$ and $(\text{P})_N$ respectively.

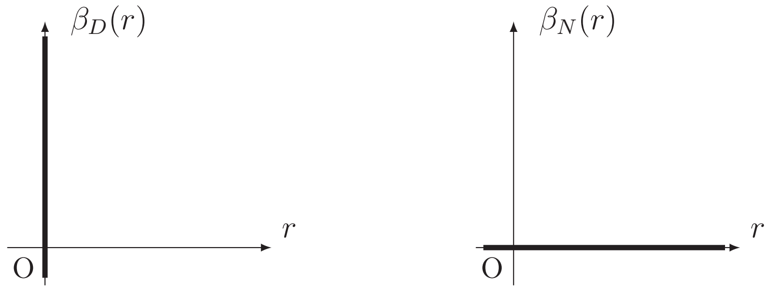

Figure 1: The homogeneous Dirichlet and Neumann boundary condition

In what follows, we always assume that the initial data belong to

$$L^\infty_+(\Omega) := \{\, v \in L^\infty(\Omega)\,;\, v \geq 0, v \not\equiv 0\,\}.$$

Then it is well known that all non-trivial nonnegative solutions of $(\mathrm{P})_N$ blow up in finite time and to the contrary, $(\mathrm{P})_D$ admits always global solutions for small initial data.

With this fact in mind, we classify the nature of (P) into the following two categories reflecting the natures of $(\mathrm{P})_N$ and $(\mathrm{P})_D$ mentioned above.

Definition 5.1.1. (i) (P) *is N-type if and only if* $T_m(u_0) < \infty$ *for all* $u_0 \in L^\infty_+(\Omega)$.

(ii) (P) *is D-type if and only* (P) *is not N-type, that is, there exists* $u_0 \in L^\infty_+(\Omega)$ *such that* $T_m(u_0) = \infty$.

Note that it is obvious $(\mathrm{P})_D$ is D-type and $(\mathrm{P})_N$ is N-type.

We here introduce the following subclass $(\beta^\alpha(\cdot))_{\alpha\in[0,\infty]}$ of (β) by

$$\beta^0 = \beta_D, \quad \beta^\infty = \beta_N \quad \text{and for } \alpha \in (0,\infty) \text{ by} \quad \beta^\alpha(r) = \begin{cases} (-\infty, 0], & r = 0, \\ 0, & r \in (0,\alpha) \\ [0, +\infty), & r = \alpha, \\ \emptyset, & \text{otherwise.} \end{cases}$$

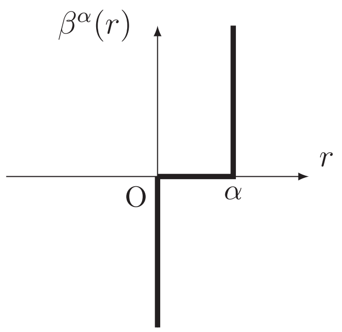

Figure 2: The graph of β_α

For the rest of this chapter, $(\mathrm{P})_\alpha$ denotes (P) with $\beta = \beta^\alpha$. We know that for all $u_0 \in L^\infty_+(\Omega)$, $(\mathrm{P})_\alpha$ possesses a unique nonnegative time local solution u satisfying the regularity in Theorem 2.2.3 (see Chapter 2).

Our main theorem can be stated as follows.

Theorem 5.1.2. *There exists a threshold value $\alpha_c \in (0,\infty)$ such that the followings hold:*

(i) *If $\alpha > \alpha_c$, then $(\mathrm{P})_\alpha$ is N-type.*

(ii) *If $\alpha < \alpha_c$, then $(\mathrm{P})_\alpha$ is D-type.*

5.2 Nonexistence of Global Solutions

We first define $\alpha_0 \in [0,\infty]$ by

$$\alpha_0 := \inf\{\,\alpha > 0\,;\,(\mathrm{P})_\alpha \text{ is N-type.}\,\},$$

where we put $\alpha_0 = \infty$ if $A := \{\,\alpha > 0\,;\,(\mathrm{P})_\alpha \text{ is N-type.}\,\}$ is an empty set.

Lemma 5.2.1. *Let λ_1 be the first eigenvalue of the Dirichlet Laplacian and let $\alpha > \lambda_1^{\frac{1}{p-2}}$, then $(\mathrm{P})_\alpha$ is N-type. In particular we have $\alpha_0 < \infty$.*

Proof. We prove the assertion by contradiction. Let

$$\alpha > \lambda_1^{\frac{1}{p-2}}, \tag{5.2.1}$$

and suppose that there exists a solution u of $(\mathrm{P})_\alpha$ satisfying the regularity in Theorem 2.2.3 with the initial data $u_0 \in L^\infty_+(\Omega)$ such that $T_m(u_0) = \infty$. Let u_1 be a solution to the following heat equation:

$$\begin{cases} \partial_t u_1 - \Delta u_1 = 0, & t > 0,\ x \in \Omega, \\ -\partial_\nu u_1 \in \beta^\alpha(u_1), & t > 0,\ x \in \partial\Omega, \\ u_1(0,x) = a^1(x), & x \in \overline{\Omega}, \end{cases} \tag{5.2.2}$$

where $a^1 \in L^\infty_+(\Omega) \cap C(\overline{\Omega})$ satisfies

$$a^1 < \alpha \quad \text{on } \overline{\Omega}, \qquad a^1 \le u_0 \quad \text{a.e. on } \Omega. \tag{5.2.3}$$

By the definition of β^α and (5.2.3), there exists $T_1 > 0$ such that (5.2.2) is equivalent to the following linear heat equation on $[0,T_1)$:

$$\begin{cases} \partial_t u_1 - \Delta u_1 = 0, & t \in (0,T_1),\ x \in \Omega, \\ \partial_\nu u_1 = 0, & t \in (0,T_1),\ x \in \partial\Omega, \\ u_1(0,x) = a^1(x), & x \in \overline{\Omega}, \end{cases} \tag{5.2.4}$$

Hence by virtue of the classical maximum principle, we see that there exist $\delta \in (0,\alpha)$ and $T_2 \in (0,T_1)$ such that

$$\delta < u_1(t,x), \qquad \forall t \in [T_2,T_1),\ \forall x \in \overline{\Omega}. \tag{5.2.5}$$

On the other hand, applying Theorem 4.1.3 with $m = 1$, $a_1 = a^1$, $a_2 = u_0$, $\gamma_1 = \gamma_2 = 0$, $\beta_1 = \beta_2 = \beta^\alpha$, $F_1 = 0$ and $F_2(r) = |r|^{p-2}r$, we obtain

$$u_1(t,x) \le u(t,x), \qquad \forall t \ge 0, \quad \text{a.e. } x \in \Omega,$$

whence follows from (5.2.5)

$$\delta < u(t,x) \qquad \forall t \geq T_2, \quad \text{a.e. } x \in \Omega,$$

We now introduce the other equation for $t \geq T_2$. Let u_2 be the solution to the following equation:

$$\begin{cases} \partial_t u_2 - \Delta u_2 = |u_2|^{p-2} u_2, & t > T_2, \ x \in \Omega, \\ -\partial_\nu u_2 \in \beta^\alpha(u_2), & t > T_2, \ x \in \partial\Omega, \\ u_2(T_2, x) \equiv \delta, & x \in \Omega, \end{cases} \tag{5.2.6}$$

and let $T_3 > T_2$ denote the maximal existence time of u_2. In the same way as above, applying Theorem 4.1.3 to $(\mathrm{P})_\alpha$ and (5.2.6) on $[T_2, T_3)$ with $m = 1$, $a_1 = \delta$, $a_2 = u(T_2)$, $\gamma_1 = \gamma_2 = 0$, $\beta_1 = \beta_2 = \beta^\alpha$ and $F_1(r) = F_2(r) = |r|^{p-2} r$, we derive

$$u_2(t,x) \leq u(t,x), \qquad \forall t \in [T_2, T_3), \quad \text{a.e. } x \in \Omega. \tag{5.2.7}$$

Note that since δ is a constant satisfying $\delta < \alpha$, by the definition of β^α we can see that there exists $T_4 \in [T_2, T_3)$ such that u_2 is independent of space variables on $[T_2, T_4]$ and satisfies

$$u_2(T_4, x) = \alpha, \qquad \forall x \in \overline{\Omega}. \tag{5.2.8}$$

Therefore, from (5.2.7) and (5.2.8), it follows that $u(t,x) = \alpha$ for all $t \geq T_4$ on $\partial\Omega$, which implies that $(\mathrm{P})_\alpha$ is equivalent to the following form:

$$\begin{cases} \partial_t u - \Delta u = |u|^{p-2} u, & t > T_4, \ x \in \Omega, \\ u = \alpha, & t > T_4, \ x \in \partial\Omega, \\ u(T_4, x) \geq \alpha, & x \in \Omega. \end{cases} \tag{5.2.9}$$

We finally consider the following nonlinear heat equation with the homogeneous Dirichlet boundary condition for $t \geq T_4$, and let u_3 be the solution to

$$\begin{cases} \partial_t u_3 - \Delta u_3 = |u_3|^{p-2} u_3, & t > T_4, \ x \in \Omega, \\ u_3 = 0, & t > T_4, \ x \in \partial\Omega, \\ u_3(T_4, x) \equiv \alpha, & x \in \Omega. \end{cases} \tag{5.2.10}$$

The maximal existence time of u_3 is denoted by $T_5 (> T_4)$. We apply Theorem 4.1.3 to two solutions of (5.2.9) and (5.2.10) on $[T_4, T_5]$ in the similar manner again, so we conclude that

$$u_3(t,x) \leq u(t,x), \qquad \forall t \in [T_4, T_5), \quad \text{a.e. } x \in \Omega,$$

whence follows

$$\|u_3(t)\|_\infty \leq \|u(t)\|_\infty, \qquad \forall t \in [T_4, T_5). \tag{5.2.11}$$

Therefore we see that $T_5 = +\infty$ due to of (5.2.11) and the assumption $T_m(u_0) = +\infty$.

On the other hand, for the eigenfunction $\phi_1(x)$ associated with the first eigenvalue λ_1 normalized by $\|\phi_1\|_1 = 1$, (5.2.1) implies that $\int_\Omega \alpha \, \phi_1(x) \, dx > \lambda_1^{\frac{1}{p-2}}$, whence it follows that u_3 blows up in finite time (see (4.2.2) and Theorem 17.1 in [55]), i.e., $T_5 < +\infty$. This leads to a contradiction. □

Lemma 5.2.2. *Let $\alpha > \alpha_0$. Then* $(\mathrm{P})_\alpha$ *is N-type.*

Proof. Let u be the solution to $(\mathrm{P})_\alpha$, and suppose that $T_m(u_0) = +\infty$ (recall that $T_m(u_0)$ is the maximal existence time of u). By the definition of α_0, we see that there exists $\underline{\alpha} \in (\alpha_0, \alpha)$ such that $(\mathrm{P})_{\underline{\alpha}}$ is N-type, i.e., for every $u_0 \in L^\infty_+(\Omega)$, all positive solutions of the following equations blow up in finite time.

$$(5.2.12)\qquad \begin{cases} \partial_t u - \Delta u = |u|^{p-2}u & t > 0,\ x \in \Omega, \\ -\partial_\nu u \in \beta^{\underline{\alpha}}(u) & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x) & x \in \Omega, \end{cases}$$

Let $\underline{u}$ and $\underline{T}_m(u_0) < \infty$ be a solution and the maximal existence time of (5.2.12) respectively, and we apply Theorem 4.1.3 with $m = 1$, $F_1(r) = F_2(r) = |r|^{p-2}r$, $a_1 = a_2 = u_0$, $\beta_1 = \beta^{\underline{\alpha}}$ and $\beta_2 = \beta^\alpha$. Since $D(\beta^{\underline{\alpha}}) = [0, \underline{\alpha}] \subset [0, \alpha] = D(\beta_\alpha)$, it is clear that (ii) of (A3) in Theorem 4.1.3 holds. As a consequence, applying Theorem 4.1.3, we derive

$$\underline{u}(t,x) \le u(t,x) \qquad \forall t \in [0, \underline{T}_m(u_0)),\ \text{a.e. } x \in \Omega.$$

which implies

$$(5.2.13)\qquad \|\underline{u}(t)\|_\infty \le \|u(t)\|_\infty \qquad \forall t \in [0, \underline{T}_m(u_0)).$$

From the above inequality, it follows that

$$T_m(u_0) \le \underline{T}_m(u_0).$$

This is a contradiction by $T_m(u_0) = +\infty$ and $\underline{T}_m(u_0) < +\infty$. □

5.3 Existence of Global Solutions

Lemma 5.3.1. $\alpha_0 > 0$.

Proof. We only need to show the existence of global solutions to $(\mathrm{P})_\alpha$ for sufficiently small $\alpha > 0$ and small initial data. Let u be the solution to $(\mathrm{P})_\alpha$ with the initial data $u_0 \in L^\infty_+(\Omega)$ satisfying $u_0 \le \alpha$, and let v be the solution to the following equation:

$$(5.3.1)\qquad \begin{cases} \partial_t v - \Delta v = |v|^{p-2}v, & t > 0,\ x \in \Omega, \\ v = \alpha, & t > 0,\ x \in \partial\Omega, \\ v(0,x) \equiv \alpha, & x \in \Omega. \end{cases}$$

The maximal existence time of v is denoted by $T^v_m \in (0, +\infty]$. We here put

$$\beta^\alpha_D(r) := \begin{cases} \mathbb{R}, & r = \alpha, \\ \emptyset, & r \ne \alpha. \end{cases}$$

Note that from $u \ge 0$ it follows that $D(\beta^\alpha) = [0, \alpha]$ and

$$r \le \alpha, \qquad \forall r \in D(\beta^\alpha).$$

Then by applying Theorem 4.1.3 with $m = 1$, $a_1 = u_0$, $a_2 = \alpha$, $\gamma_1 = \gamma_2 = 0$, $\beta_1 = \beta^\alpha$, $\beta_2 = \beta^\alpha_D$ and $F_1(r) = F_2(r) = |r|^{p-2}r$, it holds that

$$u(t,x) \le v(t,x), \qquad \forall t \in [0,T^*),\ \text{a.e. } x \in \Omega,$$

where $T^* := \min(T_m, T^v_m)$. Hence we see that

$$\|u(t)\|_\infty \le \|v(t)\|_\infty, \qquad \forall t \in [0,T),$$

whence follows

$$T^v_m \le T_m.$$

In order to prove this lemma, we are going to show $T^v_m = +\infty$. We first consider the case where $N \ge 3$. Let $w := v - \alpha$. If v solves (5.3.1), then w solves the following equation:

$$\text{(5.3.2)} \qquad \begin{cases} \partial_t w - \Delta w = |w+\alpha|^{p-2}(w+\alpha), & t>0,\ x\in\Omega, \\ w = 0, & t>0,\ x \in \partial\Omega, \\ w(0,x) = 0, & x \in \Omega. \end{cases}$$

Put

$$r_0 := \max\left\{2, \frac{N(p-2)}{2}\right\}.$$

Multiplying (5.3.2) by $|w|^{r-2}w$ for $r > r_0$, we have

$$\frac{1}{r}\frac{d}{dt}\|w(t)\|_r^r + (r-1)\int_\Omega |\nabla w|^2|w|^{r-2}dx \le \int_\Omega |w+\alpha|^{p-1}|w|^{r-1}dx.$$

Since $|a+b|^{p-1} \le C_p(|a|^{p-1} + |b|^{p-1})$ for a, $b \in \mathbb{R}$ with $C_p := 2^{p-2}$, it holds that

$$\begin{aligned} \int_\Omega |w+\alpha|^{p-1}|w|^{r-1}dx &\le C_p\int_\Omega |w|^{p+r-2}dx + C_p\int_\Omega \alpha^{p-1}\int_\Omega |w|^{r-1}dx \\ &\le C_p\|w(t)\|_{p+r-2}^{p+r-2} + C_p|\Omega|^{\frac{1}{r}}\alpha^{p-1}\|w(t)\|_r^{r-1}. \end{aligned}$$

Moreover, note that

$$\begin{aligned} (r-1)\int_\Omega |\nabla w|^2|w|^{r-2}dx &= \frac{4(r-1)}{r^2}\left\|\nabla|w|^{\frac{r}{2}}\right\|_2^2 \\ &\ge C_S\frac{4(r-1)}{r^2}\left\||w|^{\frac{r}{2}}\right\|_{2^*}^2 \\ &= C_S\frac{4(r-1)}{r^2}\|w\|^r_{\frac{rN}{N-2}}, \end{aligned}$$

where $2^* = \frac{2N}{N-2}$ and C_S is the best constant of Sobolev embedding $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$. Therefore we can deduce that

$$\text{(5.3.3)} \quad \frac{1}{r}\frac{d}{dt}\|w(t)\|_r^r + C_S\frac{4(r-1)}{r^2}\|w(t)\|^r_{\frac{rN}{N-2}} \le C_p\|w(t)\|_{p+r-2}^{p+r-2} + C_p|\Omega|^{\frac{1}{r}}\alpha^{p-1}\|w(t)\|_r^{r-1}.$$

Since $r > r_0 \geq \frac{N(p-2)}{2}$, Lemma 1.1.5 and Corollary 1.1.2 assure that for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\|w\|_{p+r-2}^{p+r-2} \leq \epsilon \|w\|_{\frac{rN}{N-2}}^r + C_\epsilon \|w\|_r^{r(1+\delta)}, \tag{5.3.4}$$

where $\delta = \frac{2(p-2)}{2r-N(p-2)}$. Hence by (5.3.3) and (5.3.4), we derive

$$\frac{1}{r}\frac{d}{dt}\|w(t)\|_r^r + \left\{ C_S \frac{4(r-1)}{r^2} - C_p \epsilon \right\} \|w(t)\|_{\frac{rN}{N-2}}^r$$
$$\leq C_p C_\epsilon \|w(t)\|_r^{r(1+\delta)} + C_p |\Omega|^{\frac{1}{r}} \alpha^{p-1} \|w(t)\|_r^{r-1}.$$

Choosing $\epsilon = \frac{2C_S(r-1)}{C_p r^2}$, noting that $\frac{rN}{N-2} > r$ and using Hölder's inequality, we obtain

$$\frac{1}{r}\frac{d}{dt}\|w(t)\|_r^r + \tilde{C}\|w(t)\|_r^r \leq C_p C_\epsilon \|w(t)\|_r^{r(1+\delta)} + C_p |\Omega|^{\frac{1}{r}} \alpha^{p-1} \|w(t)\|_r^{r-1}, \tag{5.3.5}$$

where

$$\tilde{C} := \frac{2C_S(r-1)}{C_p r^2} |\Omega|^{-\frac{2}{N}}.$$

Divide both sides of (5.3.5) by $\|w(t)\|_r^{r-1}$, then we get

$$\frac{d}{dt}\|w(t)\|_r + \tilde{C}\|w(t)\|_r \leq C_p C_\epsilon \|w(t)\|_r^{r\delta+1} + C_p |\Omega|^{\frac{1}{r}} \alpha^{p-1}. \tag{5.3.6}$$

We put

$$\epsilon_* = \left(\frac{\tilde{C}}{2C_p C_\epsilon} \right)^{\frac{1}{r\delta}}, \qquad \alpha_* = \left(\frac{\epsilon_* \tilde{C}}{4C_p |\Omega|^{\frac{1}{r}}} \right)^{\frac{1}{p-1}}. \tag{5.3.7}$$

We here claim that if $\alpha \leq \alpha_*$ then

$$\sup_{t \geq 0} \|w(t)\|_r \leq \epsilon_*.$$

Suppose that this claim dose not hold. Thus there exists some $t_0 \in (0, +\infty)$ such that

$$\|w(t_0)\|_r > \epsilon_*.$$

Moreover, since $w(0) = 0$, there exists $t_1 \in (0, T_0)$ such that

$$\begin{aligned} &\|w(t)\|_r < \epsilon_*, \qquad \forall t \in (0, T_1), \\ &\|w(t_1)\|_r = \epsilon_*. \end{aligned} \tag{5.3.8}$$

Hence we get

$$\begin{aligned} C_p C_\epsilon \|w(t)\|_r^{r\delta+1} &= C_p C_\epsilon \|w(t)\|_r^{r\delta} \|w(t)\|_r \\ &\leq \epsilon_*^{r\delta} C_p C_\epsilon \|w(t)\|_r, \qquad \forall t \in [0, t_1], \end{aligned}$$

whence follows from (5.3.6) and (5.3.7)

$$\frac{d}{dt}\|w(t)\|_r + \frac{\tilde{C}}{2}\|w(t)\|_r \le C_p|\Omega|^{\frac{1}{r}}\alpha^{p-1}, \qquad t \in [0, t_1].$$

Therefore we see that

$$\begin{aligned}
\|w(t)\|_r &\le \|w(0)\|_r e^{-\frac{\tilde{C}}{2}t} + \int_0^t C_p|\Omega|^{\frac{1}{r}}\alpha^{p-1}e^{-\frac{\tilde{C}}{2}(t-s)}ds \\
&= C_p|\Omega|^{\frac{1}{r}}\alpha^{p-1}\frac{2}{\tilde{C}}\left(1 - e^{-\frac{\tilde{C}}{2}t}\right) \\
&\le C_p|\Omega|^{\frac{1}{r}}\alpha^{p-1}\frac{2}{\tilde{C}}, \qquad t \in [0, t_1].
\end{aligned}$$

In particular, this inequality gives

$$\epsilon_* = \|w(t_1)\|_r \le \frac{\epsilon_*}{2} < \epsilon_*,$$

which is a contradiction. Thus we conclude that

$$\sup_{t\ge 0}\|w(t)\|_r \le \epsilon_*, \qquad \forall r \in (r_0, +\infty).$$

Consequently, letting $r \to +\infty$, we have

$$\sup_{t\ge 0}\|w(t)\|_\infty \le \epsilon_*$$

and $T_m^v = +\infty$ provided that $\alpha \le \alpha_*$.

As for the case where $N = 1, 2$, using Sobolev's embedding theorem

$$\begin{aligned}
&H^1(\Omega) \hookrightarrow L^s(\Omega) \qquad \forall s \in [1, +\infty), \quad (N = 2),\\
&H^1(\Omega) \hookrightarrow L^\infty(\Omega) \qquad\qquad\qquad (N = 1),
\end{aligned}$$

we obtain (5.3.3) up to constants of the second term on the left hand side. Since the above argument of this proof except the Sobolev's embedding do not depend on the dimension N, this lemma can be shown as in the case where $N \ge 3$. □

Lemma 5.3.2. *Let $\alpha < \alpha_0$. Then* $(\mathrm{P})_\alpha$ *is D-type.*

Proof. By the similar argument as the proof of Lemma 5.2.2, for fixed $\alpha > 0$, we see that there exists $\overline{\alpha} \in (\alpha, \alpha_0)$ such that $(\mathrm{P})_{\overline{\alpha}}$ is D-type. In other words, there exists $\overline{u}_0 \in L^\infty_+(\Omega)$ such that the following equation $(\mathrm{P})_{\overline{\alpha}}$ possesses a global solution $\overline{u}$ with the maximal existence time $\overline{T}_m = +\infty$:

$$\text{(5.3.9)} \qquad \begin{cases}
\partial_t u - \Delta u = |u|^{p-2}u, & t > 0,\ x \in \Omega,\\
-\partial_\nu u \in \beta^{\overline{\alpha}}(u), & t > 0,\ x \in \partial\Omega,\\
u(0, x) = \overline{u}_0(x), & x \in \Omega.
\end{cases}$$

Let u be the solution to $(\mathrm{P})_\alpha$ with the initial data $u_0 \in L^\infty_+(\Omega)$ satisfying $u_0 \leq \overline{u}_0$ and T_m be the maximal existence time of u. Since $D(\beta^\alpha) = [0, \alpha] \subset [0, \overline{\alpha}] = D(\beta^{\overline{\alpha}})$, it is clear that (ii) of (A3) in Theorem 4.1.3. Thus applying Theorem 4.1.3 with $m = 1$, $a_1 = u_0$, $a_2 = \overline{u}_0$, $\gamma_1 = \gamma_2 = 0$, $\beta_1 = \beta^\alpha$, $\beta_2 = \beta^{\overline{\alpha}}$ and $F_1(r) = F_2(r) = |r|^{p-2}r$, we have

$$u(t, x) \leq \overline{u}(t, x), \qquad \forall t \in [0, T), \quad \text{a.e. } x \in \Omega,$$

where $T := \min(T_m, \overline{T}_m) = T_m$. Hence it holds that

$$\|u(t)\|_\infty \leq \|\overline{u}(t)\|_\infty, \qquad \forall t \in [0, T_m),$$

whence follows

$$\overline{T}_m \leq T_m.$$

As a consequence, we obtain $T_m = +\infty$. □

Proof of Theorem 5.1.2. By lemma 5.2.1, 5.2.2, 5.3.1 and 5.3.2, we can deduce that theorem 5.1.2 is true with $\alpha_c = \alpha_0$. □

Chapter 6

Structural Stability

In this chapter, we revisit $(\mathrm{P})_q$:

$$(\mathrm{P})_q \qquad \begin{cases} \partial_t u - \Delta u = |u|^{p-2}u, & t > 0,\ x \in \Omega, \\ \partial_\nu u + \beta_q(u) \ni 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is bounded domain with smooth boundary $\partial\Omega$, $p > 2$ is a given number, ν denotes the unit outward normal vector on $\partial\Omega$ and $\partial_\nu u = \nabla u \cdot \nu$.

We here consider not only the single valued cases $\beta_q(r) = |r|^{q-2}r$ with $q \in (1,\infty)$ but also the multivalued cases β_1 and β_∞ corresponding to the cases $q = 1$ and $q = \infty$ respectively, given by:

$$\beta_1(r) := \begin{cases} 1 & r > 0, \\ [-1,1] & r = 0, \\ -1 & r < 0, \end{cases} \qquad \beta_\infty(r) := \begin{cases} [0,\infty) & r = 1, \\ 0 & r \in (-1,1), \\ (-\infty,0] & r = -1, \\ \emptyset & |r| > 1, \end{cases}$$

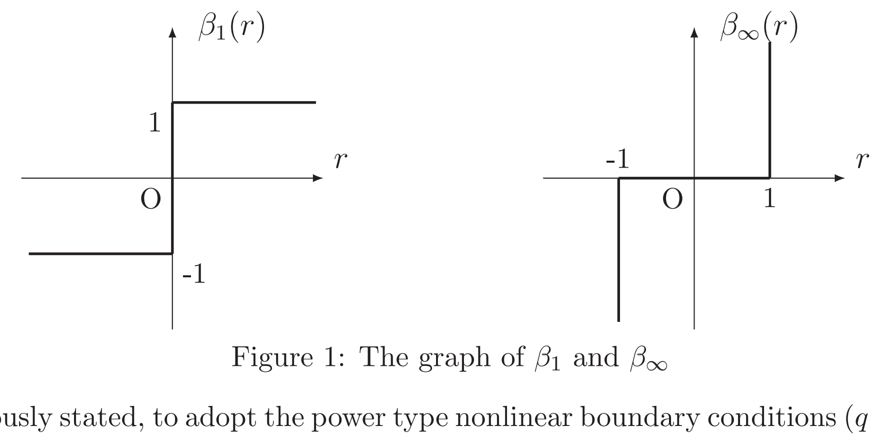

Figure 1: The graph of β_1 and β_∞

As previously stated, to adopt the power type nonlinear boundary conditions ($q \in (1,+\infty)$) is reasonable from a physical point of view (cf. Stefan-Boltzmann's radiation law).

On the other hand, the multivalued nonlinear boundary condition β_∞ also appears in the Signorini problem, which was first studied by Fichera [21]. This arises in the theory of elasticity in connection with the mathematical description of friction problems (see also [19]). We already observed that the initial-boundary value problem $(\mathrm{P})_q$ is locally well-posed for $u_0 \in L^\infty(\Omega)$ (see Theorem 2.2.1 and Theorem 2.2.3).

The main purpose of this chapter is show the continuous dependence of solutions of $(\mathrm{P})_q$ with respect to the parameter $q \in [1,\infty]$. Especially it will be shown that the solution of $(\mathrm{P})_q$ converges to the solution of $(\mathrm{P})_\infty$ or $(\mathrm{P})_1$ in a suitable sense as q converges to ∞ or 1.

To carry out this, we make the most use of the notion of Mosco convergence for convex functionals associated with the $-\Delta$ with nonlinear boundary conditions.

6.1 Mosco Convergence and Evolution Equations

First of all, we state our main result in this chapter.

Theorem 6.1.1. *Let $q_0 \in [1,\infty]$ and (q_n) be a sequence in $(1,\infty)$ satisfying $q_n \to q_0$ as $n \to \infty$. Moreover let a_n, $a \in L^\infty(\Omega)$ be the initial values for $(P)_{q_n}$ and $(P)_{q_0}$ satisfying*

$$a_n \to a \qquad \text{strongly in } L^\infty(\Omega),$$

and denote by u_n and u the solutions of $(P)_{q_n}$ and $(P)_{q_0}$ on $[0,T_0]$ given in Theorem 2.2.3 respectively. Then u_n converge to u as $n \to \infty$ in the following sense:

$$\begin{aligned} u_n &\to u && \text{strongly in } C([0,T_0];L^2(\Omega)),\\ \sqrt{t}\partial_t u_n &\to \sqrt{t}\partial_t u && \text{strongly in } L^2(0,T_0;L^2(\Omega)). \end{aligned}$$

Remark 6.1.2. *Since a_n converges to a strongly in $L^\infty(\Omega)$, we can take a common $T_0 > 0$ for all $n \in \mathbb{N}$ (see the proof of Theorem 2.2.3).*

In order to prove Theorem 6.1.1, we rely on the abstract theory of Mosco convergence of functionals and evolution equations governed by subdifferential operators in a real Hilbert space H. We first investigate the asymptotic behavior of solutions to nonlinear evolution equations associated with subdifferential operators whose functionals are Mosco convergent.

Proposition 6.1.3. *Let $(\phi^n) \subset \Phi(H)$, $\phi \in \Phi(H)$, $a_n \in \overline{D(\phi^n)}$ and $a \in \overline{D(\phi)}$, and let f_n, $f \in L^1(0,T;H)$ satisfying $\sqrt{t}f_n$, $\sqrt{t}f \in L^2(0,T;H)$. Assume that $B : H \to H$ is Lipschitz perturbation with Lipschitz constant $L > 0$. Let u_n and u be the strong solutions on $[0,T]$ of*

$$\frac{d}{dt}u_n(t) + \partial\phi^n(u_n(t)) + Bu_n(t) \ni f_n(t), \qquad u_n(0) = a_n, \tag{6.1.1}$$

$$\frac{d}{dt}u(t) + \partial\phi(u(t)) + Bu(t) \ni f(t), \qquad u(0) = a, \tag{6.1.2}$$

respectively. If $a_n \to a$ in H, $f_n \to f$ strongly in $L^1(0,T;H)$, $\sqrt{t}f_n \to \sqrt{t}f$ strongly in $L^2(0,T;H)$ and

$$\phi^n \xrightarrow{M} \phi \qquad \text{on } H, \tag{6.1.3}$$

then

$$u_n \to u \qquad \textit{strongly in } C([0,T];H),$$
$$\sqrt{t}\frac{du_n}{dt} \to \sqrt{t}\frac{du}{dt} \qquad \textit{strongly in } L^2(0,T;H).$$

Proof. Note that from Proposition 1.5.3 it follows that there exist $(\alpha, \beta) \in \partial\phi$ and $(\alpha_n, \beta_n) \in \partial\phi^n$ such that $\alpha_n \to \alpha$, $\beta_n \to \beta$ and $\phi^n(\alpha_n) \to \phi(\alpha)$. In the same way as in the proof of Proposition 1.4.1, we can assume that $B(0) = 0$ and

$$\phi(\alpha) = \min\{\phi(u)\,;\, u \in D(\phi)\} = 0,$$
$$\phi^n(\alpha_n) = \min\{\phi^n(u)\,;\, u \in D(\phi^n)\} = 0.$$

Step.1: $u_n \to u$ strongly in $C([0,T];H)$.

For the rest of this step, let A^n and A denote $\partial\phi^n$ and $\partial\phi$ respectively for simplicity. We first show this proposition for the case where $f_n = f = 0$. For fixed $\lambda > 0$, we set $y = (1 + \sqrt{\lambda}A)^{-1}a \in D(A)$ and $y_n = (1 + \sqrt{\lambda}A^n)^{-1}a_n \in D(A^n)$. Let v, v_λ, v_n, $v_{n,\lambda} \in C([0,T];H)$ be the solutions of the following Cauchy problems:

$$\frac{dv}{dt} + Av + Bv \ni 0, \qquad v(0) = y, \tag{6.1.4}$$
$$\frac{dv_\lambda}{dt} + A_\lambda v_\lambda + Bv_\lambda = 0, \qquad v_\lambda(0) = y, \tag{6.1.5}$$
$$\frac{dv_n}{dt} + A^n v_n + Bv_n \ni 0, \qquad v_n(0) = y_n, \tag{6.1.6}$$
$$\frac{dv_{n,\lambda}}{dt} + A^n_\lambda v_{n,\lambda} + Bv_{n,\lambda} = 0, \qquad v_{n,\lambda}(0) = y_n, \tag{6.1.7}$$

where A_λ and A^n_λ denote the Yosida approximation of A and A^n respectively. In order to prove this lemma, noting

$$\begin{aligned}|u_n(t) - u(t)|_H \le{}& |u_n(t) - v_n(t)|_H + |v_n(t) - v_{n,\lambda}(t)|_H + |v_{n,\lambda}(t) - v_\lambda(t)|_H \\ &+ |v_\lambda(t) - v(t)|_H + |v(t) - u(t)|_H,\end{aligned} \tag{6.1.8}$$

we are going to derive a priori estimates for all terms on the right hand side of (6.1.8).

We first consider u_n and v_n. Let $\xi_n \in A^n u_n$ and $\zeta_n \in A^n v_n$, it follows from (6.1.1) and (6.1.6)

$$\frac{d}{dt}(u_n - v_n) + \xi_n - \zeta_n + Bu_n - Bv_n = 0.$$

Multiplying this equation by $u_n - v_n$, we have

$$\frac{1}{2}\frac{d}{dt}|u_n(t) - v_n(t)|_H^2 + (\xi_n - \zeta_n, u_n - v_n)_H + (Bu_n - Bv_n, u_n - v_n)_H = 0.$$

Since A^n is monotone operator in H and B is Lipschitz continuous, it holds that

$$|u_n(t) - v_n(t)|_H^2 \le |a_n - y_n|_H^2 + \int_0^T 2L|u_n(t) - v_n(t)|_H^2 dt,$$

which implies by Gronwall's inequality

$$|u_n(t) - v_n(t)|_H \le e^{LT}|a_n - y_n|_H \qquad \forall t \in [0, T]. \tag{6.1.9}$$

Similarly, by (6.1.2) and (6.1.4), we can deduce that

$$|u(t) - v(t)|_H \le e^{LT}|a - y|_H \qquad \forall t \in [0, T]. \tag{6.1.10}$$

By (6.1.5) and (6.1.7), we have

$$\frac{d}{dt}(v_\lambda - v_{n,\lambda}) + A_\lambda v_\lambda - A^n_\lambda v_{n,\lambda} + Bv_\lambda - Bv_{n,\lambda} = 0.$$

Integrating this equation over $[0, T]$, we see that

$$\begin{aligned}
|v_\lambda(t) - v_{n,\lambda}(t)|_H &\le |y - y_n|_H + \int_0^T |A_\lambda v_\lambda(t) - A^n_\lambda v_{n,\lambda}(t)|_H dt \\
&\quad + \int_0^T |Bv_\lambda(t) - Bv_{n,\lambda}(t)|_H dt \\
&\le |y - y_n|_H + \int_0^T |A_\lambda v_\lambda(t) - A^n_\lambda v_\lambda(t)|_H dt \\
&\quad + \int_0^T |A^n_\lambda v_\lambda(t) - A^n_\lambda v_\lambda(t)|_H dt + \int_0^T L|v_\lambda(t) - v_{n,\lambda}(t)|_H dt \\
&\le |y - y_n|_H + \|A_\lambda v_\lambda - A^n_\lambda v_\lambda\|_{L^1(0,T;H)} \\
&\quad + \left(\frac{1}{\lambda} + L\right) \int_0^T |v_\lambda(t) - v_{n,\lambda}(t)|_H dt,
\end{aligned}$$

where we use the fact the Yosida approximation is Lipschitz continuous with Lipschitz constant $1/\lambda > 0$ (see Proposition 1.3.7). By the definition of Yosida approximation and Gronwall's inequality, we get

(6.1.11)

$$\begin{aligned}
|v_\lambda(t) - v_{n,\lambda}(t)|_H &\le \left(|y - y_n|_H + \|A_\lambda v_\lambda - A^n_\lambda v_\lambda\|_{L^1(0,T;H)}\right) e^{\left(\frac{1}{\lambda}+L\right)T} \\
&\le \left(|y - y_n|_H + \frac{1}{\lambda}\|(1 + \lambda A)^{-1}v_\lambda - (1 + \lambda A^n)^{-1}v_\lambda\|_{L^1(0,T;H)}\right) e^{\left(\frac{1}{\lambda}+L\right)T}.
\end{aligned}$$

We next consider the part $|v_\lambda - v|_H$. To do this, we first try to derive a priori estimates of v_λ. By the similar argument of estimates (1.4.4) in the proof of Proposition 1.4.1, we can get

$$\sup_{t\in[0,T]} |v_\lambda(t)|_H \le |\alpha|_H + e^{LT}\left(|y|_H + |\alpha|_H + T|B\alpha|_H\right). \tag{6.1.12}$$

For $h > 0$, since A_λ is monotone, we have

$$\begin{aligned}
\frac{d}{dt}|v_\lambda(t + h) - v_\lambda(t)|^2_H &= -2\left(A_\lambda v_\lambda(t + h)A_\lambda v_\lambda(t), v_\lambda(t + h) - v_\lambda(t)\right)_H \\
&\quad - 2\left(Bv_\lambda(t + h) - Bv_\lambda(t), v_\lambda(t + h) - v_\lambda(t)\right)_H \\
&\le 2L\,|v_\lambda(t + h) - v_\lambda(t)|^2_H,
\end{aligned}$$

which implies

$$|v_\lambda(t+h) - v_\lambda(t)|_H \le e^{LT}|v_\lambda(h) - v_\lambda(0)|_H.$$

Dividing both sides by h and letting $h \to 0$, we obtain

$$\left|\frac{dv_\lambda}{dt}(t)\right|_H \le e^{LT}\left|\frac{dv_\lambda}{dt}(0)\right|_H.$$

Since $\frac{dv_\lambda}{dt} = -A_\lambda v_\lambda - Bv_\lambda$ and $A_\lambda \subset AJ_\lambda$, it follows from (6.1.12) and Proposition 1.3.7 that

$$\begin{aligned}|A_\lambda v_\lambda(t)|_H &\le L|v_\lambda(t)|_H + e^{LT}\left(|A^0 y| + L|y|_H\right)\\ &\le L\left\{|\alpha|_H + e^{LT}\left(|y|_H + |\alpha|_H + T|B\alpha|_H\right)\right\} + e^{LT}\left(|A^0 y| + L|y|_H\right).\end{aligned}$$

Note that since $y = J_{\sqrt{\lambda}}a \to a$ in H as $\lambda \downarrow 0$, the estimate $|y|_H \le |a|_H + 1$ holds for small $\lambda > 0$. Hence

$$\begin{aligned}\sup_{t\in[0,T]}|A_\lambda v_\lambda(t)|_H &\le L\left\{|\alpha|_H + e^{LT}\left(|a|_H + 1 + |\alpha|_H + T|B\alpha|_H\right)\right\}\\ &\quad + e^{LT}\left\{|A^0 y| + L(|a|_H + 1)\right\}.\end{aligned} \tag{6.1.13}$$

For $0 < \mu < \lambda$, we can deduce that (see the proof of Proposition 1.4.1)

$$\frac{1}{2}\frac{d}{dt}|v_\lambda(t) - v_\mu(t)|_H^2 \le \frac{\lambda}{4}|A_\mu v_\mu(t)|_H^2 + \frac{\mu}{4}|A_\lambda v_\lambda(t)|_H^2 + L|v_\lambda(t) - v_\mu(t)|_H^2,$$

where v_μ solves

$$\frac{dv_\mu}{dt} + A_\mu v_\mu + Bv_\mu = 0, \quad (t > 0), \qquad v_\mu(0) = y.$$

By (6.1.13), we see that

$$\begin{aligned}|v_\lambda(t) - v_\mu(t)|_H^2 \le \frac{\lambda+\mu}{2}T\Big[&L\left\{|\alpha|_H + e^{LT}\left(|a|_H + 1 + |\alpha|_H + T|B\alpha|_H\right)\right\}\\ &+e^{LT}\left\{|A^0 y| + L(|a|_H + 1)\right\}\Big]^2\\ &+ 2L\int_0^t |v_\lambda(s) - v_\mu(s)|_H^2 ds,\end{aligned}$$

which implies

$$\begin{aligned}|v_\lambda(t) - v_\mu(t)|_H^2 \le e^{2LT}\frac{\lambda+\mu}{2}T\Big[&L\left\{|\alpha|_H + e^{LT}\left(|a|_H + 1 + |\alpha|_H + T|B\alpha|_H\right)\right\}\\ &+e^{LT}\left\{|A^0 y| + L(|a|_H + 1)\right\}\Big], \qquad \forall t \in [0,T].\end{aligned}$$

Letting $\mu \downarrow 0$, we conclude that

$$
\begin{aligned}
|v_\lambda(t) - v(t)|_H &\le e^{LT}\sqrt{\frac{\lambda T}{2}}\left[L\left\{|\alpha|_H + e^{LT}\left(|a|_H + 1 + |\alpha|_H + T|B\alpha|_H\right)\right\}\right. \\
&\qquad \left. + e^{LT}\left\{|A^0 y| + L(|a|_H + 1)\right\}\right] \\
&\le e^{LT}\sqrt{\frac{\lambda T}{2}}\left[L\left\{|\alpha|_H + e^{LT}\left(|a|_H + 1 + |\alpha|_H + T|B\alpha|_H\right)\right\}\right. \\
&\qquad \left. + e^{LT}\left\{L(|a|_H + 1)\right\}\right] + e^{2LT}\sqrt{\frac{\lambda T}{2}}|A^0 y|_H.
\end{aligned}
$$

Note that $A_\lambda \subset AJ_\lambda$ and

$$
|A^0 y|_H = |A^0(1 + \sqrt{\lambda}A)^{-1}a|_H \le \frac{1}{\sqrt{\lambda}}\left|(1 - J_{\sqrt{\lambda}})a\right|_H
$$

Consequently, we obtain

$$
\begin{aligned}
|v_\lambda(t) - v(t)|_H &\le e^{LT}\sqrt{\frac{\lambda T}{2}}\left[L\left\{|\alpha|_H + e^{LT}\left(|a|_H + 1 + |\alpha|_H + T|B\alpha|_H\right)\right\}\right. \\
&\qquad \left. + e^{LT}\left\{L(|a|_H + 1)\right\}\right] + e^{2LT}\sqrt{\frac{T}{2}}|a - y|_H.
\end{aligned}
\tag{6.1.14}
$$

Similarly, we have

$$
\begin{aligned}
|v^n_\lambda(t) - v^n(t)|_H &\le e^{LT}\sqrt{\frac{\lambda T}{2}}\left[L\left\{|\alpha_n|_H + e^{LT}\left(|a_n|_H + 1 + |\alpha_n|_H + T|B\alpha_n|_H\right)\right\}\right. \\
&\qquad \left. + e^{LT}\left\{L(|a_n|_H + 1)\right\}\right] + e^{2LT}\sqrt{\frac{T}{2}}|a_n - y_n|_H.
\end{aligned}
\tag{6.1.15}
$$

From (6.1.9), (6.1.10), (6.1.11), (6.1.14), (6.1.15) and (6.1.8), it follows that

$$\begin{aligned}\sup_{t\in[0,T]} |u(t)-u_n(t)| \le e^{LT}\left(|a-y|_H+|a_n-y_n|_H\right)& \\ +e^{2LT}\sqrt{\frac{T}{2}}\left(|a-y|_H+|a_n-y_n|_H\right)& \\ +e^{LT}\sqrt{\frac{\lambda T}{2}}L(|\alpha|_H+|\alpha_n|_H)& \\ +e^{2LT}\sqrt{\frac{\lambda T}{2}}\left\{|a|_H+(1+LT)\,|\alpha|_H+1\right\}& \\ +e^{2LT}\sqrt{\frac{\lambda T}{2}}\left\{|a_n|_H+(1+LT)\,|\alpha_n|_H+1\right\}& \\ +e^{2LT}\sqrt{\frac{\lambda T}{2}}\left\{L\left(|a|_H+|a_n|_H\right)+2\right\}& \\ +e^{\left(\frac{1}{\lambda}+L\right)T}|y-y_n|_H& \\ +e^{\left(\frac{1}{\lambda}+L\right)T}\frac{1}{\lambda}\|(1+\lambda A)^{-1}v_\lambda-(1+\lambda A^n)^{-1}v_\lambda\|_{L^1(0,T;H)}.&\end{aligned}$$

Note that by Proposition 1.5.3 we know

$$(1+\lambda A^n)^{-1}w\to(1+\lambda A)^{-1}w,\qquad \forall\lambda>0,\ \forall w\in H, \tag{6.1.16}$$

and

$$\begin{aligned}|y-y_n|_H &\le |(1+\sqrt{\lambda}A^n)^{-1}a_n-(1+\sqrt{\lambda}A)^{-1}a|_H \\ &\le |(1+\sqrt{\lambda}A^n)^{-1}a_n-(1+\sqrt{\lambda}A^n)^{-1}a|_H+|(1+\sqrt{\lambda}A^n)^{-1}a-(1+\sqrt{\lambda}A)^{-1}a|_H \\ &\le |a_n-a|_H+|(1+\sqrt{\lambda}A^n)^{-1}a-(1+\sqrt{\lambda}A)^{-1}a|_H \\ &\to 0.\end{aligned}$$

Moreover, applying the Lebesgue's dominant convergence theorem, we have

$$\begin{aligned}\limsup_{n\to\infty}\|u(t)-u_n(t)\|_{C([0,T];H)} \le 2e^{LT}\left(1+e^{LT}\sqrt{\frac{T}{2}}\right)|a-y|_H& \\ +2e^{2LT}\sqrt{\frac{\lambda T}{2}}\left\{L(|a|_H+|\alpha|_H)+1\right\}& \\ +2e^{2LT}\sqrt{\frac{\lambda T}{2}}\left\{|a|_H+(1+LT)\,|\alpha|_H+1\right\},\qquad \forall\lambda>0.&\end{aligned}$$

Therefore we finally conclude that

$$\lim_{n\to\infty}\|u_n(t)-u(t)\|_{C([0,T];H)}=0.$$

For the case where $f_n \neq 0$ and $f \neq 0$, we use the density argument in order to show the desired result. Let g be a piece-wise constant function in $L^1(0,T;H)$ and $\tilde{u}_n$ and $\tilde{u}$ be solutions to the following equations respectively:

$$\frac{d\tilde{u}_n}{dt} + A^n\tilde{u}_n + B\tilde{u}_n \ni g, \qquad \tilde{u}_n(0) = a_n,$$

$$\frac{d\tilde{u}}{dt} + A\tilde{u} + B\tilde{u} \ni g, \qquad \tilde{u}(0) = a.$$

The previous result applied successively, that is, $\tilde{u}_n \to \tilde{u}$ in H uniformly on $[0,T]$, since the translated operators $A^n u - g$ and $Au - g$ satisfy (6.1.16) (see Remaque 3.12 in [9]). Hence we only need to consider the following estimate:

$$|u_n(t) - u(t)|_H \leq |u_n(t) - \tilde{u}_n(t)|_H + |\tilde{u}_n(t) - \tilde{u}(t)|_H + |\tilde{u}(t) - u(t)|_H.$$

By the monotonicity of A, we can derive

$$\frac{1}{2}\frac{d}{dt}|u(t) - \tilde{u}(t)|_H^2 \leq L|u(t) - \tilde{u}(t)|_H^2 + (f - g, u - \tilde{u})_H\,.$$

Integrating it over $[0,T]$ and applying Lemma 1.2.1, we see that

$$|u(t) - \tilde{u}(t)|_H \leq e^{LT}\|f - g\|_{L^1(0,T;H)}, \qquad \forall t \in [0,T].$$

Similarly we have

$$|u_n(t) - \tilde{u}_n(t)|_H \leq e^{LT}\|f_n - g\|_{L^1(0,T;H)}, \qquad \forall t \in [0,T].$$

Therefore, from these inequalities, it follows that

$$\|u_n - u\|_{C([0,T];H)} \leq e^{LT}\left(\|f - g\|_{L^1(0,T;H)} + \|f_n - g\|_{L^1(0,T;H)}\right) + \|\tilde{u}_n - \tilde{u}\|_{C([0,T];H)},$$

whence follows

$$\limsup_{n\to\infty} \|u_n - u\|_{C([0,T];H)} \leq 2e^{LT}\|f - g\|_{L^1(0,T;H)},$$

for any piece-wise constant function g. Since these functions are dense in $L^1(0,T;H)$, the desired result holds.

Step.2: $\sqrt{t}\dfrac{du_n}{dt} \to \sqrt{t}\dfrac{du}{dt}$ strongly in $L^2(0,T;H)$.

Multiplying (6.1.1) by $t\frac{du_n}{dt}$, we have

$$t\left|\frac{du_n}{dt}\right|_H^2 + t\left(g_n, \frac{du_n}{dt}\right)_H + t\left(Bu_n, \frac{du_n}{dt}\right)_H = t\left(f_n, \frac{du_n}{dt}\right)_H,$$

for $g_n \in \partial\phi^n(u_n)$. From Proposition 1.3.12, it follows that

$$\text{(6.1.17)} \qquad \int_0^T t\left|\frac{du_n}{dt}\right|_H^2 dt + T\phi^n(u_n(T)) + \int_0^T \left(\sqrt{t}Bu_n, \sqrt{t}\frac{du_n}{dt}\right)_H dt = \int_0^T \phi^n(u_n)dt + \int_0^T \left(\sqrt{t}f_n, \sqrt{t}\frac{du_n}{dt}\right)_H dt.$$

Since $\partial\phi^n(u_n) \ni f_n - \frac{du_n}{dt} - Bu_n$, it holds that

$$\begin{aligned}\phi^n(\alpha_n) - \phi^n(u_n) &\geq \left(f_n - \frac{du_n}{dt} - Bu_n, \alpha_n - u_n\right)_H \\ &= \frac{1}{2}\frac{d}{dt}|u_n - \alpha_n|_H^2 - (Bu_n, \alpha_n - u_n)_H + (f_n, \alpha_n - u_n)_H\end{aligned}$$

Hence since $\phi^n(\alpha_n) = 0$, we see that

$$\begin{aligned}\frac{1}{2}|u_n(T) - \alpha_n|_H^2 + \int_0^T \phi^n(u_n)dt \leq \frac{1}{2}|\alpha_n - a_n|_H^2 &- \int_0^T (Bu_n, \alpha_n - u_n)_H \, dt \\ &+ \int_0^T (f_n, \alpha_n - u_n)_H \, dt,\end{aligned}$$

which implies

$$\begin{aligned}\int_0^T \phi^n(u_n)dt \leq \frac{1}{2}&|\alpha_n - a_n|_H^2 \\ &+ \|\alpha_n - u_n\|_{C([0,T];H)} \left(\|f_n\|_{L^1(0,T;H)} + LT\|u_n\|_{C([0,T];H)}\right).\end{aligned}$$

By the above inequality and (6.1.17), we get

$$\begin{aligned}\int_0^T t\left|\frac{du_n}{dt}\right|_H^2 dt &\leq \int_0^T \left(\sqrt{t}f_n, \sqrt{t}\frac{du_n}{dt}\right)_H dt + \left|\int_0^T \left(\sqrt{t}Bu_n, \sqrt{t}\frac{du_n}{dt}\right)_H dt\right| \\ &\quad + \|\alpha_n - u_n\|_{C([0,T];H)} \left(\|f_n\|_{L^1(0,T;H)} + LT\|u_n\|_{C([0,T];H)}\right) \\ &\quad + \frac{1}{2}|\alpha_n - a_n|_H^2 \\ &\leq \frac{1}{2}\int_0^T t\left|\frac{du_n}{dt}\right|_H^2 dt + \int_0^T t|f_n(t)|_H^2 dt + \int_0^T t|Bu_n|_H^2 dt \\ &\quad + \|\alpha_n - u_n\|_{C([0,T];H)} \left(\|f_n\|_{L^1(0,T;H)} + LT\|u_n\|_{C([0,T];H)}\right) \\ &\quad + \frac{1}{2}|\alpha_n - a_n|_H^2,\end{aligned}$$

whence follows

$$\begin{aligned}\frac{1}{2}\int_0^T t\left|\frac{du_n}{dt}\right|_H^2 dt \leq \left\|\sqrt{t}f_n\right\|_{L^2(0,T;H)}^2 &+ \frac{L^2T^2}{2}\|u_n\|_{C([0,T];H)}^2 + \frac{1}{2}|\alpha_n - a_n|_H^2 \\ &+ \|\alpha_n - u_n\|_{C([0,T];H)} \left(\|f_n\|_{L^1(0,T;H)} + LT\|u_n\|_{C([0,T];H)}\right).\end{aligned}$$

By the assumptions and Step.1, we can deduce that

$$\sup_{n\in\mathbb{N}} \int_0^T t\left|\frac{du_n}{dt}\right|_H^2 dt < +\infty. \tag{6.1.18}$$

Taking a subsequence of (u_n) (denoted by (u_n) again), we have

$$\sqrt{t}\frac{du_n}{dt} \rightharpoonup \sqrt{t}\frac{du_n}{dt} \qquad \text{weakly in } L^2(0,T;H). \tag{6.1.19}$$

Hence, noting that $u_n \to u$ strongly in $C([0,T];H)$ by Step.1, we obtain

$$\int_0^T \left(\sqrt{t}Bu_n, \sqrt{t}\frac{du_n}{dt}\right)_H dt \to \int_0^T \left(\sqrt{t}Bu, \sqrt{t}\frac{du}{dt}\right)_H dt, \tag{6.1.20}$$

$$\int_0^T \left(\sqrt{t}f_n, \sqrt{t}\frac{du_n}{dt}\right)_H dt \to \int_0^T \left(\sqrt{t}fu, \sqrt{t}\frac{du}{dt}\right)_H dt. \tag{6.1.21}$$

Moreover, the assumption $\phi^n \xrightarrow{M} \phi$ on H implies

$$T\phi(u(T)) \le \liminf_{n\to\infty} T\phi^n(u_n(T)). \tag{6.1.22}$$

We next prove

$$\int_0^T \phi^n(u_n(t))dt \to \int_0^T \phi(u(t))dt. \tag{6.1.23}$$

To do this, we attempt to transform (6.1.1) and (6.1.2) by using the following functionals:

$$\psi^n(z) := \phi^n(z+\alpha_n) - \phi^n(\alpha_n),$$
$$\psi(z) := \phi(z+\alpha) - \phi(\alpha).$$

It is clear that $\psi^n(0) = \psi(0) = 0$, moreover we can see that $\psi^n \xrightarrow{M} \psi$ on H. Put $w_n := u_n - \alpha_n$ and $w := u - \alpha$. Then we get

$$(6.1.1) \quad \Leftrightarrow \quad \begin{cases} \dfrac{dw_n}{dt} + \partial\psi^n(w_n) + B(w_n+\alpha_n) \ni f_n, & t>0, \\ w_n(0) = a_n - \alpha_n =: a_n', \end{cases}$$

$$(6.1.2) \quad \Leftrightarrow \quad \begin{cases} \dfrac{dw}{dt} + \partial\psi(w) + B(w+\alpha) \ni f, & t>0, \\ w(0) = a - \alpha =: a'. \end{cases}$$

Since (6.1.23) is equivalent to

$$\int_0^T \psi^n(w_n(t))dt \to \int_0^T \psi(w(t))dt, \tag{6.1.24}$$

we only need to show (6.1.24). From (1.5.1), it follows that

$$\begin{aligned} \psi^n(w_n) + (\psi^n)^*\left(f_n - \frac{dw_n}{dt} - B(w_n+\alpha_n)\right) &= \left(f_n - \frac{dw_n}{dt} - B(w_n+\alpha_n), w_n\right)_H \\ &= (w_n, f_n)_H - \frac{1}{2}\frac{d}{dt}|w_n(t)|_H^2 \\ &\quad - (w_n, B(w_n+\alpha_n))_H. \end{aligned}$$

Integrating it over $[0,T]$, we derive

$$(6.1.25)\quad 0=\int_0^T \psi^n(w_n)dt+\int_0^T (\psi^n)^*\left(f_n-\frac{dw_n}{dt}-B(w_n+\alpha_n)\right)dt-\int_0^T (w_n,f_n)_H dt$$
$$+\frac{1}{2}|w_n(T)|_H^2-\frac{1}{2}|a_n'|_H^2+\int_0^T (w_n,B(w_n+\alpha_n))_H dt.$$

In particular, since $(\psi^n)^*\geq 0$ (note that $\psi^n(0)=0$), we see that $(\psi^n)^*(f_n-\frac{dw_n}{dt}-B(w_n+\alpha_n))\in L^1(0,T)$. Moreover by using Lemma 1.5.4 we get

$$(6.1.26)\qquad \int_0^T \psi(w(t))dt\leq \liminf_{n\to\infty}\int_0^T \psi^n(w_n(t))dt.$$

We also know

$$(6.1.27)\qquad \frac{1}{2}|w_n(T)|_H^2\to\frac{1}{2}|w(T)|_H^2,$$

$$(6.1.28)\qquad \frac{1}{2}|a_n'|_H^2\to\frac{1}{2}|a'|_H^2,$$

$$(6.1.29)\qquad \int_0^T (w_n,f_n)_H dt\to\int_0^T (w,f)_H dt,$$

$$(6.1.30)\qquad \int_0^T (w_n,B(w_n+\alpha_n))_H dt\to\int_0^T (w,B(w+\alpha))_H dt$$

For any $\delta\in(0,T)$, we obtain

$$\int_0^T (\psi^n)^*\left(f_n-\frac{dw_n}{dt}-B(w_n+\alpha_n)\right)dt\geq\int_\delta^T (\psi^n)^*\left(f_n-\frac{dw_n}{dt}-B(w_n+\alpha_n)\right)dt,$$

and note that

$$f_n-\frac{dw_n}{dt}-B(w_n+\alpha_n)\rightharpoonup f-\frac{dw}{dt}-B(w+\alpha)\qquad \text{weakly in } L^2(\delta,T;H).$$

Using Lemma 1.5.4 and Lemma 1.5.5, we have $(\psi^n)^*\xrightarrow{M}\psi^*$ and $\int_\delta^T(\psi^n)^*dt\xrightarrow{M}\int_\delta^T\psi^*dt$. Therefore it holds that for all $\delta\in(0,T)$

$$\int_\delta^T \psi^*\left(f-\frac{dw}{dt}-B(w+\alpha)\right)dt\leq\liminf_{n\to\infty}\int_\delta^T (\psi^n)^*\left(f_n-\frac{dw_n}{dt}-B(w_n+\alpha_n)\right)dt$$
$$\leq\liminf_{n\to\infty}\int_0^T (\psi^n)^*\left(f_n-\frac{dw_n}{dt}-B(w_n+\alpha_n)\right)dt,$$

whence follows

$$(6.1.31)\qquad \int_0^T \psi^*\left(f-\frac{dw}{dt}-B(w+\alpha)\right)dt\leq\liminf_{n\to\infty}\int_0^T (\psi^n)^*\left(f_n-\frac{dw_n}{dt}-B(w_n+\alpha_n)\right)dt.$$

By the similar argument, we remark that w satisfies

$$0=\int_0^T \psi(w)dt+\int_0^T \psi^*\left(f-\frac{dw}{dt}-B(w+\alpha)\right)dt-\int_0^T (w,f)_H dt \\ +\frac{1}{2}|w(T)|_H^2-\frac{1}{2}|a'|_H^2+\int_0^T (w,B(w+\alpha))_H dt. \tag{6.1.32}$$

By using Lemma 1.2.5 with (6.1.25), (6.1.32), (6.1.26), (6.1.27), (6.1.28), (6.1.29), (6.1.30) and (6.1.31), we conclude that (6.1.24) holds, that is, (6.1.23) is true.

By (6.1.17), (6.1.20), (6.1.21), (6.1.22) and (6.1.23), since u satisfies

$$\int_0^T t\left|\frac{du}{dt}\right|_H^2 dt+T\phi(u(T))+\int_0^T\left(\sqrt{t}Bu,\sqrt{t}\frac{du}{dt}\right)_H dt \\ =\int_0^T \phi(u)dt+\int_0^T\left(\sqrt{t}f,\sqrt{t}\frac{du}{dt}\right)_H dt, \tag{6.1.33}$$

it follows that from Lemma 1.2.5

$$\int_0^T t\left|\frac{du_n}{dt}\right|_H^2 dt\to\int_0^T t\left|\frac{du}{dt}\right|_H^2 dt. \tag{6.1.34}$$

Consequently, (6.1.19) and (6.1.34) imply

$$\sqrt{t}\frac{du_n}{dt}\to\sqrt{t}\frac{du}{dt}\qquad \text{strongly in } L^2(0,T;H).$$

□

6.2 Convergence of Functionals

By virtue of Proposition 6.1.3, to prove Theorem 6.1.1, it suffices to verify the Mosco convergence of the functionals associated with Laplacian under nonlinear boundary conditions. Recall that $(\mathrm{P})_q$ is reduced to the following abstract evolution equation in $L^2(\Omega)$:

$$(\mathrm{AC})_q \qquad \begin{cases} \dfrac{d}{dt}u(t)+\partial\varphi_q(u(t))+B_p(u(t))=0, & t>0,\\ u(0)=u_0, \end{cases}$$

where $B_p(r)=|r|^{p-2}r$ and

$$D(\varphi_q):=\{v\in H^1(\Omega)\,;\,u\in L^q(\partial\Omega)\},$$

$$\varphi_q(u)=\begin{cases}\dfrac{1}{2}\displaystyle\int_\Omega|\nabla u|^2dx+\frac{1}{q}\int_{\partial\Omega}|u|^q d\sigma & u\in D(\varphi_q),\\ +\infty & u\in L^2(\Omega)\setminus D(\varphi_q),\end{cases} \qquad (q\in[1,\infty))$$

$$D(\varphi_\infty):=\{v\in H^1(\Omega)\,;\,|u(x)|\le 1\ \ a.e.\ x\in\partial\Omega\}$$

$$\varphi_\infty(u)=\begin{cases}\dfrac{1}{2}\displaystyle\int_\Omega|\nabla u|^2dx & u\in D(\varphi_\infty),\\ +\infty & u\in L^2(\Omega)\setminus D(\varphi_\infty).\end{cases}$$

We obtain the following statement on the Mosco convergence of φ_q.

Theorem 6.2.1. *Let* $q_0 \in [1, \infty]$ *and* (q_n) *be a sequence in* $(1, \infty)$ *satisfying* $q_n \to q_0$ *as* $n \to \infty$. *Then*

$$\varphi_{q_n} \xrightarrow{M} \varphi_{q_0}.$$

Proof. Step 1: The case of $q_n \uparrow q_0 \in (1, \infty]$.

Let $u \in D(\varphi_{q_0})$ be fixed, and set a sequence $u_n \equiv u$. Note that $D(\varphi_{q_0}) \subset D(\varphi_{q_n})$ for all $n \in \mathbb{N}$. Using Hölder's inequality and Young's inequality, we have

$$\begin{aligned}\varphi_{q_n}(u_n) = \varphi_{q_n}(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{q_n}\int_{\partial\Omega} |u|^{q_n} d\sigma \\ &\leq \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{q_n}\left(\int_{\partial\Omega} |u|^{q_0} d\sigma\right)^{\frac{q_n}{q_0}} |\partial\Omega|^{\frac{q_0-q_n}{q_0}} \\ &\leq \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{q_0}\int_{\partial\Omega} |u|^{q_0} d\sigma + \frac{q_0 - q_n}{q_0}|\partial\Omega|,\end{aligned}$$

whence follows

$$\limsup_{n\to\infty} \varphi_{q_n}(u_n) \leq \varphi_{q_0}(u).$$

Thus (i) of Definition 1.5.1 follows.

As for the condition (ii) of Definition 1.5.1, let $(u_n) \subset L^2(\Omega)$ and $u \in L^2(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $L^2(\Omega)$. Without loss of generality, we can assume that

$$\liminf_{n\to\infty} \varphi_{q_n}(u_n) < +\infty, \tag{6.2.1}$$

otherwise (ii) always holds. By (6.2.1), there exist a constant $C > 0$ and a subsequence of (n) (which is denoted by (n) again) such that

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 dx + \frac{1}{q_n}\int_{\partial\Omega} |u_n|^{q_n} d\sigma \leq C, \qquad \forall\, n \in \mathbb{N},$$

which implies

$$\|u_n\|_{q_n,\partial\Omega} \leq (q_n C)^{\frac{1}{q_n}} \to \tilde{C} \tag{6.2.2}$$

$$\|\nabla u_n\|_2 \leq (2C)^{\frac{1}{2}}, \tag{6.2.3}$$

where

$$\tilde{C} := \begin{cases} (q_0 C)^{\frac{1}{q_0}}, & q_0 \in (1, +\infty), \\ 1, & q_0 = +\infty. \end{cases}$$

For any $r \in (1, q_n)$, it follows that from Hölder's inequality

$$\begin{aligned}\left(\int_{\partial\Omega} |u_n|^r d\sigma\right)^{\frac{1}{r}} &\leq \left(\int_{\partial\Omega} |u_n|^{q_n}\right)^{\frac{1}{q_n}} |\partial\Omega|^{\frac{q_n-r}{q_n r}} \\ &\leq (q_n C)^{\frac{1}{q_n}} |\partial\Omega|^{\frac{q_n-r}{q_n r}}, \qquad \forall\, n \in \mathbb{N}.\end{aligned}$$

Hence we see that $\|u_n\|_{r,\partial\Omega}$ is bounded for $r \in (1, q_0)$. Taking a subsequence of (n) which is denoted by (n) again, we can deduce that

$$u_n \rightharpoonup u \qquad \text{weakly in } L^r(\partial\Omega) \quad (1 < r < q_0).$$

In particular, since (u_n) is bounded in $L^1(\partial\Omega)$, from (6.2.3) it follows that (u_n) is bounded in $H^1(\Omega)$. Therefore

$$u_n \rightharpoonup u \qquad \text{weakly in } H^1(\Omega). \tag{6.2.4}$$

Moreover

$$\begin{aligned} u_n &\to u \qquad \text{strongly in } L^2(\Omega), \\ \nabla u_n &\rightharpoonup \nabla u \qquad \text{weakly in } L^2(\Omega). \end{aligned} \tag{6.2.5}$$

By the weak lower semicontinuity of norm, we can see that

$$\begin{aligned} \left(\int_{\partial\Omega} |u|^r d\sigma\right)^{\frac{1}{r}} &\le \liminf_{n\to\infty} \left(\int_{\partial\Omega} |u_n|^r d\sigma\right)^{\frac{1}{r}} \\ &\le \liminf_{n\to\infty} \left(\int_{\partial\Omega} |u_n|^{q_n} d\sigma\right)^{\frac{1}{q_n}} |\partial\Omega|^{\frac{q_n - r}{q_n r}} \\ &\le (q_0 C)^{\frac{1}{q_0}} |\partial\Omega|^{\frac{q_0 - r}{q_0 r}}. \end{aligned}$$

Taking the limit of $r \to q_0$, we have

$$\|u\|_{q_0,\partial\Omega} \le \tilde{C}.$$

Hence, by this estimate and (6.2.4), we get $u \in D(\varphi_{q_0})$ for all $q_0 \in (1, +\infty]$. Moreover, for the case where $q_0 \in (1, +\infty)$, we also see that

$$\begin{aligned} \int_{\partial\Omega} |u|^r d\sigma &\le \liminf_{n\to\infty} \int_{\partial\Omega} |u_n|^r d\sigma \\ &\le \liminf_{n\to\infty} \left(\int_{\partial\Omega} |u_n|^{q_n} d\sigma\right)^{\frac{r}{q_n}} |\partial\Omega|^{\frac{q_n - r}{q_n}} \\ &\liminf_{n\to\infty} \left\{ \frac{r}{q_n} \int_{\partial\Omega} |u_n|^{q_n} d\sigma + \frac{q_n - r}{q_n} |\partial\Omega| \right\}, \end{aligned}$$

whence follows

$$\frac{1}{r} \int_{\partial\Omega} |u|^r d\sigma \le \liminf_{n\to\infty} \frac{1}{q_n} \int_{\partial\Omega} |u_n|^{q_n} d\sigma + \frac{q_n - r}{q_n r} |\partial\Omega|.$$

Passing to the limit $r \to q_0$, we deduce that

$$\frac{1}{q_0} \int_{\partial\Omega} |u|^{q_0} d\sigma \le \liminf_{n\to\infty} \frac{1}{q_0} \int_{\partial\Omega} |u_n|^{q_n} d\sigma.$$

Therefore, by the above inequality and (6.2.5),

$$\begin{aligned}
\varphi_{q_0}(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{q_0}\int_{\partial\Omega} |u|^{q_0} d\sigma \\
&\le \liminf_{n\to\infty} \frac{1}{2}\int_\Omega |\nabla u_n|^2 dx + \liminf_{n\to\infty} \frac{1}{q_n}\int_{\partial\Omega} |u_n|^{q_n} d\sigma \\
&\le \liminf_{n\to\infty} \left(\frac{1}{2}\int_\Omega |\nabla u_n|^2 dx + \frac{1}{q_n}\int_{\partial\Omega} |u_n|^{q_n} d\sigma \right) \\
&= \liminf_{n\to\infty} \varphi_{q_n}(u_n).
\end{aligned}$$

On the other hand, for $q_0 = +\infty$, it is clear that

$$\begin{aligned}
\varphi_{q_0}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx &\le \liminf_{n\to\infty} \frac{1}{2}\int_\Omega |\nabla u_n|^2 dx \\
&\le \liminf_{n\to\infty} \left(\frac{1}{2}\int_\Omega |\nabla u_n|^2 dx + \frac{1}{q_n}\int_{\partial\Omega} |u_n|^{q_n} d\sigma \right) \\
&= \liminf_{n\to\infty} \varphi_{q_n}(u_n).
\end{aligned}$$

Consequently, (ii) of Definition 1.5.1 follows.

Step 2: The case of $q_n \downarrow q_0 \in [1, \infty)$.

We first verify the condition (i) of Definition 1.5.1. For $u \in D(\varphi_{q_0})$, we set $\alpha_n = q_n - q_0$ and $u_n = \chi_n(u)$, where $\chi_n : \mathbb{R} \to \mathbb{R}$ is defined by

$$\chi_n(r) = \begin{cases} r, & |r| \le 1/\alpha_n, \\ 1/\alpha_n, & r > 1/\alpha_n, \\ -1/\alpha_n, & r < -1/\alpha_n. \end{cases}$$

Noting that

$$u_n(x) \to u(x) \qquad \text{a.e. } x \in \Omega,$$

$$|u_n(x)|^2 \le |u(x)|^2 \in L^1(\Omega),$$

by the Lebesgue dominated convergence theorem, we see that

$$u_n \to u \qquad \text{strongly in } L^2(\Omega).$$

As a consequence by the above similar argument, we can deduce that

$$u_n \to u \qquad \text{strongly in } L^{q_0}(\partial\Omega). \tag{6.2.6}$$

Since $\chi_n'(r) = 0$ on $|r| > 1/\alpha_n$, we have

$$\int_\Omega |\nabla u_n|^2 dx = \int_\Omega |\chi_n'(u)|^2 |\nabla u|^2 dx = \int_{\{|u(x)| \le 1/\alpha_n\}} |\nabla u|^2 dx,$$

which implies

$$\int_\Omega |\nabla u_n|^2 dx \to \int_\Omega |\nabla u|^2 dx. \tag{6.2.7}$$

By the definition of χ_n and $q_0 < q_n$, we obtain

$$\frac{1}{q_n}\int_{\partial\Omega} |u_n|^{q_n} d\sigma = \frac{1}{q_n}\int_{\partial\Omega} |u_n|^{\alpha_n}|u_n|^{q_0} d\sigma \le \left|\frac{1}{\alpha_n}\right|^{\alpha_n} \frac{1}{q_0}\int_{\partial\Omega} |u_n|^{q_0} d\sigma.$$

From (6.2.6), $|1/\alpha_n|^{\alpha_n} \to 1$ and the above inequality, it follows

$$\limsup_{n\to\infty} \frac{1}{q_n}\int_{\partial\Omega} |u_n|^{q_n} d\sigma \le \frac{1}{q_0}\int_{\partial\Omega} |u|^{q_0} d\sigma. \tag{6.2.8}$$

On the other hand, Hölder's inequality and Young's inequality imply

$$\begin{aligned}\int_{\partial\Omega} |u_n|^{q_0} d\sigma &\le \left(\int_{\partial\Omega} |u_n|^{q_n} d\sigma\right)^{\frac{q_0}{q_n}} |\partial\Omega|^{\frac{q_n-q_0}{q_n}} \\ &\le \frac{q_0}{q_n}\int_{\partial\Omega} |u_n|^{q_n} d\sigma + \frac{q_n-q_0}{q_n}|\partial\Omega|,\end{aligned}$$

that is,

$$\frac{1}{q_0}\int_{\partial\Omega} |u_n|^{q_0} d\sigma - \left(\frac{1}{q_0} - \frac{1}{q_n}\right)|\partial\Omega| \le \frac{1}{q_n}\int_{\partial\Omega} |u_n|^{q_n} d\sigma.$$

Hence we get

$$\frac{1}{q_0}\int_{\partial\Omega} |u|^{q_0} d\sigma \le \liminf_{n\to\infty} \frac{1}{q_n}\int_{\partial\Omega} |u_n|^{q_n} d\sigma. \tag{6.2.9}$$

By (6.2.7), (6.2.8) and (6.2.9), we can see that

$$\varphi_{q_n}(u_n) \to \varphi_{q_0}(u),$$

which is the desired convergence.

We next check the condition (ii) of Definition 1.5.1. To do this, for $(u_n) \subset L^2(\Omega)$ and $u \in L^2(\Omega)$ satisfying $u_n \rightharpoonup u$ weakly in $L^2(\Omega)$, it suffices to show

$$\varphi_{q_0}(u) \le \liminf_{n\to\infty} \varphi_{q_n}(u_n).$$

Since if $\liminf \varphi_{q_n}(u_n) = \infty$, then the above inequality holds, we can assume that $\liminf \varphi_{q_n}(u_n)$ is finite. Therefore, up to a subsequence (which is denoted by (n) again), we can derive

$$\varphi_{q_n}(u_n) \le C$$

for some positive constant C independent of n. We get

$$\int_\Omega |\nabla u_n|^2 dx \le C, \qquad \frac{1}{q_n}\int_{\partial\Omega} |u_n|^{q_n} d\sigma \le C, \tag{6.2.10}$$

which implies

$$\frac{1}{q_0}\int_{\partial\Omega}|u_n|^{q_0}d\sigma \le \frac{1}{q_n}\int_{\partial\Omega}|u_n|^{q_n}d\sigma + \left(\frac{1}{q_0}-\frac{1}{q_n}\right)|\partial\Omega| \le C + \frac{1}{q_0}|\partial\Omega|. \tag{6.2.11}$$

For $q_0 \in (1,\infty)$, by (6.2.10) and (6.2.11), taking a subsequence of (u_n) denoted also by (u_n), we obtain

$$\begin{aligned} \nabla u_n &\rightharpoonup \nabla u &&\text{weakly in } L^2(\Omega),\\ u_n &\rightharpoonup u &&\text{weakly in } L^{q_0}(\partial\Omega), \end{aligned}$$

and we see that

$$\begin{aligned} \varphi_{q_0}(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{q_0}\int_{\partial\Omega}|u|^{q_0}d\sigma\\ &\le \liminf_{n\to\infty}\frac{1}{2}\int_\Omega|\nabla u_n|^2dx + \liminf_{n\to\infty}\frac{1}{q_0}\int_{\partial\Omega}|u_n|^{q_0}d\sigma\\ &\le \liminf_{n\to\infty}\left(\frac{1}{2}\int_\Omega|\nabla u_n|^2dx + \frac{1}{q_0}\int_{\partial\Omega}|u_n|^{q_0}d\sigma\right)\\ &\le \liminf_{n\to\infty}\left\{\frac{1}{2}\int_\Omega|\nabla u_n|^2dx + \frac{1}{q_n}\int_{\partial\Omega}|u_n|^{q_n}d\sigma + \left(\frac{1}{q_0}-\frac{1}{q_n}\right)|\partial\Omega|\right\}\\ &= \liminf_{n\to\infty}\varphi_{q_n}(u_n). \end{aligned}$$

As for the case where $q_0 = 1$ (in fact the following argument works well for $q_0 \in [1,2]$), since $\|\nabla v\|_2 + \|v\|_{1,\partial\Omega}$ is a equivalent norm of the usual H^1 norm, (6.2.10) and (6.2.11) imply that (u_n) is a bounded in $H^1(\Omega)$. Hence, by the compactness, taking a subsequence of (u_n) denoted also by (u_n), we obtain

$$\begin{aligned} u_n &\rightharpoonup u &&\text{weakly in } H^1(\Omega),\\ u_n &\to u &&\text{strongly in } L^2(\Omega),\\ u_n &\to u &&\text{strongly in } L^2(\partial\Omega). \end{aligned}$$

Since in particular u_n converges to u strongly in $L^1(\partial\Omega)$, we see that

$$\begin{aligned} \varphi_1(u) &= \frac{1}{2}\int_\Omega|\nabla u|^2dx + \int_{\partial\Omega}|u|d\sigma\\ &\le \liminf_{n\to\infty}\frac{1}{2}\int_\Omega|\nabla u_n|^2dx + \lim_{n\to\infty}\int_{\partial\Omega}|u_n|d\sigma\\ &= \liminf_{n\to\infty}\left(\frac{1}{2}\int_\Omega|\nabla u_n|^2dx + \int_{\partial\Omega}|u_n|d\sigma\right)\\ &\le \liminf_{n\to\infty}\left\{\frac{1}{2}\int_\Omega|\nabla u_n|^2dx + \frac{1}{q_n}\int_{\partial\Omega}|u_n|^{q_n}d\sigma + \left(1-\frac{1}{q_n}\right)|\partial\Omega|\right\}\\ &= \liminf_{n\to\infty}\varphi_{q_n}(u_n). \end{aligned}$$

Step 3: The general case where $q_n \to q_0 \in (1, \infty)$.
We can see that if $q_n \to q_0$, then any subsequence $(q_{n_k})_{k\in\mathbb{N}}$ of $(q_n)_{n\in\mathbb{N}}$ has a subsequence $(q_{n_{k'}})_{k'\in\mathbb{N}}$ of $(q_{n_k})_{k\in\mathbb{N}}$ such that $q_{n_{k'}} \uparrow q_0$ or $q_{n_{k'}} \downarrow q_0$. Therefore, the desired result follows from Step.1, Step.2 and Lemma 1.5.6. □

We can prove Theorem 6.1.1 by the above lemmas.

Proof of Theorem 6.1.1. Set

$$\tilde{M} := \max\left\{\|a\|_\infty,\ \sup_{n\in\mathbb{N}} \|a_n\|_\infty\right\} + 2.$$

For $v,\ w \in C([0,T]; L^2(\Omega)) \cap L^\infty(0,T;L^\infty)$, we see that

$$\begin{aligned}\|B_p(v(t)) - B_p(w(t))\|_2^2 &= \int_\Omega \left||v(t)|^{p-2}v(t) - |w(t)|^{p-2}w(t)\right|^2 dx \\ &\le (p-1)\int_\Omega \left(|v(t)|^{p-2} + |w(t)|^{p-2}\right)^2 |v(t) - w(t)|^2 dx \\ &\le (p-1)\left(\|v(t)\|_\infty^{p-2} + \|w(t)\|_\infty^{p-2}\right)^2 \|v(t) - w(t)\|_2^2,\end{aligned}$$

which implies

$$\|B_p(v(t)) - B_p(w(t))\|_2 \le (p-1)^{\frac{1}{2}} \left(\|v\|_{L^\infty(0,T;L^\infty)}^{p-2} + \|w\|_{L^\infty(0,T;L^\infty)}^{p-2}\right) \|v(t) - w(t)\|_2.$$

If v and w are solutions to $(\mathrm{P})_q$ or $(\mathrm{P})_{q_n}$, by the proof of Theorem 2.2.3, then (2.2.17) holds with $\tilde{M}$ replaced by M and we see that

$$\|B_p(v(t)) - B_p(w(t))\|_2 \le 2(p-1)^{\frac{1}{2}} \left(\tilde{M} + 1\right)^{p-2} \|v(t) - w(t)\|_2.$$

Hence applying Lemma 6.1.3 and Theorem 6.2.1, we conclude that Theorem 6.1.1 holds. □

Part II

On Some Reaction Diffusion Systems

Introduction

In this part, we mainly consider the following initial-boundary value problem for a nonlinear reaction diffusion system:

$$\text{(NR)} \qquad \begin{cases} \partial_t u_1 - \Delta u_1 = u_1 u_2 - b u_1, & t > 0,\ x \in \Omega, \\ \partial_t u_2 - \Delta u_2 = a u_1, & t > 0,\ x \in \Omega, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta |u_2|^{\gamma-2} u_2 = 0, & t > 0,\ x \in \partial\Omega, \\ u_1(x,0) = u_{10}(x) \geq 0,\ u_2(x,0) = u_{20}(x) \geq 0, & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, ν denotes the unit outward normal vector on $\partial\Omega$ and ∂_ν is the outward normal derivative, i.e., $\partial_\nu u_i = \nabla u_i \cdot \nu$ $(i = 1, 2)$. Moreover u_1, u_2 are real-valued unknown functions, a and b are given positive constants. As for the parameters appearing in the boundary condition, we assume $\alpha \in [0, \infty)$, $\beta \in (0, \infty)$ and $\gamma \in [2, \infty)$. We note that the boundary condition for u_1 becomes the homogeneous Neumann boundary condition when $\alpha = 0$, and the boundary condition for u_2 gives the Robin boundary condition when $\gamma = 2$. The initial data u_{10}, u_{20} are here assumed to be nonnegative and members of $L^\infty(\Omega)$.

This system describes diffusion phenomena of neutrons and heat in nuclear reactors by taking the heat conduction into consideration, introduced by Kastenberg and Chambré [31]. In this model u_1 and u_2 represent the neutron density and the temperature in nuclear reactors, respectively. There are many studies on this model under various linear boundary conditions, for example, [14], [15], [25], [26], [30], [57] and [61]. Many of them are concerned with the existence of positive steady-state solutions and the long-time behavior of solutions.

Our problem originates in the following rather simplified model studied by [57] :

$$\text{(II.1)} \qquad \begin{cases} \partial_t u_1 - \Delta u_1 = u_1 u_2 - b u_1, & t > 0,\ x \in \Omega, \\ \partial_t u_2 = a u_1 - c u_2, & t > 0,\ x \in \Omega, \\ u_1 = 0, & t > 0,\ x \in \partial\Omega, \\ u_1(x,0) = u_{10}(x) \geq 0,\ u_2(x,0) = u_{20}(x) \geq 0, & x \in \Omega. \end{cases}$$

In (II.1), the negative feedback $-c\,u_2$ (with $c > 0$) from the heat into itself is considered instead of the diffusion term. In Rothe's book [57], the boundedness and the convergence of solutions to equilibria of (II.1) are examined in detail.

In [25], our system is studied for the case where $\alpha = 0$ and $\gamma = 2$, i.e., with the homogeneous Neumann boundary condition on u_1 and the Robin boundary condition on

u_2:

$$\text{(II.2)} \qquad \begin{cases} \partial_t u_1 - \Delta u_1 = u_1 u_2 - b u_1, & t > 0,\ x \in \Omega, \\ \partial_t u_2 - \Delta u_2 = a u_1, & t > 0,\ x \in \Omega, \\ \partial_\nu u_1 = \partial_\nu u_2 + \beta u_2 = 0, & t > 0,\ x \in \partial\Omega, \\ u_1(x,0) = u_{10}(x),\ u_2(x,0) = u_{20}(x), & x \in \Omega. \end{cases}$$

They derived the existence and the ordered uniqueness of positive stationary solution for $N \in \{2,3,4,5\}$ and showed that this stationary solution plays as a threshold of the initial data to determine whether the corresponding solution can exist globally or not (blow-up in finite time). This problem (II.2) with $\beta = 0$ is also studied in [61], where the stability of stationary solution is discussed and some estimates of the existence time for blow-up solutions are obtained.

Another variant system with the homogeneous Dirichlet boundary condition on both u_1 and u_2 given below is studied by [26] and [30].

$$\text{(II.3)} \qquad \begin{cases} \partial_t u_1 - \Delta u_1 = u_1 u_2^p - b u_1, & t > 0,\ x \in \Omega, \\ \partial_t u_2 - \Delta u_2 = a u_1, & t > 0,\ x \in \Omega, \\ u_1 = u_2 = 0, & t > 0,\ x \in \partial\Omega, \\ u_1(x,0) = u_{10}(x),\ u_2(x,0) = u_{20}(x), & x \in \Omega. \end{cases}$$

In [26], the existence of positive stationary solution is shown for the case where $p = 1$ and $N \in \{1,2\}$ or $N \in \{2,3,4,5\}$ provided that Ω is a convex domain. Furthermore, they obtained the threshold property of the stationary solution, the same as that in [25], when Ω is ball. In [30], the existence and the ordered uniqueness of positive stationary solutions are considered for general $p > 0$ and some threshold result is obtained. Moreover the blow-up rate estimate is given for positive blowing-up solutions when Ω is ball and $p \geq 1$.

In Part II, we analyze how the nonlinear boundary condition imposed on u_2 is reflected in the nature of (NR). As already emphasized in the introduction of Part I, the importance of the study of nonlinear boundary conditions from a physical point of view could be supported by Stefan-Boltzmann's law, which says that the heat energy radiation from the surface of the body in $\mathbb{R}^N$ is proportional to the $(N+1)$-th power of temperature, which can be covered by our power model $\beta|u_2|^{\gamma-2}u_2$.

The outline of the contents of Part II is as follows. In Chapter 8, we consider the stationary problem associated with (NR) and show the existence of positive solutions. In doing this, we first note that this stationary problem does not possess the variational structure. Hence we can not rely on the standard tools in the variational calculus such as the mountain pass lemma. Instead we here apply an abstract fixed point theorem based on Krasnosel'skii [32]. To apply this fixed point theorem, we need a priori estimates of solutions in $L^\infty(\Omega)$. However, because of the presence of the nonlinear boundary condition, we can not rely on the standard linear theory for this purpose. So we here introduce a new approach which enables us to obtain strong summability of solutions on the boundary.

Next, we prove the ordered uniqueness for the positive stationary solutions of (NR). We here make the most use of the property of the first eigenfunction of $-\Delta$ with the Robin boundary condition.

In Chapter 9, we study the nonstationary problem. In the first section, we show the local (in time) well-posedness of (NR) in $L^\infty(\Omega)$ along the lines of the L^∞-energy method [45]. In the second section, we show that every positive stationary solution acts as a threshold of the initial data to separate the global existence and finite time blow-up of corresponding solutions. More precisely, if the initial data is less than or equal to positive stationary solutions, then the solution of (NR) exists globally and tends to zero as $t \to \infty$, and if the initial data is strictly larger than positive stationary solutions, then the solution of (NR) blows up in finite time. In the third section, we give another type of result concerning the existence of blow-up solutions, i.e., a sufficient condition for the initial data of Kaplan type, which is described in terms of the integral of the initial data multiplied by the first eigenfunction of $-\Delta$ with the homogeneous Dirichlet boundary condition, so that the corresponding solutions of (NR) with more general nonlinear boundary conditions blow up in finite time. Here we apply Theorem 4.1.3 and the same strategy as that used in §4.2.

In Chapter 10, we consider the asymptotic behavior of global solutions of (NR) with $\gamma = 2$, i.e., with the Robin boundary conditions on u_1 and u_2:

$$\text{(II.4)} \qquad \begin{cases} \partial_t u_1 - \Delta u_1 = u_1 u_2 - b u_1, & t > 0,\ x \in \Omega, \\ \partial_t u_2 - \Delta u_2 = a u_1, & t > 0,\ x \in \Omega, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta u_2 = 0, & t > 0,\ x \in \partial\Omega, \\ u_1(0,x) = u_{10}(x) \geq 0,\ u_2(0,x) = u_{20}(x) \geq 0, & x \in \Omega. \end{cases}$$

More precisely, we here discuss about the uniform boundedness of global solutions of (II.4). The same problem is treated in Chapter 3 for the single equation $(\mathrm{P})_q$. In other words, we look for the analogue of the result given in Chapter 3. However, we here restrict ourselves to the case where $\gamma = 2$, for a technical reason. Bounds for global solutions of this system with the homogeneous Dirichlet boundary condition is already studied by Quittner [53] for the case where $N = 2$. This strong restriction on N is due to the use of a Hardy type inequality (see [11]). Since our boundary condition is different from that of [53], making use of the good properties of the first eigenfunction of $-\Delta$ with Robin boundary condition, we can treat the cases where $N = 2, 3$.

Our proof for the boundedness of global solutions of $(\mathrm{P})_q$ deeply relies on the fact that the energy functional $J(u)$, defined by $J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{q}\int_{\partial\Omega} |u|^q d\sigma - \frac{1}{p}\int_\Omega |u|^p dx$, becomes a Lyapunov function, in other words, $(\mathrm{P})_q$ possesses a good variational structure.

There are also other approaches to this problem, e.g., in [23] the rescaling argument is introduced and in [54] the bootstrap argument based on the interpolation and the maximal regularity is used.

Unfortunately these tools are not available to our system because of the presence of the coupling terms. To cope with this difficulty, by making the most use of the special form of our system, we first show the uniform bound of solutions in the L^1-norm with the positive weight φ_1, the first eigenfunction of $-\Delta$ with the Robin boundary condition. To derive the uniform H^1-bound, we rely on some energy method with the aid of a special device (see Lemma 7.1.2). Furthermore by applying Moser's iteration scheme such as in Nakao [41], we derive the uniform L^∞-bound via H^1-bound.

Chapter 7

Preliminaries

7.1 Some Results for the Following Chapters

We here state several lemmas to prove our results for (NR). The following abstract fixed point theorem in positive cone is essential and crucial to show the existence of positive (nontrivial) stationary solutions of (NR).

Lemma 7.1.1 (Krasnosel'skii-type fixed point theorem [32], [35])**.** *Suppose that E is a real Banach space with norm $\|\cdot\|$, $K \subset E$ is a positive cone, and $\Phi : K \to K$ is a compact mapping satisfying $\Phi(0) = 0$. Assume that there exists two constants $R > r > 0$ and an element $\varphi \in K \setminus \{0\}$, such that*

$$\text{(i) } u \neq \lambda\Phi(u), \ \ \forall\lambda \in (0,1), \ \ \textit{if} \ \ u \in K \ \textit{and} \ \|u\| = r,$$

$$\text{(ii) } u \neq \Phi(u) + \lambda\varphi, \ \ \forall\lambda \geq 0, \ \ \textit{if} \ \ u \in K \ \textit{and} \ \|u\| = R.$$

Then the mapping Φ possesses at least one fixed point in $K_1 := \{u \in K;\ 0 < r < \|u\| < R\}$.

The next lemma is very simple but useful to obtain a priori estimates of the solutions of partial differential equations with Robin boundary conditions.

Lemma 7.1.2 ([20])**.** *Let λ_1 and φ_1 be the first eigenvalue and the corresponding eigenfunction for the problem:*

$$\begin{cases} -\Delta\varphi = \lambda\varphi, & x \in \Omega, \\ \partial_\nu\varphi + \gamma\varphi = 0, & x \in \partial\Omega, \end{cases} \tag{7.1.1}$$

where Ω is smooth bounded domain in $\mathbb{R}^N$ and $\gamma > 0$. Then $\lambda_1 > 0$ and there exists a constant $C_\gamma > 0$ such that

$$\varphi_1(x) \geq C_\gamma \qquad x \in \overline{\Omega}.$$

Actually, it is easy to see that $\varphi_1 > 0$ in Ω by the strong maximum principle as the same method for the eigenvalue problem with the Dirichlet Laplacian. Furthermore suppose that there exists $x_0 \in \partial\Omega$ such that $\varphi_1(x_0) = 0$. Then the boundary condition assures $\partial_\nu\varphi_1(x_0) = -\gamma\varphi_1(x_0) = 0$. On the other hand, we know $\partial_\nu\varphi_1(x_0) < 0$ by Hopf's strong maximum principle. This is contradiction, i.e., $\varphi_1(x) > 0$ on $\overline{\Omega}$.

In order to deal with the power type nonlinearities, the following inequality is fundamental.

Lemma 7.1.3. ([18]) *For any $\kappa \in [2, \infty)$, there exists $C_\kappa > 0$ such that*

$$(x - y) \cdot \left(|x|^{\kappa-2}x - |y|^{\kappa-2}y\right) \geq C_\kappa |x - y|^\kappa$$

for all x, $y \in \mathbb{R}^N$.

Lemma 7.1.4 ([45])**.** *Let $y(t)$ be a bounded measurable positive function on $[0, T]$ for any $T \in (0, T_m)$ and let $\lim_{t \to T_m} y(t) = +\infty$. Suppose that there exists a monotone nondecreasing locally Lipschitz function $g : [0, +\infty) \to [0, +\infty)$ such that*

$$\int_0^{+\infty} \frac{1}{g(\tau)} d\tau = +\infty,$$
$$\int_a^{+\infty} \frac{1}{g(\tau)} d\tau < +\infty \qquad \forall a > 0.$$

Furthermore we assume that

$$y(s) \leq y(t) + \int_t^s g(y(\tau)) d\tau \qquad a.e. \;\; t, s \in [0, T_m) \;\; \text{with} \;\; t < s.$$

Then the following estimate holds.

$$y(t) \geq G^{-1}(t - T_m) \qquad a.e. \;\; t \in [0, T_m),$$

where $G^{-1}(\cdot)$ is the inverse function of $G(w) = -\displaystyle\int_w^{+\infty} \frac{1}{g(\tau)} d\tau$.

Now, if we set $g(r) = \lambda |r|^{q-2} r$ with $q \geq 2$ and $\lambda > 0$, we can see that g satisfies the assumption required in Lemma 7.1.4. Moreover we can obtain $G^{-1}(\tau) = (\lambda (q-2) \tau)^{\frac{-1}{q-2}}$ by an elementary calculation. So, Lemma 7.1.4 implies:

Corollary 7.1.5. *Let $y(t)$ be a bounded measurable positive function on $[0, T]$ for any $T \in (0, T_m)$ and let $\lim_{t \to T_m} y(t) = +\infty$. Suppose that $y(\cdot)$ satisfies*

$$y(t) \leq y(s) + \lambda \int_s^t y^{q-1}(\tau) d\tau \qquad a.e. \; s, t \in [0, T_m) \; with \; s < t.$$

Then the following estimate holds:

$$y(t) \geq (\lambda (q-2))^{\frac{-1}{q-2}} (T_m - t)^{\frac{-1}{q-2}} \qquad a.e. \; t \in [0, T_m).$$

Chapter 8

Stationary Problem of (NR)

In this chapter, we are going to show the existence of the positive stationary solutions for (NR) and prove the ordered uniqueness of them. The stationary problem for (NR) is given by

$$\text{(S-NR)} \qquad \begin{cases} -\Delta u_1 = u_1 u_2 - b u_1, & x \in \Omega, \\ -\Delta u_2 = a u_1, & x \in \Omega, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta |u_2|^{\gamma-2} u_2 = 0, & x \in \partial\Omega. \end{cases}$$

8.1 Existence of Positive Solutions

It should be noticed that since (S-NR) has no variational structure, it is not possible to apply the variational method to (S-NR). In order to show the existence of positive stationary solutions to (NR), we rely on the abstract fixed point theorem developed by Krasnosell'skii. The crucial step in proving the existence of positive stationary solutions is how to obtain L^∞-estimates of solutions.

Theorem 8.1.1. *Let* $1 \le N \le 5$ *and suppose that either* (A) *or* (B) *is satisfied* :

$$\begin{cases} \text{(A)} \quad \gamma = 2, \quad \alpha \le 2\beta, \\ \text{(B)} \quad \gamma > 2. \end{cases}$$

Then (S-NR) *has at least one positive solution.*

We rely on Lemma 7.1.1 to prove this theorem. In order to apply Lemma 7.1.1, we here fix our setting:

$$E = C(\overline{\Omega}) \times C(\overline{\Omega}), \qquad u = (u_1, u_2)^{\mathrm{T}} \in E,$$
$$\|u\| = \|u_1\|_{C(\overline{\Omega})} + \|u_2\|_{C(\overline{\Omega})}, \qquad K = \{u \in E; u_1 \ge 0, u_2 \ge 0\}.$$

Set $\varphi = (\varphi_1, 0)^{\mathrm{T}} \in K \setminus \{0\}$, where λ_1 and φ_1 are the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem:

$$\text{(8.1.1)} \qquad \begin{cases} -\Delta \varphi = \lambda \varphi, & x \in \Omega, \\ \partial_\nu \varphi + \alpha \varphi = 0, & x \in \partial\Omega. \end{cases}$$

In chapter 8, we normalize $\varphi_1(x)$ such that $\|\varphi_1\|_{L^2} = 1$. For given $u = (u_1, u_2)^{\mathrm{T}} \in K$, let $v = (v_1, v_2)^{\mathrm{T}} = \Psi(u)$ be the unique nonnegative solution (see Brézis [10]) of

$$\begin{cases} -\Delta v_1 + bv_1 = u_1u_2, & x \in \Omega, \\ -\Delta v_2 = au_1, & x \in \Omega, \\ \partial_\nu v_1 + \alpha v_1 = \partial_\nu v_2 + \beta|v_2|^{\gamma-2}v_2 = 0, & x \in \partial\Omega. \end{cases} \tag{8.1.2}$$

It is clear that $\Psi(0) = 0$. Moreover $\Psi : K \to K$ is compact. In order to prove the compactness of Ψ, we use the next Lemma for the following problem:

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ \partial_\nu u = g, & x \in \partial\Omega. \end{cases} \tag{8.1.3}$$

Lemma 8.1.2. ([44]) *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Suppose that $f \in L^{\frac{p}{2}}(\Omega)$ and $g \in L^{p-1}(\partial\Omega)$ with $p > N \geq 2$, then there exist $\delta > 0$ and a positive constant C such that every weak solution u of* (8.1.2) *belongs to $C^{0,\delta}(\overline{\Omega})$ and satisfies*

$$\|u\|_{C^{0,\delta}(\overline{\Omega})} \leq C\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^{\frac{p}{2}}(\Omega)} + \|g\|_{L^{p-1}(\partial\Omega)}\right).$$

Since Ω is bounded and $(u_1, u_2) \in C(\overline{\Omega}) \times C(\overline{\Omega})$, it follows from elliptic estimate that $v_1 \in W^{2,p}(\Omega)$ for any p. Since $W^{2,p}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$ for $p > \frac{N}{2}$, the mapping $(u_1, u_2) \mapsto v_1$ is compact. Next we assume that $N \geq 2$ and consider the following equation:

$$\begin{cases} -\Delta v_2 = au_1 \in L^\infty(\Omega), & x \in \Omega, \\ \partial_\nu v_2 + \beta|v_2|^{\gamma-2}v_2 = 0, & x \in \partial\Omega. \end{cases}$$

Multiplying the equation by $|v_2|^{r-2}v_2$ and applying integration by parts, we get

$$(r-1)\int_\Omega |v_2|^{r-2}|\nabla v_2|^2 dx + \beta\int_{\partial\Omega} |v_2|^{r+\gamma-2} d\sigma = a\int_\Omega u_1|v_2|^{r-2}v_2 dx. \tag{8.1.4}$$

Noting that $(\|\nabla v_2\|^2_{L^2(\Omega)} + \int_{\partial\Omega} v_2^2 d\sigma)^{1/2}$ is equivalent to the usual H^1-norm by Poincaré-Friedrichs type inequality, we obtain

$$\begin{aligned} (l.h.s.) &= (r-1)\int_\Omega \left||v_2|^{\frac{r-2}{2}}|\nabla v_2|\right|^2 dx + \beta\int_{\partial\Omega} |v_2|^{r+\gamma-2} d\sigma \\ &\geq \frac{4(r-1)}{r^2}\int_\Omega \left|\nabla|v_2|^{\frac{r}{2}}\right|^2 dx + \beta\int_{\partial\Omega} |v_2|^r d\sigma - \beta|\partial\Omega| \\ &\geq C_r\left(\int_\Omega \left|\nabla|v_2|^{\frac{r}{2}}\right|^2 dx + \int_{\partial\Omega} \left||v_2|^{\frac{r}{2}}\right|^2 d\sigma\right) - \beta|\partial\Omega| \\ &\geq C_r\int_\Omega \left||v_2|^{\frac{r}{2}}\right|^2 dx - \beta|\partial\Omega| = C_r\|v_2\|^r_{L^r(\Omega)} - \beta|\partial\Omega|, \end{aligned}$$

where $C_r = \min\{\frac{4(r-1)}{r^2}, \beta\} > 0$ and we used the estimate:

$$\begin{aligned}\beta \int_{\partial\Omega} |v_2|^{r+\gamma-2} d\sigma \geq \beta \int_{\{|v_2|\geq 1\}} |v_2|^{r+\gamma-2} d\sigma &\geq \beta \int_{\{|v_2|\geq 1\}} |v_2|^r d\sigma \\ &= \beta \int_{\partial\Omega} |v_2|^r d\sigma - \beta \int_{\{|v_2|\leq 1\}} |v_2|^r d\sigma \\ &\geq \beta \int_{\partial\Omega} |v_2|^r d\sigma - \beta|\partial\Omega|.\end{aligned}$$

Hence Hölder's inequality, Young's inequality and (8.1.4) yield

$$\|v_2\|_{L^r(\Omega)} \leq \left\{ \beta|\partial\Omega| \left(\frac{C_r}{2}\right)^{-1} + \frac{1}{r}\left(\frac{C_r}{2}\right)^{-r} \|au_1\|^r_{L^r(\Omega)} \right\}^{\frac{1}{r}} \qquad \forall r < \infty.$$

Therefore by (8.1.4) we have

$$\int_{\partial\Omega} |v_2|^{r+\gamma-2} d\sigma \leq \frac{1}{\beta}\|au_1\|_{L^r(\Omega)} \left\{ \beta|\partial\Omega| \left(\frac{C_r}{2}\right)^{-1} + \frac{1}{r}\left(\frac{C_r}{2}\right)^{-r} \|au_1\|^r_{L^r(\Omega)} \right\}^{\frac{r-1}{r}} \qquad \forall r < \infty.$$

Thus we see that $v_2 \in L^r(\partial\Omega)$ for all large $r < \infty$ and we can apply Lemma 8.1.2 to get $v_2 \in C^{0,\delta}(\overline{\Omega})$ for some $\delta > 0$. Note that $C^{0,\delta}(\overline{\Omega}) \hookrightarrow C(\overline{\Omega})$ is compact. As for the case where $N = 1$, (8.1.4) with $r = 2$ gives the a priori bound for $\|v_2\|_{H^1(\Omega)}$. Since the embedding $H^1(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact, the compactness of Ψ is easily derived. Thus we see that $\Psi : K \to K$ is compact.

In order to show the existence of positive stationary solutions for (S-NR), it suffices to prove that Ψ has a fixed point in K. Therefore, to prove Theorem 8.1.1 we are going to verify conditions (i) and (ii) of Lemma 7.1.1.

We first check condition (i).

Lemma 8.1.3. *Let $r = \frac{b}{2}$, then $u \neq \lambda\Psi(u)$ for any $\lambda \in (0,1)$ and $u \in K$ satisfying $\|u\| = r$. That is, condition* (i) *of* Lemma 7.1.1 *with $\Phi = \Psi$ holds.*

Proof. We prove the statement by contradiction. Suppose that there exist $\lambda \in (0,1)$ and $u \in K$ with $\|u\| = r$ such that $u = \lambda\Psi(u)$, that is, u_1 and u_2 satisfy

$$(8.1.5) \qquad \begin{cases} -\Delta u_1 + bu_1 = \lambda u_1 u_2, & x \in \Omega, \\ -\Delta u_2 = \lambda a u_1, & x \in \Omega, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta \left|\dfrac{u_2}{\lambda}\right|^{\gamma-2} u_2 = 0, & x \in \partial\Omega. \end{cases}$$

Multiplying the first equation of (8.1.5) by u_1 and using integration by parts, we obtain

$$\begin{aligned}\|\nabla u_1\|^2_{L^2(\Omega)} + \alpha \int_{\partial\Omega} u_1^2 d\sigma + b\|u_1\|^2_{L^2(\Omega)} &= \lambda \int_\Omega u_1^2 u_2 dx \\ &\leq \|u_2\|_{L^\infty(\Omega)} \|u_1\|^2_{L^2(\Omega)} \\ &\leq \frac{b}{2}\|u_1\|^2_{L^2(\Omega)},\end{aligned}$$

where we use the fact

$$\|u_2\|_{L^\infty(\Omega)} \le \|u\| = r = \frac{b}{2}.$$

Then

$$\|\nabla u_1\|^2_{L^2(\Omega)} + \alpha \int_{\partial\Omega} u_1^2 d\sigma + \frac{b}{2}\|u_1\|^2_{L^2(\Omega)} \le 0.$$

Hence we have $u_1 = 0$. By the second equation of (8.1.5), we see that u_2 satisfies

$$\begin{cases} -\Delta u_2 = 0, & x \in \Omega, \\ \partial_\nu u_2 + \beta \left|\dfrac{u_2}{\lambda}\right|^{\gamma-2} u_2 = 0, & x \in \partial\Omega. \end{cases}$$

Multiplying this equation by u_2 and integration by parts, we obtain

$$\|\nabla u_2\|^2_{L^2(\Omega)} + \frac{\beta}{|\lambda|^{\gamma-2}} \int_{\partial\Omega} |u_2|^\gamma d\sigma = 0, \quad \text{i.e.,} \qquad \|\nabla u_2\|_{L^2(\Omega)} = 0, \qquad u_2\,|_{\partial\Omega} = 0.$$

By the use of Poincaré's inequality, we also get $u_2 = 0$. Thus $u_1 = u_2 = 0$. This contradicts the assumption $\|u\| = \frac{b}{2} > 0$. □

In order to verify condition (ii) of Lemma 7.1.1, we here claim the following lemma.

Lemma 8.1.4. *Let* $1 \le N \le 5$ *and suppose that either* (A) *or* (B) *is satisfied* :

$$\begin{cases} \text{(A)} & \gamma = 2, \quad \alpha \le 2\beta, \\ \text{(B)} & \gamma > 2. \end{cases}$$

Then there exists a constant R $(> r = \frac{b}{2})$ *such that for any* $\lambda > 0$ *and any solution* u *of* $u = \Psi(u) + \lambda\varphi$, *it holds that*

$$\|u\| < R.$$

Proof. We rewrite $u = \Psi(u) + \lambda\varphi$ in terms of each component:

$$\begin{cases} -\Delta u_1 + bu_1 = u_1u_2 + \lambda(b+\lambda_1)\varphi_1, & x \in \Omega, \\ -\Delta u_2 = au_1, & x \in \Omega, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta|u_2|^{\gamma-2}u_2 = 0, & x \in \partial\Omega. \end{cases} \tag{8.1.6}$$

In what follows, we denote by C a general constant which differs from place to place. First, we derive H^1-estimate for u_2. Replacing u_1 in the first equation of (8.1.6) by $-\frac{1}{a}\Delta u_2$, we get

$$\begin{cases} \Delta^2 u_2 - b\Delta u_2 = -u_2\Delta u_2 + \lambda a(b+\lambda_1)\varphi_1, & x \in \Omega \\ \partial_\nu u_2 + \beta|u_2|^{\gamma-2}u_2 = \partial_\nu \Delta u_2 + \alpha\Delta u_2 = 0, & x \in \partial\Omega. \end{cases} \tag{8.1.7}$$

Multiplying (8.1.7) by φ_1, using integration by parts and noting that the boundary conditions $\partial_\nu \varphi_1 + \alpha\varphi_1 = \partial_\nu u_2 + \beta|u_2|^{\gamma-2}u_2 = 0$, we have

$$\begin{aligned}
(l.h.s) &= \int_\Omega \Delta^2 u_2 \varphi_1 dx - b\int_\Omega \Delta u_2 \varphi_1 dx \\
&= -\int_\Omega \nabla(\Delta u_2)\cdot\nabla\varphi_1 dx + \int_{\partial\Omega}(\partial_\nu \Delta u_2)\varphi_1 d\sigma \\
&\quad + b\int_\Omega \nabla u_2\cdot\nabla\varphi_1 dx - b\int_{\partial\Omega}(\partial_\nu u_2)\varphi_1 d\sigma \\
&= \int_\Omega \Delta u_2 \Delta\varphi_1 dx - \int_{\partial\Omega}\Delta u_2(\partial_\nu\varphi_1)d\sigma + \int_{\partial\Omega}(\partial_\nu\Delta u_2)\varphi_1 d\sigma \\
&\quad - b\int_\Omega u_2\Delta\varphi_1 dx + b\int_{\partial\Omega}u_2(\partial_\nu\varphi_1)d\sigma - b\int_{\partial\Omega}(\partial_\nu u_2)\varphi_1 d\sigma \\
&= -\lambda_1\int_\Omega \Delta u_2\varphi_1 dx + \alpha\int_{\partial\Omega}\Delta u_2\varphi_1 d\sigma - \alpha\int_{\partial\Omega}\Delta u_2\varphi_1 d\sigma \\
&\quad + b\lambda_1\int_\Omega u_2\varphi_1 dx - \alpha b\int_{\partial\Omega}u_2\varphi_1 d\sigma + \beta b\int_{\partial\Omega}u_2^{\gamma-1}\varphi_1 d\sigma \\
&= -\lambda_1\int_\Omega u_2\Delta\varphi_1 dx + \lambda_1\int_{\partial\Omega}u_2(\partial_\nu\varphi_1)d\sigma - \lambda_1\int_{\partial\Omega}(\partial_\nu u_2)\varphi_1 d\sigma \\
&\quad + b\lambda_1\int_\Omega u_2\varphi_1 dx - \alpha b\int_{\partial\Omega}u_2\varphi_1 d\sigma + \beta b\int_{\partial\Omega}u_2^{\gamma-1}\varphi_1 d\sigma \\
&= \lambda_1(b+\lambda_1)\int_\Omega u_2\varphi_1 dx + \beta(b+\lambda_1)\int_{\partial\Omega}u_2^{\gamma-1}\varphi_1 d\sigma - \alpha(b+\lambda_1)\int_{\partial\Omega}u_2\varphi_1 d\sigma,
\end{aligned}$$

and

$$\begin{aligned}
(r.h.s) &= -\int_\Omega u_2\Delta u_2\varphi_1 dx + \lambda a(b+\lambda_1)\|\varphi_1\|^2_{L^2(\Omega)} \\
&= \int_\Omega \nabla u_2\cdot\nabla(u_2\varphi_1)dx - \int_{\partial\Omega}(\partial_\nu u_2)u_2\varphi_1 d\sigma + \lambda a(b+\lambda_1) \\
&= \int_\Omega |\nabla u_2|^2\varphi_1 dx + \int_\Omega u_2\nabla u_2\cdot\nabla\varphi_1 dx + \beta\int_{\partial\Omega}u_2^\gamma\varphi_1 d\sigma + \lambda a(b+\lambda_1) \\
&= \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{1}{2}\int_\Omega \nabla u_2^2\cdot\nabla\varphi_1 dx + \beta\int_{\partial\Omega}u_2^\gamma\varphi_1 d\sigma + \lambda a(b+\lambda_1) \\
&= \int_\Omega |\nabla u_2|^2\varphi_1 dx - \frac{1}{2}\int_\Omega u_2^2\Delta\varphi_1 dx + \frac{1}{2}\int_{\partial\Omega}u_2^2(\partial_\nu\varphi_1)d\sigma + \beta\int_{\partial\Omega}u_2^\gamma\varphi_1 d\sigma + \lambda a(b+\lambda_1) \\
&= \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx + \beta\int_{\partial\Omega}u_2^\gamma\varphi_1 d\sigma - \frac{\alpha}{2}\int_{\partial\Omega}u_2^2\varphi_1 d\sigma + \lambda a(b+\lambda_1).
\end{aligned}$$

Therefore the following equality holds.

(8.1.8)
$$\lambda_1(b+\lambda_1)\int_\Omega u_2\varphi_1 dx = \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx + a(b+\lambda_1)\lambda + \int_{\partial\Omega}\left\{\beta u_2^\gamma - \beta(b+\lambda_1)u_2^{\gamma-1} - \frac{\alpha}{2}u_2^2 + \alpha(b+\lambda_1)u_2\right\}\varphi_1 d\sigma.$$

Since (A) : $\gamma = 2$, $\alpha \le 2\beta$ or (B) : $\gamma > 2$ holds, we get

$$\inf_{u_2\ge 0}\left\{\beta u_2^\gamma - \beta(b+\lambda_1)u_2^{\gamma-1} - \frac{\alpha}{2}u_2^2 + \alpha(b+\lambda_1)u_2\right\} \ge -C > -\infty.$$

Moreover, we see that due to the boundedness of φ_1 (cf. Lemma 7.1.2)

$$\lambda_1(b+\lambda_1)\int_\Omega u_2\varphi_1 dx \ge \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx + a(b+\lambda_1)\lambda - C.$$

By Schwarz's inequality and Young's inequality, it is easy to see that

$$\begin{aligned}\int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx + a(b+\lambda_1)\lambda &\le \lambda_1(b+\lambda_1)\int_\Omega u_2\varphi_1 dx + C \\ &\le \lambda_1(b+\lambda_1)\left(\int_\Omega u_2^2\varphi_1 dx\right)^{\frac{1}{2}}\|\varphi_1\|_{L^1(\Omega)}^{\frac{1}{2}} + C \\ &\le \frac{\lambda_1}{4}\int_\Omega u_2^2\varphi_1 dx + C.\end{aligned}$$

Hence we obtain

$$\int_\Omega |\nabla u_2|^2\varphi_1 dx \le C, \qquad \int_\Omega u_2^2\varphi_1 dx \le C, \qquad \lambda \le C, \tag{8.1.9}$$

and

$$\int_\Omega u_2\varphi_1 dx \le \left(\int_\Omega u_2^2\varphi_1 dx\right)^{\frac{1}{2}}\left(\int_\Omega \varphi_1 dx\right)^{\frac{1}{2}} \le C. \tag{8.1.10}$$

Furthermore it follows from Lemma 7.1.2 and (8.1.9)

$$C_\alpha\left(\int_\Omega |\nabla u_2|^2 dx + \int_\Omega u_2^2 dx\right) \le \int_\Omega |\nabla u_2|^2\varphi_1 dx + \int_\Omega u_2^2\varphi_1 dx \le C,$$

whence follows

$$\|u_2\|_{H^1(\Omega)} \le C. \tag{8.1.11}$$

By (8.1.10) and (8.1.8), we also have

$$\int_{\partial\Omega}\left\{\beta u_2^\gamma - \beta(b+\lambda_1)u_2^{\gamma-1} - \frac{\alpha}{2}u_2^2 + \alpha(b+\lambda_1)u_2\right\}\varphi_1 d\sigma \le C. \tag{8.1.12}$$

Hence we can obtain

$$\begin{cases} \int_{\partial\Omega} u_2^\gamma \varphi_1 d\sigma \le C & (\gamma > 2 \ \text{ or } \ \gamma = 2,\ \alpha < 2\beta), \\ \int_{\partial\Omega} u_2 \varphi_1 d\sigma \le C & (\gamma = 2,\ \alpha = 2\beta). \end{cases} \tag{8.1.13}$$

Indeed, if $\gamma > 2$, then by Hölder's inequality and Young's inequality, we get

$$\begin{aligned} \beta \int_{\partial\Omega} u_2^\gamma \varphi_1 d\sigma + \alpha(b+\lambda_1) \int_{\partial\Omega} u_2 \varphi_1 d\sigma &\le C + \beta(b+\lambda_1) \int_{\partial\Omega} u_2^{\gamma-1} \varphi_1 d\sigma + \frac{\alpha}{2} \int_{\partial\Omega} u_2^2 \varphi_1 d\sigma \\ &\le C + \beta(b+\lambda_1) \left(\int_{\partial\Omega} u_2^\gamma \varphi_1 d\sigma \right)^{\frac{\gamma-1}{\gamma}} \left(\int_{\partial\Omega} \varphi_1 d\sigma \right)^{\frac{1}{\gamma}} \\ &\quad + \frac{\alpha}{2} \left(\int_{\partial\Omega} u_2^\gamma \varphi_1 d\sigma \right)^{\frac{2}{\gamma}} \left(\int_{\partial\Omega} \varphi_1 d\sigma \right)^{\frac{\gamma-2}{\gamma}} \\ &\le C + \beta(b+\lambda_1) \|\varphi_1\|_{L^\infty(\Omega)}^{\frac{1}{\gamma}} |\partial\Omega|^{\frac{1}{\gamma}} \left(\int_{\partial\Omega} u_2^\gamma \varphi_1 d\sigma \right)^{\frac{\gamma-1}{\gamma}} \\ &\quad + \frac{\alpha}{2} \|\varphi_1\|_{L^\infty(\Omega)}^{\frac{\gamma-2}{\gamma}} |\partial\Omega|^{\frac{\gamma-2}{\gamma}} \left(\int_{\partial\Omega} u_2^\gamma \varphi_1 d\sigma \right)^{\frac{2}{\gamma}} \\ &\le C + \frac{\beta}{2} \int_{\partial\Omega} u_2^\gamma \varphi_1 d\sigma, \end{aligned}$$

where we denote by $|\partial\Omega|$ a volume of $\partial\Omega$ and use the following property (see [27]):

$$\|\varphi_1\|_{L^\infty(\partial\Omega)} \le \|\varphi_1\|_{L^\infty(\Omega)}.$$

On the other hand, if $\gamma = 2$ and $\alpha < 2\beta$, then it follows from Schwarz's inequality and Young's inequality

$$\begin{aligned} &\left(\beta - \frac{\alpha}{2}\right) \int_{\partial\Omega} u_2^2 \varphi_1 d\sigma + \alpha(b+\lambda_1) \int_{\partial\Omega} u_2 \varphi_1 d\sigma \\ &\le C + \beta(b+\lambda_1) \int_{\partial\Omega} u_2 \varphi_1 d\sigma \\ &\le C + \beta(b+\lambda_1) \left(\int_{\partial\Omega} u_2^2 \varphi_1 d\sigma \right)^{\frac{1}{2}} \left(\int_{\partial\Omega} \varphi_1 d\sigma \right)^{\frac{1}{2}} \\ &\le C + \beta(b+\lambda_1) \|\varphi_1\|_{L^\infty(\Omega)} |\partial\Omega|^{\frac{1}{2}} \left(\int_{\partial\Omega} u_2^2 \varphi_1 d\sigma \right)^{\frac{1}{2}} \\ &\le C + \frac{1}{2} \left(\beta - \frac{\alpha}{2}\right) \int_{\partial\Omega} u_2^2 \varphi_1 d\sigma. \end{aligned}$$

For the case of $\gamma = 2$ and $\alpha = 2\beta$, from (8.1.12) it is clear that

$$\beta \int_{\partial\Omega} u_2 \varphi_1 d\sigma \le C.$$

Thus we obtain (8.1.13).

Now, we derive H^1-estimate for u_1. Multiplying the first equation of (8.1.6) by φ_1 and using integration by parts, we get

$$(\lambda_1 + b)\int_\Omega u_1\varphi_1 dx = \int_\Omega u_1u_2\varphi_1 dx + \lambda(\lambda_1 + b) \tag{8.1.14}$$

Similarly, multiplying the second equation of (8.1.6) by φ_1, we get

$$\lambda_1 \int_\Omega u_2\varphi_1 dx + \beta\int_{\partial\Omega} u_2^{\gamma-1}\varphi_1 d\sigma - \alpha\int_{\partial\Omega} u_2\varphi_1 d\sigma = a\int_\Omega u_1\varphi_1 dx. \tag{8.1.15}$$

Then by (8.1.14), (8.1.15), (8.1.11) and (8.1.13), we obtain

$$\int_\Omega u_1\varphi_1 dx \le C, \qquad \int_\Omega u_1u_2\varphi_1 dx \le C. \tag{8.1.16}$$

Hence, by Lemma 7.1.2, we get a priori bounds for $\int_\Omega u_1 dx$ and $\int_\Omega u_1u_2 dx$. Now we are going to establish a priori bound of u_1 in $H^1(\Omega)$ for the case of $N \in [3, 5]$. Multiplying the first equation of (8.1.6) by u_1 and using integration by parts, we obtain

$$\begin{aligned} \|\nabla u_1\|^2_{L^2(\Omega)} + \alpha\int_{\partial\Omega} u_1^2 ds + b\|u_1\|^2_{L^2(\Omega)} &= \int_\Omega u_1^2u_2 dx + \lambda(b+\lambda_1)\int_\Omega u_1\varphi_1 dx \\ &\le \int_\Omega (u_1u_2)^\theta \left(u_1^{\frac{2-\theta}{1-\theta}}u_2\right)^{1-\theta} dx + C \\ &\le \left(\int_\Omega u_1u_2 dx\right)^\theta \left(\int_\Omega u_1^{\frac{2-\theta}{1-\theta}}u_2 dx\right)^{1-\theta} + C, \end{aligned} \tag{8.1.17}$$

where we apply Hölder's inequality with exponent $(\frac{1}{\theta}, \frac{1}{1-\theta})$ for the first term on the right hand side. Here we take $\theta = \frac{6-N}{4} \in (0, 1)$, then by applying Hölder's inequality with exponent $(\frac{2N}{N+2}, \frac{2N}{N-2})$,

$$\left(\int_\Omega u_1^{\frac{2-\theta}{1-\theta}}u_2 dx\right)^{1-\theta} = \left(\int_\Omega u_1^{\frac{N+2}{N-2}}u_2 dx\right)^{\frac{N-2}{4}} \le \|u_1\|^{\frac{N+2}{4}}_{L^{2^*}(\Omega)}\|u_2\|^{\frac{N-2}{4}}_{L^{2^*}(\Omega)}.$$

where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. Using Sobolev's embedding $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and (8.1.11), we obtain

$$\|u_1\|^{\frac{N+2}{4}}_{L^{2^*}(\Omega)}\|u_2\|^{\frac{N-2}{4}}_{L^{2^*}(\Omega)} \le C\|u_1\|^{\frac{N+2}{4}}_{H^1(\Omega)}.$$

Since $(\|\nabla u_1\|^2_{L^2(\Omega)} + \alpha\int_{\partial\Omega} u_1^2 ds + b\|u_1\|^2_{L^2(\Omega)})^{1/2}$ is equivalent to the usual H^1-norm of u_1 due to trace inequality and Poincaré-Friedrichs type inequality, as a consequence we have

$$\|u_1\|^2_{H^1(\Omega)} \le C\|u_1\|^{\frac{N+2}{4}}_{H^1(\Omega)} + C.$$

Since $N \in [3,5]$, we have $\frac{N+2}{4} < 2$. Hence it follows from Young's inequality

$$\|u_1\|_{H^1(\Omega)}^2 \leq C\|u_1\|_{H^1(\Omega)}^{\frac{N+2}{4}} + C \leq \frac{1}{2}\|u_1\|_{H^1(\Omega)}^2 + C.$$

Thus we derive

$$\|u_1\|_{H^1(\Omega)} \leq C. \tag{8.1.18}$$

Next, we derive L^∞-estimates for u_1 as for the case $N \in [3,5]$. From Sobolev's embedding $H^1(\Omega) \hookrightarrow L^{\frac{10}{3}}(\Omega)$, we can see that u_1, $u_2 \in L^{\frac{10}{3}}(\Omega)$ and $u_1u_2 \in L^{\frac{5}{3}}(\Omega)$. We get $u_1 \in W^{2,\frac{5}{3}}(\Omega)$ by the elliptic estimate for the first equation of (8.1.6). Moreover, $u_1 \in L^5(\Omega)$ by Sobolev's embedding $W^{2,\frac{5}{3}}(\Omega) \hookrightarrow L^5(\Omega)$. Then by Hölder's inequality,

$$\int_\Omega u_1^2u_2^2dx \leq \left(\int_\Omega u_1^{2\cdot\frac{5}{2}}dx\right)^{\frac{2}{5}}\left(\int_\Omega u_2^{2\cdot\frac{5}{3}}dx\right)^{\frac{3}{5}},$$

we can see that $u_1u_2 \in L^2(\Omega)$. By the same reason as before, we know that $u_1 \in W^{2,2}(\Omega) \hookrightarrow L^{10}(\Omega)$. By Hölder's inequality, we have $u_1u_2 \in L^{\frac{5}{2}}(\Omega)$. Hence applying elliptic estimate and Sobolev's embedding again, we get $u_1 \in W^{2,\frac{5}{2}}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \in [1,\infty)$. Therefore $u_1u_2 \in L^{\frac{10q}{3q+10}}(\Omega)$ and $u_1 \in W^{2,\frac{10q}{3q+10}}(\Omega)$. Choosing $q > 10$, we have

$$\|u_1\|_{L^\infty(\Omega)} \leq C_1,$$

where we use the Sobolev's embedding $W^{2,\frac{10q}{3q+10}}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $q > 10$.

Thus we obtain L^∞-estimate of u_1 for the case of $N \in [3,5]$. About the regularity for u_2, it suffices to consider the following problem for given $u_1 \in L^\infty(\Omega)$:

$$\begin{cases} -\Delta u_2 = au_1 \in L^\infty(\Omega), & x \in \Omega, \\ \partial_\nu u_2 + \beta|u_2|^{\gamma-2}u_2 = 0, & x \in \partial\Omega. \end{cases}$$

Therefore we can derive L^∞-estimate for u_2, i.e.,

$$\|u_2\|_{L^\infty(\Omega)} \leq C_2$$

by the same arguments as for the compactness of Ψ applying Lemma 8.1.2. Choosing $R > C_1 + C_2$, we can see that the conclusion of this lemma holds.

As for the case $N = 1, 2$, it suffices to obtain L^∞-estimate for each component. First, let $N = 2$. Choosing $\theta = \frac{1}{2}$ in (8.1.17), we see that it follows from Sobolev's embedding

$H^1(\Omega) \hookrightarrow L^p(\Omega)$ (for all $p \in [1, \infty)$)

$$\begin{aligned}\|\nabla u_1\|_{L^2(\Omega)}^2 + \alpha \int_{\partial\Omega} u_1^2 ds + b\|u_1\|_{L^2(\Omega)}^2 &= \int_\Omega u_1^2 u_2 dx + \lambda(b+\lambda_1) \int_\Omega u_1 \varphi_1 dx \\ &\le \int_\Omega (u_1 u_2)^{\frac{1}{2}} \left(u_1^3 u_2\right)^{\frac{1}{2}} dx + C \\ &\le \left(\int_\Omega u_1 u_2 dx\right)^{\frac{1}{2}} \left(\int_\Omega u_1^3 u_2 dx\right)^{\frac{1}{2}} + C \\ &\le C \left(\int_\Omega u_1^3 u_2 dx\right)^{\frac{1}{2}} + C \\ &\le C\|u_1\|_{L^6(\Omega)}^{\frac{3}{2}} \|u_2\|_{L^2(\Omega)}^{\frac{1}{2}} + C \\ &\le C\|u_1\|_{H^1(\Omega)}^{\frac{3}{2}} + C.\end{aligned}$$

Here we note that we have already had H^1-estimate for u_2 without restrictions on the space dimension. Thus we also get H^1-estimate for u_1. In the similar way as for the previous case $N \in [3, 5]$, we can derive L^∞-estimates for u_1 and u_2.

Let $N = 1$ and $\Omega = (a_0, b_0)$ with $a_0 < b_0$. Since $u_1 \in C(\overline{\Omega})$, there exists $x_0 \in \overline{\Omega}$ such that

$$u_1(x_0) = \min_{x \in \overline{\Omega}} u_1(x).$$

Furthermore, since it holds that $\|u_1\|_{L^1(\Omega)} \le C$ for any space dimension, we have

$$\min_{x \in \overline{\Omega}} u_1(x) \le \frac{1}{|\Omega|} \int_\Omega u_1 dx \le C.$$

Here by the fundamental theorem of calculus,

$$u_1(x) = u_1(x_0) + \int_{x_0}^x u_1'(\xi) d\xi.$$

Therefore we get the following inequality:

$$\|u_1\|_{L^\infty(\Omega)} \le \int_{a_0}^{b_0} |u_1'(\xi)| d\xi + |u_1(x_0)| \le \|u_1'\|_{L^1(\Omega)} + C. \tag{8.1.19}$$

From (8.1.19), Schwarz's inequality and Young's inequality, we see that

$$\begin{aligned}\|u_1'\|_{L^2}^2 + \alpha \int_{\partial\Omega} u_1^2 ds + b\|u_1\|_{L^2}^2 &= \int_\Omega u_1^2 u_2 dx + \lambda(b+\lambda_1) \int_\Omega u_1 \varphi_1 dx \\ &\le \|u_1\|_{L^\infty} \int_\Omega u_1 u_2 dx + C \\ &\le C\left(\|u_1'\|_{L^1} + C\right) + C \\ &\le C\|u_1'\|_{L^2} + C \le \frac{1}{2}\|u_1'\|_{L^2}^2 + C.\end{aligned}$$

Hence we obtain a priori bound for $\|u_1\|_{H^1(\Omega)}$. Since Sobolev's embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ holds for $N = 1$, we obtain the desired estimates. □

Proof of Theorem 8.1.1. By applying Lemma 8.1.3, Lemma 8.1.4 and Lemma 7.1.1, we can verify that Theorem 8.1.1 holds. □

Remark 8.1.5. If $\alpha = 0$, for $\gamma \in (1, 2)$ we can derive H^1-estimate for u_2 by taking H^1 norm of u_2 as $\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^1(\partial\Omega)}$ in the proof of Lemma 8.1.4. In fact, it is easy to see that this norm is equivalent to the usual $H^1(\Omega)$ norm by Lemma 1.1.10. Therefore it is easy to see that Theorem 8.1.1 holds in the case of $\alpha = 0$, $\beta > 0$ and $\gamma > 1$.

8.2 Ordered Uniqueness

Next, we discuss the ordered uniqueness of the positive solutions for (S-NR). The proof of the ordered uniqueness is based on the simplicity of the first eigenvalue and the positivity of the corresponding eigenfunction.

Theorem 8.2.1. *Let* (u_1, u_2) *and* (v_1, v_2) *be two positive solutions of* (S-NR) *satisfying* $u_1 \le v_1$ *or* $u_2 \le v_2$*. Then* $u_1 \equiv v_1$ *and* $u_2 \equiv v_2$*.*

Proof. Suppose that $u_1 \not\equiv v_1$ or $u_2 \not\equiv v_2$. Without loss of generality, we only have to consider the case where $u_2 \not\equiv v_2$ and $u_2 \le v_2$. In fact, if $u_1 \le v_1$, by the second equation of (S-NR) we have

$$-\Delta(u_2 - v_2) = a(u_1 - v_1) \le 0. \tag{8.2.1}$$

Multiplying (8.2.1) by $[u_2 - v_2]^+ := \max\{u_2 - v_2, 0\}$ and using integration by parts, we obtain

$$\|\nabla[u_2 - v_2]^+\|^2_{L^2(\Omega)} + \beta \int_{\partial\Omega} [u_2 - v_2]^+ \left(|u_2|^{\gamma-2}u_2 - |v_2|^{\gamma-2}v_2\right) d\sigma \le 0. \tag{8.2.2}$$

Note that by Lemma 7.1.3

$$\begin{aligned}\int_{\partial\Omega} [u_2 - v_2]^+ \left(|u_2|^{\gamma-2}u_2 - |v_2|^{\gamma-2}v_2\right) d\sigma &= \int_{\{u_2 \ge v_2\}} (u_2 - v_2) \left(|u_2|^{\gamma-2}u_2 - |v_2|^{\gamma-2}v_2\right) d\sigma \\ &\ge \int_{\{u_2 \ge v_2\}} C_\gamma (u_2 - v_2)^\gamma d\sigma \\ &= C_\gamma \int_{\partial\Omega} \left([u_2 - v_2]^+\right)^\gamma d\sigma.\end{aligned}$$

By this inequality and (8.2.2), we get

$$\|\nabla[u_2 - v_2]^+\|^2_{L^2(\Omega)} + C_\gamma \int_{\partial\Omega} \left([u_2 - v_2]^+\right)^\gamma d\sigma \le 0.$$

Therefore we have

$$\nabla[u_2 - v_2]^+ = 0,$$

$$[u_2 - v_2]^+\big|_{\partial\Omega} = 0.$$

Hence we deduce $[u_2 - v_2]^+ \equiv 0$, i.e., $u_2 \le v_2$.

Next we consider the following eigenvalue problems:

$$\begin{cases} -\Delta w + (b - u_2(x))\, w = \mu' w & \text{in } \Omega, \\ \partial_\nu w + \alpha w = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.2.3}$$

and

$$\begin{cases} -\Delta w + (b - v_2(x))\, w = \eta' w & \text{in } \Omega, \\ \partial_\nu w + \alpha w = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.2.4}$$

If necessary, we take some nonnegative constant $L \ge 0$ and add both sides of equations of (8.2.3) and (8.2.4) by L, and we can assume $U(x) := b - u_2(x) + L \ge 1$ and $V(x) := b - v_2(x) + L \ge 1$. Thus we consider the following problems in stead of (8.2.3) and (8.2.4):

$$\begin{cases} -\Delta w + U(x) w = \mu w & \text{in } \Omega, \\ \partial_\nu w + \alpha w = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.2.5}$$

and

$$\begin{cases} -\Delta w + V(x) w = \eta w & \text{in } \Omega, \\ \partial_\nu w + \alpha w = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.2.6}$$

By applying the compactness argument for the associate Rayleigh's quotients of (8.2.5) and (8.2.6) , we know that the smallest positive eigenvalues of (8.2.5) and (8.2.6) are attained and we denote them by μ_0 and η_0. Moreover, thanks to $u_2 \not\equiv v_2$ and $u_2 \le v_2$, we see that $\eta_0 < \mu_0$. On the other hand, since (u_1, u_2) and (v_1, v_2) are positive stationary solutions for (S-NR), $u_1 > 0$ and $v_1 > 0$ satisfy

$$\begin{cases} -\Delta u_1 + (b - u_2(x) + L)\, u_1 = L u_1 & \text{in } \Omega, \\ \partial_\nu u_1 + \alpha u_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta v_1 + (b - v_2(x) + L)\, v_1 = L v_1 & \text{in } \Omega, \\ \partial_\nu v_1 + \alpha v_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

By the fact that the eigenvalue corresponding to the positive eigenfunction is the smallest one, we deduce $\mu_0 = L = \eta_0$. This contradicts $\eta_0 < \mu_0$. Thus the proof is completed. □

Chapter 9

Nonstationary Problem of (NR)

In this chapter, we investigate the large time behavior of solutions to (NR) and prove that the positive stationary solution plays a role of threshold to classify initial data into two groups; namely corresponding solutions of (NR) blow up in finite time or exist globally. Moreover we discuss the existence of blow-up solutions to (NR) with nonlinear boundary conditions as an application of Theorem 4.1.3 in Part I.

9.1 Local Well-posedness

First we state the local well-posedness of problem (NR).

Theorem 9.1.1. *Assume* $(u_{10}, u_{20}) \in L^\infty(\Omega) \times L^\infty(\Omega)$. *Then there exists* $T > 0$ *such that* (NR) *possesses a unique solution* $(u_1, u_2) \in (L^\infty(0,T; L^\infty(\Omega)) \cap C([0,T); L^2(\Omega)))^2$ *satisfying*

$$\sqrt{t}\partial_t u_1, \sqrt{t}\partial_t u_2, \sqrt{t}\Delta u_1, \sqrt{t}\Delta u_2 \in L^2(0,T; L^2(\Omega)). \tag{9.1.1}$$

Furthermore, if the initial data is nonnegative, then the local solution (u_1, u_2) *for* (NR) *is nonnegative.*

In order to prove this theorem, we rely on L^∞-energy method developed in [45] (see Chapter 2). This theory is very useful to show the existence of strong solutions with bounded initial data.

Proof of Theorem 9.1.1. (Existence and regularity) We consider the following approximate problem:

$$\begin{cases} \partial_t u_1 - \Delta u_1 = [u_1]_M [u_2]_M - b u_1, & t > 0,\ x \in \Omega, \\ \partial_t u_2 - \Delta u_2 = a u_1, & t > 0,\ x \in \Omega, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta |u_2|^{\gamma-2} u_2 = 0, & t > 0,\ x \in \partial\Omega, \\ u_1(x,0) = u_{10}(x),\ u_2(x,0) = u_{20}(x), & x \in \Omega, \end{cases} \tag{9.1.2}$$

where $M > 0$ is a given constant and the cut-off function $[u]_M$ is defined by

$$[u]_M = \begin{cases} M, & u \geq M, \\ u, & |u| \leq M, \\ -M, & u \leq -M. \end{cases}$$

Since $u \mapsto [u]_M$ is Lipschitz continuous from $L^2(\Omega)$ into itself, it is well known that (9.1.2) has a unique global solution (u_1, u_2) satisfying (9.1.1) by applying the abstract theory on maximal monotone operators developed by H. Brézis [10].

By multiplying the first equation of (9.1.2) by $|u_1|^{r-2}u_1$ and using integration by parts,

$$\frac{1}{r}\frac{d}{dt}\|u_1(t)\|_{L^r}^r + (r-1)\int_\Omega |\nabla u_1|^2 u_1^{r-2} dx + \alpha \int_{\partial\Omega} |u_1|^r d\sigma \leq \int_\Omega |u_1|^r |u_2| dx - b\int_\Omega |u_1|^r dx.$$

Hence

$$\frac{1}{r}\frac{d}{dt}\|u_1(t)\|_{L^r}^r \leq \|u_2(t)\|_{L^\infty}\|u_1(t)\|_{L^r}^r.$$

Divide both sides by $\|u_1\|_{L^r}^{r-1}$ and integrate with respect to t on $[0, t]$, then we get

$$\|u_1(t)\|_{L^r} \leq \|u_{10}\|_{L^r} + \int_0^t \|u_1(\tau)\|_{L^r}\|u_2(\tau)\|_{L^\infty} d\tau.$$

Letting r tend to ∞ (Lemma 1.2.2), we derive

$$\|u_1(t)\|_{L^\infty} \leq \|u_{10}\|_{L^\infty} + \int_0^t \|u_1(\tau)\|_{L^\infty}\|u_2(\tau)\|_{L^\infty} d\tau.$$

Similarly, we can get the following L^∞ estimate for u_2 ;

$$\|u_2(t)\|_{L^\infty} \leq \|u_{20}\|_{L^\infty} + \int_0^t a\|u_1(\tau)\|_{L^\infty} d\tau.$$

Therefore setting $y(t) = \|u_1(t)\|_{L^\infty(\Omega)} + \|u_2(t)\|_{L^\infty(\Omega)}$, we get

$$y(t) \leq y(0) + \int_0^t \left(y^2(\tau) + ay(\tau)\right) d\tau.$$

Thus applying Lemma 1.2.3, we find that there exists a number $T > 0$ depending only on $\|u_{10}\|_{L^\infty(\Omega)}$ and $\|u_{20}\|_{L^\infty(\Omega)}$ such that

$$y(t) \leq y(0) + 1 \qquad \text{a.e. } t \in [0, T].$$

In other words, we get

$$\|u_1(t)\|_{L^\infty(\Omega)} + \|u_2(t)\|_{L^\infty(\Omega)} \leq \|u_{10}\|_{L^\infty(\Omega)} + \|u_{20}\|_{L^\infty(\Omega)} + 1 \qquad \text{a.e. } t \in [0, T].$$

Hence choosing $M > \|u_{10}\|_{L^\infty(\Omega)} + \|u_{20}\|_{L^\infty(\Omega)} + 1$, we can see that (u_1, u_2) gives a solution for (NR) on $[0, T]$ by the definition of the cut-off function $[u]_M$. Note that even though

$\|u_1(t)\|_{L^r}^{r-1}$ attains zero, we can justify this argument by Proposition 1 in [40]. To get the regularity estimate of the solution for (NR) is standard, so we omit the details.

(Uniqueness) Let (u_1, u_2) and (v_1, v_2) be two solutions to (NR) with initial data (u_{10}, u_{20}) and (v_{10}, v_{20}) respectively. We set $w_1 = u_1 - v_1$ and $w_2 = u_2 - v_2$. From (NR), we have

$$\partial_t w_1 - \Delta w_1 = w_1 u_2 + v_1 w_2 - b w_1, \tag{9.1.3}$$

$$\partial_t w_2 - \Delta w_2 = a w_1, \tag{9.1.4}$$

$$\partial_\nu w_1 + \alpha w_1 = \partial_\nu w_2 + \beta \left(|u_2|^{\gamma-2} u_2 - |v_2|^{\gamma-2} v_2 \right) = 0, \qquad \text{on } \partial\Omega.$$

We multiply (9.1.3) and (9.1.4) by w_1 and w_2 respectively, integrate over Ω and use integration by parts. Then we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w_1(t)\|_{L^2(\Omega)}^2 + \|\nabla w_1\|_{L^2(\Omega)}^2 + \alpha \int_{\partial\Omega} w_1^2 d\sigma \\
&\le \int_\Omega w_1^2 u_2 dx + \int_\Omega v_1 w_1 w_2 dx \\
&\le \|u_2\|_{L^\infty(0,T;L^\infty(\Omega))} \int_\Omega w_1^2 dx + \|v_1\|_{L^\infty(0,T;L^\infty(\Omega))} \int_\Omega w_1 w_2 dx \\
&\le C \left(\|w_1(t)\|_{L^2(\Omega)}^2 + \|w_2(t)\|_{L^2(\Omega)}^2 \right),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w_2(t)\|_{L^2(\Omega)}^2 + \|\nabla w_2\|_{L^2(\Omega)}^2 + \beta \int_{\partial\Omega} \left(|u_2|^{\gamma-2} u_2 - |v_2|^{\gamma-2} v_2 \right) (u_2 - v_2)\, d\sigma \\
&\le a \int_\Omega w_1 w_2 dx \\
&\le \frac{a}{2} \left(\|w_1(t)\|_{L^2(\Omega)}^2 + \|w_2(t)\|_{L^2(\Omega)}^2 \right).
\end{aligned}$$

Noting that

$$\int_{\partial\Omega} \left(|u_2|^{\gamma-2} u_2 - |v_2|^{\gamma-2} v_2 \right) (u_2 - v_2)\, d\sigma \ge \int_{\partial\Omega} C_\gamma |w_2|^\gamma d\sigma \ge 0$$

by Lemma 7.1.3, we can get the following differential inequality:

$$\frac{d}{dt} \left(\|w_1(t)\|_{L^2(\Omega)}^2 + \|w_2(t)\|_{L^2(\Omega)}^2 \right) \le C \left(\|w_1(t)\|_{L^2(\Omega)}^2 + \|w_2(t)\|_{L^2(\Omega)}^2 \right),$$

whence, from Gronwall's inequality,

$$\left(\|w_1(t)\|_{L^2(\Omega)}^2 + \|w_2(t)\|_{L^2(\Omega)}^2 \right) \le \left(\|u_{10} - v_{10}\|_{L^2(\Omega)}^2 + \|u_{20} - v_{20}\|_{L^2(\Omega)}^2 \right) e^{Ct} \qquad t \in [0, T).$$

This yields the uniqueness of the solution for (NR).

(Nonnegativity) Multiplying the first equation of (NR) by $u_1^- := \max\{-u_1, 0\}$, we get

$$\int_\Omega \partial_t u_1 u_1^- dx - \int_\Omega \Delta u_1 u_1^- dx \geq - \int_\Omega |u_1^-|^2 |u_2| dx - b \int_\Omega u_1 u_1^- dx.$$

Here, we can see that

$$\int_\Omega \partial_t u_1 u_1^- dx = \int_{\{u_1 \leq 0\}} \partial_t u_1 (-u_1) dx = -\frac{1}{2}\frac{d}{dt} \int_{\{u_1 \leq 0\}} (-u_1)^2 dx = -\frac{1}{2}\frac{d}{dt} \int_\Omega \left(u_1^-\right)^2 dx,$$

and

$$\begin{aligned} -\int_\Omega \Delta u_1 u_1^- dx &= \int_\Omega \nabla u_1 \cdot \nabla u_1^- dx + \alpha \int_{\partial\Omega} u_1 u_1^- d\sigma \\ &= -\int_\Omega |\nabla u_1^-|^2 dx - \alpha \int_{\{u_1 \leq 0\}} u_1^2 d\sigma = -\int_\Omega |\nabla u_1^-|^2 dx - \alpha \int_{\partial\Omega} \left(u_1^-\right)^2 d\sigma. \end{aligned}$$

Therefore we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_1^-(t)\|^2_{L^2(\Omega)} + \|\nabla u_1^-\|^2_{L^2(\Omega)} + \alpha \int_{\partial\Omega} \left(u_1^-\right)^2 d\sigma &= \int_\Omega |u_1^-|^2 |u_2| dx - b\|u_1^-(t)\|^2_{L^2(\Omega)} \\ &\leq \|u_2\|_{L^\infty(0,T;L^\infty(\Omega))} \|u_1^-(t)\|^2_{L^2(\Omega)}. \end{aligned}$$

Applying Gronwall's inequality, we obtain

$$\|u_1^-(t)\|^2_{L^2(\Omega)} \leq \|u_1^-(0)\|^2_{L^2(\Omega)} e^{2\|u_2\|_{L^\infty(0,T;L^\infty(\Omega))} t} \qquad t \in [0, T),$$

where T is maximal existence time for (NR). Since $u_{10} \geq 0$, i.e., $\|u_1^-(0)\|_{L^2(\Omega)} = 0$, it holds that

$$u_1^-(t) = 0 \qquad a.e. \text{ in } \Omega \quad \forall t \in [0, T).$$

Hence $u_1 \geq 0$. Similarly, multiplying the second equation of (NR) by $-u_2^-$, we get

$$\frac{1}{2}\frac{d}{dt}\|u_2^-(t)\|^2_{L^2(\Omega)} + \|\nabla u_2^-\|^2_{L^2(\Omega)} + \beta \int_{\partial\Omega} |u_2|^{\gamma-2} |u_2^-|^2 d\sigma = -a \int_\Omega u_1 u_2^- dx \leq 0.$$

Therefore $\|u_2^-(t)\|^2_{L^2(\Omega)} \leq \|u_2^-(0)\|^2_{L^2(\Omega)} = 0$, i.e., $u_2 \geq 0$. □

9.2 Threshold Property

Finally, we study the threshold property and prove that every positive stationary solution for (NR) gives a threshold for the blow up of solutions in the following sense.

Theorem 9.2.1. *Let $(\overline{u}_1, \overline{u}_2)$ be a positive stationary solution of* (NR)*, then the followings hold.*

(1) *Let $0 \leq u_{10}(x) \leq \overline{u}_1(x)$, $0 \leq u_{20}(x) \leq \overline{u}_2(x)$, then the solution (u_1, u_2) of* (NR)

exists globally. In addition, if $0 \le u_{10}(x) \le l_1\overline{u}_1(x)$*,* $0 \le u_{20}(x) \le l_2\overline{u}_2(x)$ *for some* $0 < l_1 < l_2 \le 1$*, then*

$$\lim_{t\to+\infty}(u_1(t,x),u_2(t,x)) = (0,0) \quad \text{pointwise on } \overline{\Omega}.$$

(2) *Assume further* $\gamma = 2$*,* $\alpha \le 2\beta$ *and let* $u_{10}(x) \ge l_1\overline{u}_1(x)$*,* $u_{20}(x) \ge l_2\overline{u}_2(x)$ *for some* $l_1 > l_2 > 1$*, then the solution* (u_1, u_2) *of* (NR) *blows up in finite time.*

Remark 9.2.2. The second assertion of Theorem 9.2.1 is also announced in [25] for the case where $\alpha = 0$ and $\gamma = 2$. However it seems that their proof contains some serious gaps.

We first prepare the following comparison theorem.

Lemma 9.2.3 (Comparison theorem)**.** *If* (u_{10}, u_{20})*,* (v_{10}, v_{20}) *are two initial data for* (NR) *satisfying*

$$0 \le u_{10} \le v_{10}, \quad 0 \le u_{20} \le v_{20} \qquad \text{on } \overline{\Omega},$$

then the corresponding solutions (u_1, u_2)*,* (v_1, v_2) *remain in the initial data order in time interval where the solutions exist, i.e.,* $u_1(t,x) \le v_1(t,x)$ *and* $u_2(t,x) \le v_2(t,x)$ *a.e.* $x \in \Omega$ *as long as* (u_1,u_2) *and* (v_1,v_2) *exist.*

Proof. Let $w_1 = u_1 - v_1$, $w_2 = u_2 - v_2$. By (NR) we have

$$\begin{cases} \partial_t w_1 - \Delta w_1 = w_1 u_2 + v_1 w_2 - b w_1, & t \in (0,T_m),\ x \in \Omega, \\ \partial_t w_2 - \Delta w_2 = a w_1, & t \in (0,T_m),\ x \in \Omega, \\ \partial_\nu w_1 + \alpha w_1 = \partial_\nu w_2 + \beta\left(|u_2|^{\gamma-2}u_2 - |v_2|^{\gamma-2}v_2\right) = 0, & t \in (0,T_m),\ x \in \partial\Omega, \\ w_1(x,0) \le 0, \quad w_2(x,0) \le 0, & x \in \Omega \end{cases} \tag{9.2.1}$$

where $T_m > 0$ is the maximum existence time for (u_1, u_2) and (v_1, v_2). We set

$$w^+ = w \vee 0, \quad w^- = (-w) \vee 0,$$

where $a \vee b = \max\{a, b\}$. It is easy to see that w^+, $w^- \ge 0$ and

$$w = w^+ - w^-, \quad |w| = w^+ + w^-.$$

Multiplying the first equation of (9.2.1) by w_1^+, we get

$$\int_\Omega \partial_t w_1 w_1^+ dx - \int_\Omega \Delta w_1 w_1^+ dx = \int_\Omega w_1 u_2 w_1^+ dx + \int_\Omega v_1 w_2 w_1^+ dx - b\int_\Omega w_1 w_1^+ dx.$$

Here, we see that

$$\int_\Omega \partial_t w_1 w_1^+ dx = \int_{\{w_1 \ge 0\}} \partial_t w_1 w_1 dx = \frac{1}{2}\frac{d}{dt}\int_{\{w_1\ge 0\}} w_1^2 dx = \frac{1}{2}\frac{d}{dt}\int_\Omega \left(w_1^+\right)^2 dx.$$

Similarly,

$$\begin{aligned} -\int_\Omega \Delta w_1 w_1^+ dx &= \int_\Omega \nabla w_1 \cdot \nabla w_1^+ dx + \alpha\int_{\partial\Omega} w_1 w_1^+ d\sigma \\ &= \int_{\{w_1\ge 0\}} |\nabla w_1|^2 dx + \alpha \int_{\{w_1 \ge 0\}} w_1^2 d\sigma = \int_\Omega |\nabla w_1^+|^2 dx + \alpha\int_{\partial\Omega}\left(w_1^+\right)^2 d\sigma. \end{aligned}$$

Hence noting that $v_1 \geq 0$, we obtain for any $T \in (0, T_m)$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left(w_1^+\right)^2 dx + \int_\Omega |\nabla w_1^+|^2 dx + \alpha \int_{\partial\Omega} \left(w_1^+\right)^2 d\sigma \\
&= \int_\Omega w_1 u_2 w_1^+ dx + \int_\Omega v_1 w_2 w_1^+ dx - b\int_\Omega w_1 w_1^+ dx \\
&= \int_\Omega \left(w_1^+ - w_1^-\right) u_2 w_1^+ dx + \int_\Omega v_1 \left(w_2^+ - w_2^-\right) w_1^+ dx - b\int_\Omega \left(w_1^+\right)^2 dx \\
&\leq \|u_2\|_{L^\infty(0,T;L^\infty(\Omega))} \int_\Omega \left(w_1^+\right)^2 dx + \|v_1\|_{L^\infty(0,T;L^\infty(\Omega))} \int_\Omega w_1^+ w_2^+ dx \\
&\leq C\left(\|w_1^+(t)\|^2_{L^2(\Omega)} + \|w_2^+(t)\|^2_{L^2(\Omega)}\right).
\end{aligned}
$$

Hence we get

$$
\frac{1}{2}\frac{d}{dt}\|w_1^+(t)\|^2_{L^2(\Omega)} \leq C\left(\|w_1^+(t)\|^2_{L^2(\Omega)} + \|w_2^+(t)\|^2_{L^2(\Omega)}\right). \tag{9.2.2}
$$

Next we do the same calculation for the second equation of (9.2.1) and get

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega \left(w_2^+\right)^2 dx + \int_\Omega |\nabla w_2^+|^2 dx - \int_{\partial\Omega} (\partial_\nu w_2) w_2^+ d\sigma \leq \frac{a}{2}\left(\|w_1^+(t)\|^2_{L^2(\Omega)} + \|w_2^+(t)\|^2_{L^2(\Omega)}\right),
$$

and

$$
\begin{aligned}
-\int_{\partial\Omega} (\partial_\nu w_2) w_2^+ d\sigma &= \beta \int_{\partial\Omega} \left(|u_2|^{\gamma-2}u_2 - |v_2|^{\gamma-2}v_2\right) w_2^+ d\sigma \\
&= \beta \int_{\{u_2 \geq v_2\}} \left(|u_2|^{\gamma-2}u_2 - |v_2|^{\gamma-2}v_2\right)(u_2 - v_2)\, d\sigma \geq 0.
\end{aligned}
$$

Therefore

$$
\frac{1}{2}\frac{d}{dt}\|w_2^+(t)\|^2_{L^2(\Omega)} \leq \frac{a}{2}\left(\|w_1^+(t)\|^2_{L^2(\Omega)} + \|w_2^+(t)\|^2_{L^2(\Omega)}\right). \tag{9.2.3}
$$

Thus by (9.2.2), (9.2.3) and Gronwall's inequality, we get

$$
\|w_1^+(t)\|^2_{L^2(\Omega)} + \|w_2^+(t)\|^2_{L^2(\Omega)} \leq \left(\|w_1^+(0)\|^2_{L^2(\Omega)} + \|w_2^+(0)\|^2_{L^2(\Omega)}\right) e^{Ct} \qquad \forall t \in [0, T_m).
$$

Since $w_1^+(0) = w_2^+(0) = 0$, the above inequality means $w_1^+ = w_2^+ = 0$. Hence, we have the desired result. □

Proof of Theorem 9.2.1. (1) If $0 \leq u_{10} \leq \overline{u}_1$ and $0 \leq u_{20} \leq \overline{u}_2$, then since $(\overline{u}_1, \overline{u}_2)$ is a global solution for (NR), $0 \leq u_1(t,x) \leq \overline{u}_1(x)$ and $0 \leq u_2(t,x) \leq \overline{u}_2(x)$ follow directly from Lemma 9.2.3. That is, we have

$$
\sup_{t\in[0,T)} \|u_i(t,\cdot)\|_{L^\infty(\Omega)} \leq \|\overline{u}_i\|_{L^\infty(\Omega)} \qquad (i = 1, 2).
$$

Hence the solution (u_1, u_2) exists globally.

In addition, let $u_{10}(x) \le l_1\overline{u}_1(x)$, $u_{20}(x) \le l_2\overline{u}_2(x)$ for some $0 < l_1 < l_2 \le 1$. Since the comparison theorem holds, without loss of generality, we can assume that $u_{10}(x) = l_1\overline{u}_1(x)$, $u_{20}(x) = l_2\overline{u}_2(x)$ and $l_1 < l_2 \le 1$. We here note that $\delta u_1 := u_1(t+h) - u_1(t)$ and $\delta u_2 := u_2(t+h) - u_2(t)$ for $h > 0$ satisfy the following equations:

(9.2.4)
$$\begin{cases} \partial_t\,(\delta u_1) - \Delta\,(\delta u_1) = (\delta u_1)\,u_2(t+h) + u_1(t)\,(\delta u_2) - b\,(\delta u_1)\,, \\ \partial_t\,(\delta u_2) - \Delta\,(\delta u_2) = a\,(\delta u_1)\,, \\ \partial_\nu\,(\delta u_1) + \alpha\,(\delta u_1) = \partial_\nu\,(\delta u_2) + \beta\left(|u_2(t+h)|^{\gamma-2}u_2(t+h) - |u_2(t)|^{\gamma-2}u_2(t)\right) = 0, \\ \delta u_1(0) = u_1(0+h) - u_1(0), \quad \delta u_2(0) = u_2(0+h) - u_2(0). \end{cases}$$

Multiplying the first and second equation of (9.2.4) by $[\delta u_1]^+$ and $[\delta u_2]^+$ respectively and using integration by parts and repeating the same argument as for (9.2.2), we obtain the following inequality:

$$\|[\delta u_1]^+\|^2_{L^2(\Omega)} + \|[\delta u_2]^+\|^2_{L^2(\Omega)} \le \left(\|[\delta u_1(0)]^+\|^2_{L^2(\Omega)} + \|[\delta u_2(0)]^+\|^2_{L^2(\Omega)}\right) e^{Ct} \qquad \forall t \in [0,\infty).$$

We divide both sides of this inequality by h^2:

$$\left\|\left[\frac{\delta u_1}{h}\right]^+\right\|^2_{L^2(\Omega)} + \left\|\left[\frac{\delta u_2}{h}\right]^+\right\|^2_{L^2(\Omega)} \le \left(\left\|\left[\frac{\delta u_1(0)}{h}\right]^+\right\|^2_{L^2(\Omega)} + \left\|\left[\frac{\delta u_2(0)}{h}\right]^+\right\|^2_{L^2(\Omega)}\right) e^{Ct}.$$

Since we know that u_1, u_2 is differentiable on *a.e.* t by the regularity results of Theorem 9.1.1, by letting $h \searrow 0$, we obtain

$$\|[\partial_t u_1]^+\|^2_{L^2} + \|[\partial_t u_2]^+\|^2_{L^2} \le \left(\|[\partial_t u_1(0)]^+\|^2_{L^2} + \|[\partial_t u_2(0)]^+\|^2_{L^2}\right) e^{Ct} \qquad a.e.\ t \in [0,\infty).$$

We here note that since $(l_1\overline{u}_1, l_2\overline{u}_2)$ is strict upper solution for (S-NR), it holds that

$$\begin{aligned} \partial_t u_1(0) &= \Delta u_{10} + u_{10}u_{20} - bu_{10} \\ &= l_1\Delta\overline{u}_1 + l_1l_2\overline{u}_1\overline{u}_2 - bl_1\overline{u}_1 \\ &\le l_1\left(\Delta\overline{u}_1 + \overline{u}_1\overline{u}_2 - b\overline{u}_1\right) = 0, \\ \partial_t u_2(0) &= \Delta u_{20} + au_{10} \\ &= l_2\Delta\overline{u}_2 + al_1\overline{u}_1 \\ &< l_2\left(\Delta\overline{u}_2 + a\overline{u}_1\right) = 0, \end{aligned}$$

which imply that $[\partial_t u_1(0)]^+ = [\partial_t u_2(0)]^+ = 0$. Hence we find that $\partial_t u_1 \le 0$ and $\partial_t u_2 \le 0$, i.e., $u_1(t,x)$ and $u_2(t,x)$ are monotone decreasing in t for a.e. $x \in \Omega$. Thus

$$\lim_{t\to\infty}\,(u_1(t,x), u_2(t,x)) =: (\tilde{u}_1(x), \tilde{u}_2(x))$$

exists and satisfies $(0,0) \le (\tilde{u}_1, \tilde{u}_2) \le (l_1\overline{u}_1, l_2\overline{u}_2) < (\overline{u}_1, \overline{u}_2)$. Now we prove that $(\tilde{u}_1, \tilde{u}_2)$ is a nonnegative stationary solution of (NR). First we note that

(9.2.5) $\quad u_i(t) \to \tilde{u}_i \quad$ *strongly* in $L^p(\Omega)$ as $k \to \infty \quad \forall p \in (1,\infty) \qquad (i = 1,2)$.

In fact, since $|u_i(x,t) - \tilde{u}_i(x)|^p \to 0$ *a.e.* $x \in \Omega$ as $t \to \infty$ and $|u_i(x,t) - \tilde{u}_i(x)|^p \le 2^p|\overline{u}_i(x)|^p \le 2^p\|\overline{u}_i\|^p_{L^\infty(\Omega)}$ *a.e.* $x \in \Omega$, Lebesgue's dominant convergence theorem assures (9.2.5). Next multiplying the first and the second equations of (NR) by $\partial_t u_1$ and $\partial_t u_2$ respectively, we get

$$\|\partial_t u_1(t)\|^2_{L^2(\Omega)} + \frac{d}{dt}\left\{\frac{1}{2}\|\nabla u_1(t)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u_1(t)\|^2_{L^2(\partial\Omega)} + \frac{b}{2}\|u_1(t)\|^2_{L^2(\Omega)}\right\}$$
$$= \int_\Omega u_1 u_2 \partial_t u_1 dx \le 0,$$

$$\|\partial_t u_2(t)\|^2_{L^2(\Omega)} + \frac{d}{dt}\left\{\frac{1}{2}\|\nabla u_2(t)\|^2_{L^2(\Omega)} + \frac{\beta}{\gamma}\|u_2(t)\|^\gamma_{L^\gamma(\partial\Omega)}\right\} = a\int_\Omega u_1 \partial_t u_2 dx \le 0.$$

Then integration of these over $(0,T)$ for any $T > 0$ gives

$$\int_0^\infty \|\partial_t u_1(t)\|^2_{L^2(\Omega)} dt + \int_0^\infty \|\partial_t u_2(t)\|^2_{L^2(\Omega)} dt \le C_0, \tag{9.2.6}$$

$$\sup_{t>0}\left\{\|u_1(t)\|^2_{H^1(\Omega)} + \|u_2(t)\|^2_{H^1(\Omega)}\right\} \le C_0, \tag{9.2.7}$$

where C_0 is a positive constant depending on $\|u_{10}\|_{H^1(\Omega)}$, $\|u_{20}\|_{H^1(\Omega)}$ and $\|u_{20}\|_{L^\gamma(\partial\Omega)}$. Hence since $u_i \in L^\infty(0,\infty;L^\infty(\Omega))$ $(i = 1,2)$, from equation (NR), we derive

$$\int_n^{n+1}\left\{\|\partial_t u_1(t)\|^2_{L^2(\Omega)} + \|\partial_t u_2(t)\|^2_{L^2(\Omega)}\right\} dt \to 0 \qquad \text{as} \qquad n \to \infty, \tag{9.2.8}$$

$$\sup_n \int_n^{n+1}\left\{\|\Delta u_1(t)\|^2_{L^2(\Omega)} + \|\Delta u_2(t)\|^2_{L^2(\Omega)}\right\} dt \le C_0. \tag{9.2.9}$$

Furthermore, since $\|u_2(t)\|_{L^\infty(\partial\Omega)} \le \|u_2(t)\|_{L^\infty(\Omega)}$ (see [27]), we obtain

$$\sup_{t>0}\|u_2(t)\|_{L^\infty(\partial\Omega)} \le \|\overline{u}_2\|_{L^\infty(\Omega)}. \tag{9.2.10}$$

Here we put

$$u_i^n(x,t) = u_i(x,n+t) \in \mathscr{H} := L^2(0,1;L^2(\Omega)) \quad t \in (0,1) \qquad (i = 1,2). \tag{9.2.11}$$

Then $u_i^n(t)$ satisfy

$$\begin{cases} \partial_t u_1^n(t) - \Delta u_1^n(t) = u_1^n(t)u_2^n(t) - bu_1^n(t), & t \in (0,1),\ x \in \Omega, \\ \partial_t u_2^n(t) - \Delta u_2^n(t) = au_1^n(t), & t \in (0,1),\ x \in \Omega, \\ \partial_\nu u_1^n(t) + \alpha u_1^n(t) = \partial_\nu u_2^n(t) + \beta|u_2^n(t)|^{\gamma-2}u_2^n(t) = 0, & t \in (0,1),\ x \in \partial\Omega. \end{cases} \tag{9.2.12}$$

Then, by virtue of (9.2.5), (9.2.7), (9.2.8), (9.2.9) and (9.2.10), there exists a subsequence of $\{u_i^n(t)\}$ denoted again by $\{u_i^n(t)\}$ such that

(9.2.13) $\partial_t u_i^n(t) \to 0$ *strongly* in $\mathscr{H}$ as $n \to \infty$,

(9.2.14) $u_i^n(t) \to \tilde{u}_i(t) \equiv \tilde{u}_i$ *strongly* in $\mathscr{H}$ as $n \to \infty$,

(9.2.15) $u_1^n(t)u_2^n(t) \to \tilde{u}_1(t)\tilde{u}_2(t) \equiv \tilde{u}_1\tilde{u}_2$ *strongly* in $\mathscr{H}$ as $n \to \infty$,

(9.2.16) $\Delta u_i^n(t) \rightharpoonup \Delta\tilde{u}_i(t) \equiv \Delta\tilde{u}_i$ *weakly* in $\mathscr{H}$ as $n \to \infty$,

(9.2.17) $u_i^n(t) \to \tilde{u}_i(t) \equiv \tilde{u}_i$ *strongly* in $L^2(0,1;L^2(\partial\Omega))$ as $n \to \infty$,

(9.2.18) $|u_2^n(t)|^{\gamma-2}u_2^n(t) \rightharpoonup |\tilde{u}_2|^{\gamma-2}\tilde{u}_2$ *weakly* in $L^2(0,1;L^2(\partial\Omega))$ as $n \to \infty$,

(9.2.19) $\partial_\nu u_i^n(t) \rightharpoonup \partial_\nu \tilde{u}_i$ *weakly* in $L^2(0,1;L^2(\partial\Omega))$ as $n \to \infty$.

Thus $\tilde{u}_1$ and $\tilde{u}_2$ satisfy

$$\begin{cases} -\Delta\tilde{u}_1 = \tilde{u}_1\tilde{u}_2 - b\tilde{u}_1, & x \in \Omega, \\ -\Delta\tilde{u}_2 = a\tilde{u}_1, & x \in \Omega, \\ \partial_\nu\tilde{u}_1 + \alpha\tilde{u}_1 = \partial_\nu\tilde{u}_2 + \beta|\tilde{u}_2|^{\gamma-2}\tilde{u}_2 = 0, & x \in \partial\Omega. \end{cases}$$

(2) Let $\gamma = 2$ and $\alpha \le 2\beta$. By the comparison theorem, we can assume without loss of generality that $u_{10}(x) = l_1\overline{u}_1(x)$, $u_{20}(x) = l_2\overline{u}_2(x)$ for some $l_1 > l_2 > 1$. Suppose that the solution (u_1, u_2) of (NR) exists globally, i.e.,

$$\sup_{t\in[0,T]} \|u_i(t,\cdot)\|_{L^\infty(\Omega)} < \infty, \qquad (i = 1,2) \quad \forall\, T > 0. \tag{9.2.20}$$

Now we are going to construct a subsolution. For this purpose, we first note that there exists a sufficiently small number $\varepsilon > 0$ such that

$$\begin{cases} a(l_2 - l_1)\overline{u}_1 + \varepsilon l_2\overline{u}_2 < 0 & \text{on } \overline{\Omega}, \\ \varepsilon + (1 - l_2)\overline{u}_2 < 0 & \text{on } \overline{\Omega}. \end{cases} \tag{9.2.21}$$

Here we used the fact that $\overline{u}_1(x) > 0$, $\overline{u}_2(x) > 0$ on $\overline{\Omega}$, which is assured by Hopf's type maximum principle. Let $u_1^*(t,x) = l_1 e^{\varepsilon t}\overline{u}_1(x)$ and $u_2^*(t,x) = l_2 e^{\varepsilon t}\overline{u}_2(x)$. Then using (9.2.21), we get

$$\begin{aligned} \partial_t u_1^* - \Delta u_1^* - u_1^*u_2^* + bu_1^* &= \varepsilon l_1 e^{\varepsilon t}\overline{u}_1 - l_1 e^{\varepsilon t}\Delta\overline{u}_1 - l_1 e^{\varepsilon t}\overline{u}_1 l_2 e^{\varepsilon t}\overline{u}_2 + bl_1 e^{\varepsilon t}\overline{u}_1 \\ &= \varepsilon l_1 e^{\varepsilon t}\overline{u}_1 + l_1 e^{\varepsilon t}\left(\overline{u}_1\overline{u}_2 - b\overline{u}_1\right) - l_1 e^{\varepsilon t}\overline{u}_1 l_2 e^{\varepsilon t}\overline{u}_2 + bl_1 e^{\varepsilon t}\overline{u}_1 \\ &\le \varepsilon l_1 e^{\varepsilon t}\overline{u}_1 + l_1 e^{\varepsilon t}\overline{u}_1\overline{u}_2 - l_1 l_2 e^{\varepsilon t}\overline{u}_1\overline{u}_2 \\ &= \left\{\varepsilon + (1 - l_2)\,\overline{u}_2\right\} l_1 e^{\varepsilon t}\overline{u}_1 < 0, \\ \partial_t u_2^* - \Delta u_2^* - au_1^* &= \varepsilon l_2 e^{\varepsilon t}\overline{u}_2 - l_2 e^{\varepsilon t}\Delta\overline{u}_2 - al_1 e^{\varepsilon t}\overline{u}_1 \\ &= \varepsilon l_2 e^{\varepsilon t}\overline{u}_2 + l_2 e^{\varepsilon t}a\overline{u}_1 - al_1 e^{\varepsilon t}\overline{u}_1 \\ &= \left\{\varepsilon l_2\overline{u}_2 + a\,(l_2 - l_1)\,\overline{u}_1\right\} e^{\varepsilon t} < 0, \end{aligned}$$

where we used the fact that $(\overline{u}_1, \overline{u}_2)$ satisfies

$$\begin{cases} -\Delta\overline{u}_1 = \overline{u}_1\overline{u}_2 - b\overline{u}_1, \\ -\Delta\overline{u}_2 = a\overline{u}_1. \end{cases}$$

Moreover $\partial_\nu u_1^* + \alpha u_1^* = 0$, $\partial_\nu u_2^* + \beta u_2^* = 0$ on $\partial\Omega$ and $u_1^*(0,x) = l_1\overline{u}_1(x)$, $u_2^*(0,x) = l_2\overline{u}_2(x)$. Hence by the comparison principle, we have

$$l_1 e^{\varepsilon t}\overline{u}_1(x) = u_1^*(t,x) \le u_1(t,x), \quad l_2 e^{\varepsilon t}\overline{u}_2(x) = u_2^*(t,x) \le u_2(t,x). \tag{9.2.22}$$

Multiplication of equations of (NR) by φ_1 and integration by parts yield

$$\frac{d}{dt}\left(\int_\Omega u_1\varphi_1 dx\right) + (b+\lambda_1)\int_\Omega u_1\varphi_1 dx = \int_\Omega u_1 u_2 \varphi_1 dx, \tag{9.2.23}$$

$$\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) + \lambda_1\int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{\partial\Omega} u_2\varphi_1 d\sigma = a\int_\Omega u_1\varphi_1 dx, \tag{9.2.24}$$

where λ_1 and φ_1 are the first eigenvalue and the corresponding eigenfunction for (8.1.1). We here normalize φ_1 so that $\|\varphi_1\|_{L^1(\Omega)} = 1$. Substituting (9.2.24) and $u_1 = \frac{1}{a}(\partial_t u_2 - \Delta u_2)$ in (9.2.23) and using integration by parts, we get

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) + \lambda_1\int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{\partial\Omega} u_2\varphi_1 d\sigma\right\} \\ &\quad + (b+\lambda_1)\left\{\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) + \lambda_1\int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{\partial\Omega} u_2\varphi_1 d\sigma\right\} \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega u_2^2\varphi_1 dx + \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx + \left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 d\sigma, \end{aligned} \tag{9.2.25}$$

where we used the fact that

$$\begin{aligned} -\int_\Omega (\Delta u_2)u_2\varphi_1 dx &= \int_\Omega \nabla u_2\cdot\nabla(u_2\varphi_1)dx - \int_{\partial\Omega}(\partial_\nu u_2)u_2\varphi_1 d\sigma \\ &= \int_\Omega |\nabla u_2|^2\varphi_1 dx + \int_\Omega u_2\nabla u_2\cdot\nabla\varphi_1 dx + \beta\int_{\partial\Omega} u_2^2\varphi_1 d\sigma \\ &= \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{1}{2}\int_\Omega \nabla u_2^2\cdot\nabla\varphi_1 dx + \beta\int_{\partial\Omega} u_2^2\varphi_1 d\sigma \\ &= \int_\Omega |\nabla u_2|^2\varphi_1 dx - \frac{1}{2}\int_\Omega u_2^2\Delta\varphi_1 dx - \frac{\alpha}{2}\int_{\partial\Omega} u_2^2\varphi_1 d\sigma + \beta\int_{\partial\Omega} u_2^2\varphi_1 d\sigma \\ &= \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx + \left(\beta-\frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 d\sigma. \end{aligned}$$

We here assume $\beta - \alpha > 0$. From (9.2.22), it follows that

$$\begin{aligned}
&\frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx - (b+\lambda_1)\lambda_1\int_\Omega u_2\varphi_1 dx \\
=&\frac{\lambda_1}{4}\int_\Omega u_2^2\varphi_1 dx + \lambda_1\int_\Omega\left\{\frac{1}{4}u_2 - (b+\lambda_1)\right\}u_2\varphi_1 dx \\
\geq&\frac{\lambda_1}{4}\int_\Omega u_2^2\varphi_1 dx + \lambda_1\int_\Omega\left\{\frac{1}{4}u_2^* - (b+\lambda_1)\right\}u_2\varphi_1 dx \\
\geq&\frac{\lambda_1}{4}\int_\Omega u_2^2\varphi_1 dx + \lambda_1\int_\Omega\left\{\frac{1}{4}me^{\varepsilon t} - (b+\lambda_1)\right\}u_2\varphi_1 dx,
\end{aligned}$$

where $m := \min_{x\in\overline{\Omega}} l_2\overline{u}_2(x) > 0$. Hence there exists $t_1 > 0$ such that

$$\frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx - (b+\lambda_1)\lambda_1\int_\Omega u_2\varphi_1 dx \geq \frac{\lambda_1}{4}\int_\Omega u_2^2\varphi_1 dx \qquad \forall\, t \geq t_1. \tag{9.2.26}$$

Similarly, since

$$\begin{aligned}
&\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 d\sigma - (b+\lambda_1)(\beta-\alpha)\int_{\partial\Omega} u_2\varphi_1 d\sigma \\
=&\frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 d\sigma + \int_{\partial\Omega}\left\{\frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)u_2 - (b+\lambda_1)(\beta-\alpha)\right\}u_2\varphi_1 d\sigma \\
\geq&\frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 d\sigma + \int_{\partial\Omega}\left\{\frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)me^{\varepsilon t} - (b+\lambda_1)(\beta-\alpha)\right\}u_2\varphi_1 d\sigma,
\end{aligned}$$

there exists $t_2 > 0$ such that

$$\begin{aligned}
&\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 d\sigma - (b+\lambda_1)(\beta-\alpha)\int_{\partial\Omega} u_2\varphi_1 d\sigma \\
&\qquad \geq \frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 d\sigma \qquad \forall\, t \geq t_2.
\end{aligned} \tag{9.2.27}$$

Therefore by (9.2.26), (9.2.27) and (9.2.25), we have

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right)\right\} + (b+2\lambda_1)\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) + (\beta-\alpha)\frac{d}{dt}\left(\int_{\partial\Omega} u_2\varphi_1 d\sigma\right) \\
\geq&\frac{1}{2}\frac{d}{dt}\left(\int_\Omega u_2^2\varphi_1 dx\right) + \frac{\lambda_1}{4}\int_\Omega u_2^2\varphi_1 dx + \frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 d\sigma \qquad \forall\, t \geq t_3,
\end{aligned} \tag{9.2.28}$$

where $t_3 := t_1 \vee t_2$. Now we integrate (9.2.28) with respect to t over $[t_3, t]$ to get

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 d\sigma d\tau\right\} \\
\geq&\frac{1}{2}\int_\Omega u_2^2\varphi_1 dx - (b+2\lambda_1)\int_\Omega u_2\varphi_1 dx - \frac{1}{2}\int_\Omega u_2^2(t_3)\varphi_1 dx \\
&+\frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)\int_{t_3}^t\int_{\partial\Omega} u_2^2\varphi_1 d\sigma d\tau + \int_\Omega \partial_t u_2(t_3)\varphi_1 dx,
\end{aligned} \tag{9.2.29}$$

where we neglected positive terms. Moreover we can see that there exists $t_4 > t_3$ such that

$$\begin{aligned}\frac{1}{2}\int_\Omega u_2^2\varphi_1 dx-&(b+2\lambda_1)\int_\Omega u_2\varphi_1 dx\\ &-\frac{1}{2}\int_\Omega u_2^2(t_3)\varphi_1 dx+\int_\Omega \partial_t u_2(t_3)\varphi_1 dx \geq \frac{1}{4}\int_\Omega u_2^2\varphi_1 dx \end{aligned} \tag{9.2.30}$$

for $t \geq t_4$ by the same argument as before. Therefore from (9.2.29) and (9.2.30), we have

$$\begin{aligned}\frac{d}{dt}&\left\{\int_\Omega u_2\varphi_1 dx+(\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 d\sigma d\tau\right\}\\ &\geq \frac{1}{4}\int_\Omega u_2^2\varphi_1 dx+\frac{1}{2}\left(\beta-\frac{\alpha}{2}\right)\int_{t_3}^t\int_{\partial\Omega} u_2^2\varphi_1 d\sigma d\tau.\end{aligned} \tag{9.2.31}$$

Since $\|\varphi_1\|_{L^1(\Omega)} = 1$, by Schwarz's inequality, we get

$$\frac{1}{4}\int_\Omega u_2^2\varphi_1 dx \geq \frac{1}{4}\left(\int_\Omega u_2\varphi_1 dx\right)^2,$$

and

$$\begin{aligned}&\frac{1}{2}\left(\beta-\frac{\alpha}{2}\right)\int_{t_3}^t\int_{\partial\Omega} u_2^2\varphi_1 d\sigma d\tau\\ \geq& \frac{1}{2}(\beta-\frac{\alpha}{2})\frac{1}{\|\varphi_1\|_{L^\infty(\Omega)}|\partial\Omega|}\frac{1}{t-t_3}\left\{\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 d\sigma d\tau\right\}^2\\ =& \frac{1}{2}\frac{\beta-\frac{\alpha}{2}}{\|\varphi_1\|_{L^\infty(\Omega)}|\partial\Omega|(\beta-\alpha)^2}\frac{1}{t-t_3}\left\{(\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 d\sigma d\tau\right\}^2.\end{aligned}$$

By the above inequalities and (9.2.31), for $t \geq t_5 := t_4 \vee (t_3+1)$, we finally get

$$\begin{aligned}&\frac{d}{dt}\left\{\int_\Omega u_2\varphi_1 dx+(\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 d\sigma d\tau\right\}\\ \geq&\frac{1}{4}\int_\Omega u_2^2\varphi_1 dx+\frac{1}{2}\left(\beta-\frac{\alpha}{2}\right)\int_{t_3}^t\int_{\partial\Omega} u_2^2\varphi_1 d\sigma d\tau\\ \geq&\frac{1}{4}\left(\int_\Omega u_2\varphi_1 dx\right)^2+\frac{1}{2}\frac{\beta-\frac{\alpha}{2}}{\|\varphi_1\|_{L^\infty(\Omega)}|\partial\Omega|(\beta-\alpha)^2}\frac{1}{t-t_3}\left\{(\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 d\sigma d\tau\right\}^2\\ \geq&C\frac{1}{t-t_3}\left\{\left(\int_\Omega u_2\varphi_1 dx\right)^2+\left((\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 d\sigma d\tau\right)^2\right\}\\ \geq&C\frac{1}{t-t_3}\left\{\int_\Omega u_2\varphi_1 dx+(\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 d\sigma d\tau\right\}^2,\end{aligned}$$

where C denotes some general positive constant independent of t. Set $y(t) := \int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{t_3}^t \int_{\partial\Omega} u_2\varphi_1 d\sigma d\tau$, then the above inequality yields the following:

$$\begin{cases} \dfrac{d}{dt}y(t) \geq \dfrac{C}{t-t_3}y^2(t) \quad t \geq t_5, \\ y(t_5) > 0. \end{cases}$$

We can see that there exists $T^* > t_5$ such that

$$\lim_{t\to T^*} y(t) = +\infty. \tag{9.2.32}$$

In order to show the existence of T^* satisfying (9.2.32), it suffices to consider the following ordinary differential equation:

$$\begin{cases} \dfrac{d}{dt}\tilde{y}(t) = \dfrac{C}{t-t_3}\tilde{y}^2(t) \quad t \geq t_5, \\ \tilde{y}(t_5) > 0. \end{cases}$$

Since $\frac{d}{dt}\tilde{y}(t) > 0$ for all $t \geq t_5$ and $\tilde{y}(t_5) > 0$, it is clear that $\tilde{y}(t) > 0$ for all $t \geq t_5$. Divide both sides by $\tilde{y}^2(t)$ and integrate with respect to t on $[t_5, t]$, then we have

$$\frac{1}{\tilde{y}^2(t)}\frac{d}{dt}\tilde{y}(t) = \frac{C}{t-t_3},$$

$$\int_{\tilde{y}(t_5)}^{\tilde{y}(t)} \frac{1}{y^2}dy = C\log\frac{t-t_3}{t_5-t_3},$$

$$-\frac{1}{\tilde{y}(t)} + \frac{1}{\tilde{y}(t_5)} = C\log\frac{t-t_3}{t_5-t_3}.$$

Therefore we have

$$\tilde{y}(t) = \frac{1}{\frac{1}{\tilde{y}(t_5)} - C\log\frac{t-t_3}{t_5-t_3}}.$$

Hence there exists $\tilde{T} > t_5$ satisfying

$$\frac{1}{\tilde{y}(t_5)} - C\log\frac{\tilde{T}-t_3}{t_5-t_3} = 0$$

such that

$$\lim_{t\to\tilde{T}} \tilde{y}(t) = +\infty.$$

Thus (9.2.32) holds by comparison theorem for ordinary differential equations. This contradicts the assumption that (u_1, u_2) exists globally.

For the case of $\frac{\alpha}{2} \leq \beta \leq \alpha$, we can prove the same result with a slight modification. Actually, we get from (9.2.25)

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) + \lambda_1 \int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{\partial\Omega} u_2\varphi_1 d\sigma\right\}\\ &\quad + (b+\lambda_1)\left\{\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) + \lambda_1\int_\Omega u_2\varphi_1 dx\right\}\\ &\geq \frac{1}{2}\frac{d}{dt}\int_\Omega u_2^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx.\end{aligned}$$

Using (9.2.26) and integrating above inequality with respect to t over $[t_1, t]$, we have

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) - \int_\Omega \partial_t u_2(t_1)\varphi_1 dx + (\beta-\alpha)\int_{\partial\Omega} u_2\varphi_1 d\sigma - (\beta-\alpha)\int_{\partial\Omega} u_2(t_1)\varphi_1 d\sigma\\ \geq&\frac{1}{2}\int_\Omega u_2^2\varphi_1 dx - \frac{1}{2}\int_\Omega u_2^2(t_3)\varphi_1 dx - (b+2\lambda_1)\int_\Omega u_2\varphi_1 dx + (b+2\lambda_1)\int_\Omega u_2(t_1)\varphi_1 dx.\end{aligned}$$

Repeating the same arguments as for (9.2.26), we see that there exists $t_6 \geq t_1$ such that

$$\begin{aligned}&\frac{1}{2}\int_\Omega u_2^2\varphi_1 dx - \frac{1}{2}\int_\Omega u_2^2(t_3)\varphi_1 dx - (b+2\lambda_1)\int_\Omega u_2\varphi_1 dx\\ &\quad + \int_\Omega \partial_t u_2(t_1)\varphi_1 dx + (\beta-\alpha)\int_{\partial\Omega} u_2(t_1)\varphi_1 d\sigma\\ \geq&\ \frac{1}{4}\int_\Omega u_2^2\varphi_1 dx\end{aligned}$$

for all $t \geq t_6$. From these inequalities and Schwarz's inequality, it holds that

$$\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) \geq \frac{1}{4}\left(\int_\Omega u_2\varphi_1 dx\right)^2 \qquad \forall\ t \geq t_6.$$

Therefore we can get the following differential inequality:

$$\begin{cases}\dfrac{d}{dt}y(t) \geq y^2(t) & t \geq t_6,\\ y(t_6) > 0,\end{cases}$$

where $y(t) = \int_\Omega u_2\varphi_1 dx$. It is easy to see that there exists $T^{**} > t_6$ such that

$$\lim_{t\to T^{**}} y(t) = +\infty.$$

This leads to a contradiction. □

Remark 9.2.4. Since the blow-up result is proved by contradiction, there is no knowing if $\|u_1(t)\|_{L^\infty}$ and $\|u_2(t)\|_{L^\infty}$ blow up simultaneously. However we can show by another argument that L^∞-norms of u_1 and u_2 blow up at the same time, i.e., there exists $T > 0$ such that

$$\lim_{t\to T}\|u_1(t)\|_{L^\infty(\Omega)} = \infty \quad \text{and} \quad \lim_{t\to T}\|u_2(t)\|_{L^\infty(\Omega)} = \infty.$$

In fact, multiplying the first equation of (NR) by $|u_1|^{r-2}u_1$ and using integration by parts and similar calculation in the proof of Theorem 9.1.1, we obtain

$$\frac{d}{dt}\|u_1(t)\|_{L^r(\Omega)} \le \|u_2(t)\|_{L^\infty(\Omega)}\|u_1(t)\|_{L^r(\Omega)} \qquad \forall t \in [0,T). \tag{9.2.33}$$

From the second equation of (NR), we also have

$$\|u_2(t)\|_{L^\infty(\Omega)} \le \|u_{20}\|_{L^\infty(\Omega)} + a\int_0^t \|u_1(\tau)\|_{L^\infty(\Omega)}d\tau \qquad \forall t \in [0,T). \tag{9.2.34}$$

Suppose that

$$\lim_{t\to T}\|u_1(t)\|_{L^\infty(\Omega)} = \infty \qquad \text{and} \qquad M_2 := \sup_{0\le t\le T}\|u_2(t)\|_{L^\infty(\Omega)} < \infty,$$

then it follows from (9.2.33)

$$\frac{d}{dt}\|u_1(t)\|_{L^r} \le M_2\|u_1(t)\|_{L^r(\Omega)} \qquad \forall t \in [0,T).$$

By Gronwall's inequality, we get

$$\|u_1(t)\|_{L^r(\Omega)} \le \|u_{10}\|_{L^r(\Omega)}e^{M_2 t} \le \|u_{10}\|_{L^r(\Omega)}e^{M_2 T} \qquad \forall t \in [0,T).$$

Letting r tend to ∞, we obtain

$$\|u_1(t)\|_{L^\infty(\Omega)} \le \|u_{10}\|_{L^\infty(\Omega)}e^{M_2 T} \qquad \forall t \in [0,T),$$

which contradicts the fact $\lim_{t\to T}\|u_1(t)\|_{L^\infty(\Omega)} = \infty$. Next, suppose that

$$M_1 := \sup_{0\le t\le T}\|u_1(t)\|_{L^\infty(\Omega)} < \infty \qquad \text{and} \qquad \lim_{t\to T}\|u_2(t)\|_{L^\infty(\Omega)} = \infty,$$

then by (9.2.34) we see that

$$\|u_2(t)\|_{L^\infty(\Omega)} \le \|u_{20}\|_{L^\infty(\Omega)} + aM_1 T \qquad \forall t \in [0,T).$$

Letting t tend to T, we get contradiction. Thus we see that u_1 and u_2 blow up at the same time.

9.3 Blowing-up Solutions of (NR)

In this section, we exemplify the applicability of Theorem 4.1.3 in Part I for systems of parabolic equations. We consider the following reaction diffusion system, which consists of two equations possessing a nonlinear coupling term between two real-valued unknown functions, which is a generalized system of (NR).

$$\text{(NR)}^* \qquad \begin{cases} \partial_t u_1 - \Delta u_1 = u_1u_2 - bu_1, & t>0,\ x\in\Omega, \\ \partial_t u_2 - \Delta u_2 = au_1, & t>0,\ x\in\Omega, \\ \partial_\nu u_1 + \alpha_1|u_1|^{\gamma_1-2}u_1 = \partial_\nu u_2 + \alpha_2|u_2|^{\gamma_2-2}u_2 = 0, & t>0,\ x\in\partial\Omega, \\ u_1(0,x) = u_{10}(x) \ge 0,\ u_2(0,x) = u_{20}(x) \ge 0, & x\in\Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. ν denotes the unit outward normal vector on $\partial\Omega$ and ∂_ν is outward normal derivative, i.e., $\partial_\nu u_i = \nabla u_i \cdot \nu$ $(i = 1, 2)$. Moreover u_1, u_2 are real-valued unknown functions, a and b are given positive constants. As for the parameters appearing in the boundary condition, we assume $\alpha_i \in [0, \infty)$, $\gamma_i \in (1, \infty)$ $(i = 1, 2)$. We note that the boundary condition for u_i becomes the homogeneous Neumann boundary condition when $\alpha_i = 0$, and the Robin boundary condition when $\alpha_i > 0$ and $\gamma_i = 2$. We further assume that the given initial data u_{10}, u_{20} are nonnegative and belong to $L^\infty(\Omega)$. In the former section, we dealt with the case where $\alpha_1 = \alpha \in [0, \infty)$, $\alpha_2 = \beta \in (0, \infty)$, $\gamma_1 = 2$ and $\gamma_2 = \gamma \in [2, \infty)$. As for the case where $\gamma_i \neq 2$, note that it is not clear whether there are solutions which blow up in finite time.

Nevertheless it is possible to show that (NR)* with $\gamma_i \neq 2$ admits blow-up solutions by applying the same strategy as that in Chapter 4. Along the same lines as before, we first consider the following Dirichlet problem for (NR).

$$
(\mathrm{NR})^D \qquad \begin{cases} \partial_t u_1 - \Delta u_1 = u_1 u_2 - b u_1, & t > 0,\ x \in \Omega, \\ \partial_t u_2 - \Delta u_2 = a u_1, & t > 0,\ x \in \Omega, \\ u_1 = u_2 = 0, & t > 0,\ x \in \partial\Omega, \\ u_1(0, x) = u_{10}(x) \geq 0,\ u_2(0, x) = u_{20}(x) \geq 0, & x \in \Omega. \end{cases}
$$

We first note that for every $U_0 := (u_{10}, u_{20}) \in \mathbb{L}^\infty_+(\Omega) := \{\, (u_1, u_2); u_i \geq 0, u_i \in L^\infty(\Omega)\ (i = 1, 2)\,\}$, (NR) or (NR)^D possess a unique solution $U(t) := (u_1(t), u_2(t)) \in \mathbb{L}^\infty_+(\Omega)$ satisfying the blow-up alternative with respect to L^∞-norm such as in Proposition 9.1.1. We are going to show this result for a more general equation:

$$
(\mathrm{NR})^\beta \qquad \begin{cases} \partial_t u_1 - \Delta u_1 = u_1 u_2 - b u_1, & t > 0,\ x \in \Omega, \\ \partial_t u_2 - \Delta u_2 = a u_1, & t > 0,\ x \in \Omega, \\ \partial_\nu u_1 + \beta_1(u_1) = \partial_\nu u_2 + \beta_2(u_2) = 0, & t > 0,\ x \in \partial\Omega, \\ u_1(0, x) = u_{10}(x) \geq 0,\ u_2(0, x) = u_{20}(x) \geq 0, & x \in \Omega, \end{cases}
$$

where $\beta_i : \mathbb{R}^1 \to 2^{\mathbb{R}^1}$ are maximal monotone operators $(i = 1, 2)$. To do this, we can repeat much the same arguments as those in the proof of Proposition 2.2.3.

Let $H := L^2(\Omega) \times L^2(\Omega)$ with inner product $(U, V)_H := (u_1, v_1)_{L^2} + (u_2, v_2)_{L^2}$ for $U = (u_1, u_2)$, $V = (v_1, v_2)$, and put $|\nabla U|^2 = |\nabla u_1|^2 + |\nabla u_2|^2$. Let $j_i : \mathbb{R}^1 \to (-\infty, +\infty]$ be lower semi-continuous convex functions such that $\partial j_i = \beta_i$ $(i = 1, 2)$. For the Dirichlet (resp. Neumann) boundary condition, we put $j_i(0) = 0$ and $j_i(r) = +\infty$ for $r \neq 0$ (resp. $j_i(r) = 0,\ \forall r \in \mathbb{R}^1$).

Then we define

$$
\varphi(U) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega (|\nabla U(x)|^2 + |U(x)|^2) dx + \sum_{i=1}^2 \int_{\partial\Omega} j_i(u_i(x)) d\sigma & U \in D(\varphi), \\ +\infty & U \in H \backslash D(\varphi), \end{cases}
$$

where $D(\varphi) := \{U; u_i \in H^1(\Omega)\ j_i(u_i) \in L^1(\Omega)\ (i = 1, 2)\}$. For the homogeneous Dirichlet (resp. Neumann) boundary condition case, we take $D(\varphi) = H^1_0(\Omega) \times H^1_0(\Omega)$ (resp.

$H^1(\Omega) \times H^1(\Omega))$. Then we have

$$\begin{cases} \partial\varphi(U) = (-\Delta u_1 + u_1, -\Delta u_2 + u_2), \\ D(\partial\varphi) = \{U = (u_1, u_2)\,;\, u_i \in H^2(\Omega),\ -\partial_\nu u_i(x) \in \beta_i(u_i(x))\ (i = 1, 2) \quad \text{a.e. on } \partial\Omega\}. \end{cases}$$

Furthermore the elliptic estimate (2.2.10) with u replaced by u_i $(i = 1, 2)$ holds true for all $U \in D(\partial\varphi)$.

Then by putting $B(U) := (-u_1\, u_2 + (b-1)\, u_1, -u_2 - a\, u_1)$, $(\mathrm{NR})^\gamma$ can be reduced to the following abstract evolution equation in H.

$$(\mathrm{CP})^\beta \qquad \begin{cases} \dfrac{d}{dt}U(t) + \partial\varphi(U(t)) + B(U(t)) \ni 0, \quad t > 0, \\ U(0) = U_0 = (u_{10}, u_{20}). \end{cases}$$

In order to apply "L^∞-Energy Method", we again introduce the following cut-off functions $I_{K_{i,M}}(\cdot)$ $(i = 1, 2)$:

$$I_{K_{i,M}}(U) := \begin{cases} 0, & U \in K_{i,M} := \{\, U = (u_1, u_2) \in H\,;\, |u_i(x)| \le M \quad a.e.\ x \in \Omega\,\}, \\ +\infty, & U \in H \setminus K_{i,M}, \end{cases}$$

and put

$$\varphi_M(U) := \varphi(U) + I_{K_{1,M}}(U) + I_{K_{2,M}}(U).$$

Then we get

$$\partial\varphi(U) = \partial\varphi(U) + \partial I_{1,M}(U) + \partial I_{2,M}(U) \qquad \forall U \in D(\partial\varphi) \cap K_{1,M} \cap K_{2,M}.$$

Consider the following auxiliary equation:

$$(\mathrm{CP})^\beta_M \qquad \begin{cases} \dfrac{d}{dt}U(t) + \partial\varphi_M(U(t)) + B(U(t)) \ni 0, \quad t > 0, \\ U(0) = U_0, \end{cases}$$

where we choose $M > 0$ such that

$$M = \|U_0\|_{L^\infty} + 2 := \|u_{10}\|_{L^\infty} + \|u_{20}\|_{L^\infty} + 2.$$

Then as in the proof of Proposition 2.2.3, we can easily show that $(\mathrm{CP})^\gamma_M$, which is equivalent to the following $(\mathrm{NR})^\gamma_M$, admits a unique global solution $U(t) = (u_1(t), u_2(t))$.

$$(\mathrm{NR})^\beta_M \qquad \begin{cases} \partial_t u_1 - \Delta u_1 + \beta_M(u_1) = u_1 u_2 - b u_1, & t > 0,\ x \in \Omega, \\ \partial_t u_2 - \Delta u_2 + \beta_M(u_2) = a u_1, & t > 0,\ x \in \Omega, \\ \partial_\nu u_1 + \beta_1(u_1) = \partial_\nu u_2 + \beta_2(u_2) = 0, & t > 0,\ x \in \partial\Omega, \\ u_1(0, x) = u_{10}(x) \ge 0,\ u_2(0, x) = u_{20}(x) \ge 0, & x \in \Omega. \end{cases}$$

Then in parallel with (2.2.18), multiplying the first and second equations of $(\mathrm{NR})^\gamma_M$ by $|u_1|^{r-2}u_1$ and $|u_2|^{r-2}u_2$, we can obtain

$$\|U(t)\|_{L^\infty} \le \|U_0\|_{L^\infty} + \int_0^t \ell(\|U(s)\|_{L^\infty})ds \quad \text{with } \ell(r) = ar + r^2,$$

where $\|U\|_{L^\infty} = \|(u_1, u_2)\|_{L^\infty} := \|u_1\|_{L^\infty} + \|u_2\|_{L^\infty}$. Then we can repeat the same arguments as those in the proof of Proposition 2.2.3. Furthermore multiplying the first and second equations of $(\mathrm{NR})^D$ by $u_1^- := \max(-u_1, 0)$ and $u_2^- := \max(-u_2, 0)$, we can easily deduce

$$\frac{1}{2}\frac{d}{dt}(\|u_1^-(t)\|_{L^2}^2 + \|u_2^-(t)\|_{L^2}^2) \le \|u_2\|_{L^\infty}\|u_1^-(t)\|_{L^2}^2 + a\,\|u_1^-(t)\|_{L^2}\|u_2^-(t)\|_{L^2}$$
$$\le (\|u_2\|_{L^\infty} + a)\,(\|u_1^-(t)\|_{L^2}^2 + \|u_2^-(t)\|_{L^2}^2).$$

Then by Gronwall's inequality, we get $u_1^-(t) = u_2^-(t) = 0$ for all t, i.e., (u_1, u_2) is a non-negative solution (see [33]). (The non-negativity of solutions can be also derived from application of Theorem 4.1.3 for $(\mathrm{NR})^\beta$ with the coupling term $u_1\,u_2$ replaced by $u_1^+\,u_2$.)

Here we prepare the following lemma concerning the existence of blow-up solutions of $(\mathrm{NR})^D$.

Proposition 9.3.1. *Assume that (u_{10}, u_{20}) belongs to $\mathbb{L}_+^\infty(\Omega)$ and satisfies*

$$\text{(9.3.1)}\quad \int_\Omega (a\,u_{10}(x) + b\,u_{20}(x) - \frac{1}{2}u_{20}^2(x))\,\phi_1(x)\,dx \ge 0, \qquad \int_\Omega u_{20}(x)\,\phi_1(x)\,dx > 2(b + \lambda_1).$$

Then the solution $U(t) = (u_1(t), u_2(t))$ of $(\mathrm{NR})^D$ blows up in finite time. Here λ_1 and ϕ_1 are the first eigenvalue and its associate normalized positive eigenfunction of (4.2.1).

Proof. Suppose that $U(t)$ is a global solution. Then multiplying the first and second equations of $(\mathrm{NR})^D$ by ϕ_1, we obtain

$$\text{(9.3.2)}\qquad \frac{d}{dt}\left(\int_\Omega u_1\varphi_1 dx\right) + (b + \lambda_1)\left(\int_\Omega u_1\phi_1 dx\right) = \int_\Omega u_1 u_2 \phi_1 dx,$$

$$\text{(9.3.3)}\qquad \frac{d}{dt}\left(\int_\Omega u_2\phi_1 dx\right) + \lambda_1 \int_\Omega u_2\phi_1 dx = a\int_\Omega u_1\phi_1 dx.$$

Following [53], we set

$$y(t) := \int_\Omega u_2(t)\phi_1 dx, \quad z(t) := y'(t) + (b + \lambda_1)y(t) - \frac{1}{2}\int_\Omega u_2^2(t)\phi_1 dx.$$

Then by (9.3.3) and (9.3.2), we get

$$y''(t) = -\lambda_1 y'(t) + a\int_\Omega u_1'(t)\phi_1 dx$$
$$\text{(9.3.4)}\qquad = -\lambda_1 y'(t) - (b + \lambda_1)\int_\Omega a u_1\phi_1 dx + \int_\Omega a u_1 u_2 \phi_1 dx.$$

We substitute $au_1 = \partial_t u_2 - \Delta u_2$ in (9.3.4), then by integration by parts we have

$$y''(t) + (b+2\lambda_1)y'(t) + \lambda_1(b+\lambda_1)y(t) = \frac{1}{2}\frac{d}{dt}\left(\int_\Omega u_2^2\phi_1 dx\right) + \int_\Omega |\nabla u_2|^2\phi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\phi_1 dx,$$

whence follows

$$z'(t) \ge -\lambda_1 z(t).$$

Therefore we get $z(t) \geq z(s)e^{-\lambda_1(t-s)}$ for $0 < s < t$. Here (9.3.3) and (9.3.1) yield

$$\begin{aligned} z(s) &= y'(s) + (b+\lambda_1)\, y(s) - \frac{1}{2}\int_\Omega u_2^2(s)\phi_1 dx \\ &= \int_\Omega (a\, u_1(s) + b\, u_2(s) - \frac{1}{2}u_2^2(s))\, \phi_1 dx \\ &\to \int_\Omega (a\, u_{10} + b\, u_{20} - \frac{1}{2}u_{20}^2)\, \phi_1 dx \geq 0 \quad \text{as } s \to 0, \end{aligned}$$

since $u_1(t), u_2(t) \in C([0,1]; L^2(\Omega)) \cap L^\infty(0,1; L^\infty(\Omega))$. Hence we see that $z(t) \geq 0$ for all $t > 0$, i.e., we have

$$\begin{aligned} y'(t) &\geq -(b+\lambda_1)\, y(t) + \frac{1}{2}\int_\Omega u_2^2(t)\phi_1 dx \\ &\geq -(b+\lambda_1)\, y(t) + \frac{1}{2}\, y^2(t) \\ &\geq \frac{1}{2}\, y(t)(y(t) - 2(b+\lambda_1)). \end{aligned} \tag{9.3.5}$$

Then (9.3.5) assures that $y(t)$ blows up in finite time if $y(0) > 2(b+\lambda_1)$. □

In order to make it clear that solutions of parabolic systems differ according to their boundary conditions imposed, we here denote the unique solutions of $(\mathrm{NR})^\beta$ and $(\mathrm{NR})^D$ by $U^\beta(t) = (u_1^\beta(t), u_2^\beta(t))$ and $U^D(t) = (u_1^D(t), u_2^D(t))$ with the same initial data $U_0 \in \mathbb{L}_+^\infty(\Omega)$, respectively.

We are going to compare $U^\beta(t)$ with $U^D(t)$ by applying Theorem 4.1.3. for $U_1 = U^D,\ U_2 = U^\beta$. Let

$$m = 2;\ a_{i,j}^1 = a_{i,j}^2 = \delta_{i,j};\quad a_1^1 = a_2^1 = u_{10},\ a_1^2 = a_2^2 = u_{20};\quad \gamma_1^1 = \gamma_2^1 = \gamma_1^2 = \gamma_2^2 = 0;$$

$$F_1^1(U) = F_2^1(U) = F^1(U) := u_1 u_2 - b u_1,\ F_2^1(U) = F_2^2(U) = F^2(U) := a u_1;$$

$$\beta_1^1(r) = \beta_1^2(r) = \beta^D(r), \quad \beta_2^i(r) = \begin{cases} \alpha_i\, |r|^{\gamma_i - 2} r & \text{for } r > 0, \\ (-\infty, 0] & \text{for } r = 0, \\ \emptyset & \text{for } r < 0, \end{cases} \qquad (i = 1, 2),$$

where γ^D is the maximal monotone graph defined by (4.2.3). Then (A1), (A2) and (i) of (A4) are obviously satisfied. Moreover as in the proof of Proposition 4.2.1, we can see that u_1^D and u_2^D obey the homogeneous Dirichlet boundary condition, and that $-\partial_\nu u_1^\beta \in \beta_2^1(u_1^\beta)$ and $-\partial_\nu u_2^\beta \in \beta_2^2(u_2^\beta)$ hold, since u_1^β and u_2^β are non-negative solutions. Therefore $D(\beta_1^1) = D(\beta_1^2) = D(\beta^D) = \{0\}$ and $D(\beta_2^1) = D(\beta_2^2) = [0, \infty)$ assure (iii) of (A3).

Hence to apply Theorem 4.1.3, it suffices to check (ii) of (A4), i.e., $F^1(U) = u_1 u_2 - b u_1,\ F^2(U) = a u_1$ satisfies (SC). Since $F^1, F^2 \in C^1(\mathbb{R}^2)$, (2.2.8) is obvious. As for (2.2.7), we get

$$\frac{\partial}{\partial u_1} F^2(U) = a > 0, \quad \frac{\partial}{\partial u_2} F^1(U) = u_1 \geq 0.$$

Consequently, applying Theorem 4.1.3, we conclude

$$T_m(U^\beta) \le T_m(U^D) \quad \text{and}$$
$$0 \le u_1^D(t,x) \le u_1^\beta(t,x), \quad 0 \le u_2^D(t,x) \le u_2^\beta(t,x) \quad \forall t \in [0, T_m(U^\beta)) \quad a.e.\ x \in \Omega.$$

Thus by virtue of Proposition 9.3.1, we have the following corollary.

Corollary 9.3.2. *Assume that* (u_{10}, u_{20}) *belongs to* $\mathbb{L}^\infty_+(\Omega)$ *and satisfies* (9.3.1). *Then the unique solution* $U(t) = (u_1(t), u_2(t))$ *of* (NR)* *blows up in finite time.*

Remark 9.3.3. *The existence of* (u_{10}, u_{20}) *satisfying* (9.3.1) *is assured when* $a > 0$. *For instance, if* $u_{10} \ge \frac{1}{2a} u_{20}^2$ *and* u_{20} *is sufficiently large, then* (9.3.1) *is satisfied.*

For the case where $a = 0$, *however, there is no initial data* (u_{10}, u_{20}) *satisfying* (9.3.1). *In fact,* $a = 0$ *implies that* $\sup_{t \ge 0} \|u_2(t)\|_{L^\infty} \le \|u_{20}\|_{L^\infty}$, *then* $u_1(t)$ *satisfies* $\partial_t u_1 - \Delta u_1(t) \le \|u_{20}\|_{L^\infty} u_1(t)$, *whence follows* $\|u_1(t)\|_{L^\infty} \le \|u_{10}\|_{L^\infty}\, e^{\|u_{20}\|_{L^\infty} t}$. *Consequently every local solution can be continued globally.*

Remark 9.3.4. *The assertion of Corollary 9.3.2 holds true for more general equation* (NR)$^\beta$, *provided that* $0 \in \beta_i(0)$ $(i = 1, 2)$ *is satisfied.*

Chapter 10

Bounds for Global Solutions of (NR)

We are concerned with a bound of global solutions to (NR) with Robin boundary conditions. In the previous chapter, we showed the global existence of solutions to (NR) for small data. More precisely, if the initial data is smaller than or equal to a positive stationary solution, then the corresponding solution exists globally and converges to the trivial solution. In this chapter, we are going to show every global solution of (NR) with Robin boundary conditions (γ=2) is bounded uniformly in time.

10.1 Existence of Local Solutions

Throughout this chapter, $\|\cdot\|$ denotes the norm in $H^1(\Omega)$. We also simply write $u(t)$ instead of $u(t,\cdot)$. In this section, we mention the local well-posedness. The local well-posedness of (NR) in $L^\infty(\Omega)$ is proved in Chapter 9 as Theorem 9.1.1. In order to treat the case where the data belong to $H^1(\Omega)$, we need to fix some abstract setting. Let $H := L^2(\Omega) \times L^2(\Omega)$ and for $u = (u_1, u_2) \in H$ we put

$$D(\phi) := \{ u \in H\,;\, u_1, u_2 \in H^1(\Omega),\ u_2 \in L^\gamma(\partial\Omega) \},$$

$$\phi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega (|\nabla u_1(x)|^2 + b|u_1(x)|^2 + |\nabla u_2(x)|^2)dx \\ \qquad\qquad + \displaystyle\int_{\partial\Omega} \left(\frac{\alpha}{2}|u_1(x)|^2 + \frac{\beta}{\gamma}|u_2(x)|^\gamma \right) d\sigma & \text{if } u \in D(\phi), \\ +\infty & \text{if } u \notin D(\phi). \end{cases}$$

Then ϕ is a lower semi-continuous convex function from H into $[0,\infty)$ and its subdifferential $\partial\phi$ is given by

$$\partial\phi(u) = \{ w \in H\,;\, w = (-\Delta u_1 + bu_1, -\Delta u_2) \} \quad \forall u \in D(\partial\phi),$$

$$D(\partial\phi) = \{ u = (u_1, u_2)\,;\, u_1, u_2 \in H^2(\Omega),\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta\,|u_2|^{\gamma-2} u_2 = 0 \}.$$

Then we have

Theorem 10.1.1. *Let $N \leq 5$ and $(u_{10}, u_{20}) \in D(\phi)$, then there exists $T = T(\phi(u_0)) > 0$ such that* (NR) *possesses a unique solution $(u_1, u_2) \in (C([0,T]; L^2(\Omega)))^2$ satisfying*

$$\partial_t u_1, \partial_t u_2, \Delta u_1, \Delta u_2 \in L^2(0,T; L^2(\Omega)). \tag{10.1.1}$$

Furthermore, if the initial data is nonnegative, then the local solution (u_1, u_2) for (NR) *is nonnegative.*

Proof. Put $u(t) = (u_1(t), u_2(t))$ and

$$B(u) := \{\, b \in H \,;\, b = (-u_1 u_2, -a u_1) \,\},$$

then (NR) can be reduced to the following abstract evolution equation in H:

$$\frac{d}{dt}u(t) + \partial\phi(u(t)) + B(u(t)) = 0, \quad u(0) = (u_{10}, u_{20}). \tag{10.1.2}$$

We are going to apply Proposition 1.4.4. To do this, we have to check three assumptions. The compactness assumption (A1) requires that the set $\{\, u \in H \,;\, \phi(u) + |u|_H^2 \leq L \,\}$ is compact in H for all $L > 0$, which is assured by the Rellich-Kondrachov theorem. The demiclosedness assumption (A2) on $B(u)$ is assured by the continuity of the mapping $(u_1, u_2) \mapsto (-u_1 u_2, -a u_1)$ in $\mathbb{R}^2$.

The last assumption to check is the boundedness assumption (A3):

$$|B(u)|_H^2 \leq k\,|\partial\phi(u)|_H^2 + \ell(\phi(u) + |u|_H) \quad \forall u \in D(\partial\phi), \tag{10.1.3}$$

where $k \in [0,1)$ and $\ell(\cdot) : [0,\infty) \to [0,\infty)$ is a monotone increasing function. We note that

$$|B(u)|_H^2 \leq \|u_1\|_4^2 \|u_2\|_4^2 + a^2\|u_1\|_2^2, \quad \exists C > 0 \text{ such that } C(\|u_1\|^2 + \|u_2\|^2) \leq \phi(u) + 1. \tag{10.1.4}$$

Hence for $N \leq 4$, (10.1.3) holds true with $k = 0$ and $\ell(r) = Cr^2$.

As for the case where $N = 5$, Gagliardo-Nirenberg interpolation inequality gives

$$\|v\|_4 \leq C\|v\|_{H^2}^{\frac{1}{4}} \|v\|^{\frac{3}{4}}.$$

Then by Young's inequality, (10.1.3) is satisfied with $\ell(r) = Cr^3$. Thus the local existence part is verified.

To prove the uniqueness part, let $u^1 = (u_1^1, u_2^1)$, $u^2 = (u_1^2, u_2^2)$ be solutions of (NR) and put $\delta u_i = u_i^1 - u_i^2$ $(i = 1,2)$. Then δu_i satisfy

$$\partial_t \delta u_1 - \Delta\delta u_1 + b\delta u_1 = \delta u_1 u_2^1 + \delta u_2 u_1^2, \tag{10.1.5}$$

$$\partial_t \delta u_2 - \Delta\delta u_2 = a\delta u_1, \tag{10.1.6}$$

$$\partial_\nu \delta u_1 + \alpha\delta u_1 = \partial_\nu \delta u_2 + \beta(|u_2^1|^{\gamma-2}u_2^1 - |u_2^2|^{\gamma-2}u_2^2) = 0. \tag{10.1.7}$$

Multiplying (10.1.5) by δu_1 and (10.1.6) by δu_2, we have by (10.1.7)

$$\frac{1}{2}\frac{d}{dt}\|\delta u_1(t)\|_2^2 + \|\nabla\delta u_1\|_2^2 + \alpha\|\delta u_1\|_{2,\partial\Omega}^2 + b\|\delta u_1\|_2^2 \leq \int_\Omega (|\delta u_1|^2\,|u_2^1| + |\delta u_1|\,|\delta u_2|\,|u_1^2|)\,dx, \tag{10.1.8}$$

$$\frac{1}{2}\frac{d}{dt}\|\delta u_2(t)\|_2^2 + \|\nabla\delta u_2\|_2^2 + \beta\int_{\partial\Omega}(|u_2^1|^{\gamma-2}u_2^1 - |u_2^2|^{\gamma-2}u_2^2)\,\delta u_2\,d\sigma \leq a\int_\Omega |\delta u_1|\,|\delta u_2|\,dx. \tag{10.1.9}$$

Let $N \le 5$, then since $H^1(\Omega)$ and $H^2(\Omega)$ are embedded in $L^{\frac{10}{3}}(\Omega)$ and $L^{10}(\Omega)$ respectively, by Young's inequality we find that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}\int_\Omega |\delta u_i|\,|\delta u_j|\,|w|\,dx &\le C\,\|\delta u_i\|\,\|\delta u_j\|_2\,\|w\|_{H^2(\Omega)}\\ &\le \varepsilon\,(\|\nabla \delta u_i\|_2^2 + \|\delta u_i\|_2^2) + C_\varepsilon \|\delta u_j\|_2^2 \|w\|_{H^2(\Omega)}^2.\end{aligned}$$

Hence, by adding (10.1.8) and (10.1.9), we obtain

$$\frac{d}{dt}(\|\delta u_1(t)\|_2^2 + \|\delta u_2(t)\|_2^2) \le C(\|u_2^1\|_{H^2(\Omega)}^2 + \|u_1^2\|_{H^2(\Omega)}^2 + 1)\,(\|\delta u_1(t)\|_2^2 + \|\delta u_2(t)\|_2^2),$$

Thus since $u_2^1, u_1^2 \in L^2(0,T;H^2(\Omega))$, the uniqueness follows from Gronwall's inequality. The nonnegativity of solutions can be proved by exactly the same argument as in the proof of Theorem 9.1.1 in Chapter 9. □

10.2 Main Result and Proof

In what follows we always consider the case where $\gamma = 2$ and we are concerned with global solutions of (II.4). We put $H^1 = \{(w_1, w_2) \in H^1(\Omega) \times H^1(\Omega)\,;\, w_1, w_2 \ge 0, w_1, w_2 \not\equiv 0\}$ and $V = \{(w_1, w_2) \in L^\infty(\Omega) \times L^\infty(\Omega)\,;\, w_1, w_2 \ge 0, w_1, w_2 \not\equiv 0\}$. Our main theorem can be stated as follows.

Theorem 10.2.1. *Let $N = 2, 3$ and $\alpha \le 2\beta$. Assume that $(u_{10}, u_{20}) \in H^1$ and (u_1, u_2) is the corresponding global solution of* (II.4) *satisfying the same regularity given in Theorem 10.1.1. Then there exist constants $M_i = M_i(\|u_{10}\|, \|u_{20}\|) > 0$ $(i = 1, 2)$ such that*

$$\sup_{t\ge 0} \|u_1(t)\| \le M_1, \quad \sup_{t\ge 0} \|u_2(t)\| \le M_2. \tag{10.2.1}$$

Moreover if $(u_{10}, u_{20}) \in V$ and (u_1, u_2) is the corresponding global solution of (II.4) *satisfying the same regularity given in Theorem* **??**. *Then there exist constants ${M'}_i = {M'}_i(\|u_{10}\|_\infty, \|u_{20}\|_\infty) > 0$ $(i = 1, 2)$ such that*

$$\sup_{t\ge 0} \|u_1(t)\|_\infty \le {M'}_1, \quad \sup_{t\ge 0} \|u_2(t)\|_\infty \le {M'}_2. \tag{10.2.2}$$

We divide the proof into several steps. We first derive the L^1-estimate of the solutions. In this step, we rely on the properties of the first eigenvalue and the corresponding eigenfunction of $-\Delta$ with the Robin boundary conditions (Lemma 7.1.2). The second step is to derive uniform L^2-estimates and third one is to derive uniform H^1-estimates. In the last step, we get uniform L^∞ bounds for global solutions of (II.4) applying Moser's iteration scheme (see [1] and [41]).

(1) *Uniform estimates in L^1*

Let λ_1 and φ_1 be the first eigenvalue and the corresponding eigenfunction of (7.1.1) respectively. We here normalize φ_1 so that $\|\varphi_1\|_1 = 1$. Multiplying φ_1 by the first and second equations of (II.4), we get

$$\Big(\int_\Omega u_1\varphi_1 dx\Big)_t + (b+\lambda_1)\int_\Omega u_1\varphi_1 dx + (\alpha-\gamma)\int_{\partial\Omega} u_1\varphi_1 d\sigma = \int_\Omega u_1u_2\varphi_1 dx, \tag{10.2.3}$$

$$\Big(\int_\Omega u_2\varphi_1 dx\Big)_t + \lambda_1\int_\Omega u_2\varphi_1 dx + (\beta-\gamma)\int_{\partial\Omega} u_2\varphi_1 d\sigma = a\int_\Omega u_1\varphi_1 dx. \tag{10.2.4}$$

Multiplying (10.2.3) by a and substituting (10.2.4) and equation (II.4) to the second term of the left-hand side and the right-hand side respectively, we have

(10.2.5)

$$a\Big(\int_\Omega u_1\varphi_1 dx\Big)_t + (b+\lambda_1)\Big(\Big(\int_\Omega u_2\varphi_1 dx\Big)_t + \lambda_1\int_\Omega u_2\varphi_1 dx + (\beta-\gamma)\int_{\partial\Omega} u_2\varphi_1 d\sigma\Big)$$
$$+a(\alpha-\gamma)\int_{\partial\Omega} u_1\varphi_1 d\sigma = \int_\Omega (\partial_t u_2 - \Delta u_2)\,u_2\varphi_1 dx.$$

Then differentiating (10.2.4) with respect to t once and substituting (10.2.5) to the right-hand side, we obtain

$$\Big(\int_\Omega u_2\varphi_1 dx\Big)_{tt} + (b+2\lambda_1)\Big(\int_\Omega u_2\varphi_1 dx\Big)_t + \lambda_1(b+\lambda_1)\int_\Omega u_2\varphi_1 dx$$
$$+ a(\alpha-\gamma)\int_{\partial\Omega} u_1\varphi_1 d\sigma + (\beta-\gamma)\Big(\int_{\partial\Omega} u_2\varphi_1 d\sigma\Big)_t + (\beta-\gamma)(b+\lambda_1)\int_{\partial\Omega} u_2\varphi_1 d\sigma$$
$$= \int_\Omega (\partial_t u_2 - \Delta u_2)\,u_2\varphi_1 dx$$

(10.2.6)

$$= \frac{1}{2}\Big(\int_\Omega u_2^2\varphi_1 dx\Big)_t + \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx + \Big(\beta - \frac{\gamma}{2}\Big)\int_{\partial\Omega} u_2^2\varphi_1 d\sigma.$$

Finally choosing $\gamma = \frac{\alpha+2\beta}{2} > 0$, we deduce

(10.2.7)

$$\Big(\int_\Omega u_2\varphi_1 dx\Big)_{tt} + (b+2\lambda_1)\Big(\int_\Omega u_2\varphi_1 dx\Big)_t + \lambda_1(b+\lambda_1)\int_\Omega u_2\varphi_1 dx$$
$$-\frac{\alpha}{2}\Big(\int_{\partial\Omega} u_2\varphi_1 d\sigma\Big)_t - \frac{\alpha}{2}\lambda_1\int_{\partial\Omega} u_2\varphi_1 d\sigma \ge \frac{1}{2}\Big(\int_\Omega u_2^2\varphi_1 dx\Big)_t + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx.$$

We now set

$$y(t) := w'(t) + (b+\lambda_1)\,w(t) - \frac{1}{2}\int_\Omega u_2^2\,\varphi_1\,dx - \frac{\alpha}{2}\int_{\partial\Omega} u_2\,\varphi_1\,d\sigma, \quad w(t) := \int_\Omega u_2\,\varphi_1\,dx.$$

Since $\partial_t u_2 \in L^2(0,T;L^2(\Omega))$ implies that there exists $s_0 \in (0,1)$ such that $|y(s_0)| < \infty$. Then (10.2.7) yields

$$y'(t) \ge -\lambda_1\,y(t), \quad \text{hence} \quad y(t) \ge y(s_0)\,e^{-\lambda_1(t-s_0)} \ge -|y(s_0)| =: -C_0 \quad \forall t \ge s_0.$$

Hence by virtue of Schwarz's inequality and Young's inequality, we get

$$\begin{aligned}-C_0 \le y(t) &= w'(t) + (b+\lambda_1)\, w(t) - \frac{1}{2}\int_\Omega u_2^2\varphi_1\, dx - \frac{\alpha}{2}\int_{\partial\Omega} u_2\varphi_1\, d\sigma \\ &\le w'(t) + (b+\lambda_1)\, w(t) - \frac{1}{2} w^2(t) \\ &\le w'(t) - \frac{1}{4}\, w^2(t) + (b+\lambda_1)^2 \qquad \forall t \ge s_0,\end{aligned}$$

i.e.,

$$w'(t) \ge \frac{1}{4} w^2(t) - C_1, \quad C_1 := C_0 + (b+\lambda_1)^2 > 0 \quad \forall t \ge s_0, \tag{10.2.8}$$

whence follows

$$w(t) \le 2C_1^{\frac{1}{2}} =: C_2 \qquad \forall t \ge s_0, \tag{10.2.9}$$

Indeed, if there exists $t_1 \ge s_0$ such that

$$\frac{1}{4} w^2(t_1) - C_1 > 0, \tag{10.2.10}$$

then from (10.2.8), (10.2.10) we can deduce that there exists $t_2 > t_1$ such that

$$\lim_{t\to t_2} w(t) = +\infty,$$

which contradicts the assumption that $w(t)$ exists globally. Thus (10.2.9) holds and the following global bound for $w(t)$ is established.

$$\sup_{t\ge 0}\int_\Omega u_2\,\varphi_1\, dx \le \overline{C}_2 := \max\Big(C_2, \max_{0\le s\le s_0} w(s)\Big). \tag{10.2.11}$$

Next we derive a uniform estimate for $\int_\Omega u_1\varphi_1 dx$. Using the facts that $u_1 = \frac{1}{a}(\partial_t u_2 - \Delta u_2)$ and (u_1, u_2) are nonnegative in (10.2.3), we can get

$$\begin{aligned}\frac{d}{dt}\Big(\int_\Omega u_1\varphi_1 dx\Big) &\ge -(b+\lambda_1)\int_\Omega u_1\varphi_1\, dx \\ &= -(b+\lambda_1)\frac{1}{a}\int_\Omega (\partial_t u_2 - \Delta u_2)\varphi_1\, dx \\ &= -\frac{b+\lambda_1}{a} w'(t) - \frac{(b+\lambda_1)\lambda_1}{a} w(t) + \frac{(b+\lambda_1)\alpha}{2a}\int_{\partial\Omega} u_2\varphi_1 d\sigma \\ &\ge -\frac{b+\lambda_1}{a} w'(t) - \frac{(b+\lambda_1)\lambda_1}{a} w(t).\end{aligned}$$

For $\eta \in (0,1)$, integrating this inequality over $(t, t+\eta)$ and using (10.2.11), we obtain

$$\begin{aligned}\Big[\int_\Omega u_1\varphi_1 dx\Big]_t^{t+\eta} &\ge -\frac{b+\lambda_1}{a}\,(w(t+\eta) - w(t)) - \frac{(b+\lambda_1)\lambda_1}{a}\int_t^{t+\eta} w(\tau)\, d\tau \\ &\ge -\frac{b+\lambda_1}{a}\,\overline{C}_2 - \frac{(b+\lambda_1)\lambda_1}{a}\,\overline{C}_2 =: -C_3,\end{aligned}$$

where $C_3 > 0$ is independent of t and η. This implies that

$$\int_\Omega u_1(t)\varphi_1\,dx \le C_3 + \int_\Omega u_1(t+\eta)\varphi_1\,dx. \tag{10.2.12}$$

Integrating (10.2.12) over $\eta \in (0,1)$ and using integration by parts, we get

$$\begin{aligned}\int_\Omega u_1(t)\varphi_1 dx &\le C_3 + \int_0^1\int_\Omega u_1(t+\eta)\varphi_1\,dx\,d\eta \\ &= C_3 + \int_t^{t+1}\int_\Omega u_1(\tau)\varphi_1\,dx\,d\tau \\ &= C_3 + \frac{1}{a}\int_t^{t+1}\int_\Omega (\partial_t u_2 - \Delta u_2)\varphi_1\,dx\,d\tau \\ &= C_3 + \frac{1}{a}\left(w(t+1)-w(t)\right) + \frac{\lambda_1}{a}\int_t^{t+1} w(\tau)\,d\tau - \frac{\alpha}{2a}\int_t^{t+1}\int_{\partial\Omega} u_2\varphi_1\,d\sigma\,d\tau \\ &\le C_3 + \frac{1+\lambda_1}{a}\overline{C}_2 =: C_4,\end{aligned}$$

which concludes that

$$\sup_{t\ge 0}\int_\Omega u_1\varphi_1 dx \le C_4. \tag{10.2.13}$$

Thus, from (10.2.11), (10.2.13) and Lemma 7.1.2, we can derive the following estimates:

$$\sup_{t\ge 0}\|u_1(t)\|_1 \le C_5, \quad \sup_{t\ge 0}\|u_2(t)\|_1 \le C_6. \tag{10.2.14}$$

(2) *Uniform estimates in L^2*

We here try to get L^2 uniform bounds of solutions of (II.4). Since (10.2.3) gives

$$\int_\Omega u_1u_2\varphi_1 dx \le \frac{d}{dt}\Big(\int_\Omega u_1\varphi_1\,dx\Big) + (b+\lambda_1)\int_\Omega u_1\varphi_1\,dx,$$

it follows from (10.2.13) that

$$\sup_{t\ge 0}\int_t^{t+1}\int_\Omega u_1u_2\,dx\,d\tau \le C_7. \tag{10.2.15}$$

Multiplying the second equation of (II.4) by u_2 and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|u_2(t)\|_2^2 + \|\nabla u_2(t)\|_2^2 + \beta\|u_2(t)\|_{2,\partial\Omega}^2 = a\int_\Omega u_1u_2\,dx,$$

where $\|v\|_{2,\partial\Omega}^2 = \int_{\partial\Omega} v^2 d\sigma$. Hence by virtue of Poincaré - Friedrichs' inequality $C_F\|v\|_2^2 \le (\|\nabla v\|_2^2 + \beta\|v\|_{2,\partial\Omega}^2)$, we have

$$\frac{1}{2}\frac{d}{dt}\|u_2(t)\|_2^2 + C_F\|u_2(t)\|_2^2 \le a\int_\Omega u_1u_2\,dx. \tag{10.2.16}$$

Applying Gronwall's inequality to (10.2.16), we get

$$\|u_2(t)\|_2^2 \le e^{-2C_F t}\|u_{20}\|_2^2 + \int_0^t 2a\Big(\int_\Omega u_1u_2\,dx\Big)e^{-2C_F(t-\tau)}\,d\tau. \tag{10.2.17}$$

In order to obtain uniform bounds of L^2-norm for u_2 with respect to t, we need to confirm that the second term of right hand side of (10.2.17) is bounded. For any $t \ge 0$, we can express $t = n + \varepsilon$ with some $n \in \mathbb{N} \cup \{0\}$ and $\varepsilon \in [0, 1)$. Then, by virtue of (10.2.15), it follows that

$$\begin{aligned}
&\int_0^t \Big(\int_\Omega u_1u_2dx\Big)e^{-2C_F(t-\tau)}\,d\tau \\
&= \int_{t-1}^t \Big(\int_\Omega u_1u_2\,dx\Big)e^{-2C_F(t-\tau)}\,d\tau + \int_{t-2}^{t-1} \Big(\int_\Omega u_1u_2\,dx\Big)e^{-2C_F(t-\tau)}\,d\tau \\
&\quad + \cdots + \int_{t-n}^{t-(n-1)} \Big(\int_\Omega u_1u_2\,dx\Big)e^{-2C_F(t-\tau)}\,d\tau + \int_0^{t-n} \Big(\int_\Omega u_1u_2\,dx\Big)e^{-2C_F(t-\tau)}\,d\tau \\
&\le e^{-0}\int_{t-1}^t \Big(\int_\Omega u_1u_2\,dx\Big)d\tau + e^{-2C_F}\int_{t-2}^{t-1} \Big(\int_\Omega u_1u_2\,dx\Big)d\tau \\
&\quad + \cdots + e^{-2(n-1)C_F}\int_{t-n}^{t-(n-1)} \Big(\int_\Omega u_1u_2\,dx\Big)d\tau + e^{-2nC_F}\int_0^{t-n} \Big(\int_\Omega u_1u_2\,dx\Big)d\tau \\
&\le C_7\Big(1 + e^{-2C_F} + e^{-4C_F} + \cdots + e^{-2nC_F}\Big) \\
&= C_7\frac{1 - e^{-2(n+1)C_F}}{1 - e^{-2C_F}} \le \frac{C_7}{1 - e^{-2C_F}}.
\end{aligned}$$

Therefore we obtain from (10.2.17)

$$\|u_2(t)\|_2^2 \le e^{-2C_F t}\|u_{20}\|_2^2 + \frac{2aC_7}{1 - e^{-2C_F}} \qquad \forall t \ge 0.$$

This implies that there exists $C_8 > 0$ such that

$$\sup_{t\ge 0}\|u_2(t)\|_2 \le C_8. \tag{10.2.18}$$

Note that the above argument can be done without any restriction on dimension N.

We next derive a uniform L^2-estimate of u_1 for $N \le 3$. Multiplying the first equation of (II.4) by u_1 and using integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|u_1(t)\|_2^2 + \|\nabla u_1(t)\|_2^2 + \alpha\|u_1(t)\|_{2,\partial\Omega}^2 + b\|u_1(t)\|_2^2 = \int_\Omega u_1^2u_2\,dx.$$

We here adopt $(\|\nabla v\|_2^2 + b\,\|v\|_2^2)^{1/2}$ as the H^1 norm for u_1. By using Hölder's inequality, the interpolation inequality and the embedding theorem ($\|v\|_6 \le C_9\|v\|$), it holds that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_1(t)\|_2^2 + \|u_1(t)\|^2 &\le \int_\Omega u_1^2 u_2\, dx \\ &\le \|u_1(t)\|_4^2\|u_2(t)\|_2 \\ &\le \|u_1(t)\|_1^{\frac{1}{5}}\|u_1(t)\|_6^{\frac{9}{5}}\|u_2(t)\|_2 \\ &\le C_5^{\frac{1}{5}} C_8 C_9^{\frac{9}{5}}\|u_1(t)\|^{\frac{9}{5}} \le \frac{1}{2}\|u_1(t)\|^2 + C_{10},\end{aligned}$$

which implies

$$\frac{1}{2}\frac{d}{dt}\|u_1(t)\|_2^2 + \frac{1}{2}\|u_1(t)\|^2 \le C_{10}.$$

Hence we obtain

$$\|u_1(t)\|_2^2 \le e^{-t}\|u_{10}\|_2^2 + 2C_{10}\left(1 - e^{-t}\right),$$

i.e.,

$$\sup_{t\ge 0}\|u_1(t)\|_2 \le C_{11}. \tag{10.2.19}$$

(3) Uniform estimates in H^1

Now we are in the position to derive a uniform H^1 bounds of solutions of (II.4). Multiplying the second equation of (II.4) by $-\Delta u_2$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla u_2(t)\|_2^2 + \beta\|u_2(t)\|_{2,\partial\Omega}^2) + \|\Delta u_2(t)\|_2^2 \\ &\qquad = -a\int_\Omega u_1 \Delta u_2\, dx \le \frac{1}{2}\|\Delta u_2(t)\|_2^2 + \frac{a^2}{2}\|u_1(t)\|_2^2.\end{aligned}$$

Here we define the H^1-norm of u_2 by

$$\|u_2\|^2 := \|\nabla u_2(t)\|_2^2 + \beta\|u_2(t)\|_{2,\partial\Omega}^2.$$

Then it holds that $C_F\|u_2\|^2 \le \|\Delta u_2\|_2^2$, since

$$(C_F)^{\frac{1}{2}}\|u_2\|_2\,\|u_2\| \le \|\nabla u_2\|_2^2 + \beta\|u_2(t)\|_{2,\partial\Omega}^2 = (-\Delta u_2, u_2)_{L^2} \le \|\Delta u_2\|_2\|u_2\|_2.$$

Hence we obtain

$$\frac{d}{dt}\|u_2(t)\|^2 + C_F\|u_2(t)\|^2 \le a^2\, C_{11}^2,$$

whence follows

$$\sup_{t\ge 0}\|u_2(t)\| \le C_{12}. \tag{10.2.20}$$

In order to derive the uniform H^1-estimate for u_1, we prepare the following functional $\phi_1(u_1)$:

$$\phi_1(u_1) := \frac{1}{2}(\|\nabla u_1\|_2^2 + \alpha\,\|u_1\|_{2,\partial\Omega}^2 + b\,\|u_1\|_2^2) \qquad u_1 \in H^1(\Omega).$$

Then it is easy to see

(10.2.21)

$$\phi_1(u_1) \geq \frac{1}{2}\|u_1\|^2 \geq \frac{b}{2}\|u_1\|_2^2,$$

$$\| - \Delta u_1 + b\, u_1\|_2\|u_1\|_2 \geq |(-\Delta u_1 + b\, u_1, u_1)_{L^2}| = 2\,\phi_1(u_1) \geq 2\,\sqrt{\phi_1(u_1)}\sqrt{\frac{b}{2}}\|u_1\|_2,$$

whence follows

$$2b\,\phi_1(u_1) \leq \| - \Delta u_1 + b\, u_1\|_2^2. \tag{10.2.22}$$

Multiplication of the first equation of (II.4) by $-\Delta u_1 + bu_1$ and integration over Ω yield

$$\begin{aligned}(\partial_t u_1, -\Delta u_1 + b\, u_1)_{L^2} + \|-\Delta u_1 + b\, u_1\|_2^2 &= (u_1 u_2, -\Delta u_1 + b\, u_1)_{L^2} \\ &\leq \frac{1}{2}(\|u_1 u_2\|_2^2 + \|-\Delta u_1 + b\, u_1\|_2^2).\end{aligned} \tag{10.2.23}$$

Here we note

$$(\partial_t u_1, -\Delta u_1 + b\, u_1)_{L^2} = \frac{d}{dt}\phi_1(u_1(t)).$$

Hence, in view of (10.2.23) and (10.2.22), we obtain

$$\frac{d}{dt}\phi_1(u_1(t)) + b\,\phi_1(u_1(t)) \leq \frac{1}{2}\|u_1 u_2\|_2^2.$$

Here by Hölder's inequality, (10.2.18), (10.2.19), (10.2.20),(10.2.21) and Young's inequality, we get

$$\begin{aligned}\|u_1 u_2\|_2^2 = \int_\Omega u_1^2\, u_2^2\, dx = \int_\Omega u_1^{\frac{1}{2}} u_2^{\frac{1}{2}} u_1^{\frac{3}{2}} u_2^{\frac{3}{2}}\, dx \\ \leq \Big(\int_\Omega u_1 u_2\, dx\Big)^{\frac{1}{2}} \Big(\int_\Omega u_1^3 u_2^3\, dx\Big)^{\frac{1}{2}} \\ \leq C_{11}^{\frac{1}{2}} C_8^{\frac{1}{2}}\,\|u_1(t)\|_6^{\frac{3}{2}}\,\|u_2(t)\|_6^{\frac{3}{2}} \\ \leq b\,\phi_1(u_1(t)) + C_{13}.\end{aligned}$$

Hence it follows that

$$\frac{d}{dt}\phi_1(u_1(t)) + \frac{b}{2}\phi_1(u_1(t)) \leq \frac{C_{13}}{2}.$$

Therefore, applying Gronwall's inequality, we deduce

$$\phi_1(u_1(t)) \leq \phi_1(u_1(0))\, e^{-\frac{b}{2}t} + \frac{C_{13}}{b}.$$

which implies that

$$\sup_{t\geq 0}\|u_1(t)\| \leq C_{14}. \tag{10.2.24}$$

(4) *Uniform estimates in L^∞*

Since Theorem 10.1.1 assures that there exists $s_1 \in (0,1)$ such that $u(s_1) \in H^1(\Omega)$ and $\|u(t)\|_\infty$ is bounded on $[0, s_1]$, we can assume without loss of generality that $(u_{10}, u_{20}) \in H^1 \cap V$. To derive L^∞ bounds via H^1 bounds, we rely on the Alikakos - Moser's iteration scheme (Lemma 3.2.9), which plays an essential role in our argument.

In order to apply Lemma 3.2.9, we deform (II.4) in the following way:

$$\partial_t u_1 - \Delta u_1 + u_1 = u_1 u_2 - b\, u_1 + u_1, \tag{10.2.25}$$

$$\partial_t u_2 - \Delta u_2 + u_2 = a\, u_1 + u_2. \tag{10.2.26}$$

Hereafter we employ the usual H^1 norm $(\|\nabla v\|_2^2 + \|v\|_2^2)^{1/2}$ for u_1 and u_2. Multiplying (10.2.25) by $|u_1|^{r-2}u_1$ $(r \geq 2)$ and using integration by parts, we obtain

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\|u_1(t)\|_r^r + (r-1)\int_\Omega |\nabla u_1|^2 |u_1|^{r-2}\, dx + \int_{\partial\Omega} |u_1|^r\, d\sigma + \|u_1(t)\|_r^r \\ = \int_\Omega u_1^r u_2\, dx - b\, \|u_1(t)\|_r^r + \|u_1(t)\|_r^r.\end{aligned}$$

Hence we have

$$\frac{1}{r}\frac{d}{dt}\|u_1(t)\|_r^r + (r-1)\int_\Omega |\nabla u_1|^2 |u_1|^{r-2}\, dx + \|u_1(t)\|_r^r \leq \int_\Omega |u_1|^r |u_2|\, dx + \|u_1(t)\|_r^r.$$

Moreover we note

$$\begin{aligned}(r-1)\int_\Omega |\nabla u_1|^2 |u_1|^{r-2}\, dx + \|u_1(t)\|_r^r &= \frac{4(r-1)}{r^2}\int_\Omega \left|\nabla |u|^{\frac{r}{2}}\right|^2 dx + \|\, |u_1(t)|^{\frac{r}{2}}\|_2^2 \\ &\geq \frac{4(r-1)}{r^2}\|\, |u_1(t)|^{\frac{r}{2}}\|^2,\end{aligned}$$

where we used the fact that $r \geq 2$ implies $\frac{4(r-1)}{r^2} \in (0,1]$ to the last inequality. Hence we obtain

$$\frac{1}{r}\frac{d}{dt}\|u_1(t)\|_r^r + \frac{4(r-1)}{r^2}\||u_1(t)|^{\frac{r}{2}}\|^2 \leq \int_\Omega |u_1|^r |u_2|\, dx + \|u_1(t)\|_r^r. \tag{10.2.27}$$

By using Hölder's inequality, interpolation inequality, Sobolev's embedding theorem and Young's inequality, we can get

$$\begin{aligned}\int_\Omega |u_1|^r |u_2|\, dx &\leq \|u_1(t)\|_{\frac{3r}{2}}^r\, \|u_2(t)\|_3 \\ &\leq \|u_1(t)\|_r^{\frac{r}{2}}\, \|u_1(t)\|_{3r}^{\frac{r}{2}}\, \|u_2(t)\|_3 \\ &\leq \|u_2(t)\|_3\, \|u_1(t)\|_r^{\frac{r}{2}}\, \|\, |u_1(t)|^{\frac{r}{2}}\|_6 \\ &\leq C_{15}\, \|u_1(t)\|_r^{\frac{r}{2}}\, \|\, |u_1(t)|^{\frac{r}{2}}\| \\ &\leq \frac{2(r-1)}{r^2}\|\, |u_1(t)|^{\frac{r}{2}}\|^2 + \frac{C_{15}^2\, r^2}{8(r-1)}\|u_1(t)\|_r^r.\end{aligned}$$

Since $r \geq 2$, it is easy to see that $\frac{r^2}{8(r-1)} \leq r$. Then, from these observations, (10.2.27) leads to

$$\frac{1}{r}\frac{d}{dt}\|u_1(t)\|_r^r + \frac{2(r-1)}{r^2}\| \, |u_1(t)|^{\frac{r}{2}}\|^2 \leq C_{15}^2 \, r \, \|u_1(t)\|_r^r + \|u_1(t)\|_r^r,$$

that is,

$$\frac{d}{dt}\|u_1(t)\|_r^r + \| \, |u_1(t)|^{\frac{r}{2}}\|^2 \leq C_{16} \, r^2 \big(\|u_1(t)\|_r^r + 1\big). \tag{10.2.28}$$

Here we used the fact that $1 \leq \frac{2(r-1)}{r}$ provided that $r \geq 2$. Then $u_1(t)$ satisfies (3.2.23) with $c_1 = 1$, $c_2 = C_{16}$, $\theta_1 = 0$ and $\theta_2 = 2$. Thus applying Lemma 3.2.9 to (10.2.28), we see that there exists $C_{17} > 0$ such that

$$\sup_{t \geq 0} \|u_1(t)\|_\infty \leq C_{17}. \tag{10.2.29}$$

Finally, applying the same argument as above for $u_2(t)$, we have

$$\frac{1}{r}\frac{d}{dt}\|u_2(t)\|_r^r + \frac{4(r-1)}{r^2}\| \, |u_2(t)|^{\frac{r}{2}}\|^2 \leq a \int_\Omega u_1 u_2^{r-1} dx + \|u_2(t)\|_r^r. \tag{10.2.30}$$

Since $\frac{r-1}{r} \leq 1$ and $\frac{1}{r} \leq 1$, due to (10.2.29) we can deduce

$$\begin{aligned} a \int_\Omega u_1 u_2^{r-1} \, dx &\leq a \, C_{17} \, \|u_2(t)\|_{r-1}^{r-1} \\ &\leq a \, C_{17} \Big\{ \frac{r-1}{r}\|u_2(t)\|_r^r + \frac{1}{r}|\Omega| \Big\} \\ &\leq a \, C_{17} \Big(\|u_2(t)\|_r^r + |\Omega| \Big), \end{aligned}$$

which implies

$$\frac{1}{r}\frac{d}{dt}\|u_2(t)\|_r^r + \frac{4(r-1)}{r^2}\| \, |u_2(t)|^{\frac{r}{2}}\|^2 \leq C_{18} \Big(\|u_2(t)\|_r^r + 1 \Big),$$

for some $C_{18} > 0$. Since $2 \leq \frac{4(r-1)}{r}$, we conclude that

$$\frac{d}{dt}\|u_2(t)\|_r^r + 2 \, \| \, |u_2(t)|^{\frac{r}{2}}\|^2 \leq C_{18} \, r \Big(\|u_2(t)\|_r^r + 1 \Big). \tag{10.2.31}$$

Then we can apply Lemma 3.2.9 to (10.2.31) with $c_1 = 2$, $c_2 = C_{18}$, $\theta_1 = 0$ and $\theta_2 = 1$. Thus there exists $C_{19} > 0$ such that

$$\sup_{t \geq 0} \|u_2(t)\|_\infty \leq C_{19}. \tag{10.2.32}$$

These a priori bounds (10.2.29) and (10.2.32) complete the proof.

□

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