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博 士 論 文 審 査 報 告 書
Screening Results Report

論　文　題　目
Thesis Theme

A study on the qualitative theory of solutions for some parabolic equations with nonlinear boundary conditions

非線形境界条件を伴う放物型方程式の解の定性理論の研究

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When we formulate mathematical models describing actual nonlinear phenomena, it is very important to single out right nonlinear structures in domains where the phenomena take place, while it is sometimes crucial to pay a special attention of the choice of boundary conditions. Let us take the diffusion phenomena of heat, for example. Then we should notice that the standard boundary conditions such as of Dirichlet, Neumann, or Robin type have a physical meaning only when an artificial control of the heat flux on the boundary is possible. This is not the case with systems in a large scale, where such a control on the boundary is unavailable. For instance, Stefan-Boltzmann's law shows that the heat energy radiation out of the surface of the body (the boundary of the domain) is proportional to the fourth power of the difference of temperatures between the inside and outside of the body (across the boundary of the domain) in three dimensional Euclidean space.

Therefore, nonlinear boundary conditions are sometimes more natural rather than the standard linear boundary conditions such as the homogeneous Dirichlet, Neumann, or Robin boundary conditions from physical point of view. In spite of such importance, however, there are few results even on special parabolic partial differential equations with nonlinear boundary conditions of radiation type. The first study on the Cauchy problem of the heat equation with nonlinear Neumann boundary condition was given by H. Brézis in 1971, where the well-posedness of the problem was proved by a new class of maximal monotone operator in the framework of Kōmura's theory of nonlinear semigroups in the Hilbert space. In this work, dissipative structure of the system plays an essential role, and accordingly, studies on parabolic equations with nonlinear boundary conditions are not fully pursued for non-dissipative systems which may admit blow-up solutions.

In this thesis the author studies the Cauchy problem for a class of nonlinear parabolic equations with nonlinear boundary conditions in a bounded domain with regular boundary in the Euclidean space, where nonlinear interactions in the equations may not have dissipative structure and boundary conditions are of radiation type.

This thesis is composed of two parts and ten chapters. Part I is composed of Chapters 1-6 and Part II is composed of Chapters 7-10.

In Part I, nonlinear heat equations of Fujita type are studied with normal derivative given by a power of the temperature on the boundary. The purpose of Part I is to prove the existence and uniqueness of local solutions, the existence and nonexistence of global solutions, uniform bounds of global solutions, comparison theorems, and structural stability of nonlinear boundary conditions.

In Chapter 1, preliminaries are summarized on maximal monotone operators, subdifferentials, and Mosco convergence of nonlinear functionals.

In Chapter 2, the existence of local solutions is proved by L^∞-energy method and non-monotone perturbation theory for nonlinear parabolic equations derived by subdifferential of functionals. These functionals are new. Here, nonlinear boundary conditions are described by surface integral over the boundary of integrated power of the temperature.

In Chapter 3, space-time uniform bounds of global solutions are obtained by the Moser type estimates on these new functionals. The proof does not depend on the homogeneity of functionals. Argument of this type is also new.

In Chapter 4, comparison theorems of new type is formulated and proved for general parabolic systems with linear and nonlinear boundary conditions. These new comparison theorems are applicable to comparison of two solutions with different boundary conditions. As an application, the existence of blow-up solutions satisfying nonlinear boundary conditions is established.

In Chapter 5, a special one-parameter family of nonlinear boundary condition is introduced to describe the existence of blow-up solutions for large parameters and of global small-amplitude solutions for small parameters. The existence of unique threshold dividing both regimes of parameters is also proved. This result gives a clear picture of nonlinear boundary conditions which connects Dirichlet and Neumann conditions by one-parameter.

In Chapter 6, structural stability is discussed in terms of the continuous dependence of solutions with respect to continuous powers describing boundary condition. Mosco convergence of the associated convex functionals is proved.

In Part II, reaction diffusion systems of the neutron density and temperature are studied with normal derivative given by a power of the temperature on the boundary. The purpose of Part II is to prove the existence of positive solutions to the stationary problem and the associated orderd uniqueness, the local well-posedness of the Cauchy problem in L^∞, and the existence of global solutions and blow-up solutions.

In Chapter 7, preliminary results on fixed point theories and Grönwall type argument are summarized to be used in the subsequent chapters.

In Chapter 8, the existence of positive solutions to the stationary problem is proved by Kranosel'ski's fixed point theorem in the space of nonnegative continuous functions on the closure of the domain. Variational methods are not applicable to this reaction diffusion systems due to the lack of variational structure. A new approach is introduced to L^∞-estimates of solutions with control of contribution by the boundary.

In Chapter 9, the Cauchy problem for the reaction diffusion systems with nonlinear boundary conditions are studied. The local well-posedness of the Cauchy problem is proved by the L^∞-energy method. Regarding global behavior of solutions, it is shown that every positive stationary solution separates the set of positive initial data into two subsets:

(i) The set of the data that lead to global positive solutions vanishing as time tends to infinity.

(ii) The set of the data that lead to solutions of finite lifespan with diverging weighted heat summation as time tends to lifespan.

In Chapter 10, the uniform boundedness of global solutions is discussed. It is proved that every global solutions of the Cauchy problem for the reaction diffusion systems with Robin boundary condition is bounded in space-time. The proof depends on the dissipative structure of the system.

All theorems of the applicant in this thesis are new and original.

In conclusion, this thesis makes a clear account of several powerful methods for the study of nonlinear parabolic equations with nonlinear boundary conditions that was not treated with a few exceptions. The committee of referees accepts that the thesis deserves Doctor's Degree (Science) conferral.

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