Essays in Nonlinear Economic Dynamics
非線形経済動学論集

by

Masanori Yokoo

Department of Economics, Okayama University

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Contents

Preface

During the last couple of decades it has been widely recognized in the literature of dynamic economics, at least from theoretical viewpoints, that *nonlinearity* can play a crucial role in generating a large number of diverse and complicated types of behavior even in a simple deterministic economic system. One of the important notions representing this diversity and complexity is *chaos*, although it is ambiguous in its definition. As it turns out, in spite of a large variety of modelled economic situations, these economic models have many features in the mechanisms of generating such complicated dynamics in common. The thesis, consisting of five essays, is primarily concerned with nonlinear dynamics in several economic situations, with a special emphasis on the mechanisms of the occurrence of periodic and chaotic behavior due to nonlinearities inherent in the economic systems.

The first essay *'Chaotic Dynamics in a Two-dimensional Overlapping Generations Model'* analyzes the global dynamics in a two-dimensionally extended OLG model. Using a perturbation argument, we derive a sufficient condition under which the model exhibits a horseshoe (topological chaos) due to a transverse homoclinic orbit associated with the steady state. In a parametric example with a CES production function, we show that for a large set of parameter values the economic system exhibits complex dynamics such as strange attractors and infinitely many co-existing periodic attractors of arbitrarily large period.

In the second essay *'Complex Dynamics in a Cobweb Model with Adaptive Production Adjustment'*, we consider a simple nonlinear cobweb model in which cautious suppliers slowly adjust the production amount toward the target level in each time period. We show that the model exhibits topological chaos as well as observable chaos. Numerical simulations are carried out to suggest that the faster the suppliers adjust the production amount and the more inelastic the demand is, the more likely the market behaves chaotically.

By introducing a kind of behavioral heterogeneity into the model considered in the second essay, the third essay *'Stability, Chaos and Multiple Attractors: A Single Agent Makes a Difference'* examines whether even arbitrarily slight heterogeneity can drastically affect the qualitative dynamic property of the market. As for our model, the answer is affirmative. We consider two types of producers, cautious adapters and naive optimizers, which differ in the speed of production adjustment (or cautiousness). In a market solely occupied by naive optimizers, a single cautious adapter stabilizes the otherwise exploding market. On the other hand, in a market of cautious adapters, a single naive optimizer may destabilize the market; without him the market has at most one periodic attractor, but with him there may appear infinitely many co-existing periodic attractors due to the Newhouse phenomenon associated with homoclinic bifurcations.

The fourth essay *'Threshold Nonlinearities and Asymmteric Endogenous Business Cycles'* provides a model of endogenous business cyles in the presence of knowledge spillovers and a time-to-build restriction. There are two key assumptions of the model: (i) the payoff to each firm depends on the aggregate level of knowledge and (ii) innovation of a project is time-consuming, that is, the firm that decides to innovate has to forgo the opportunity to produce in that period. The resulting dynamic process is characterized by a piecewise linear difference equation with a discontinuity at some threshold level. We show that the model can exhibit asymmetric periodic cycles of arbitrary period, which mimic asymmetric business cycles that appear to keep switching between different regimes over time. The dynamic property is also shown to be characterized by the expansion rate that gives the period of a cycle and the probability of the economy being in the expansion phase over the course of one cycle.

The last short essay *'A Note on Heterogeneity-Induced Chaos'* examines the effect of behavioral heterogeneity on the qualitative dynamic properties of a business cycle model with knowledge spillovers and a time-to-build restriction. We introduce behavioral heterogeneity into the model presented in the previous chapter to show that even arbitrarily weak heterogeneity in technologies of firms can cause chaotic dynamics for that model which generates only regular motions in the absence of heterogeneity. We also give a parametric example with uniformly distributed heterogeneity to show that the model exhibits observable chaos for a large set of parameter values.

I would like to thank Tamotsu Onozaki, Gernot Sieg and Junichiro Ishida for permission to include the results of their joint works with me in the thesis. I am grateful to my supervisor, Professor Ryo Nagata, for his encouragement during the past five years when I was a doctoral graduate student at Waseda University. I am indebted to the committee members for the thesis, Professors Kazuyuki Sasakura, Yoriaki Fujimori and Takashi Oginuma, for their many valuable comments. I also gratefully acknowledge Professor (emeritus) Tatsuji Owase for his supervision during the past decade, even after his retirement, and for leading me to this exciting research field of nonlinear economic dynamics. Finally, I dedicate this thesis to my brother Tsuyoshi and my parents, in gratitude for their long-time support.

Chapter 1

Chaotic Dynamics in a Two-Dimensional Overlapping Generations Model

abstract

This paper[1] investigates the global dynamics of a two-dimensional Diamond type overlapping generaions model extended to allow for government intervention. Using a singular perturbation method, we identify conditions under which transverse homoclinic points to the golden rule steady state are generated. For a parametric example with a CES production function, the occurrence of complicated dynamics (e.g. strange attractors) associated with homoclinic bifurcations is demonstrated.

[1]This essay is based on the paper that appeared in the Journal of Economic Dynamics and Control, 2000, with some minor modifications. I would like to thank Tatsuji Owase and Akitaka Dohtani for helpful suggestions and comments to an earlier draft. I am also indebted to two anonymous referees and one of the editors of the Journal of Economic Dynamics and Control for several valuable suggestions.

1.1 Introduction

In the recent literature on economic dynamics it has been widely recognized that a variety of fluctuating patterns in economic variables can emerge even in deterministic systems. In particular, beginning in the 1980's, many economic models exhibiting periodic as well as more complicated motion such as so-called *chaos* have been studied.[2] Overlapping generations (OLG) models have played an important role in the development of chaotic nonlinear business cycle theory compatible with the competitive framework. See Benhabib and Day (1982) and Grandmont (1985) for one-dimensional (1-D) OLG models, which are early examples of the use of chaos in economics.

In this paper we study the dynamics of a standard discrete-time two-dimensional (2-D) OLG model with a Cobb-Douglas utility function, inelastic labor supply, productive capital, and the government following a balanced budget policy. Our model is based on that of Farmer (1986), which is, in turn, a version of the seminal model of Diamond (1965) extended to two dimensions. Farmer aimed to derive, using local bifurcation theory, a necessary condition for his 2-D OLG model to generate persistent cycles on an invariant closed curve around the golden rule steady state. He showed that such cycles appear, in his setting, only if the net worth of the government is positive at the golden rule steady state.[3] Our main interest, however, is in the global and complicated behavior of the model, rather than the local and regular behavior.

Several numerical results suggesting the occurrence of chaos in 2-D OLG models have appeared in the economic literature; Medio (1992), Böhm

[2]See e.g. Boldrin and Woodford (1990), Hommes (1991), Lorenz (1993) for general surveys and examples of chaos in endogenous economic dynamics.

[3]For other related production 2-D OLG models exhibiting periodic or quasi-periodic fluctuations, see e.g. Reichlin (1986) and also Jullien (1988).

(1993), and Medio and Negroni (1996) have provided interesting simulation results for 2-D OLG models in various settings. Such numerical experiments have given rise to the question whether these models displaying seemingly complicated behavior would *really* be chaotic in a certain strict sense. While dynamical systems theory explains that so-called *homoclinic points*, and in particular their creation and destruction are responsible for chaotic dynamics,[4] it is, in general, very hard to detect them in higher-dimensional concrete systems.

De Vilder (1996; also 1995) has offered, however, a promising approach for studying higher-dimensional nonlinear systems; he has presented a 'computer assisted proof' that an explicit 2-D OLG model with elastic labor supply and Leontief technology, essentially based on that of Reichlin (1986), can really exhibit complicated dynamics generated by *homoclinic bifurcation*[5] associated with the stable and unstable manifolds of the autarkic steady state. Although his method is based on numerical computations, the proof itself is rigorous because of his accurate estimation of computational errors.

As a complementary approach to that of de Vilder, we use a *singular perturbation method* suggested by Marotto (1979); see also van Strien (1981). Without requiring a computer and numerical specification of parameter values, this approach allows us to rigorously establish the occurrence of *horseshoes* (topological chaos). These horseshoes are assured by the presence of a transverse homoclinic point to the golden rule steady state for our 2-D model with a small constant rate of savings. This is done by finding a trans-

[4]See e.g. Palis and Takens (1993) for recent dynamical systems theory with an emphasis on homoclinic bifurcations.

[5]For an excellent analysis concerning homoclinic and heteroclinic bifurcations in a 2-D cobweb model with heterogeneous beliefs, see Brock and Hommes (1997), who proved, using a geometric configuration of the stable and unstable manifolds in a 'limiting case' where all agents choose the optimal predictor, that their model undergoes homoclinic bifurcations.

verse homoclinic point (or a 'snap-back repeller') for the reduced singular (i.e., 1-D) system and then by perturbing the latter again into the corresponding nonsingular 2-D (but nearly 1-D) system without destroying the transverse homoclinic point.[6] Moreover, in analyzing a parametric example with a CES production function, the perturbation technique is used to detect not only horseshoes but also *strange attractors* (observable chaos) as well as *infinitely many coexisting periodic attractors* which are created by homoclinic bifurcation.

This paper is organized as follows: Section 2 introduces the basic model. In Section 3 conditions and implications of the existence of transverse homoclinic points are discussed. In Section 4 we consider the complicated dynamics of a parametric example with a CES production function. In Section 5 some concluding remarks are given. Proofs of lemmas and propositions are assembled in the Appendix.

1.2 Basic model

We introduce a 2-D version of a Diamond type OLG model, which is essentially based on that of Farmer (1986). See also Azariadis (1993) for intensive studies of models of this type and Jullien (1988) for a similar OLG model with productive capital and money.

We are concerned with a discrete-time OLG economy with productive

[6]Various strategies to establish the occurrence of chaos in discrete-time 2-D models are presented by several authors: Jullien (1988) restricts his 2-D OLG model with real money balances onto a 1-D invariant curve on which chaotic behavior is possible. In Hommes (1991), a return map technique reduces a 2-D piecewise linear inventory cycle model to circle maps which exhibit (quasi-)periodic as well as chaotic attractors; in de Vilder (1995), a similar method is applied to a 2-D OLG model with an investment constraint. Dohtani et al. (1996) use a 1-D reduction and perturbation method, similar to ours, to prove the occurrence of topological chaos in a 2-D discrete version of Kaldor type business cycle model. For a 2-D 'addiction' model, Feichtinger et al. (1997) use a 1-D reduction method to show the existence of horseshoes, and then use a perturbation method to show that the horseshoes are preserved for nearby 2-D systems.

capital and government intervention. The population is constant over time. The representative consumer lives for two periods and he supplies his labor inelastically only in youth. In order to emphasize the role of the production side, the consumer at period t is assumed to have a simple linearly homogeneous Cobb-Douglas utility function with constant weight $s \in (0,1)$ on his consumption in old age:

$$u(c_{1,t}, c_{2,t+1}) := a c_{1,t}^{1-s} c_{2,t+1}^{s}, \qquad a > 0 \tag{1.1}$$

where $c_{i,j}$, $i = 1, 2$, denotes the quantity of consumption by the young and old at period j, respectively. Given the wage rate at period t, w_t, and the gross real interest rate at the next period, r_{t+1}, utility maximizing behavior yields the savings function represented by

$$\begin{aligned} S(w_t) &:= \{z_t \in [0, w_t] \mid \max_{0 \le z_t \le w_t} u(w_t - z_t, r_{t+1} z_t)\} \\ &= s w_t. \end{aligned} \tag{1.2}$$

By our choice of the utility function, this savings function is independent of the interest rate, and the parameter s can be referred to as the *savings rate*, i.e., the propensity to save out of wage income in youth.

The representative firm is characterized by a well-behaved production function $f(k_t)$ defined on $\mathbb{R}_+$, where $k_t \in \mathbb{R}_+$ stands for the capital-labor ratio at period t. Using capital and labor, the firm produces a single perishable commodity (e.g. rice) which depreciates totally in one period. We assume that the production function f satisfies the following:

Condition (A):

(A.1) f is C^2 on $\mathbb{R}_+$,

(A.2) $f(0) = 0,\ f(x) > 0,\ f'(x) > 0$, and $f''(x) < 0$ for all $x > 0$,

(A.3) $f'(0) > 1$, and $\lim_{x \to +\infty} f'(x) \to 0$,

(A.4) $f'(x) = x \Leftrightarrow x = 1$, (normalization)

(A.5) $f''(1) < -2$,

(A.6) the elasticity of marginal production function $\eta(x) := -\dfrac{xf''(x)}{f'(x)}$ is strictly increasing with respect to the capital-labor ratio.

Conditions (A.1) through (A.3) are standard in economics. Condition (A.4) is justified by Condition (A.3). Conditions (A.5) and (A.6) look a bit unusual. These last two conditions will be discussed later.

Competition implies that the marginal product of each factor is equal to its factor price (w_t or r_t), that is,

$$r_t = f'(k_t) \tag{1.3}$$

$$w_t = f(k_t) - k_t f'(k_t). \tag{1.4}$$

According to Farmer (1986), we will assume that the government follows a policy maintaining a zero budget deficit at all times. This implies, from the government budget constraint, that

$$b_{t+1} = r_t b_t, \tag{1.5}$$

where $b_t \in \mathbb{R}$ denotes the debt-labor ratio (i.e., the government debt per worker) at time t. Requiring that the asset market be cleared, we have

$$k_{t+1} + b_{t+1} = S(w_t). \tag{1.6}$$

Combining equations (1.3) to (1.6), we obtain a second order difference equation which characterizes the system

$$B(k_t, k_{t+1}) = f'(k_t)B(k_{t-1}, k_t), \tag{1.7}$$

where

$$B(k_t, k_{t+1}) := s(f(k_t) - k_t f'(k_t)) - k_{t+1}. \tag{1.8}$$

This expression represents the net indebtedness of the government to the private sector.

Note that b_t is negative if $B(k_{t-1}, k_t) < 0$, i.e., if the economywide capital stock exceeds net private ownership. In this case the government is a net creditor to the private sector.

Let

$$g(x) := xf'(x), \quad w(x) := f(x) - xf'(x), \quad \text{and} \quad h(x, y) := w(y) - f'(y)w(x), \tag{1.9}$$

then (1.7) is transformed into

$$k_{t+1} - g(k_t) - sh(k_{t-1}, k_t) = 0. \tag{1.10}$$

We see that $f(k_t) \equiv g(k_t) + w(k_t)$. Given the capital-labor ratio k_t, $w(k_t)$ represents the competitive wage rate or the share of labor in output per worker $f(k_t)$, while $g(k_t)$ can be viewed as the share of capital. We will call this function g the *capital-share function*.

Setting $k_t = x_t$ and $k_{t+1} = y_t$, we obtain a second order difference equation with one parameter equivalent to (1.7):

$$(x_{t+1}, y_{t+1}) = F_s(x_t, y_t), \tag{1.11}$$

where $F_s(x, y) = (y, g(y) + sh(x, y))$.

Suppose now that not only the initial capital-labor ratio, $k_0 \in \mathbb{R}_+$, but also the initial debt-labor ratio, $b_0 \in \mathbb{R}$, are given historically. This pair of initial states (k_0, b_0) determines $(k_0, k_1) = (x_0, y_0)$ by $k_1 = sw(k_0) - f'(k_0)b_0$. Because of the nonnegativity of the capital-labor ratio, we have to restrict ourselves to the set of initial states whose iterates by F_s will never leave the nonnegative quadrant, i.e.,

$$X_s := \{(x, y) \in \mathbb{R}_+^2 \mid F_s^n(x, y) \in \mathbb{R}_+^2 \quad \text{for all} \quad n \geq 0\}. \tag{1.12}$$

Hence, for each parameter value $s \in (0, 1)$, the difference equation (1.11) induces a map from X_s to itself:

$$F_s : X_s \subset \mathbb{R}_+^2 \to X_s. \tag{1.13}$$

The set of initial states X_s might be very small. Nevertheless, as Lemma 2 below indicates, this is not the case at least when the parameter s is small.

Before ending this section, we will give some remarks on steady states (i.e., fixed points) of (1.11). The set of steady states of (1.11), denoted by $\mathrm{Fix}(F_s)$, is defined as

$$\mathrm{Fix}(F_s) := \{(k^*, k^*) \in \mathbb{R}_+^2 \mid B(k^*, k^*)(f'(k^*) - 1) = 0\}. \tag{1.14}$$

Since $f'(k^*) = 1$ if and only if $k^* = 1$ by Condition (A.4), the *golden rule steady state* $p = (1, 1) \in \mathbb{R}_+^2$ is well-defined and independent of the parameter s. Since $B(1, 1) = s(f(1) - 1) - 1$, the government is a net creditor at the golden rule steady state whenever the propensity to save is sufficiently small. In this paper we do not take account of balanced steady states[7] satisfying $B(k^*, k^*) = sw(k^*) - k^* = 0$.

[7] The autarkic steady state $(0, 0) \in \mathrm{Fix}(F_s)$ may well be very influential for the complicated global dynamics in our model. See de Vilder (1996) for the creation of homoclinic points to the autarkic steady state.

1.3 Characterization of global dynamics

3.1 Preliminaries

In this subsection, we briefly discuss some notions and implications of homoclinic points and homoclinic bifurcations, which will be used in what follows. Guckenheimer and Holmes (1983) and Palis and Takens (1991) offer mathematical treatments of the subject. For a discussion of these topics in an economic context, the reader is referred to de Vilder (1995, 1996) and Brock and Hommes (1997).

For simplicity, we mostly treat here differentiable invertible maps (i.e. diffeomorphisms), but some similar results are provided even for differentiable noninvertible maps. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable invertible map and $p \in \mathbb{R}^2$ be a hyperbolic fixed point for F (i.e., where the Jacobian matrix evaluated at p, D_pF, has no eigenvalues with norm 1). If p is a periodic point of period k, then we may replace F by $G = F^k$. For a small neighbourhood U of p, the *local stable and unstable manifolds* of p are defined as

$$\begin{aligned} W^s_{loc}(p) &= \{x \in U \mid \lim_{k\to\infty} F^k(x) \to p\}, \\ W^u_{loc}(p) &= \{x \in U \mid \lim_{k\to\infty} F^{-k}(x) \to p\}, \end{aligned}$$

respectively. Even if F is not invertible, such invariant manifolds do exist. The *global stable and unstable manifolds* of p are then defined as

$$\begin{aligned} W^s(p) &= \bigcup_{n=0}^{\infty} F^{-n}(W^s_{loc}(p)), \\ W^u(p) &= \bigcup_{n=0}^{\infty} F^{n}(W^u_{loc}(p)), \end{aligned}$$

respectively. Note that if F is not invertible, then $W^s(p)$ and $W^u(p)$ may no longer be manifolds in the global sense (see e.g. Palis and Takens 1993 for more information). A point $q \in W^s(p) \cap W^u(p) \setminus \{p\}$ is said to be a *homoclinic point* to p. If $W^s(p)$ and $W^u(p)$ intersect transversely at this homoclinic point q, then we say that q is a *transverse homoclinic point* to p, and the orbit of q, $O(q) := \{F^i(q)\}_{i \in Z}$, is called a transverse homoclinic orbit. If $W^s(p)$ and $W^u(p)$ intersect tangentially at this homoclinic point q, then we say that q is a *homoclinic tangency.* The Homoclinic Point Theorem[8] assures that a transverse homoclinic orbit to a hyperbolic fixed point p implies the existence of a *horseshoe* near the homoclinic orbit. This is defined as a Cantor set which is invariant under (some iterate of) the map $G = F^n$ and on which G is topologically equivalent to a shift map with a countable infinity of periodic orbits, an uncountable infinity of aperiodic orbits, topological transitivity, and sensitive dependence on initial conditions. In addition, horseshoes as well as transverse homoclinic orbits to a hyperbolic fixed point (more generally, hyperbolic sets) have a kind of semi-local structural stability, that is, they are persistent against small perturbations of the map.

The existence of horseshoes does not imply that a typical trajectory exhibits complicated long-run dynamical behavior, since a horseshoe will not attract nearby points in the phase space. Topological chaos in the sense of horseshoes is, therefore, not observable in general.[9] To describe the asymptotic behavior which can be observed in the long run, we need some notions concerning 'attractors'. A compact invariant set A of the map F is called an *attractor* if it contains a dense orbit (i.e., $F|A$ is topologically transitive) and

[8]See e.g. Smale (1967), Guckenheimer and Holmes (1983), Palis and Takens (1993) for various versions of this theorem; for noninvertible maps, see e.g. Marotto (1978).

[9]This does not mean that topological chaos would be empirically insignificant at least in the short run. In fact, it is very likely to generate long-lasting complicated *transient* motion, which will be often sustained under the influence of very small noise. See e.g. Dohtani et al. (1996) for the effect of noise on a chaotic Kaldor type business cycle model.

its basin of attraction, i.e., a set of points x such that $\operatorname{dist}(F^n(x), A) \to 0$ as $n \to \infty$, has nonempty interior. An attractor A of the map F is said to be *strange* if $F|A$ has sensitive dependence on initial conditions. Contrary to the case of the horseshoe, chaotic dynamics will be observed in the long run for a large set of initial states if strange attractors are present in the system.

Now consider a one-parameter family of maps $\{F_\mu : \mu \in I \subset \mathbb{R}\}$ with a hyperbolic saddle $p = p(\mu)$. We say that the family of maps $\{F_\mu\}$ exhibits a *homoclinic bifurcation*, associated with p, at $\mu = 0$ if

(i) for $\mu < 0$, $W^s(p)$ and $W^u(p)$ have no intersection;

(ii) for $\mu = 0$, $W^s(p)$ and $W^u(p)$ have a tangency at $q \neq p$;

(iii) for $\mu > 0$, $W^s(p)$ and $W^u(p)$ have a transverse intersection.

Furthermore, if we can choose a μ-dependent local coordinate (x, y) near q so that $W^s(p)$ is given by $y = 0$ and $W^u(p)$ by

$$y = ax^2 + b\mu, \quad a \neq 0 \quad \text{and} \quad b \neq 0,$$

then we say that the tangency q is *quadratic* ($a \neq 0$) and *unfolds generically* ($b \neq 0$). Some results of dynamical systems theory guarantee that for the families of maps $\{F_\mu\}$ exhibiting a generically unfolding quadratic homoclinic tangency at $\mu = 0$, several interesting dynamical complexities arise for μ-values near $\mu = 0$ such as

- creation (or destruction) of horseshoes;
- coexistence of infinitely many periodic attractors or repellers (Newhouse 1979);
- existence of strange attractors or repellers for a positive Lebesgue measure set of μ-values (Mora and Viana 1993).

1.3.1 Chaotic dynamics

In the present section we try to identify a sufficient condition under which the dynamics of F_s in (1.11) are *topologically chaotic* in the sense that the system has a horseshoe. To this end, we will use a perturbation method (see e.g. Marotto 1979, van Strien 1981, Palis and Takens 1993) to detect a transverse homoclinic orbit to the golden rule steady state when the system is 2-D but nearly 1-D.

As a first step, it is convenient to consider the extreme case in which $s = 0$ in (1.11). This corresponds to a world in which generations do not virtually overlap because the representative consumer consumes all his income in youth and nothing in old age. Consequently, all the capital needed for production is owned by the government, i.e., $k_t = -b_t$. In this limiting case, the two-dimensional map (1.11) collapses, formally, to a *singular map* from $\mathbb{R}^2_+$ onto the graph of g, i.e., $C = \{(x, y) \in \mathbb{R}^2_+ \mid y = g(x),\ x \geq 0\}$, as follows:

$$F_0 : \mathbb{R}^2_+ \to C \subset \mathbb{R}^2_+; \quad F_0(x, y) = (y, g(y)), \tag{1.15}$$

whose dynamics are clearly governed by an equivalent one-dimensional map

$$g : \mathbb{R}_+ \to \mathbb{R}_+. \tag{1.16}$$

Hence, the system F_s in (1.11) with small $s > 0$ can be regarded as a perturbation of F_0 in (1.15).

By Condition (A), the following properties of the map g can easily be checked:

Lemma 1: *Under Condition (A), the following statements hold:*

(L1.1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ *is* C^1*;*

(L1.2) $g(0) = 0$ *and* $g'(0) > 1$*;*

(L1.3) $g(1) = 1$ *and* $g(x) > (<)\, x$ *as* $x < (>)\, 1$ *(*$x = 1$ *is a fixed point);*

(L1.4) $g'(1) < -1$ *(the fixed point* $x = 1$ *is a repeller);*

(L1.5) there exist unique points q *and* $\theta \in \mathbb{R}_{++}$ *such that*

(L1.5.1) $g'(\theta) = 0$ *with* $g'(x) > \ (<)0$ *as* $x < \ (>)\theta$ *(unimodality),*

(L1.5.2) $g(q) = 1$*, and*

(L1.5.3) $0 < q < \theta < 1 < g(\theta)$ *and* $0 < g^2(\theta) < 1$.

Assertion (1.4) in Lemma 1 follows from Condition (A.5), and (1.5) essentially follows from (A.6). From Lemma 1, we see that the graph of the marginal production function $y = f'(x)$ satisfying Condition (A) must have two and only two intersections with the hyperbola of $y = 1/x$, at $x = 1$ and at $x = q$. On the interval $[q, 1]$ the graph of f' is located above the hyperbola, and below otherwise. A typical situation with a 'reversed sigmoidal' marginal production function[10] is depicted in Fig.1.

It is worth noting that the capital-share function g is not monotone[11] with respect to the capital-labor ratio, whereas both the wage function w ($w'(x) = -xf''(x) > 0$ for $x > 0$) and the production function f are strictly increasing (see Fig.2). Nonmonotonicity of the function g requires that the elasticity of marginal production function η straddles unity because for $x > 0$, $\eta(x) = 1$ if and only if $g'(x) = f'(x)(1 - \eta(x)) = 0$. In case that Condition (A.6) is not satisfied, the function g may have more than one

[10] Fig.1 and Fig.2. are drawn based on a CES production function satisfying Condition (A), which will be discussed in Section 4.

[11] For a similar 2-D OLG model, Jullien (1988) assumed that the function g is nondecreasing.

hump. Moreover, even if neither Condition (A.5) nor (A.6) is satisfied, g cannot be a monotone function as long as $\sup_{x\geq 0}\eta(x) > 1$.

insert << Fig.1 >> about here

insert << Fig.2 >> about here

In order to guarantee the existence of bounded and positive-valued equilibrium paths in terms of the capital-labor ratio $\{k_t\}_{t\geq 0}$ for a *large* set of initial states, it is meaningful to show that, on the strictly positive phase plane, we can find a compact region M such that the forward orbit of every initial state in M cannot escape from there. We will then show that F_s has a *trapping region* in $\mathbb{R}^2_{++}$.

Lemma 2: *Suppose Condition (A) holds. Then there exist a compact region $M \subset \mathbb{R}^2_{++}$ ($p \in \mathrm{int}\, M$) and a number $\varepsilon > 0$ such that for every $s \in (0, \varepsilon)$ the following assertions hold:*

(L2.1) *M is a trapping region for F_s, i.e., $F_s(M) \subset \mathrm{int}\, M$,*

(L2.2) *$F_s|M : M \to M$ is a C^1-diffeomorphism onto its image,*

(L2.3) *p is a hyberbolic saddle, i.e., the Jacobian matrix D_pF_s evaluated at p has two real eigenvalues $\lambda_1(s)$ and $\lambda_2(s)$ with $|\lambda_1(s)| > 1 > |\lambda_2(s)| > 0$,*

(L2.4) *p is dissipative, i.e., $|\det D_pF_s| = |\lambda_1(s)\lambda_2(s)| < 1$.*

Of course, for every $s \in (0, \varepsilon)$, the trapping region M is contained in X_s.

We first attempt to identify conditions under which F_0 has a transverse homoclinic point to p. To do this, it suffices, using the argument presented by Marotto (1979), to identify conditions under which the 1-D map g has a so-called *snap-back repeller*, introduced by Marotto (1978), for $x = 1$:

Lemma 3: (Marotto 1979, Lemma 2.2) *If g has a snap-back repeller, then $F_0(x, y) = (y, g(y))$ has a transverse homoclinic point.*

Note that the fixed point $x = 1$ of g is a hyperbolic repeller, i.e., $|g'(1)| > 1$, from (1.4) in Lemma 1. In order to show that the fixed point $x = 1$ of g is a snap-back repeller, it is then sufficient to find a point $z \in \mathbb{R}_+$ $(z \neq 1)$ which has an orbit $O(z) = O^+(z) \cup O^-(z)$ satisfying the following:

(S1) $O^+(z) = \{x_i \in \mathbb{R}_+ \mid x_0 = z,\ g^m(x_0) = 1 \quad \text{for some } m \geq 1, \text{ and } \quad x_{i+1} = g(x_i) \quad \text{for} \quad i \geq 0\}$,

(S2) $O^-(z) = \{x_{-i} \in \mathbb{R}_+ \mid x_0 = z,\ x_{-i} \to 1 \quad \text{as} \quad i \to \infty, \text{ and } \quad x_{-i} = g(x_{-i-1}) \quad \text{for} \quad i \geq 0\}$,

(S3) $g'(x) \neq 0$ for each $x \in O(z)$.

In addition to Condition (A), we impose further conditions on g:

Condition (B):

(B.1) $g^2(\theta) < q$,

(B.2) $g^2(x) \neq x$ for any $x \in (\theta, 1)$,

where θ and q are unique points with $g'(\theta) = 0$ and $g(q) = 1$ $(0 < q < \theta < 1)$ as given in Lemma 1.

Note that the statement of (B.1) can be replaced by $g^3(\theta) < 1$. Condition (B.2) requires that the map g has no periodic point of period two on the interval $(\theta, 1)$.

We can see that, under Conditions (A) and (B), the map g has a snap-back repeller, which implies that the singular map F_0 has a transverse homoclinic point.

Lemma 4: *Under Conditions (A) and (B), the map* $F_0(x, y) = (y, g(y))$ *in (1.15) has a transverse homoclinic point to the golden rule steady state* p.

This situation is depicted in Fig.3. The stable 'manifold' for F_0, $W^s(p, F_0)$, consists of horizontal line segments passing through points which are eventually mapped onto the golden rule steady state p. In particular, $W^s(p, F_0)$ contains a horizontal line segment $\gamma^s = \{(x, y) \in \mathbb{R}^2_+ \mid x \in [g^2(\theta), g(\theta)],\ y = 1\}$ passing through p. Furthermore, the unstable manifold $W^u(p, F_0)$ contains an arc on the graph C of g, $\gamma^u = \{(x, y) \in C \subset \mathbb{R}^2_+ \mid x \in [g^2(\theta), g(\theta)],\ y = g(x)\}$, because each point on this arc γ^u has a backward orbit converging to p (see the proof of Lemma 4 in the Appendix).

insert << Fig.3 >> about here

By the perturbation argument of invariant manifolds (see e.g. Palis and Takens 1993, Appendix 1 and Appendix 4), we can perturb the singular map F_0 with a transverse homoclinic point, by making the parameter s slightly bigger than zero, so that every nearby nonsingular map F_s retains a transverse homoclinic point. Hence, by applying the Homoclinic Point Theorem, we obtain the following result:

Proposition 1: *Under Conditions (A) and (B), there exists* $\varepsilon > 0$ *such that for every* $s \in (0, \varepsilon)$*,* F_s *has a horseshoe* $\Lambda_s \subset M$ *with* $p \in \Lambda_s$*, where* $M \subset \mathbb{R}^2_{++}$ *is a trapping region for* F_s*.*

1.3.2 Asymptotic behavior near homoclinic orbits

Before proceeding to the next section, we will briefly discuss the asymptotic behavior for a large set of initial states when homoclinic orbits exist *and* the dissipativity of the system is strong. Since $M \subset \mathbb{R}^2_{++}$ in Proposition 1 is a trapping region containing a horseshoe Λ_s, every positive orbit of a point starting in M is indeed bounded and does not leave M. But this fact does not imply that the asymptotic behavior of every such orbit would be approximated by $F_s|\Lambda_s$ because Λ_s itself is not an attractor even though it may be a part of such a set. Some points in M might settle down to periodic attractors. We can, however, show at least that there is an open set surrounded by the segments of the stable and unstable manifolds of p such that every point in the set is drawn near the unstable manifold by the iteration process provided the dissipativity is strong enough,[12] i.e., provided $|\det D_x F_s| < 1$ for every $x \in M$. More precisely, there is an open set $U_s \subset M$ (depending on s) such that the ω-limit set of each point $x \in U_s$, $\omega(x) := \{y \in M \mid \exists n_i \to +\infty;\ F_s^{n_i}(x) \to y\}$, is contained in the closure of the unstable manifold $\overline{W^u(p)}$ of the golden rule p. This implies that certain attractors are contained in $\overline{W^u(p)}$:

Proposition 2: *Let Conditions (A) and (B) hold, and let* $M \subset \mathbb{R}^2_{++}$ *be a compact region as in Proposition 1. Then there is* $\varepsilon' > 0$ *such that for every*

[12] A similar result for a different 2-D OLG model with strong dissipativity has been shown by de Vilder (1995).

$s \in (0, \varepsilon')$, M contains an open set U_s with $\omega(x) \subset \overline{W^u(p)}$ for every $x \in U_s$.

1.4 Example with CES production function

1.4.1 Horseshoes and homoclinic tangles

In this section we give an example with a CES (Constant Elasticity of Substitution) production function $f_\beta(x)$ with two parameters: one is the distribution factor, $\alpha \in (0,1)$, and the other is the substitution factor, $\beta \in (-1,0) \cup (0,\infty)$, which is related to the elasticity of factor substitution. The CES production function is assumed to be of the following form:

$$\begin{aligned} f_\beta(x) &:= \frac{1}{\alpha}\left[1-\alpha+\alpha x^{-\beta}\right]^{-\frac{1}{\beta}} \\ &= \frac{x}{\alpha\,[\,(1-\alpha)x^\beta+\alpha]^{\frac{1}{\beta}}}, \end{aligned} \tag{1.17}$$

where the elasticity of substitution is given by $(1+\beta)^{-1}$. The first and second derivative of f_β and the elasticity of f'_β are calculated as follows:

$$\begin{aligned} f'_\beta(x) &= [\alpha+(1-\alpha)x^\beta]^{-\frac{\beta+1}{\beta}}, \\ f''_\beta(x) &= -\frac{(1-\alpha)(1+\beta)x^{\beta-1}}{[\alpha+(1-\alpha)x^\beta]^{\frac{2\beta+1}{\beta}}}, \\ \eta_\beta(x) &= -\frac{xf''_\beta(x)}{f'_\beta(x)} = \frac{(1-\alpha)(1+\beta)x^\beta}{\alpha+(1-\alpha)x^\beta}. \end{aligned}$$

One can easily see that if $\beta > (\alpha+1)/(1-\alpha) > 1$ then the production function f_β satisfies Condition (A). The functions $g_\beta(x) := xf'_\beta(x)$ and $h_\beta(x,y) := w_\beta(y) - f'_\beta(y)w_\beta(x)$ are then represented by

$$g_\beta(x) = x[\alpha+(1-\alpha)x^\beta]^{-\frac{1+\beta}{\beta}}, \quad \text{and} \tag{1.18}$$

$$h_\beta(x,y) = \frac{(1-\alpha)[y^{\beta+1}[\alpha+(1-\alpha)x^\beta]^{\frac{\beta+1}{\beta}} - x^{\beta+1}]}{\alpha[\alpha+(1-\alpha)x^\beta]^{\frac{\beta+1}{\beta}}[\alpha+(1-\alpha)y^\beta]^{\frac{\beta+1}{\beta}}}. \tag{1.19}$$

The point θ at which g_β attains its maximum can be calculated by solving $\eta_\beta(\theta) = 1$:

$$\theta = \theta(\beta) = \left(\frac{\alpha}{\beta(1-\alpha)}\right)^{\frac{1}{\beta}}.$$

Note that $\theta(\beta) \in (0,1)$ whenever $\beta > \alpha/(1-\alpha)$ and that $\theta(\beta) \to 1$ as $\beta \to \infty$.

The dynamics of this economic system can then be characterized by

$$F_{s,\beta}(x,y) = (\, y,\, g_\beta(y) + sh_\beta(x,y)\,). \tag{1.20}$$

It can be shown that g_β satisfies Condition (B) for every sufficiently large β (see Appendix). According to Proposition 1, we can therefore state the following proposition:

Proposition 3: *Fix $\alpha \in (0,1)$. Then there exists $\beta^* > (\alpha+1)/(1-\alpha)$ such that, given $\beta \geq \beta^*$, $F_{s,\beta}$ in (1.20) has a horseshoe for every $s \in (0,\varepsilon)$ for some $\varepsilon = \varepsilon(\beta) > 0$.*

Proposition 3 says that if the elasticitiy of substitution between capital and labor is sufficiently small and the representative consumer consumes “too much” out of his wage income in youth, then the economic system given by (1.20) may give rise to topological chaos.

For example, given $\alpha = 0.5$ fixed, one can numerically derive that $\beta^* = 4.85$ is sufficient. So, given $\beta \geq 4.85$, topological chaos occurs for every sufficiently small s. The parameter s may have to be very small indeed for chaos to occur. However, the simulation results illustrate that the system $F_{s,\beta}$ can have a transverse homoclinic point to the golden rule even for relatively large values of s: Fig.4 illustrates the case of so-called *homoclinic tangles* for the parameter set $(\alpha, \beta, s) = (0.5, 7.5, 0.1)$. These appear as a result of the infinitely many intersections of the stable and unstable manifolds of the saddle type golden rule[13] and the resulting wild oscillations of these manifolds.

insert << Fig.4 >> about here

1.4.2 Homoclinic bifurcations and complicated dynamics

The singular perturbation argument developed in Section 3 to detect horseshoes can be extended to establish the occurrence of homoclinic bifurcations and the resulting complex dynamics for (1.20) (see e.g. van Strien (1981) for a similar argument). This allows us to find β-values[14] for which $F_{s,\beta}$ has a quadratic homoclinic tangency which unfolds generically, associated with the hyperbolic and dissipative saddle golden rule steady state p for small s. Applying the theorems of Newhouse (1979, Theorem 3) and Mora and Viana (1993, Theorem A) (see also Palis and Takens 1993) yields the following proposition:

Proposition 4: *Fix* $\alpha \in (0,1)$ *arbitrarily. Then there exists* $\varepsilon > 0$ *such that for each* $s \in (0, \varepsilon)$ *and for some* $\hat{\beta} = \hat{\beta}(s)$, $F_{s,\hat{\beta}}$ *has a quadratic homoclinic tangency, unfolding generically, associated with the golden rule steady state*

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[13]For $\alpha = \frac{1}{2}$ and $\beta \in (3, 9)$, one may easily check that the golden rule steady state $p = (1,1)$ of $F_{s,\beta}$ in (1.20) is a hyperbolic saddle if $0 < s < s_{PD} = \frac{\beta - 3}{3(1+\beta)}$, and it is a hyperbolic attractor if $s_{PD} < s < s_{NS} = \frac{2}{1+\beta}$, where s_{PD} denotes the period-doubling bifurcation point and s_{NS} denotes the Neimark-Sacker (or Hopf) bifurcation point ($\beta \neq 5, 7$). See Yokoo (1996) for more details about local bifurcations of the golden rule steady state of this parametric model.

[14]Some numerical approximations of homoclinic bifurcation values for model (1.20) can be found in Yokoo (1997).

p. Thus the following assertions hold for all $\delta > 0$*:*

(i) *Coexistence of Infinitely Many Periodic Attractors: There is a non-trivial subinterval* $I \subset [\hat{\beta} - \delta, \hat{\beta} + \delta]$ *and a dense subset* $J \subset I$ *such that for each* $\beta \in J$, $F_{s,\beta}$ *has infinitely many coexisting periodic attractors of arbitrarily large period (The Newhouse Phenomenon).*

(ii) *Abundance of Strange Attractors: There is a positive Lebesgue measure set of* β*-values* $E \subset [\hat{\beta} - \delta, \hat{\beta} + \delta]$ *such that* $F_{s,\beta}$ *exhibits a strange attractor for each* $\beta \in E$*.*

While the occurrence of horseshoes in Proposition 3 does not assure the observability of chaotic behavior in $F_{s,\beta}$ in the long run, the second assertion in Proposition 4 *does*, even for a measure-theoretically large set of parameter values. In this sense, the occurrence of observable chaos is one of the typical dynamical phenomena for system (1.20).

However, the coexistence of infinitely many periodic attractors demonstrated in the first assertion of Proposition 4 might be a *rare* phenomenon; the set of parameter values for which $F_{s,\beta}$ has infinitely many coexisting periodic attractors is conjectured to be of measure zero (see Tedeschini-Lalli and Yorke 1986). Nevertheless, as is known (see e.g. van Strien 1981, Guckenheimer and Holmes 1983, chapter 6), Newhouse Phenomena cannot occur for the singular (i.e., 1-D) map $F_{0,\beta}$. This fact implies that, however small the savings rate $s > 0$ may be, there will be a big qualitative difference in the global dynamics between the singular ($s = 0$) and nonsingular ($s > 0$) maps.

1.4.3 Multiple attractors for a large savings rate: numerical observation

Even though the infinitely many attractors associated with homoclinic bifurcations may hardly occur, the coexistence of a finite number of attractors is certainly a common feature of nonlinear systems (see e.g. Hommes 1991). In this subsection, we observe using computer simulations that the system $F_{s,\beta}$ in (1.20) can simultaneously exhibit stationary, periodic, and chaotic attractors for a certain set of parameter values with the savings rate relatively large[15].

Let us fix the parameter values as follows:

$$\alpha = 0.5, \quad \beta = 6.5, \quad \text{and} \quad s = 0.23.$$

Then at least three coexisting attractors can be observed numerically (see Fig.5). Plotting a bifurcation diagram[16] by computer helps us find these attractors (Fig.6). In fact, from Fig.6 we can see multiple attractors appearing in an overlapping way around $s = 0.23$. The first attractor is the attracting golden rule steady state $p = (1, 1)$. Recall that, as mentioned previously in the footnote, if the savings rate s lies between the period-doubling and Neimark-Sacker bifurcation points, i.e., if $s \in (s_{PD}, s_{NS}) = (\frac{\beta-3}{3(1+\beta)}, \frac{2}{1+\beta}) \approx (0.156, 0.267)$, then the golden rule p is an attractor. The second is a periodic attractor of period eight. The third is (probably) a strange attractor, which may be called a "three-piece" strange attractor because it seems to consist of three isolated pieces.

insert << Fig.5 >> about here

[15] Note that the perturbation argument described above is no longer valid here. In fact, the golden rule stesdy state turns out to be a sink, rather than a dissipative saddle, for our choice of parameter set.

[16] The initial points are randomized for every increment of the parameter value s.

insert << Fig.6 >> about here

Since an attractor will attract all its nearby points, it is interesting to know by which attractor the initial points randomly given on the phase plane are attracted; Fig.7 depicts how the basins of attraction of these three attractors share the phase plane. The boundary of the closure of a basin is called a *basin boundary*. One can see that the structure of some basin boundaries looks very complicated. Such so-called *fractal basin boundaries* may arise due to homoclinic or heteroclinic bifurcations to some periodic points; Brock and Hommes (1997) have presented a computer assisted proof that in a cobweb model with heterogeneous beliefs, fractal basin boundaries are created by heteroclinic bifurcation between the stable and unstable manifolds of two different saddles of period four. Fractal basin boundaries may obstruct the precise prediction of *final states* for given initial states (see e.g. McDonald et al. (1985) for more details about these topics). In this sense, the complexity of basin boundaries provides another type of unpredicability different from that of chaos, defined as the sensitive dependence on initial conditions.

insert << Fig.7 >> about here

1.5 Concluding remarks

We have investigated the dynamics of a simple 2-D OLG model with production and government intervention. Using a singular perturbation technique, we have derived conditions under which topological chaos occurs due to transverse homoclinic orbits to the golden rule steady state when the constant propensity to save is small enough, or in other words, when the 2-D system is nearly 1-D. It turns out that a high elasticity of marginal production function may lead to strong nonlinearity in the capital-share function, which is responsible for the chaotic dynamics of (at least) nearly 1-D systems. We have also given a useful parametric example with a CES production function which exhibits observable chaos associated with homoclinic bifurcations for a large set of parameter values. From a methodological viewpoint, there are several advantages in the perturbation method presented here. This method allows us to prove the existence of transverse and/or tangential homoclinic points in 2-D or even much higher-dimensional systems without the use of a computer,[17] provided the systems can be transformed into tractable 1-D. Furthermore, it may require less specification of function forms or parameter values than other computer assisted methods. Of course, we should also point out that our method, so far, does not take account of the global dynamics far from nearly 1-D. For instance, geometric structures and generation mechanisms for fractal basin boundaries as observed by computer simulations in our OLG model are not well analyzed yet. This will be an important topic for future research.

1.5.1 Appendix

Proof of Lemma 1: (L1.1): Obvious from (A.1). (L1.2): From (A.3). (L1.3): From (L1.2) and (A.4). (L1.4): Since $g'(x) = f'(x) + xf''(x)$, it follows from (A.4) and (A.5) that $g'(1) = 1 + f''(1) < -1$. (L1.5): Note first that $g'(x) = f'(x)(1 - \eta(x))$. Since $\eta(1) = 2$ and $\eta(0) = 0$, and η is strictly increasing by (A.6), there is a unique point $\theta \in (0,1)$ such that $\eta(\theta) = 1$, implying (L1.5.1). Since g is unimodal with its global maximum at $\theta \in (0,1)$ and $g(x) > 0$ for all $x > 0$, it follows that $g(0) = 0 < \theta < 1 < g(\theta)$ and $0 < g^2(\theta) < 1$. Thus there is a unique point q with $g(q) = 1$, which proves

[17] Of course, there may be benefits from combining our reduction and perturbation method with numerical methods: see e.g. Yokoo (1998) for the occurrence of chaos in an n-dimensional production OLG model with adaptive expectations.

(L1.5.2) and (L1.5.3). Q.E.D.

Proof of Lemma 2: (L2.1): First, consider the case when $s = 0$. Putting $a = g^2(\theta)$ and $b = g(\theta)$, we have $0 < a < 1 < b$. We prove the case when $a < \theta$ (the argument for the case when $\theta \leq a < 1$ is similar). Note that $[a, b]$ is invariant under g, i.e., $g([a, b]) = [a, b]$. Given $\varepsilon_0 \in (0, a)$, we can choose $\varepsilon_1 > 0$ such that $g(b + \varepsilon_1) > a - \varepsilon_0$. Let $I_y := [a - \varepsilon_0, b + \varepsilon_1] \subset \mathbb{R}_{++}$, then $g(I_y) \subset \operatorname{int} I_y$. Similarly, given $\varepsilon_0' \in (\varepsilon_0, a)$, we can choose $\varepsilon_1' > \varepsilon_1$ such that $g(b + \varepsilon_1') > a - \varepsilon_0'$. Then $I_x := [a - \varepsilon_0', b + \varepsilon_1'] \subset \mathbb{R}_{++}$ satisfies $I_y \subset \operatorname{int} I_x$ and $g(I_x) \subset \operatorname{int} I_x$. Let $M \subset \mathbb{R}^2_{++}$ be a compact rectangle defined by $M := I_x \times I_y$, then $F_0(M) = I_y \times g(I_y) \subset \operatorname{int} I_x \times \operatorname{int} I_y = \operatorname{int} M$. By definition of M, $p = (1, 1) \in \operatorname{int} M$. Since $h(x, y)$ is continuous on the compact set M, we see that, for any sufficiently small $s > 0$, $F_s(M) \subset \operatorname{int} M$.

(L2.2): Since $h(x, y)$ is obviously C^1 on $\mathbb{R}^2_{++}$, $F_s(x, y) = (y, g(y) + sh(x, y))$ is C^1 on $\mathbb{R}^2_{++}$. Therefore it is sufficient to show that F_s is injective (one-to-one) on $\mathbb{R}^2_{++}$ and that the Jacobian matrix of F_s is nonsingular, i.e., $\det D_x F_s \neq 0$ for $x \in \mathbb{R}^2_{++}$.

Suppose that F_s is not injective. Then there exist two distinct points $a = (a_1, a_2) \in \mathbb{R}^2_{++}$ and $b = (b_1, b_2) \in \mathbb{R}^2_{++}$ $(a \neq b)$ such that $F_s(a_1, a_2) = F_s(b_1, b_2)$. This implies that $a_2 = b_2$ and $h(a_1, a_2) = h(b_1, b_2)$. Thus $h(a_1, b_2) = h(b_1, b_2)$, which implies $w(a_1) = w(b_1)$. But the function $w(x)$ is strictly increasing, since $w'(x) = -xf''(x) > 0$ for $x > 0$. Hence $a_1 = b_1$, which contradicts the hypothesis.

On the other hand, the Jacobian matrix of F_s at every point $z = (x, y) \in$

$\mathbb{R}^2_{++}$ is given by

$$D_zF_s = \begin{bmatrix} 0 & 1 \\ sh_1(x,y) & g'(y) + sh_2(x,y) \end{bmatrix}. \tag{1.21}$$

Hence we have $\det D_zF_s = -sh_1(x,y) = -sxf'(y)f''(x) > 0$.

(L2.3) and (L2.4): Let $\lambda_1(s)$ and $\lambda_2(s)$ with $|\lambda_1(s)| \geq |\lambda_2(s)|$ be the two eigenvalues of (1.21) at $p = (1,1)$, then $\lim_{s\to 0} \lambda_1(s) = g'(1) < -1$ and $\lim_{s\to 0} \lambda_2(s) = 0$. By continuity of $\lambda_i(s)$ $(i = 1,2)$ with respect to s and by $|\det D_pF_s| = |\lambda_1(s)\lambda_2(s)| > 0$ $(s \neq 0)$, the claims follow. Q.E.D.

Proof of Lemma 4: According to Lemma 3, it is sufficient to prove that the 1-D map g has a snap-back repeller for $x = 1$. By Condition (B.1) and (1.5.3) in Lemma 1, the ordering $0 < g^2(\theta) < q < \theta < 1 < g(\theta)$ holds. Let $I := [g^2(\theta), g(\theta)]$, $I_1 := [g^2(\theta), \theta]$, $I_2 := [\theta, 1]$, and $I_3 := [1, g(\theta)]$ be intervals with $I = \cup_{i=1}^3 I_i$, then g is stricity increasing on I_1 and g is strictly decreasing on $I_2 \cup I_3$. We claim that

(C1) every $x \in I$ has a backward orbit which is contained in I and converges to 1, and

(C2) every $x \in I_2 \cup I_3 \setminus \{\theta, g(\theta)\}$ has a backward orbit which is contained in $I_2 \cup I_3 \setminus \{\theta, g(\theta)\}$ and converges to 1.

(C1): since $g(I_2) = I_3$ and $g(I_3) = g^2(I_2) = I_1 \cup I_2$, it follows that for every $x \in I$ there is a point $y \in I_2$ with $g^m(x) = y$ for some $2 \geq m \geq 0$. Hence it suffices to verify that every $x \in I_2$ has a backward orbit for g^2 which is contained in I_2 and converges to 1. Condition (B.2), together with $(g^2)'(1) = (g'(1))^2 > 1$ and $g^2(\theta) < \theta$, implies that $g^2(x) < x$ holds for every $x \in I_2 \setminus \{1\}$. Since $g^2(x_0) < x_0$ and $g^2(1) = 1 > x_0$ for every $x_0 \in I_2 \setminus \{1\}$, it follows that there is $x_{-1} \in (x_0, 1)$ with $x_0 = g^2(x_{-1})$, and, inductively, that there is a strictly increasing sequence of points $\{x_0, x_{-1}, x_{-2}, ...\} \subset I_2$ such that $g^2(x_{-i-1}) = x_i$ for $i \geq 0$ and $x_{-i} \to 1$ as $i \to \infty$. This proves (C1). (C2): recall that g is strictly monotone on $I_2 \cup I_3$ and maps I_2 onto I_3 homeomorphically; then (C2) is obvious from the proof of (C1).

On the other hand, it follows by Condition (B.1) and (1.5.3) in Lemma 1, that $g^2(\theta) < q$ and $g^2(1) = 1 > q$. So there is a point $q' \in I_2 \setminus \{\theta, 1\}$ such that $g^2(q') = q$. By (C2), q' has a backward orbit $O^-(q') \subset I_2 \cup I_3 \setminus \{\theta, g(\theta)\}$ satisfying (S2). Since the forward orbit of q', $O^+(q') = \{q', g(q') \in I_3, g^2(q') = q, g^3(q') = 1\}$, satisfies (S1) and does not contain θ, the so obtained orbit of q', $O(q') = O^+(q') \cup O^-(q')$ satisfies (S1), (S2), and (S3). Q.E.D.

Proof of Proposition 1: By Lemma 4, some compact parts $\gamma^{s/u}$ of the stable und unstable manifolds $W^{s/u}(p, F_0)$ have a transverse intersection. In particular, $W^u(p, F_0)$ contains a parabolic arc $\gamma^u = \{(x, y) \in \mathbb{R}^2_+ \,|\, x \in [g^2(\theta), g(\theta)],\, y = g(x)\}$ and $W^s(p, F_0)$ contains a compact horizontal line segment $\gamma^s = \{(x, y) \in \mathbb{R}^2_+ \,|\, x \in [g^2(\theta), g(\theta)],\, y = 1\}$. Clearly, γ^u and γ^u have a transverse intersection at $(q, 1) \in \mathbb{R}^2_+$. By the perturbation argument of invariant manifolds for noninvertible maps (see Palis and Takens 1993, Appendix 1 and 4), we see that the compact arc of $W^u(p, F_0)$ and the compact arc consisting of regular points[18] of $W^s(p, F_0)$ vary continuously on the map in the C^1 sense. This means that, for every sufficiently small $s > 0$, some compact arcs $\hat{\gamma}^{s/u}$ sufficiently C^1-close to $\gamma^{s/u}$ are contained in $W^{s/u}(p, F_s)$, respectively. Since transverse intersections are stable in the

[18]See Appendix 4 in Palis and Takens (1993). Let $K \subset W^s(p)$ be any compact set. Then, for some n, $F^n(K) \subset W^s_{loc}(p)$. We say that the points of K are *regular* points of $W^s(p)$ if for each $x \in K$, $Im(D_x F^n) + T_{F^n(x)}(W^s_{loc}(p)) = \mathbb{R}^2$. So the arc γ^s contains only regular points of $W^s(p)$.

C^1 sense, $\hat{\gamma}^s$ and $\hat{\gamma}^u$ above do have a transverse intersection. Since for all sufficiently small $s > 0$, the fixed point p is hyperbolic and the homoclinic orbit to p, obtained above, is contained in $\operatorname{int} M$ by Lemma 2, the map F_s has, by the Homoclinic Point Theorem, a horseshoe $\Lambda_s \subset M$ for each $s \in (0, \varepsilon)$, for some $\varepsilon > 0$. Q.E.D.

Proof of Proposition 2: Note that $\det D_x F_s = -sxf'(y)f''(x)$. We can pick a small number $\delta > 0$ so that $1 > \delta \cdot \max_{(x,y)\in M} |xf'(y)f''(x)| > 0$. Let $\varepsilon > 0$ be as in Proposition 1 and let $\varepsilon' = \min\{\varepsilon, \delta\}$; then for every $s \in (0, \varepsilon')$, M is a trapping region for F_s and $F_s|M$ satisfies the following

- $W^s(p) \cap W^u(p) \setminus \{p\} \neq \phi$,
- $W^u(p) \subset M$, and
- $|\det D_x F_s| < 1$ for every $x \in M$.

We can take a bounded region $U_s \subset M$ whose boundary consists of segments of $W^s(p)$ and $W^u(p)$, and now apply Proposition 1 in Appendix 3 in Palis and Takens (1993). Q.E.D.

Proof of Proposition 3 : We know that if $\beta > (1+\alpha)/(1-\alpha)$ then f_β satisfies Condition (A). By Proposition 1 in Section 3, all we need is to verify that the function g_β satisfies Condition (B) for every sufficiently large β. In what follows, we assume $\beta > (1+\alpha)/(1-\alpha)$.
(B.1): We first note that $f_\beta(0) = g_\beta(0) = 0$ and $f'_\beta(0) = \alpha^{-(\beta+1)/\beta}$. Since $w(x) = f_\beta(x) - g_\beta(x)$ is strictly incresing and f_β is strictly concave, it follows that $g_\beta(x) < f_\beta(x) < x/\alpha^{1+\frac{1}{\beta}}$ for $x > 0$ and $0 < \alpha^{1+\frac{1}{\beta}} < q(\beta)$, where $g_\beta(q(\beta)) = 1$ and $q(\beta) \in (0, \theta(\beta))$. Since $\lim_{\beta\to\infty} \alpha^{1+\frac{1}{\beta}} = \alpha$, it is

sufficient to show that $\lim_{\beta\to\infty} g_\beta^2(\theta(\beta)) \to 0$. Considering that

$$g_\beta(\theta(\beta)) = \frac{1}{\alpha(1-\alpha)^{\frac{1}{\beta}}\beta^{\frac{1}{\beta}}(1+\frac{1}{\beta})^{1+\frac{1}{\beta}}} \longrightarrow \frac{1}{\alpha}(>1) \quad \text{as} \quad \beta\to\infty,$$

and that $\lim_{\beta\to\infty} g_\beta(x) \to 0$ holds for each $x > 1$, we obtain the last claim.

(B.2): Since

$$g_\beta''(x) = \frac{(1-\alpha)(1+\beta)x^{\beta+1}[\beta(1-\alpha)x^\beta - \alpha(1+\beta)]}{[\alpha+(1-\alpha)x^\beta]^{\frac{3\beta+1}{\beta}}},$$

it follows that g_β has a unique inflection point

$$\tilde{\theta}(\beta) = \left(\frac{\alpha(1+\beta)}{\beta(1-\alpha)}\right)^{\frac{1}{\beta}} > \theta(\beta),$$

so that $g_\beta''(x) \geq (<)0$ as $x \geq (<)\,\tilde{\theta}(\beta)$ and the critical point $\theta(\beta)$ is quadratic, i.e., $g_\beta''(\theta(\beta)) \neq 0$.

Let us define a continuous piecewise linear map $\psi_{a,b} : \mathbb{R}\to\mathbb{R}$ with two parameters $a > 0$ and $b > 0$ by

$$\psi_{a,b}(x) := \begin{cases} -a(x-1)+1 =: l_1(x), & \text{for } x \leq 1 \\ -b(x-1)+1 =: l_2(x), & \text{for } x > 1. \end{cases}$$

By virtue of the uniqueness of the inflection point, one can check that if we let us define $a(\beta)$ and $b(\beta)$ as

$$a(\beta) := \min\left\{|g_\beta'(1)|, \left|\frac{g_\beta(\theta(\beta))-1}{\theta(\beta)-1}\right|\right\}$$

and

$$b(\beta) := \min\left\{|g_\beta'(1)|, \left|\frac{g_\beta^2(\theta(\beta))-1}{g_\beta(\theta(\beta))-1}\right|\right\},$$

then $\psi_{a(\beta),b(\beta)}(x) = l_1(x) < g_\beta(x)$ for $x \in (\theta(\beta), 1)$ and $\psi_{a(\beta),b(\beta)}(x) = l_2(x) > g_\beta(x)$ for $x \in (1, g_\beta(\theta(\beta)))$. We claim that if $a(\beta)b(\beta) > 1$ then $g_\beta^2(x) < x$ for all $x \in (\theta(\beta), 1)$. To see this, note first that for $x \in (\theta(\beta), 1)$, we have $\psi^2_{a(\beta),b(\beta)}(x) = l_2 \circ l_1(x) > l_2 \circ g_\beta(x) > g_\beta^2(x)$. And note that if $a(\beta)b(\beta) > 1$ and $x < 1$, then $\psi^2_{a(\beta),b(\beta)}(x) - x = l_2 \circ l_1(x) - x = (1 - a(\beta)b(\beta))(1 - x) < 0$. Combining these inequalities, we get the claim.

To complete the proof, it is then sufficient to show that $a(\beta)b(\beta) > 1$ for β large enough. But this follows from the fact that

$$\lim_{\beta \to \infty} |g'_\beta(1)| = \lim_{\beta \to \infty} |\alpha - \beta(1 - \alpha)| \to \infty, \qquad \lim_{\beta \to \infty} |\theta(\beta) - 1| = 0,$$

$$\lim_{\beta \to \infty} |g_\beta(\theta(\beta)) - 1| = \frac{1 - \alpha}{\alpha}, \quad \text{and} \qquad \lim_{\beta \to \infty} |g_\beta^2(\theta(\beta)) - 1| = 1.$$

Q.E.D.

Proof of Proposition 4: Given $\alpha \in (0, 1)$, let $\beta \geq \beta^*$, where β^* is as given in Proposition 3. Note first that the unstable manifold $W^u(p, F_{0,\beta})$ contains a compact parabolic arc $\gamma^u(\beta) = \{(x, y) \in \mathbb{R}^2_+ \,|\, x \in [q(\beta), 1],\, y = g_\beta(x)\}$. Next, we can observe that there is a sequence of points, depending upon β, which are eventually mapped to the fixed point $x = 1$:

$$\begin{aligned} Q(\beta) = \{q_i(\beta) \in [0, \theta(\beta)] \quad | \quad & q_i(\beta) = g_\beta(q_{i+1}(\beta)) \text{ for } i \in \mathbb{N}, \\ q \quad = \quad & q_1 > q_2 > \cdots;\, q_i \to 0 \text{ as } i \to \infty\}. \end{aligned}$$

Since for every $x \in (0, 1)$, $g_\beta(x) \to x/\alpha$ as $\beta \to \infty$ and so the increasing part of the graph of g_β converges to the line segment $y = x/\alpha$ as $\beta \to \infty$, we have $q_i(\beta) \in Q(\beta) \to \alpha^i$ as $\beta \to \infty$. On the other hand, by the proof of Proposition 3, we have $g_\beta^2(\theta(\beta)) \to 0$ as $\beta \to 0$. Hence, given $\beta_1 \geq \beta^*$, there is $k \in \mathbb{N}$ with $g^2_{\beta_1}(\theta(\beta_1)) > q_k(\beta_1)$, and there is $\beta_2 > \beta_1$ with

$g_{\beta_2}^2(\theta(\beta_2)) < q_k(\beta_2)$. So $g_{\beta_3}^2(\theta(\beta_3)) = q_k(\beta_3)$ for some $\beta_3 \in (\beta_1, \beta_2)$. Consequently, $W^s(p, F_{0,\beta})$ contains a horizontal compact line segment $\gamma^s(\beta)$ such that (i) $\gamma^u(\beta_1)$ and $\gamma^s(\beta_1)$ have no intersection, (ii) $\gamma^u(\beta_2)$ and $\gamma^s(\beta_2)$ have two transverse intersections, and (iii) $\gamma^u(\beta_3)$ and $\gamma^s(\beta_3)$ have a quadratic tangency at $(\theta(\beta_3), g_{\beta_3}(\theta(\beta_3))) \in \mathbb{R}_+^2$.

By the same perturbation argument used in the proof of Proposition 1, for every sufficiently small $s > 0$, the stable and unstable manifolds $W^{s/u}(p, F_{s,\beta})$ contain arcs $\hat{\gamma}^{s/u}(\beta)$, sufficiently C^r-close ($r \geq 1$) to $\gamma^{s/u}(\beta)$. These satisfy the 'inevitable tangency' condition (see Takens 1992 for weakened generic conditions for real-analytic diffeomorphisms):

(i) $\hat{\gamma}^s(\beta_1) \subset W^s(p, F_{s,\beta_1})$ and $\hat{\gamma}^u(\beta_1) \subset W^u(p, F_{s,\beta_1})$ have no intersection;

(ii) $\hat{\gamma}^s(\beta_2) \subset W^s(p, F_{s,\beta_2})$ and $\hat{\gamma}^u(\beta_2) \subset W^u(p, F_{s,\beta_2})$ have two transverse intersections.

Hence, for $s > 0$ small enough, we get a homoclinic bifurcation value $\beta = \hat{\beta}(s) \in (\beta_1, \beta_2)$ at which $\hat{\gamma}^s(\hat{\beta}) \subset W^s(p, F_{s,\hat{\beta}})$ and $\hat{\gamma}^u(\hat{\beta}) \subset W^u(p, F_{s,\hat{\beta}})$ have a homoclinic tangency.

Since Takens' generic conditions (inevitable tangency, analyticity of the map, and non-constantness of $-\log(\lambda_1(\beta))/\log(\lambda_2(\beta))$ with respect to β, where $\lambda_1(\beta)$ and $\lambda_2(\beta)$ are eigenvalues of $D_pF_{s,\beta}$) are satisfied, the homoclinic tangency obtained above is quadratic and unfolds generically. Recall that the fixed point p is hyperbolic and dissipative for every sufficiently small $s > 0$ by Lemma 2. Now apply Theorem 3 from Newhouse (1979) for the first assertion of Proposition 4, and Theorem A from Mora and Viana (1993) for the second assertion. Q.E.D.

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Chapter 2

Complex Dynamics in a Cobweb Model with Adaptive Production Adjustment

abstract

The present paper[1] considers a nonlinear cobweb model in which 'cautious' suppliers gradually adjust their production amount over time toward the target level based on the naive expectation formation rule. We show that for a large set of parameter values the cobweb market exhibits topological chaos (horseshoes) as well as observable chaos (strange attractors) associated with the homoclinic orbits. Some numerical simulations are carried out to suggest that the faster suppliers adjust their production and the more inelastic demand is, the more likely the market behaves chaotically.

[1]This essay is based on the joint paper with Tamotsu Onozaki and Gernot Sieg, which is published in the Journal of Economic Behavior and Organization, 2000, with some modifications. I thank an anonymous referee and the editor of the Journal of Economic Behavior and Organization for helpful comments.

2.1 Introduction

Once time is taken into consideration in modelling e.g. agricultural economies, it is natural to postulate that there is some time lag between the period when the decision how much to produce is done by the farmer and the period when the product becomes ready for sale. Consequently, the farmer has to forecast the future price upon which he determines the amount of production. In the literature, one prevalent way of modelling how the anticipated prices are formed is the rational (including perfect foresight) or model-consistent expectations. There is some empirical evidence, however, that the assumption of such model-consistent expectations are not so plausible. To name a few, Ito (1990) and Hey (1994) test the hypothesis that expectations are model-consistent and reject it.

As for alternative ways of modelling expectations, some experimental evidence suggests that agents use past market prices to forecast and follow rules of thumb. Williams (1987), e.g., shows that the adaptive expectation hypothesis (including naive forecasts, which are a special case of adaptive expectations) describes expectations in an experimental double-auction market better than the extrapolative one. This result seems consistent with our intuition on the possible common behavior of the farmer who uses the most recently received price as his prediction for the next period.

Dynamics of a cobweb market under such naive expectations have been well documented in the formal cobweb literature. See Kaldor (1934), Leontief (1934) and Ezekiel (1938), among others, for an early stage of the literature. In models with a normal upward sloping supply curve and a downward sloping demand curve only three types of simple dynamics are possible: convergence to an equilibrium, two-period cycles or exploding oscillations. If expectations are not naive but adaptive, price behavior in the model with linear supply and demand is also simple [Nerlove (1958)]. In reality, as empirical evidence suggests, the behavior of prices in agricultural markets is not so simple, whereas many agricultural markets are regulated to stabilize prices. See for example Finkenstädt (1995) for volatile price movements of egg, potato and pig in Northern Germany.

¿From theoretical viewpoints, Artstein (1983), Jensen/Urban (1984), Lichtenberg/Ujihara (1989) and Day/Hanson (1991) show that complex price behavior is possible if at least either demand or supply is non-monotonic. Hommes (1991, 1994), Finkenstädt/ Kuhbier (1992) and Finkenstädt (1995) find chaotic behavior in normal markets with adaptive expectations when both supply and demand are monotonic but at least one of them is nonlinear.[2] This literature assumes that farmers make a best (optimizing) response given current expectations.

The present paper provides an alternative rule which specifies that farmers adjust partially in the direction of the best current response under naive expectations, keeping both the demand and supply curves monotonic. Such adjustment is a behavioral response to uncertainty and adjustment costs. In order to show that our model exhibits topological and observable chaos, we exploit some mathematical results concerning homoclinic points. The occurrence of topological chaos is proved by applying the classical Homoclinic Point Theorem which asserts that a transverse homoclinic orbit implies a horseshoe. Under some differentiability, this result is a little sharper than that by the Li-Yorke Theorem in regard to continuous maps on interval. Furthermore, the occurrence of observable chaos (i.e., strange attractors) for a large set of parameter values is shown, under some minor assumption, with the aid of a recent result concerning the homoclinic bifurcation

[2]See Lorenz (1993) for a general introduction to complex economic behavior due to various kind of nonlinearities.

by Mora/Viana (1993). In this way, we detect chaotic behavior in a theoretically large and empirically relevant region of price elasticities of demand and adjustment speeds. The faster suppliers adjust their production and the more inelastic the demand is, the more likely the market behaves chaotically.

2.2 The basic model

At period t, a supplier decides his production x_{t+1} for period $t+1$. As he knows well, even a production plan that maximizes profits may turn out to be a disaster in reality. He calculates the profit maximum $\tilde{x}_{t+1}$ and uses it as a target of adjustment. The calculation is done subject to the quadratic cost function $\frac{b}{2}x^2,\ b > 0$ and naive price expectation, which means that his price expectation for the next period is equal to the current price p_t. The resulting amount is

$$\tilde{x}_{t+1} = \frac{p_t}{b}. \tag{2.1}$$

He has to adjust cautiously since every theoretically advantageous change may or may not enlarge real profits. Therefore, he is assumed not to produce $\tilde{x}_{t+1}$ immediately but to adjust adaptively his last period's production in the direction of $\tilde{x}_{t+1}$. This is a simple hedging rule in the uncertain real world and is expressed as the equation:

$$x_{t+1} = x_t + \alpha\left(\tilde{x}_{t+1} - x_t\right), \tag{2.2}$$

where $\alpha \in (0,1)$ is the speed of adjustment. This equation, which can be rationalized by adjustment costs, is one of the earliest ways of incorporating adaptive processes explicitly into economic models [Nerlove (1958)] and is often used in econometric studies of macroeconomic behaviors.[3]

[3]For a survey of adaptive behavior, see Day (1998).

In order to bridge the gap between a single supplier and the market as a whole, we suppose that all n suppliers are homogeneous and behave identically. Therefore, the aggregate supply X is given by

$$X_t = nx_t. \tag{2.3}$$

We assume the following monotonic inverse demand function with constant price elasticity of $1/\beta$ $(\beta > 0)$:

$$p_t = \frac{c}{Y_t^{\beta}}, \tag{2.4}$$

where Y_t is demand at period t and c is a positive shift parameter, which can be regarded as the extent of the market.

Finally, price is set so that the market clears at each period:

$$Y_t = X_{t.} \tag{2.5}$$

Summarizing the model, we substitute (2.1), (2.3), (2.4) and (2.5) into (2.2) and obtain the one-dimensional, discrete-time dynamical equation:

$$X_{t+1} = (1-\alpha)X_t + \frac{\alpha cn}{bX_t^{\beta}}. \tag{2.6}$$

To make the analysis below easier, let us consider the variable transformation:

$$X_t := \left(\frac{b}{cn}\right)^{-\frac{1}{1+\beta}} z_t.$$

Substituting into (2.6), we get

$$z_{t+1} = (1-\alpha)\, z_t + \frac{\alpha}{z_t^{\beta}}. \tag{2.7}$$

Because the transformation is linear, X_t behaves periodically if z_t does so, and X_t behaves aperiodically if z_t does so, regardless of the number of suppliers n, the slope of marginal cost b and the extent of the market c. These parameters change the value of z_t into that of X_t through the scalar $\left(\frac{b}{cn}\right)^{-\frac{1}{1+\beta}}$. Essential parameters for the qualitative behavior of our model are the adjustment speed of production α and price elasticity of demand $1/\beta$. In what follows, we concentrate only on the dynamics of (2.7).

2.3 Analysis of the model

Our model (2.7) can be reformulated by the two-parameter family of maps $f_{\alpha,\beta} : \mathbb{R}_{++} \to \mathbb{R}_{++}$ as

$$f_{\alpha,\beta}(z) = (1-\alpha)z + \frac{\alpha}{z^{\beta}}, \qquad (\alpha,\beta) \in (0,1) \times (0,\infty), \tag{2.8}$$

which are also expressed as f_β or simply f. Note that for any pair $(\alpha,\beta) \in (0,1) \times (0,\infty)$, $z^* = 1$ is the unique fixed point for f, i.e.,

$$f(z^*) = z^* \Leftrightarrow z^* = 1.$$

The first and second derivatives are calculated as

$$\begin{aligned} f'(z) &= 1 - \alpha - \frac{\alpha\beta}{z^{1+\beta}} \\ f''(z) &= \frac{\alpha\beta(1+\beta)}{z^{2+\beta}} > 0 \qquad z \in \mathbb{R}_{++}, \end{aligned}$$

which imply that f is a strictly convex and unimodal function on $\mathbb{R}_{++}$ (see Fig.1) with its minimum at the critical point

$$z = \left[\frac{\alpha\beta}{1-\alpha}\right]^{\frac{1}{1+\beta}} =: \theta(\alpha,\beta) \ \ (= \theta(\beta) = \theta).$$

The fixed point is a repeller if

$$f^{'}(1) = 1 - \alpha - \alpha\beta < -1,$$

which is equivalently rewritten as

$$\beta > \frac{2-\alpha}{\alpha}. \tag{2.9}$$

*** *Insert Fig.1 about here* ***

In the following subsections we present two propositions concerning the complex behavior of our model. The proofs and the related fundamental notions of symbolic dynamics are given in the appendix.

2.3.1 Existence of horseshoes

First we present a proposition which states that for every sufficiently large β, the map f_β exhibits a *horseshoe* . By a horseshoe we mean here a compact invariant set on which some iterate of f_β is topologically conjugate to the one-sided full-shift on two symbols. The existence of a horseshoe is assured by that of a transverse homoclinic point. We say that a map exhibits *topological chaos* either if it has a horseshoe or, alternatively, if the topological entropy of the map is positive.[4] Although a map restricted on horseshoes behaves in a complicated way, the existence of horseshoes itself does not assure complex dynamics in the long run; the economic system may eventually settle down to a periodic motion even if horseshoes are present. Nevertheless, horseshoes may often generate long-lasting complicated transient dynamics,

[4] See, e.g. Block/Coppel (1992) for details about topological entropy.

and even small external shocks are likely to give rise to erratic motions of a system which are otherwise periodic in the long run. Finding horseshoes in our model is, therefore, not insignificant even from an empirical viewpoint.

Our result is summarized as follows:

Proposition 1: *Fix an $\alpha \in (0,1)$ arbitrarily. Then there exists a number $\bar{\beta} = \bar{\beta}(\alpha) > \frac{2-\alpha}{\alpha}$ such that f_β in (2.8) has a horseshoe for each $\beta \geq \bar{\beta}$.*

One of the important features of a horseshoe (more precisely, a hyperbolic set) is the stability of the associated map against C^r –perturbations ($r \geq 1$) [see e.g. de Melo/van Strien (1993), p.225, Theorem 2.3]. Roughly speaking, once the economic system described by f_β possesses horseshoes, they will be preserved despite small changes in the underlying economic structure.

2.3.2 Existence of strange attractors

Next we present a proposition which states that, under some generic condition, our model frequently exhibits *observable chaos* in the sense of *strange attractors*. While horseshoes do not assure complex dynamics in the long run, strange attractors do assure that we can observe erratic behavior for some large set of initial conditions. Hence, frequent occurrence of observable chaos seems to be useful in explaining irregular behavior of economic time series.

To state our proposition, we first introduce some basic notions.

Definition: A compact invariant set $\mathcal{A} \subset \mathbb{R}$ for f is called an *attractor* if its stable set $W^s(\mathcal{A}) = \{z \in \mathbb{R} : \lim_{n\to\infty} d(f^n(z), \mathcal{A}) = 0\}$ contains a nonempty interior and f has a dense orbit in $\mathcal{A}$. An attractor $\mathcal{A}$

is here said to be *strange* if it contains a dense orbit with positive Lyapunov exponent, i.e., there is a point $z \in \mathcal{A}$ for which $\overline{\{f^n(z)\}_{n\geq 0}} = \mathcal{A}$ and $\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} \log|f'(f^{i-1}(z))| > 0$.

Strange attractors will appear via *homoclinic bifurcation*, that is, when a non-degenerate homoclinic tangency unfolds generically. In other words, they will appear when the critical point $\theta(\beta)$ with $f''(\theta) \neq 0$ which is contained in a homoclinic orbit to the repeller $z^* = 1$ for some $\beta = \beta^*$ passes through z^* at non-zero speed for some iterate of f_β as β varies. Formally, $\frac{d}{d\beta} f_\beta^n(\theta(\beta)) \neq 0$ at $\beta = \beta^*$ for some n with $f_{\beta^*}^n(\theta(\beta^*)) = 1$. See Mora/Viana(1993) for a full explanation.

The following result shows that our model may exhibit observable chaos for measure-theoretically large sets of parameter values.

Proposition 2: *For any* $\alpha \in (0,1)$, *there is generically a positive measure set of parameter values of* β, $E \subset \mathbb{R}_+$, *such that for every* $\beta \in E$ *the map* f_β *exhibits a strange attractor.*

2.4 Numerical simulations of the model

In this section we perform some numerical simulations and show that those are supported by the theoretical results in the previous section.

First we depict the standard, one-parameter bifurcation plot of (2.8) with respect to β in Fig.2 and the corresponding topological entropy (TE) and Lyapunov exponent (LE) in Fig.3 and 4.[5] Intuitively, topological entropy measures the exponential growth rate of the number of foldings of the graph of the nth iterate of a map. By definition $TE \geq 0$, and if $TE > 0$ then the

[5] To calculate topological entropy, we utilized the algorithm presented by Block/Keesling/Li/Peterson (1989).

map exhibits topological chaos. From Fig.3 we can observe that topological chaos occurs in our model for values of β larger than $\beta_{TE>0} \approx 3.0008$. Proposition 1 states that our model may exhibit topological chaos when β satisfies the condition (2.9). We can confirm this as follows: the value of α used in the calculation of TE is 0.7 and substituting into $\beta = (2-\alpha)/\alpha$ gives $\widehat{\beta} \approx 1.857 < \beta_{TE>0}$ which is found in Fig.2.[6]

*** *Put Fig.2-4 side by side vertically with the same scale about here* ***

The Lyapunov exponent expresses the exponential rate of divergence between two arbitrarily close orbits as time elapses. If $LE > 0$ then the map exhibits observable chaos in the sense that it has strange attractors, and no stable-periodic orbit has a positive Lyapunov exponent. Values of β such that Lyapunov exponents have a positive sign in Fig.4 correspond to those of shaded area in Fig.2, and observable chaos occurs for such values of β.

As stated in the previous section, the observable motions may be indeed periodic even if topological chaos is present. Comparing Fig.2 and 3, it is realized that in the region of $TE > 0$ there exist windows of periodic behavior. On the other hand, if there appear periodic motions then $LE \leq 0$. Therefore, we can classify chaos to be present in our model as follows:

$$TE > 0 : \text{topological chaos} \quad \begin{cases} LE > 0 : \text{observable chaos}, \\ LE \leq 0 : \text{windows (latent chaos)}. \end{cases}$$

[6] $\widehat{\beta}$ is exactly the period-doubling bifurcation value at which a single stable fixed point splits into a stable period-2 cycle.

In addition, the Schwarzian derivative of f at z is

$$Sf(z) = -\frac{\alpha\beta(1+\beta)\left[\alpha\beta(\beta-1) + 2(1-\alpha)(2+\beta)z^{1+\beta}\right]}{2\left[(\alpha-1)z^{2+\beta} + \alpha\beta z\right]^2},$$

and it is negative if (2.9) is satisfied. Therefore, f has at most one periodic attractor.[7]

As mentioned above, the essential parameters in our model are α and β. Thus a question arises here: what happens to the above bifurcation plot if α also varies? To answer this, we draw two kinds of 2-parameter diagrams after the manner of Gallas/Nusse (1996): one is an *iso-period plot* and the other is an *observable chaos plot*. The former is the union of all iso-period-p plots for $p \in [1, \bar{p}] \subset \mathbb{N}$ (in this paper $\bar{p} = 64$). And each iso-period-p plot is made of the set in the parameter space such that for each element in this set the trajectory through some fixed initial point x_0 converges to a stable period-p cycle. The latter consists of the set in the parameter space such that for each element in this set the orbit through some fixed initial point x_0 is observably chaotic in the sense that it has a positive Lyapunov exponent.

*** *Insert Fig.5, 6 about here* ***

We consider the region $S = \{(\alpha, \beta) \mid 0 < \alpha < 1,\ 0 < \beta \leq 10\}$. The resulting iso-period plot is shown in Fig.5. In this figure, the green area exhibits pairs of parameter values for which every trajectory converges to a unique stable fixed point. The blue area consists of pairs of parameter

[7] See, e.g. Devaney (1989, pp.69f.).

values for which every trajectory converges to a period-2 cycle. The light-blue area corresponds to a period-3 cycle, the yellow area corresponds to a period-4 cycle, the magenta area corresponds to a period-6 cycle, and the red area corresponds to a period-8 cycle, etc. We emphasize that the border between the green area and the blue area is expressed by the equation $\beta = (2 - \alpha)/\alpha$, the upper region of which satisfies (2.9); therefore, the fixed point of the model is unstable there.

The resulting chaos plot is presented in Fig.6. The black area in this figure is the set of parameters for which our model exhibits observable chaos in the sense of a positive Lyapunov exponent. This figure implies that observable chaos occurs when α and β are large. In other words, the faster suppliers adjust their production and the more inelastic demand is, the more likely the market behaves chaotically.

Finally it should be stressed that these figures suggest that periodic behavior and observably chaotic behavior are complementary in the sense that the union of the colored area in Fig.5 and the black region in Fig.6 is equal to the whole region of S. But unfortunately, as Gallas/Nusse (1996) point out, there exists no theoretical result which assures this fact.

2.5 Concluding remarks

We have investigated the dynamics of a nonlinear cobweb model where suppliers adjust cautiously to hedge against the uncertain world. If suppliers adapt slowly, they may stabilize the market. Adaptive adjustment could be a reasonable strategy to prevent large price fluctuations. Whether the cautious behavior stabilizes the market effectively depends on how much consumers change their demand as price changes.

It is well-known that price elasticities of demand for essential goods like food are relatively low. In fact, according to estimates by Pagoulatos/Sorensen (1989), the majority of U.S. food and tobacco industries has price elasticities of less than 1/5. In such markets, our model predicts that chaos, observable or unobservable, will occur even if adjustment is rather cautious.

The main difference between our model and existing ones consists of the adaptive adjustment hypothesis. Future research is required to better understand the adaptive behavior of economic agents. Our model is so simple that we consider it a mere stepping-stone; nevertheless, it shows that the adaptive adjustment approach is a promising research agenda for explaining complex economic phenomena.

2.6 Appendix

2.6.1 Some fundamental notions of symbolic dynamics

Let Σ_2 denote the set of all infinite sequences $s = (s_1, s_2, s_3, ...)$, where $s_i = 0$ or 1 for $i \in \mathbb{N}$. We define a metric on Σ_2 by the function

$$d(s,t) = \sum_{i=1}^{\infty} \frac{|s_i - t_i|}{2^i}, \qquad s,\, t \in \Sigma_2.$$

The metric space (Σ_2, d) is then compact, totally disconnected, and perfect, i.e., it is a Cantor set. The *shift* map $\sigma : \Sigma_2 \to \Sigma_2$ is defined by $\sigma((s_1, s_2, ...)) = (s_2, s_3, ...)$, which is referred to as the *one-sided shift on two symbols.*

The shift map $\sigma : \Sigma_2 \to \Sigma_2$ has the following properties:

(i) Σ_2 contains a countably infinite set of periodic orbits;

(ii) Σ_2 contains an uncountably infinite set of aperiodic orbits;

(iii) the set of periodic points is dense in Σ_2;

(iv) $\sigma : \Sigma_2 \to \Sigma_2$ is *topologically mixing*, i.e., for every pair of nonempty open sets $U, V \subset \Sigma_2$, there is $m \geq 1$ such that $\sigma^n(U) \cap V \neq \phi$ for all $n \geq m$;

(v) $\sigma : \Sigma_2 \to \Sigma_2$ is *expansive*, i.e., there is $\delta > 0$ such that for any $s, t \in \Sigma_2\,(s \neq t)$, there is $m \geq 1$ with $d(\sigma^m(s), \sigma^m(t)) \geq \delta$.

By definition, topological mixing property implies *topological transitivity*, and expansiveness implies *sensitive dependence on initial conditions.*

Let X and Y be metric spaces, and let $f : X \to X$ and $g : Y \to Y$ be continuous maps. The map f is said to be *topologically conjugate* (or *equivalent*) to g if there exists a homeomorphism $h : X \to Y$ (one-to-one, onto, continuous map with continuous inverse) such that

$$h \circ f(x) = g \circ h(x) \qquad \text{for every } x \in X,$$

i.e., the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow[g]{} & Y \end{array}$$

commutes. The map f is said to be *topologically semi-conjugate* to g if there is an endomorphism h of X *onto* Y (i.e., continuous map of X onto Y) such that $h \circ f = g \circ h$.

Let us next consider a C^1-map f of (an interval of) $\mathbb{R}$ into itself. Let $p \in \mathbb{R}$ be a repelling hyperbolic fixed point (or repeller) of f, i.e., $f(p) = p$ and $|f'(p)| > 1$. Let $q \in \mathbb{R}\,(q \neq p)$ be a point such that $f^n(p) = q$ for some $n \in \mathbb{N}$ and that there exists a sequence $\{q_{-i}\}_{i=0}^{\infty}$ with the property that $q = q_0$, $f(q_{-i-1}) = q_{-i}\,(i \geq 0)$ and $q_{-i} \to p\,(i \to \infty)$. The sequence $\mathcal{HO}_f(q,p) := \{q, f(q), f^2(q), ..., f^n(q) = p\} \cup \{q_{-i}\}_{i=0}^{\infty}$ is said to be a *homoclinic orbit* of

q to p. An element of the homoclinic orbit $x \in \mathcal{HO}_f(q,p)$ is a homoclinic point to p. A homoclinic orbit $HO_f(q,p)$ is said to be *transverse* if $f'(z) \neq 0$ for every $z \in \mathcal{HO}_f(q,p)$.

Finally we introduce an important theorem which is utilized to prove proposition 1.

Theorem (Homoclinic Point Theorem for C^1-Map on Interval):[8] *Let $f: \mathbb{R} \to \mathbb{R}$ be a C^1-map and $p \in \mathbb{R}$ be a repelling hyperbolic fixed point. If f has a transverse homoclinic orbit to p, then there exist a number $n \in \mathbb{N}$ and a compact set $\Lambda \subset \mathbb{R}$ such that*

(i) $f^n(\Lambda) = \Lambda$;

(ii) $p \in \Lambda$;

(iii) $f^n|\Lambda : \Lambda \to \Lambda$ *is topologically conjugate to the one-sided full-shift on two symbols $\sigma : \Sigma_2 \to \Sigma_2$, i.e., there is a homeomorphism $h : \Lambda \to \Sigma_2$ with $h \circ f^n|\Lambda = \sigma \circ h$.*

The set Λ here is called a *horseshoe,* and if the map f has such a set, we say that the map f has a horseshoe. By topological conjugacy, $f^n|\Lambda$ inherits complicated dynamical properties of the shift map $\sigma|\Sigma_2$ described above. Furthermore, if f is a continuous map on interval and has a periodic point of period three, then for some n and some invariant set Λ for f^n, $f^n|\Lambda$ is topologically *semi*-conjugate to $\sigma|\Sigma_2$. See Block/Coppel (1991) for more detail.

[8] For the proof of this theorem see e.g. Devaney (1989). For another representation of the theorem using a subshift of finite type on n symbols, see Yokoo (1997).

2.6.2 Proof of Proposition 1

By the Homoclinic Point Theorem in the previous subsection, it suffices to show that the map f has a transverse homoclinic orbit. To show this we have to make some preparations.

Lemma 1: *The following statements hold:*

(i) $0 < f(\theta) < 1 < \theta$;

(ii) there is a unique point $q = q(\beta) > \theta$ *such that* $f(q) = 1$.

Proof. (i) From (2.9), we get the last inequality. Since f has its global minimum at θ, we get the second inequality (see Fig.1). (ii) Since $f(\theta) < 1$ and $f(x) > 1$ for large x, there is $q > \theta$ with $f(q) = 1$. Since $f(x)$ is strictly increasing on the interval (θ, ∞), the uniqueness follows (see Fig.1). Q.E.D.

Let us define a piecewise linear map $L_\beta : \mathbb{R} \to \mathbb{R}$ by

$$L_\beta(z) := \begin{cases} l_1(z) = f'_\beta(1)(z-1)+1, & z \le 1, \\ l_2(z) = -(z-1)+1, & z > 1. \end{cases}$$

Clearly, $L(1) = 1$ and $L^{-n}(z) \to 1$ $(n \to \infty)$ for each $z \in \mathbb{R}$ (see Fig.7).

*** *Insert Fig.7 about here* ***

Lemma 2: *There is a number* $\beta_1 > (2-\alpha)/\alpha$ *such that for every* $\beta \geq \beta_1$ *and for every* $z \in I := [f_\beta(\theta), \theta]$ *there is a unique sequence* $\{z_{-i}\}_{i=0}^{\infty} \subset I$ *such that* $z_0 = z$, $f(z_{-i-1}) = z_{-i}$ *for* $i \geq 0$, *and* $z_{-i} \to 1$ *as* $i \to \infty$.

Proof. Since f_β is one-to-one on the interval $[f_\beta(\theta), \theta]$, the conclusion holds if given sufficiently large β, $f_\beta(z) > l_1(z)$ for $z \in [f_\beta(\theta), 1)$ and $f_\beta(z) < l_2(z)$ for $z \in (1, \theta]$. By the strict convexity of f and by the construction of L, it is sufficient to show that for β large enough, the inequality $f(\theta) < l_2(\theta)$, i.e.,

$$f_\beta(\theta(\beta)) + \theta(\beta) < 2$$

holds. This is verified by the fact

$$\begin{aligned}
\lim_{\beta\to\infty} \theta(\beta) &= \lim_{\beta\to\infty} \left[\frac{\alpha\beta}{1-\alpha}\right]^{\frac{1}{\beta+1}} = 1, \\
\lim_{\beta\to\infty} f_\beta(\theta(\beta)) &= \lim_{\beta\to\infty} \left[(1-\alpha)\theta(\beta) + \alpha\theta(\beta)^{-\beta}\right] \\
&= 1-\alpha
\end{aligned}$$

which completes the proof. Q.E.D.

Lemma 3: *There is a number* $\beta_2 > (2-\alpha)/\alpha$ *such that for every* $\beta \geq \beta_2$ *the following inequality holds:*

$$0 < f_\beta(\theta(\beta)) < 1 < \theta(\beta) < q(\beta) < f_\beta^2(\theta(\beta)).$$

Proof. By Lemma 1, it suffices to show that for any arbitrarily large β, the following inequality holds:

$$q < f^2(\theta). \tag{2.10}$$

Let us define a function l by

$$l(z) := f'(1)(z-1) + 1 = (1-\alpha-\alpha\beta)(z-1) + 1.$$

Again by the strict convexity of f, we obtain

$$l \circ f_\beta(\theta(\beta)) < f_\beta^2(\theta(\beta)).$$

Note that $f_\beta(z) > (1-\alpha)z$ holds for every $z \in \mathbb{R}_{++}$(see Fig.1), so we have

$$q(\beta) < \frac{1}{1-\alpha}.$$

To obtain the inequality (2.10), it is therefore sufficient to show

$$\frac{1}{1-\alpha} \leq l \circ f_\beta(\theta(\beta))$$

for any sufficiently large β. Noting that

$$l \circ f_\beta(\theta(\beta)) = (1-\alpha-\alpha\beta)\left[(1-\alpha)\theta(\beta) + \alpha\theta(\beta)^{-\beta} - 1\right] + 1,$$

we get $\lim_{\beta\to\infty} l \circ f_\beta(\theta(\beta)) = \infty$. So the lemma follows. Q.E.D.

Proof of Proposition 1. Let $\overline{\beta} = \max\{\beta_1, \beta_2\}$ and pick $\beta \geq \overline{\beta}$ arbitrarily. By Lemma 3, there is a point $q' \in (f(\theta), 1)$ such that $f(q') = q$. By Lemma 2, there is a sequence $\{q'_{-i}\}_{i=0}^\infty \subset I$ such that $q'_0 = q'$, $f(q'_{-i-1}) = q'_{-i}$ $(i \geq 0)$, and $q'_{-i} \to 1$ $(i \to \infty)$. Hence, together with $f^2(q') = 1$, q' is a homoclinic point to the repeller 1. Clearly, for any homoclinic point $z \in \mathcal{HO}_f(q', 1) = \{q', f(q'), f^2(q') = 1\} \cup \{q'_{-i}\}_{i=0}^\infty$, we have $z \neq \theta$ and so $f'(z) \neq 0$, which implies that the homoclinic orbit of q' to 1, $\mathcal{HO}_f(q', 1)$, is transverse. By the Homoclinic Point Theorem, the statement is proved. Q.E.D.

2.6.3 Proof of Proposition 2

To find the abundance of strange attractors for the family of maps $\{f_\beta\}_\beta$, we exploit the theorem by Mora/Viana (1993, Theorem C).

Proof of Proposition 2. We show that the (non-degenerate) critical point $\theta(\beta)$ is contained in the homoclinic orbit of the repelling fixed point $z^* = 1$ for some sequence of β-values. Note first that the critical point θ is always non-degenerate, i.e., $f''(\theta) \neq 0$, since $f''(x) > 0$ for all $x \in \mathbb{R}_{++}$.

We can observe that there is a sequence of eventually fixed points depending smoothly on β,

$$Q(\beta) = \{q_i(\beta) : q_i(\beta) = f_\beta(q_{i+1}(\beta)) \text{ for } i \in \mathbb{N},\ q = q_1 < q_2 < \cdots < q_n < \cdots\},$$

where $q_i(\beta) \to (1-\alpha)^{-i} < \infty$ as $\beta \to \infty$ for every $i \in \mathbb{N}$.

Let us fix $\alpha \in (0,1)$ arbitrarily and take $\beta = \beta_1$ as in Lemma 2. Then, from the observation above, there is $q_i(\beta_1) \in Q(\beta_1)$ such that $f^2_{\beta_1}(\theta(\beta_1)) < q_i(\beta_1)$. Since $f^2_\beta(\theta(\beta)) \to \infty$ as $\beta \to \infty$ by the proof of Lemma 3, and $q_i(\beta) \to (1-\alpha)^{-i}$ as $\beta \to \infty$, there is $\beta^* > \beta_1$ such that $f^2_{\beta^*}(\theta(\beta^*)) = q_i(\beta^*)$, which implies that the backward orbit of $\theta(\beta^*)$ converges to the repelling fixed point $z^* = 1$ and there is an integer n such that $f^n_{\beta^*}(\theta(\beta^*)) = 1$ and $f^m_{\beta^*}(\theta(\beta^*)) \neq 1$ for $m < n$. Hence, the non-degenerate critical point $\theta(\beta^*)$ lies in a homoclinic orbit to the fixed point $z^* = 1$ (homoclinic tangency). Since $\frac{\partial}{\partial\beta} f(x,\beta) = 0$ if and only if $x = 1$, we may generically assume that $\frac{d}{d\beta} f^n_\beta(\theta(\beta)) \neq 0$ at $\beta = \beta^*$, which implies that, in our case, this homoclinic tangency unfolds generically. By Theorem C in Mora/Viana(1993), the statement of Proposition 2 follows. Q.E.D.

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Fig.2: Bifurcation diagram with respect to β $(1.5 \leq \beta \leq 4.7)$ with $\alpha = 0.7$.

Fig.3: Topological entropy corresponding to Fig.2 with $\alpha = 0.7$. A positive value for topological entropy indicates topological chaos.

Fig.4: Lyapunov exponent corresponding to Fig.2 with $\alpha = 0.7$. A positive value of the Lyapunov exponent indicates observable chaos.

Fig.5: Iso-period plot for the map (2.8). The green area corresponds to period-1 cycles (a stable fixed point), the dark blue area to period-2 cycles, the yellow area to period-4 cycles, and the red area to period-8. The blue and magenta areas correspond to period-3 and period-6 cycles.

Fig.6: Observable chaos plot for the map (2.8) The black area is the set of (α, β) for which the model has a strange attractor.

Fig.7: Graph of the map (2.8) and the piecewise linear map $L_\beta(z)$.

Chapter 3

Stability, Chaos and Multiple Attractors: A Single Agent Makes a Difference

abstract

This paper[1] provides an example in which a slight behavioral heterogeneity may fundamentally change the qualitative properties of a nonlinear cobweb market with a quadratic cost function and an isoelastic demand function. We consider two types of producers; cautious adapters and naive optimizers. In a market of naive optimizers a single cautious adapter stabilizes the otherwise exploding market. In a market of cautious adapters a single naive optimizer may destabilize the market; without him there exists at most one periodic attractor in the market but with him there may appear many (and even infinitely many) coexisting periodic attractors.

[1]This essay is based on the joint paper with Tamotsu Onozaki and Gernot Sieg, to appear in the Journal of Economic Dynamics and Control. I would like to thank Atsuro Sannami for helpful discussions. Thanks also go to two anonymous referees and one of the editors of the Journal of Economic Dynamics and Control for constructive suggestions to an earlier version of the paper.

3.1 Introduction

A producer can choose his own way from the many available techniques to adjust production capacity and many different types of behavior coexist in reality. However, monotypic behavior dominates in economic theories. A representative agent typifies preferences and technologies as well as rational behavior of the whole society of agents.

One possible argument in favor of simplifying a model by assuming a 'representative rational agent' apparatus is that all the different behavior has already died out and only the representative agent survives [Lucas (1986)]. Evolutionary economics shows, however, that survival probabilities depend on the environment of agents and the selection mechanism [Axelrod (1984)]. Different types of behavior can survive simultaneously.

Another possible defense of assuming a representative agent is that a majority of the agents behaves in the same way and their behavior determines the dynamics of the market. In the stock market, however, a small group of risky traders could disturb the behavior of stock prices. The type of the market may determine whether the behavior of a majority determines the market outcome or whether the outcome depends on a minority of agents. The 'representative rational agent' is a theoretical apparatus that works with certainty only when all agents behave in the same way.

There are different behavioral techniques available and a dynamical process of switching to successful technologies seems to be plausible. However, even a unique superior technology does not necessarily extinguish all different types of technologies. At least one producer may sometimes behave differently. If the market is still in a phase of transition, this producer still uses the 'old' technology because he is a late adopter. If we are in a steady state, this producer tries a ‘new’ technology to improve profits. Thus heterogeneity, or diversity, of agents’ behavior is a natural feature in our daily life, but not in traditional economics. Only recently, dynamical economics has considered heterogeneous agents [Gallegati/Kirman (1999), Delli-Gatti/Gallegati/Kirman (2000), Den-Haan (2001) and Kirman/Zimmermann (2001)]. The literature separates three different kinds of heterogeneity; personal characteristics like preferences or income, the way expectations are formed, and behavioral rules that agents use due to their bounded rationality. Some important results are already available.

In a growth model, agents with heterogenous preferences for income are examined by Cardak (1999); in a periodical economy with progressive tax system heterogeneity in the rate of impatience is studied by Sarte (1997) and in an overlapping generation model heterogeneity in income and talent is analyzed by Chiu (1998). Heterogeneous general preferences in a perfect-foresight equilibrium of a finance-constrained economy allows Hopf cycles to be entirely consistent with a wide range of elasticities of substitution [Barinci (2001)].

Brock/Hommes (1997), Gaunersdorfer (2000) and Goeree/Hommes (2000) study dynamical models where agents update their expectations according to an observed measure such as net profits. Bomfin (2001) shows that if some agents solve their inference problems based on simple forecasting rules of thumb, there is a significant effect on the aggregate properties of the economy. In a cobweb model, where two different forecasting procedures are considered, either one destabilizes the price dynamics if it is uniformly adopted by all firms; or the price equilibrium becomes locally stable if firms are heterogeneous, and the two rules are suitably mixed within the population [Franke/Nesemann (1999)].

Day/Huang (1990), Chiarella (1992), Lux (1995) and Lux/Marchesi (2000) study how heterogeneous behavior of traders generates complex motion of financial markets. Cooper (1998) considers heterogeneity of agents in a standard stochastic growth model by assuming that agents react with different probabilities to current values of relevant state variables.

In the present paper we would like to investigate whether a slight behavioral heterogeneity could be a factor that generates complex dynamics of a market. We consider a nonlinear cobweb market with a quadratic cost function and an isoelastic demand function. Two types of producers are considered; 'naive optimizers' and 'cautious adapters'. A naive optimizer produces the profit-maximizing quantity instantaneously, while a cautious adapter adjusts his output toward the profit-maximizing quantity as a target.

We show that a single agent may change the complexity of the market behavior. If there is no adapter and demand is inelastic enough for the market to explode, a single adapter can stabilize the market in the sense that it would not explode, but only by causing chaos. On the other hand, when there are exclusively adapters, there exists at most one periodic attractor for the market. If a single optimizer appears, then there may appear many and even infinitely many coexisting periodic attractors of arbitrarily large period.

3.2 Model

In this section, we derive a two-dimensional nonlinear cobweb model including naive optimizers and cautious adapters from a general, N-dimensional model including N-types of agents.

3.2.1 Description of behavior

Before presenting the general model, we start by defining precisely the notion of 'naive optimizer' and 'cautious adapter'.

Let us consider the market of a perishable commodity with one period of production. The market is competitive and clears each period.[2] A fictitious auctioneer therefore sets the price whereas suppliers set the quantities supplied. A supplier decides in period t the production x_{t+1} for period $t+1$. A quadratic cost function $ax^2/2,\ a > 0$ describes the technology. The expected profit-maximizing quantity on condition that price expectations are naive (or static) is

$$\tilde{x}_{t+1} = \frac{p_t}{a}.$$

If a supplier produces this quantity instantaneously, i.e.

$$x_{t+1} = \tilde{x}_{t+1}, \tag{3.1}$$

we call him a naive optimizer (or simply, optimizer).

On the other hand, as considered in Onozaki/Sieg/Yokoo (2000), a cautious adapter (or simply, adapter) is not confident that the same price will prevail in the next period and adjusts his last period's production x_t only partially in the direction of $\tilde{x}_{t+1}$. Thus his adjustment behavior is described by

$$x_{t+1} = x_t + \alpha \left(\tilde{x}_{t+1} - x_t \right), \tag{3.2}$$

where $\alpha \in [0,1]$ is the speed of adjustment. A cautious adapter may follow a rule of thumb when adjusting capacities. Furthermore, risk aversion or adjustment costs may make a supplier act as an adapter. Furthermore, if a supplier expects competitors to adjust also their quantities, it is quite

[2] A more realistic setup includes adjustment costs of price changes. These adjustment costs may stabilize the market.

sensible to adjust only partially. Therefore, $1-\alpha$ can be interpreted as the degree of inertia or the supplier's level of caution. If $\alpha = 1$, then Eq.(3.2) is identical to Eq.(3.1), which means that naive behavior is a special case of adaptive behavior where $\alpha = 1$.

3.2.2 General model

Let us consider a general model where there are N (a positive integer) types of adapters.[3] All groups of suppliers share the same cost function considered above. Production $x_{i,t+1}$ in period $t+1$ of the i-th type of suppliers is determined by

$$x_{i,t+1} = (1-\alpha_i)x_{i,t} + \frac{\alpha_i p_t}{a}, \quad i = 1, 2, \cdots, N$$

where $\alpha_i \in [0,1]$ is a speed of adjustment of the i-th type of suppliers. Therefore, the aggregate supply per capita X_t in period t is given by

$$X_t = \sum_{i=1}^{N} n_i x_{i,t} \quad \text{with} \quad n_i \in [0,1] \quad \text{and} \quad \sum_{i=1}^{N} n_i = 1$$

where n_i is the relative size of the i-th group of suppliers.

We assume an inverse demand function which is isoelastic with a price elasticity of $1/\sigma$:

$$p_t = \frac{b}{Y_t^{\sigma}}, \quad b > 0, \quad \sigma > 0.$$

Equating the aggregate supply and demand, $X_t = Y_t$, gives an N - dimensional discrete-time dynamical system:

$$x_{i,t+1} = (1-\alpha_i)x_{i,t} + \frac{b\alpha_i}{a\left(\sum_{i=1}^{N} n_i x_{i,t}\right)^{\sigma}}, \quad i = 1, 2, \cdots, N.$$

[3] Another possible introduction of heterogeneity is to assume that cardinality of types is a continuum represented as the unit interval $[0,1]$.

Applying a variable transformation,

$$z_{i,t} = \left(\frac{a}{b}\right)^{\frac{1}{1+\sigma}} x_{i,t},$$

we obtain the final form:

$$z_{i,t+1} = (1-\alpha_i)z_{i,t} + \frac{\alpha_i}{\left(\sum_{i=1}^{N} n_i z_{i,t}\right)^{\sigma}}, \quad i = 1, 2, \cdots, N. \tag{3.3}$$

3.2.3 Reduced models

Using the general model (3.3), we can derive a 'standard', homogeneous cobweb model and an adaptive, homogeneous cobweb model [Onozaki/Sieg/Yokoo (2000)].

Let us assume $N = 1$ and $\alpha = 1$, which implies that there are exclusively naive optimizers. Then the model (3.3) reduces to a first-order difference equation

$$z_{t+1} = \frac{1}{z_t^{\sigma}}, \tag{3.4}$$

which preserves the properties of the standard cobweb model. The behavior of Eq.(3.4) depends on price elasticity; if price elasticity is greater than one ($\sigma < 1$), trajectories generated by Eq.(3.4) converge to a stable fixed point $z^* = 1$. If price elasticity is equal to one ($\sigma = 1$), trajectories exhibit period-2 cycles. However, if price elasticity is less than one ($\sigma > 1$), trajectories oscillate and explode to infinity.

Now let us assume $N = 1$ and $\alpha \in [0, 1)$, which implies that there are exclusively cautious adapters. Then the model (3.3) reduces to a first-order difference equation

$$z_{t+1} = (1-\alpha)z_t + \frac{\alpha}{z_t^{\sigma}}. \tag{3.5}$$

Some dynamical properties of it are studied by Onozaki/Sieg/Yokoo (2000) and Onozaki/ Sawada (2001) to show that if $\sigma < (2-\alpha)/\alpha$, trajectories converge to a unique stable fixed point $z^* = 1$. The fixed point undergoes a period-doubling bifurcation at $\sigma = (2-\alpha)/\alpha$. If $\sigma > (2-\alpha)/\alpha$, trajectories may exhibit periodic cycles or chaos. Because $(2-\alpha)/\alpha > 1$, the last inequality implies that $\sigma > 1$. We can state that adaptive behavior prevents the unstable market from going to infinity, only by causing periodic cycles or chaotic behaviors. Adaptive behavior stabilizes the market in this sense.

However, the assumption that all agents behave homogeneously is unrealistic. To get a better picture of a cobweb market, we concentrate on a simple type of heterogeneity. We consider a model that is a little more general than Eqs.(3.4) and (3.5) by including both cautious adapters and naive optimizers. Reducing the N-dimensional model to a two-dimensional model makes not only analytical treatment but also the graphical depiction much easier and makes it possible to show the difference between one-dimensional and two-dimensional model.

It is easy to derive a two-type suppliers model from the general model (3.3). Let us suppose $N = 2$, denote an adapter by $i = 1$ and an optimizer by $i = 2$. The relative size n_1 of the adapters' group is replaced by m, so that the relative size of the optimizers' group is $1 - m$. Since an optimizer produces the profit-maximizing amount immediately, his adjustment speed α_2 is obviously unity. Letting $z_{1,t} = u_t \in \mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ and $z_{2,t} = v_t \in \mathbb{R}_{++}$ gives

$$u_{t+1} = (1-\alpha)u_t + \frac{\alpha}{[mu_t + (1-m)v_t]^\sigma}, \tag{3.6}$$

$$v_{t+1} = \frac{1}{[mu_t + (1-m)v_t]^\sigma}, \tag{3.7}$$

where $\alpha \in (0, 1)$ expresses the adjustment speed of adapters.

3.3 Analysis of the model

The main purpose of this section is to show that heterogeneity in agents' production adjustment behavior can give rise to qualitatively different periodical features than those of the homogeneous production case.

Eliminating u's from Eqs.(3.6) and (3.7), we obtain the following second-order difference equation:

$$v_{t+2}^{-\frac{1}{\sigma}} = (1-m)v_{t+1} - (1-m)(1-\alpha)v_t + (1-\alpha)v_{t+1}^{-\frac{1}{\sigma}} + \alpha m v_{t+1}. \tag{3.8}$$

Letting

$$x_t = v_t^{-\frac{1}{\sigma}} \quad \text{and} \quad x_{t+1} = y_t,$$

Eq.(3.8) can then be transformed into the two-dimensional dynamical system $F : V \subset \mathbb{R}^2_{++} \to V$ defined by

$$(x_{t+1}, y_{t+1}) = F(x_t, y_t), \tag{3.9}$$

where

$$F(x,y) = (y, f(y) + (1-m)\, h(x,y)) \tag{3.10}$$

with

$$\begin{aligned} f(y) &= (1-\alpha)y + \alpha y^{-\sigma}, \\ h(x,y) &= (1-\alpha)\left[y^{-\sigma} - x^{-\sigma}\right], \quad \text{and} \\ V = V_{\sigma,m} &= \{(x,y) \in \mathbb{R}^2_{++} : x > 0,\ y > (1-m)x^{-\sigma}\}. \end{aligned}$$

In order to indicate the dependence of F and f on the parameter σ and m, we sometimes write these as $F_{\sigma,m}$ and f_σ. In this section, the parameter $\alpha \in (0,1)$ is assumed to be always arbitrarily fixed in $(0,1)$.

Note here that unless $m = 1$, the dynamical system $F : V \to V$ is a diffeomorphism onto its image $\mathrm{Im}(F) = F(V) \subset V$. In fact, for $m \neq 1$, the map F is injective (i.e., one-to-one) on V and the determinant $\det DF$ of the Jacobian matrix DF of the map F is non-zero everywhere on V, that is, $\det DF(x,y) = -(1-m)(1-\alpha)\sigma x^{-\sigma-1} < 0$ for any $(x,y) \in V$.

3.3.1 Local stability analysis

The map $F_{\sigma,m}$ given by (3.10) has a unique fixed point $p = (1,1)$, independent of parameter values. We examine the stability of the fixed point p. The Jacobian matrix $DF_{\sigma,m}$ of $F_{\sigma,m}$ evaluated at p is given by

$$DF_{\sigma,m}(p) = \begin{bmatrix} 0 & 1 \\ (1-m)(1-\alpha)\sigma & 1-\alpha-\alpha\sigma-(1-m)(1-\alpha)\sigma \end{bmatrix}. \quad (3.11)$$

The characteristic polynomial $P(\lambda)$ associated with the Jacobian matrix (3.11) is represented as

$$P(\lambda) = \lambda^2 - \mathrm{tr}DF_{\sigma,m}(p) \cdot \lambda + \det DF_{\sigma,m}(p), \quad (3.12)$$

where $\mathrm{tr}DF_{\sigma,m}(p) = 1-\alpha-\alpha\sigma-(1-m)(1-\alpha)\sigma$ is the trace of (3.11) and $\det DF_{\sigma,m}(p) = -(1-m)(1-\alpha)\sigma$ is the determinant of (3.11). Since $P(0) = \det DF_{\sigma,m}(p) = -(1-m)(1-\alpha)\sigma \leq 0$ for $m \in [0,1]$ (equality holds for $m = 1$) and $P(1) = \alpha(1+\sigma) > 0$, the Jacobian matrix $DF_{\sigma,m}(p)$ always has a non-negative real eigenvalue whose absolute value is less than unity. Apparently, the fixed point p cannot undergo either a Neimark-Sacker (also known as Hopf) bifurcation, occurring with a pair of complex eigenvalues with absolute value 1, or other local bifurcations with eigenvalue 1 such as a saddle-node bifurcation. The only possible bifurcation is a period-doubling (also known as flip) bifurcation that occurs when one of the eigenvalues is -1. The period-doubling bifurcation curve can be given by a simple explicit equation[4]

$$P(-1) = 2 - \alpha(1+\sigma) - 2(1-m)(1-\alpha)\sigma = 0. \qquad (3.13)$$

The fixed point p is called *hyperbolic* if $DF_{\sigma,m}(p)$ has no eigenvalues with absolute value of unity, and *dissipative* if $|\det DF_{\sigma,m}(p)| < 1$. As is easily seen, if $P(-1) > 0$ then p is locally stable, and if $P(-1) < 0$ then the fixed point p is a saddle point. To be more precise:

LEMMA 1: *If the inequalities*

$$\frac{2-\alpha}{\alpha + 2(1-m)(1-\alpha)} < \sigma < \frac{1}{(1-m)(1-\alpha)} \qquad (3.14)$$

hold, then the fixed point p is a dissipative hyperbolic saddle.

PROOF: The first inequality is equivalent to $P(-1) < 0$, which implies that p is a hyperbolic saddle. The second is just the dissipativity condition: $|\det DF_{\sigma,m}(p)| = (1-m)(1-\alpha)\sigma < 1$. Q.E.D.

3.3.2 Preparatory arguments

In order to study the heterogeneous cases, i.e., $m \in (0,1)$, it is useful to first examine the homogeneous case where there are only adapters in the market, i.e., $m = 1$. In this case, the map $F_{\sigma,m}$ given by (3.10) reduces to the singular (thus non-invertible) map $F_{\sigma,1} : \mathbb{R}^2_{++} \to \mathbb{R}^2_{++}$, given by

$$F_{\sigma,1}(x,y) = (y, f_\sigma(y)).$$

The map $F_{\sigma,1}$ is clearly equivalent to the one-dimensional map f_σ in the sense that f_σ on $\mathbb{R}_{++}$ is topologically conjugate to the map $F_{\sigma,1}$ restricted

[4] This curve is visible as the border between the red area and the orange areas in Fig.6 and 7.

onto its image $\mathrm{Im}(F_{\sigma,1})$ through the conjugacy $\varphi(x) = (x, f_\sigma(x))$. The dynamics of f_σ is studied in Onozaki/Sieg/Yokoo (2000) where f_σ is shown to be strictly convex and unimodal with its global minimum at

$$\theta = \theta(\sigma) = \left(\frac{\alpha\sigma}{1-\alpha}\right)^{\frac{1}{1+\sigma}}.$$

That is, $f'_\sigma(\theta) = 0$ and $f''_\sigma(x) > 0$ for every $x > 0$. Furthermore, the Schwarzian derivative of f_σ is given by[5]

$$\begin{aligned} Sf_\sigma(x) &= \frac{f'''_\sigma(x)}{f'_\sigma(x)} - \frac{3}{2}\left(\frac{f''_\sigma(x)}{f'_\sigma(x)}\right)^2 \\ &= -\frac{\alpha\sigma(1+\sigma)\left[\alpha\sigma(\sigma-1) + 2(1-\alpha)(2+\sigma)x^{1+\sigma}\right]}{2\left[(\alpha-1)x^{2+\sigma} + \alpha\sigma x\right]^2}. \quad (3.15) \end{aligned}$$

We see that

$$Sf_\sigma(x) < 0 \qquad \text{for} \quad \sigma \geq 1 \quad \text{and} \quad x > 0.$$

Let us denote the global stable (unstable) manifold of the fixed point $p = (1,1)$ of the map $F_{\sigma,m}$ by $W^s_{\sigma,m}(p)$ ($W^u_{\sigma,m}(p)$, respectively). We will abuse this notation for the singular case when $m = 1$.

For σ large, the unstable manifold $W^u_{\sigma,1}(p)$ on the (x,y)–plane is simply an arc consisting of a compact part of $\mathrm{Im}(F_{\sigma,1})$, which is a part of the graph of f_σ.

LEMMA 2: *Let σ^* be a σ-value such that $f^2_{\sigma^*}(\theta(\sigma^*)) = \theta(\sigma^*)$ and $\theta(\sigma) < f^2_\sigma(\theta(\sigma))$ for $\sigma > \sigma^*$ (such $\sigma^* > (2-\alpha)/\alpha$ exists), then for $\sigma > \sigma^*$ and $m = 1$, the unstable manifold of the fixed point p is given by $W^u_{\sigma,1}(p) = \{(x,y) : x \in [f_\sigma(\theta), f^2_\sigma(\theta)],\, y = f_\sigma(x)\}$.*

[5] See Onozaki/Sieg/Yokoo (2000), p. 108.

Proof: See Appendix.

When σ is large, the stable manifold $W^s_{\sigma,1}(p)$ on the (x,y)–plane consists of infinitely many horizontal lines containing $y = 1$ and $y = \hat{y}_i(\sigma)$ (depending on σ) such that $f^n_\sigma(\hat{y}_i) = 1$ for some integer $n = n(i)$ indexed by i. That is, $\hat{y}_i$ is a preimage of 1 by the one-dimensional map f. We can see that $W^u_{\sigma,1}(p)$ attains a tangential intersection with a horizontal line $y = \hat{y}_i(\sigma)$ belonging to $W^s_{\sigma,1}(p)$, which then unfolds into two transverse intersections as σ increases. Note here that the map f has two inverses, say f_L^{-1} and f_R^{-1} giving the preimages of a point y, one on the left of θ and one on the right of θ, respectively. That is, $f_L^{-1}(y) = f^{-1}(y) \cap (0, \theta]$ and $f_R^{-1}(y) = f^{-1}(y) \cap (\theta, \infty)$. Thus $f^{-1}(1) = 1 \cup f_R^{-1}(1)$ with $f_L^{-1}(1) = 1$, and so on.

Lemma 3: *For $m = 1$, there exist horizontal lines $\{y = \hat{y}_i(\sigma)\}_i \subset W^s_{\sigma,1}(p)$ (depending smoothly on σ) such that for some $\sigma_1 = \sigma_1(i)$ and $\sigma_2 = \sigma_2(i)$ with $\sigma_1 < \sigma_2$:*

(P1) *$W^u_{\sigma_1,1}(p)$ and $y = \hat{y}_i(\sigma_1)$ have no intersection;*

(P2) *$W^u_{\sigma_2,1}(p)$ and $y = \hat{y}_i(\sigma_2)$ have two transverse homoclinic intersections, and*

(P3) *$W^u_{\sigma_H,1}(p)$ and $y = \hat{y}_i(\sigma_H)$ have a quadratic homoclinic tangency for some $\sigma_H = \sigma_{H(i)} \in (\sigma_1, \sigma_2)$.*

Proof: See Appendix.

It is worth explaining here in more detail the occurrence of homoclinic bifurcations associated with p for $m = 1$, that is, how the homoclinic points of p are created for the singular case as the parameter σ increases. This will be helpful in understanding the more complicated situations occurring for $m < 1$.

As long as $f^2(\theta) < f_R^{-1}(1)$, no homoclinic point of the fixed point p exists, i.e., $W^s_{\sigma,1}(p) \cap W^u_{\sigma,1}(p) \setminus \{p\} = \phi$. See Fig.1(a).

As σ increases, we can observe that $f^2(\theta)$ increases toward infinity while $f_R^{-1}(1)$ tends to $1/(1-\alpha)$. Thus the first homoclinic point of p appears at some σ-value $\sigma_{H1}(> \sigma^*)$ at which $f^2(\theta) = f_R^{-1}(1)$, or, equivalently, $f^3(\theta) = 1$. At $\sigma = \sigma_{H1}$, the unstable manifold $W^u_{\sigma_{H1},1}(p)$ and the horizontal line $y = f(\theta)$ contained in the stable manifold $W^s_{\sigma_{H1},1}(p)$ has a quadratic homoclinic tangency at the point $(\theta, f(\theta))$. Besides the homoclinic tangency, there suddenly appear a lot of other homoclinic points of p at $\sigma = \sigma_{H1}$: $W^s_{\sigma_{H1},1}(p)$ now contains in the interval $(f(\theta), f^2(\theta))$ infinitely many horizontal lines $y = \hat{y}_i$ (i.e., $\hat{y}_i \in (f(\theta), f^2(\theta))$), each of which intersects the arc $W^u_{\sigma_{H1},1}(p)$ transversely in two homoclinic points. Also there appears a sequence of horizontal lines $\{y = \hat{y}_j\}_j$ of $W^s_{\sigma,1}(p)$ in the interval $(f(\theta) - \varepsilon, f(\theta))$ for any $\varepsilon > 0$ such that $\hat{y}_j \to f(\theta)$ $(j \to \infty)$ from below. See Fig.1(b).

Although the homoclinic tangency itself is fragile in the sense that it is destroyed even by a slight increase in σ, such a small increase in σ will create new transverse homoclinic points and can create a new homoclinic tangency of p (say at σ_{Hj} with $\sigma_{Hj} > \sigma_{H1}$) by making the arc $W^u_{\sigma_{Hj},1}(p)$ tangent to some other horizontal line $y = \hat{y}_j$ of $W^s_{\sigma_{Hj},1}(p)$. See Fig.1(c). At such values of σ_{Hj} of homoclinic tangencies, we have $f(\theta) = \hat{y}_j$, or $f^{n_j}(\theta) = 1$ for some higher integer n_j. As σ increases further from σ_{H1} up to, say, $\sigma_{H2}\,(> \sigma_{H1})$ at which $f^2(\theta) = f_R^{-1} \circ f_R^{-1}(1)$ or $f^4(\theta) = 1$, this process of creating a new homoclinic tangency will occur infinitely many times.

*** Fig.1(a)–(c) a bout here ***

3.3.3 Some implications of homoclinic bifurcation

It has been widely known in economic literature that complex dynamics can arise via homoclinic bifurcations; see, e.g., Palis/Takens (1993) for a detailed mathematical treatment of this subject. We will show that the economic model given by Eq.(3.9) exhibits a homoclinic bifurcation, that is, the model has a hyperbolic fixed point of saddle type whose stable and unstable manifolds have homoclinic tangencies that unfold generically. Roughly speaking, we mean by the 'generic unfolding' of a homoclinic tangency that as the parameter σ varies, the unstable manifold cuts across the stable manifold with 'non-zero relative speed' at the point of tangency. As a result, the system is shown to exhibit complex dynamics such as strange attractors (observable chaos), infinitely many periodic attractors (due to the persistence of homoclinic tangencies), creation of horseshoes, and cascades of period-doubling bifurcations.

Now we will perturb the singular map $F_{\sigma,1}$ into non-singular maps by making m slightly smaller.

Lemma 4 (Homoclinic Bifurcations): *There exists $\varepsilon \in (0,1)$ such that for any $m \in (\varepsilon, 1)$ and for some $\sigma = \hat{\sigma}_H = \hat{\sigma}_H(m)$, the map $F_{\hat{\sigma}_H,m}$ has the following properties:*

(i) *the fixed point p is a dissipative hyperbolic saddle;*

(ii) *the stable manifold $W^s_{\hat{\sigma}_H,m}(p)$ and the unstable manifold $W^u_{\hat{\sigma}_H,m}(p)$ have a quadratic homoclinic tangency that unfolds generically with re-*

spect to σ.

PROOF: See Appendix.

The relation between the stable and unstable manifolds of the fixed point p is depicted in Fig.2 for m (relatively) close to 1 and σ big. In Fig.2, the unstable manifolds $W^u_{\sigma,m}(p)$ is shown to have many homoclinic intersections with the stable manifold $W^s_{\sigma,m}(p)$. Although the figure is reminiscent of ,e.g., Fig.1(c) for the singular case of $m = 1$, there are some differences in the shapes of the stable and unstable manifolds of p. While the stable manifold of p for $m = 1$ consists of infinitely many horizontal lines, they become connected with each other in some way, by folding, for $m < 1$ as depicted in Fig.2. Unlike the case of $m = 1$, the unstable manifold of p for $m < 1$ now tangles in a very complicated way in the presence of transverse homoclinic orbits (this situation is sometimes referred to as 'homoclinic tangles'), but it cannot have self-intersections because the map $F_{\sigma,m}$ is injective for $m \neq 1$.

*** Fig.2 about here ***

The dynamical complexities stated in the following proposition are due to homoclinic bifurcations. For complex dynamics due to homoclinic bifurcations in an overlapping generations model, see e.g. de Vilder (1996).

PROPOSITION 1 (COMPLEX DYNAMICS): *Take $m \in (\varepsilon, 1)$ and $\hat{\sigma}_H$ as in Lemma 4. Let $\delta > 0$ be a sufficiently small number and let the interval $I = (\hat{\sigma}_H - \delta, \hat{\sigma}_H + \delta)$. Then the following holds:*

(i) *There exists an interval $H \subset I$ such that for each $\sigma \in H$, $F_{\sigma,m}$ has a horseshoe. That is, there exists an $F_{\sigma,m}$-invariant set Λ_σ on which $F_{\sigma,m}$ has infinitely many saddle-type periodic orbits of arbitrarily large period;*

(ii) *There exists a set of σ-values $E \subset I$ with positive Lebesgue measure such that for each σ, $F_{\sigma,m}$ exhibits an Hénon-like strange attractor;*

(iii) *There exists a sequence $\{\sigma_n\} \subset I$ with $\sigma_n \to \hat{\sigma}_H$ as $n \to \infty$ such that for each σ_n, $F_{\sigma_n,m}$ exhibits a period-doubling bifurcation;*

(iv) *For each $k \geq 1$, there exists an interval $J_k \subset I$ such that for each $\sigma \in J_k$, $F_{\sigma,m}$ has at least k coexisting periodic attractors. Furthermore, there exist infinitely many subintervals $I_n \subset I$ and a dense subset $M_n \subset I_n$ such that for each $\sigma \in M_n$, $F_{\sigma,m}$ has infinitely many periodic attractors of arbitrarily large period (the Newhouse phenomenon).*

PROOF: See, e.g., Palis/Takens (1993, Chapter 2) for (i), Mora/Viana (1993) or Palis/Takens (1993, Chapter 7) for (ii), Yorke/Alligood (1983) or Palis/Takens (1993, Chapter 3) for (iii), and Robinson (1983) or Palis/Takens (1993, Chapter 6) for (iv).

3.3.4 A single heterogeneous agent makes a difference

Once a single agent (to be more precise, a sufficiently small fraction of agents) of a different type is put into a homogeneous group, what will happen in the market? We will show that such a single heterogeneous agent may drastically change the qualitative dynamical feature of a market.

First, we show that in a market of naive optimizers a single cautious adapter can stabilize the otherwise exploding market. If there is no adapter and demand is inelastic enough, the market with optimizers explodes. However, when a single adapter appears, the market does not explode anymore. A single adapter can stabilize the market in the sense that the trajectories do not explode but are trapped into a compact region in the positive quadrant.

PROPOSITION 2 (A SINGLE CAUTIOUS ADAPTER MAKES A DIFFERENCE): (i) *For* $m = 0$ *and* $\sigma > 1$*, the trajectory generated by Eqs. (3.6) and (3.7) for any initial condition* $(u_0, v_0) \in \mathbb{R}^2_{++}$ *explodes unless* $v_0 = 1$. (ii) *On the other hand, for* $m \in (0, 1]$*, every trajectory starting from* $\mathbb{R}^2_{++}$ *is trapped into a compact region in* $\mathbb{R}^2_{++}$.

PROOF: See Appendix.

Although the trajectories are trapped in a compact region, they may fluctuate chaotically. Therefore, an independent observer may not judge such a chaotic market to be 'stable'. However, a single agent changes the qualitative behavior of the market from explosion to chaos or periodic cycles. Furthermore, as numerically shown in the next section (Fig.4), the trapping region shrinks as the relative size of cautious adapters increases.

Conversely, if a single naive optimizer appears in a market where there exist exclusively cautious adapters, then there may appear many and even infinitely many coexisting periodic attractors in the market. Multiplicity of attractors cannot occur in a market solely occupied by cautious adapters.

PROPOSITION 3 (A SINGLE NAIVE OPTIMIZER MAKES A DIFFERENCE): (i) *For* $m = 1$*, there exists at most one periodic attractor for the map* $F_{\sigma,1}$.

(ii) *On the other hand, for any $m < 1$ sufficiently close to 1 and for any integer $k \geq 1$, there exists an interval J_k of σ-values such that for each $\sigma \in J_k$, $F_{\sigma,m}$ has at least k coexisting periodic attractors. Furthermore, for $m < 1$ sufficiently close to 1, there exist intervals $\{I_i\}_{i=1}^{\infty}$ of σ-values and dense subsets $\{M_i \subset I_i\}$ such that for each $\sigma \in M_i$, $F_{\sigma,m}$ exhibits infinitely many coexisting periodic attractors of arbitrarily large period.*

PROOF: See Appendix.

3.4 Numerical simulations

In this section we present some results of numerical simulations of Eq.(3.9) mainly to visualize our theoretical results.[6]

First, we depict a strange attractor of the model for the parameter constellation $(\alpha, \sigma, m) = (0.1, 24, 0.97)$ in Fig.3.

*** Fig.3 about here ***

A one-parameter bifurcation diagram with respect to m is depicted in Fig.4. From PROPOSITION 2 we can state that if there appears one cautious adapter in an otherwise unstable cobweb market ($m = 0$ and $\sigma > 1$) then the market will not explode anymore but behave chaotically or periodically. The figure, which is calculated for $m \geq 0.01$, illustrates that the trajectories of the model including cautious adapters is trapped into a range of $(0, e^{13})$. Without cautious adapters the market oscillates and explodes. Furthermore,

[6] In this section the parameters α, σ and m are choosen quite arbitrarily to visualize our theoretical results.

it is observed that as the relative size m of cautious adapters increases, the amplitude of trajectories shrinks. In these senses, adaptive behavior stabilizes a cobweb market.

*** Fig.4 about here ***

A one-parameter bifurcation diagram with respect to σ is depicted in Fig.5. From Eq.(3.13) the fixed point p undergoes a period-doubling bifurcation at $\hat{\sigma} = (2-\alpha)/[\alpha + 2(1-m)(1-\alpha)] \approx 12.338$, which is shown in the figure.

*** Fig.5 about here ***

Next we turn to two-parameter bifurcation diagrams, i.e., Fig.6 and 7. Each color in the figures corresponds to period's number of cycles as displayed in the table of Fig.6. The red area exhibits pairs of parameter values for which trajectories converge to a unique stable fixed point. The orange area consists of pairs of parameter values for which trajectories converge to a period-2 cycle. As stated above, between these two areas lies the period-doubling bifurcation curve given by Eq.(3.13). The mustard-colored area corresponds to a period-3 cycle, the yellow area corresponds to a period-4 cycle, the emeraldine area corresponds to a period-6 cycle, and the light-blue area corresponds to a period-8 cycle, etc. The black area corresponds to a cycle of period-16 up to -64 and the white area corresponds to a cycle of period-over-65 or an aperiodic (including chaotic) orbit. For almost all the set of parameters belonging to the white area, our model exhibits observable chaos in the sense of a positive Lyapunov exponent.

*** Fig.6–7 about here ***

¿From both figures it is observed that the white area shrinks as the relative size m of adapters increases, which implies that the set of parameters generating chaotic behavior of the market shrinks as m increases. This is the other aspect of the statement that adaptive behavior stabilizes a cobweb market. Furthermore, by taking notice of α and σ we can get the same observation as in Onozaki/Sieg/Yokoo (2000): The faster the speed of adjustment and the less elastic the demand, the more likely the market behaves chaotically.

Fig.8 exhibits a supplementary result to PROPOSITION 3 that a single naive optimizer makes a difference. The figure depicts the basins of attraction for a parameter constellation $(\alpha, \sigma, m) = (0.5, 6, 0.961)$. Although there exists at most one periodic attractor for the market without naive optimizers ($m = 1$), it is observed that the emergence of a relatively small fraction of optimizers ($m = 0.961$) causes two coexisting periodic attractors. The basins of attraction have a fractal structure as exhibited in the figure, due to the existence of homoclinic tangles associated with the horseshoes. Any point (x, y) of the phase space either belongs to one of the two basins or is a point on the basin boundary. Every trajectory starting from an initial point that does not belong to the boundary converges to either period-10 (the black area) or period-18 (the white area) cycles. The coexistence of periodic attractors is impossible if the market consists only of cautious adapters.

*** Fig.8 about here ***

3.5 Conclusion

We have investigated the dynamics of a nonlinear, two-dimensional cobweb model which contains two types of heterogeneous agents; cautious adapters and naive optimizers. Even a single heterogeneous agent may change the qualitative behavior of the market. If there are exclusively naive optimizers and demand is inelastic enough for the market to explode, a single adapter can stabilize the market in the sense that it would not explode, but possibly by causing chaos. On the other hand, when there are exclusively cautious adapters, there exists at most one periodic attractor for the market. If a single naive optimizer appears in such a market, then there may appear many (and even infinitely many) coexisting periodic attractors of arbitrarily large period.

Onozaki/Sieg/Yokoo (2000) hypothesis states that in a market with exclusively adaptive agents, low price elasticities and fast adjustment may cause the market to behave chaotically. In this paper, we extend this hypothesis so as to hold for a market with heterogenous agents. More importantly, however, market behavior is not necessarily determined by the behavior of a majority of agents but even a single heterogenous agent may have a profound impact on the qualitative behavior of a market. Therefore, a statement that the theoretical concept of homogenous agents is an appropriate approximation of the reality of coexisting heterogenous agents, which is common in traditional economic theory, seems questionable. Heterogeneity, or diversity, of agents may be the mother of rich dynamics and therefore possibly the source of stability, oscillation and chaos.

3.6 Appendix

3.6.1 Proof of Lemma 2

An exercise shows that if $(1-\alpha)/\alpha < \sigma \leq (2-\alpha)/\alpha$, then $1 < f^2(\theta) < \theta$ [see Onozaki/Sawada(2001)]. Also, since $f^2(\theta) \to \infty$ and $\theta \to 1$ as $\sigma \to \infty$ [see Lemma 3 in Onozaki/Sieg/Yokoo(2000)], there exists a σ-value $\sigma^* > (2-\alpha)/\alpha\ (> 1)$ as in the statement of the lemma. Let $\sigma > \sigma^*$ be given. Then we have $f(\theta) < 1 < \theta < f^2(\theta)$. Since $f([1,\theta]) = [f(\theta), 1]$ and $f([f(\theta), 1]) = [1, f^2(\theta)]$, the set $W^u_{\sigma,1}(p)$ is proved to be the unstable manifold of p if it is shown that for any $z \in (1,\theta)$, there exists $n = n(z) \geq 1$ such that $f^{2n}(z) > \theta$. Clearly, it suffices to show that $f^2(x) > x$ for any $x \in (1,\theta]$. If this does not hold, then there exists $q \in (1,\theta]$ such that $f^2(q) \leq q$. Noting that $df^2(1)/dx = (f'(1))^2 > 1$ and $f^2(1) = 1$, we can assume $f^2(q) = q$ without loss of generality. For simplicity of notation, let $g(x) = f^2(x)$. Note that g is strictly increasing on $[1,\theta]$ as $dg(x)/dx = f'(f(x))f'(x) > 0$ for $x \in [1,\theta)$. Again since $dg(1)/dx > 1$ and $g(1) = 1$, there exists $r \in (1,q)$ such that $r < g(r) < g(q) = q$. Thus

$$0 < \frac{g(q) - g(r)}{q - r} < 1.$$

And since $q < \theta < g(\theta)$, we also have

$$1 < \frac{g(\theta) - g(q)}{\theta - q}.$$

Therefore, by the mean value theorem, g' attains a (local) minimum at some $c \in (1,\theta)$, i.e., $g''(c) = 0$ and $g'''(c) \geq 0$. Remember that if the Schwarzian derivative of f, Sf, is negative, then $S(f \circ f) = Sg$ is also negative [cf.

Singer(1978)]. Since $Sf(x) < 0$ for $x > 0$, we then must have

$$Sf^2(c) = Sg(c) = \frac{g'''(c)}{g'(c)} < 0,$$

which contradicts $g'(c) > 0$ and $g'''(c) \geq 0$. Q.E.D.

3.6.2 Proof of Lemma 3

Let $\sigma_1 > \sigma^*$ be given, where σ^* is as in Lemma 2. We can see that there is an integer $\bar{k} \geq 1$ such that for $k \geq \bar{k}$, $\theta(\sigma_1) < f^2_{\sigma_1}(\theta(\sigma_1)) < f^{-k}_{\sigma_1,R}(1) \equiv \tilde{y}_k(\sigma_1)$, where $f^{-k}_{\sigma,s}$ means the k-th composite of $f^{-1}_{\sigma,s}$ ($s = R$, L) and $\tilde{y}_k(\sigma)$ is, by the implicit function theorem, a smooth function of $\sigma > (1-\alpha)/\alpha$ with $d\hat{y}_k(\sigma)/d\sigma > 0$. Since $f^2_\sigma(\theta(\sigma)) \to \infty$ and $\hat{y}_k(\sigma) \to 1/(1-\alpha)^k$ ($\sigma \to \infty$), there is $\sigma_2 (> \sigma_1)$ such that $f^2_{\sigma_2}(\theta(\sigma_2)) > \tilde{y}_k(\sigma_2) > \theta(\sigma_2)$. This implies that $f_{\sigma_1}(\theta(\sigma_1)) > f^{-1}_{\sigma_1,L}(\tilde{y}_k(\sigma_1)) \equiv \hat{y}_k(\sigma_1)$ (where $\hat{y}_k(\sigma)$ is a smooth function of σ) and $f_{\sigma_2}(\theta(\sigma_2)) < f^{-1}_{\sigma_2,L}(\tilde{y}_k(\sigma_2)) = \hat{y}_k(\sigma_2)$. Since the horizontal line $y = \hat{y}_k(\sigma_j)$ obtained above belongs to $W^s_{\sigma_j,1}(p)$ because $f^{k+1}_{\sigma_j}(\hat{y}_k) = 1$ and since the point $(\theta(\sigma_j), f_{\sigma_j}(\theta(\sigma_j)))$ belongs to $W^u_{\sigma_j,1}(p)$ ($j = 1, 2$) by Lemma 2, the assertions (P1) and (P2) immediately follow. (P3) follows from (P1) and (P2) by continuity. Q.E.D.

3.6.3 Proof of Lemma 4

(i): By Lemma 1, if

$$\frac{2-\alpha}{\alpha} < \sigma < \frac{1}{(1-m)(1-\alpha)},$$

then the fixed point p of $F_{\sigma,m}$ is a dissipative hyperbolic saddle. Thus if $\sigma > (2-\alpha)/\alpha$, then for $m \in (\varepsilon_1, 1)$ where $\varepsilon_1 = \varepsilon_1(\sigma) = 1 - \frac{1}{\sigma(1-\alpha)}$, p is a dissipative hyperbolic saddle.

(ii): This part is proved if the following is verified:

Claim: Let σ_1 and σ_2 be as in Lemma 3. Then there exists $\varepsilon \in (0,1)$ such that for every $m \in (\varepsilon, 1)$, the map $F_{\sigma,m}$ has arcs $\gamma^s_{\sigma,m} \subset W^s_{\sigma,m}(p)$ and $\gamma^u_{\sigma,m} \subset W^u_{\sigma,m}(p)$ satisfying the following:

(a) $\gamma^s_{\sigma_1,m} \cap \gamma^u_{\sigma_1,m} = \phi$,

(b) $\gamma^s_{\sigma_2,m}$ and $\gamma^u_{\sigma_2,m}$ have two transverse intersections, and

(c) For some $\sigma_H \in (\sigma_1, \sigma_2)$, $\gamma^s_{\sigma_H,m}$ and $\gamma^u_{\sigma_H,m}$ have a quadratic homoclinic tangency that unfolds generically with respect to σ.

Proof of Claim: Since $\sigma_2 > \sigma_1 > (2-\alpha)/\alpha$, the fixed point p of the non-singular map $F_{\sigma,m}$ for every $\sigma \in [\sigma_1, \sigma_2]$ and $m \in (\varepsilon_1(\sigma_2), 1)$ is a dissipative hyperbolic saddle from the result of part (i) above. Thus, by continuous dependence of the unstable manifold of a hyperbolic fixed point on $F_{\sigma,m}$ in the C^2 topology, the map $F_{\sigma,m}$ has an arc $\gamma^u_{\sigma,m} \subset W^u_{\sigma,m}(p)$ which is C^2-close to $W^u_{\sigma,1}(p)$ (obtained in Lemma 2) for each $\sigma \in [\sigma_1, \sigma_2]$ and for m close to 1. Furthermore, note that each horizontal line $\{y = \hat{y}_i\} \subset W^s_{\sigma,1}(p)$ in the proof of Lemma 3 consists of regular points: for every $x \in \{y = \hat{y}_i\} \subset \mathbb{R}^2$ and for n such that $F^n_{\sigma,1}(x) = p$, it holds that $\mathrm{Im}(DF^n_{\sigma,1}(x)) + T_p(\{y=1\}) = \mathbb{R}^2$, that is, the vectors $(\Pi_{j=1}^{n-1} f'(f^{j-1}(\hat{y}_i)), \Pi_{j=1}^{n} f'(f^{j-1}(\hat{y}_i)))$ and $(1,0)$ span $\mathbb{R}^2$ because $f^{j-1}(\hat{y}_i) \neq \theta$ for $j = 1, \ldots, n$. Thus by Proposition 1 in Appendix 4 in Palis and Takens (1993, p.182), the non-singular map $F_{\sigma,m}$ for m sufficiently close to 1 has an arc $\gamma^s_{\sigma,m} \subset W^s_{\sigma,m}(p)$ which is C^2-close to a suitable compact line segment of the horizontal line $y = \hat{y}_i$. By stability of transversality and by Lemma 3, (a) and (b) follow, which are the situation of 'inevitable tangency', a part of Takens' weakened generic conditions for real-analytic families of diffeomorphisms [see Takens (1992)]. Evidently, the existence of $\sigma_{\hat{H}}$ for which $\gamma^s_{\sigma_{\hat{H}},m}$ and $\gamma^u_{\sigma_{\hat{H}},m}$ have a quadratic homoclinic tangency follows from (a) and (b). By Takens' weakened generic conditions, we immediately have the generic unfolding of the homoclinic tangency: in fact, the ratio $-\log(|\lambda_2(\sigma)|)/\log(|\lambda_1(\sigma)|)$ of eigenvalues λ_1 and λ_2 of $DF_{\sigma,m}(p)$ is clearly non-constant with respect to σ. This proves (c) and thus (ii) of Lemma 4. Q.E.D.

3.6.4 Proof of Proposition 2

(i): Since $v_t = v_0^{(-\sigma)^t}$, we have $\limsup_{t\to\infty} v_t = \infty$ for $v_0 > 0$ and $v_0 \neq 1$. If $v_0 = 1$, then $v_t = 1$ for $t \geq 0$ and $u_{t+1} = (1-\alpha)u_t + \alpha$, implying $u_t \to 1$ as $t \to +\infty$ for any u_0.

(ii): Suppose first that the (positive) sequence $\{v_t\}$ generated by Eqs.(3.6) and (3.7) is eventually uniformly bounded from above in the sense that $\limsup_{t\to\infty} v_t \leq \bar{v}$ for some $0 < \bar{v} < \infty$, independent of $(u_0, v_0) \in \mathbb{R}^2_{++}$. In other words, there exist an integer K (depending on v_0 and u_0) and a uniform $\bar{v} < +\infty$ such that $0 < v_t \leq \bar{v}$ for any $t \geq K$. If so, then $u_{t+1} = (1-\alpha)u_t + \alpha v_{t+1} \leq (1-\alpha)u_t + \alpha\bar{v}$ for $t \geq K$. Thus, for $t > K$, $u_t \leq (1-\alpha)^{t-K}u_K + \alpha\bar{v}\sum_{i=0}^{t-K-1}(1-\alpha)^i \leq (1-\alpha)^{t-K}u_K + \bar{v}$, which implies $\limsup_{t\to\infty} u_t \leq \bar{v}$. Evidently, if v_t (and thus also u_t) is eventually uniformly bounded from above, then $\liminf_{t\to\infty} u_t \geq \underline{u}$ and $\liminf_{t\to\infty} v_t \geq \underline{v}$ for some uniform $\underline{u} > 0$ and $\underline{v} > 0$. The rectangle defined by $R = [\bar{v}, \underline{u}] \times [\bar{v}, \underline{v}] \subset \mathbb{R}^2_{++}$ is then a required compact region.

Therefore, it suffices to show that $\{v_t\}$ is eventually uniformly bounded from above. If it does not hold, then, for any $\varepsilon > 0$, there exists $L = L(\varepsilon, u_0, v_0)$ such that $v_{L+3} > \varepsilon^{-1/\sigma}$ or, equivalently, $0 < mu_{L+2} + (1-m)v_{L+2} < \varepsilon$, which implies that $u_{L+2} < \varepsilon/m$ and $v_{L+2} < \varepsilon/(1-m)$. Thus we get

$$(1-\alpha)u_{L+1} + \frac{\alpha}{[mu_{L+1} + (1-m)v_{L+1}]^{\sigma}} < \frac{\varepsilon}{m} \quad \text{and} \tag{3.16}$$

$$\frac{1}{[mu_{L+1} + (1-m)v_{L+1}]^{\sigma}} < \frac{\varepsilon}{1-m}. \tag{3.17}$$

From (3.16) we obtain

$$u_{L+1} < \frac{\varepsilon}{m(1-\alpha)}. \tag{3.18}$$

Rearranging (3.17) gives

$$\left[\frac{1-m}{\varepsilon}\right]^{\frac{1}{\sigma}} < mu_{L+1} + (1-m)v_{L+1}. \tag{3.19}$$

Combining (3.18) and (3.19) we obtain

$$v_{L+1} > \frac{1}{1-m}\left[\frac{1-m}{\varepsilon}\right]^{\frac{1}{\sigma}} - \frac{\varepsilon}{(1-\alpha)(1-m)} \equiv \Delta(\varepsilon). \tag{3.20}$$

From (3.18) and (3.20), it follows that

$$\begin{aligned}
\frac{\varepsilon}{m(1-\alpha)} &> u_{L+1} = (1-\alpha)u_L + \frac{\alpha}{[mu_L + (1-m)v_L]^{\sigma}} \\
&= (1-\alpha)u_L + \alpha v_{L+1} \\
&> (1-\alpha)u_L + \alpha\Delta(\varepsilon) \\
&\geq \alpha\Delta(\varepsilon).
\end{aligned}$$

Hence we obtain $\varepsilon > m\alpha(1-\alpha)\Delta(\varepsilon)$. Since $\Delta(\varepsilon) \to +\infty$ as $\varepsilon \to 0$, we get a contradiction. This completes the proof. Q.E.D.

3.6.5 Proof of Proposition 3

(i): Since f_σ is unimodal and the Schwarzian derivative of f_σ given by (3.15) is negative (i.e., $Sf_\sigma(x) < 0$) for $\sigma \geq 1$ and $x > 0$, f_σ has, by Singer's theorem [Singer (1978)], at most one periodic attractor for $\sigma \geq 1$, and so does $F_{\sigma,1}$. For $\sigma \in (0,1)$, the unique fixed point $x = 1$ of f_σ has been shown to be globally attracting [see Onozaki/Sawada(2001)], so the fixed point $p = (1,1)$ of $F_{\sigma,1}$ is also globally attracting.

(ii): See (iv) in Proposition 1. Q.E.D.

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Chapter 4

Threshold Nonlinearities and Asymmetric Endogenous Business Cycles

abstract

This paper[1] presents a model of endogenous business cycles in the presence of knowledge spillovers and a time-to-build restriction. There are two key assumptions which virtually characterize the model: (i) the payoff to each firm depends on the aggregate state of knowledge; and (ii) the innovation of a project is time-consuming. Under those assumptions, a simple structure is shown to be described by a piecewise linear difference equation with a discontinuity. We show that the resulting dynamics generated by such a dynamical system leads to an asymmetric periodic cycle of arbitrary period, which appears to switch repeatedly between different regimes. The model presents a simple theoretical ground for a dynamical system with threshold nonlinearities.

[1]This essay is based on joint works with Junichiro Ishida. See Ishida/Yokoo (2002). Also see Ishida/Yokoo (2001) for an earlier draft with slightly different settings in the model.

4.1 Introduction

It is often pointed out that many economic variables, such as GDP, cannot simply be characterized by a symmetric stochastic process. There is ample evidence that business cycles are asymmetric in nature. To name a few, Hamilton (1989), Beaudry and Koop (1993) and McQueen and Thorley (1993) report that economic booms tend to be more persistent than recessions in the United State. Caner and Hansen (2001) find some evidence that the unemployment rate is better characterized by a stationary threshold specification. Also, examining international evidence, Hess and Iwata (1997) and Razzak (2001) find that the nature of business cycles is asymmetric in many developed countries. Those findings seem to suggest that modelling economic fluctuations as a symmetric process, either stationary or nonstationary, potentially trivializes a critical aspect of business cycles.

While many important contributions are made as far as statistical methodologies are concerned, not much is explored theoretically for the underlying mechanism which gives rise to a dynamical system with threshold nonlinearities. The paper intends to fill this gap. We construct a model in which the resulting dynamics is characterized by a piecewise linear difference equation with a discontinuity, the type of dynamics which is found routinely, for instance, in the field of neural networks but rarely in the field of economics. Nevertheless, the dynamics generated by such a dynamical system is potentially important in economics since it leads to an asymmetric periodic cycle in which two states of the economy alternate each other asymmetrically. The resulting dynamics thus appears to switch repeatedly between different regimes, as often modelled in time series econometrics. The model presents a simple theoretical ground to capture this type of (endogenous) asymmetric dynamics.

Two assumptions virtually characterize the model and lead to the emergence of asymmetric dynamics. First, we consider the effect of knowledge spillovers and assume that firms are more productive when more firms engage in innovations. Second, while each firm can innovate and upgrade its project, this innovation process is time-consuming in that a firm which chooses to innovate its project must stay inactive for that period. This is a sort of time-to-build restriction which eventually amounts to a choice between instant gratification and future prosperity: each firm must choose either to adopt a low-quality project immediately or to wait for a while and adopt a high-quality project in the future. The assumption has certain generality since this problem is inherent in many economic situations.

Under those assumptions, a simple structure is shown to exhibit perpetual fluctuations between two phases, referred to as the expansion and the contraction phase. The intuition behind this is fairly simple. It is more costly for firms to forgo production opportunities when they are more productive. When the opportunity cost of forgoing production is sufficiently high, firms would rather choose not to innovate. The expansion phase, however, may not be sustainable in the long run. As more firms choose not to innovate, aggregate output starts to decline and the economy eventually spirals into the contraction phase. By the same logic, the contraction phase may not be sustainable in the long run either. In the contraction phase, the opportunity cost of forgoing production is low, and firms would be willing to wait and innovate. As more firms choose to innovate, aggregate output indeed starts to rise and the economy eventually moves back into the expansion phase.

The paper is related in its scope to Benhabib and Rustichini (1991), Asea and Zak (1999) and Kitagawa and Shibata (2001) which incorporate the time-to-build restriction to generate endogenous cycles. Within the framework in which any investment requires some gestation period before it becomes productive, for instance, Kitagawa and Shibata (2001) show that period-n cycles arise when the gestation period exceeds the agents' time horizon. It should be pointed out, however, that the underlying mechanism behind endogenous cycles in our model is completely different from theirs. The critical assumption in our model which directly leads to endogenous cycles is that each firm can choose the quality of its project while, in theirs, there is no variation in the quality of investment and each investment requires the same gestation period.[2]

The rest of the paper is organized as follows. The model is briefly outlined in section 2, and is analyzed in section 3. The dynamics of the model is examined and the possibility of endogenous cycles is explored in section 4. Finally, some concluding remarks are made in section 5.

4.2 The Model

4.2.1 Environment

Time is discrete and extends from zero to infinity. We consider an economy with a continuum of firms, each indexed by i, with unit measure. At the end of each period, a constant fraction $1-\delta$, where $\delta \in (0,1)$, of (randomly chosen) firms disappear and are replaced by new ones. Each firm is risk neutral and attempts to maximize the discounted sum of expected profits with $\gamma \in (0,1)$ being the subjective discount factor. Define $\beta \equiv \gamma\delta$ as the effective discount factor.

[2] In Kitagawa and Shibata (2001), endogenous cycles arise due to the long gestation of investment; in our model, they arise due to the tradeoff between the quality of investment and its opportunity cost.

In any given period, each firm is either active or inactive and engages in at most one project at a time. Each project is characterized by its quality, either high or low, which determines the cost of production. The cost of production is k if the project is of low quality and zero if it is of high-quality. Upon entering the market, each new firm has a project at hand which is of low quality and must decide either to adopt the project as it is or to innovate it. If a firm decides to adopt (i.e., not to innovate), it immediately becomes active and produces until it disappears; if a firm decides to innovate, it can upgrade the project to high quality in the next period while it must stay inactive for that period.[3] Let $n_t \in [0,1]$ denote the number (measure) of firms which decide to innovate in period t.

4.2.2 Production

We assume that knowledge accumulates through innovations. To capture this, we assume that the productivity of each firm depends on the number of firms with a high-quality (innovated) project at the beginning of each period, a type of assumption similar in nature to the one adopted by Durlauf (1991,1993), Gale (1996) and many others.[4] Let $x_t \in [0,1]$ denote the number of firms with a high-quality project at the beginning of period t. The law of motion for x_t is then given by

$$x_{t+1} = \delta(x_t + n_t). \tag{4.1}$$

Each firm is equally productive in any given period. Let $y_t(i)$ denote the output level of firm i in period t. We specify that, when it is active,

[3] We assume that each firm can innovate its project only once.

[4] The nature of complementarities is intertemporal under this setup. Durlauf (1991,1993) argues that intertemporal complementarities capture the idea of learning-by-doing. It should be noted, however, that the intertemporal nature of complementarities is not critical to the results of the model. An important point is that the aggregate level of innovative activity affects the productivity of each firm in some sense.

each firm can produce $y_t(i) = h(x_t)$ through the (normalized) production function $h : [0, 1] \to [0, 1]$. This function $h(x)$ is assumed to be continuous and strictly increasing in x with $h(0) = \theta \in [0, 1)$ and $h(1) = 1$.

The payoff to each firm is its profit. Let $\pi_t(i)$ denote firm i's profit in period t, which is given by $h(x_t) - k$ if the firm's project is of low quality and $h(x_t)$ if it is of high quality. Each new firm chooses whether to adopt or innovate so as to maximize the discounted sum of expected profits.

4.3 Analysis

It is straightforward to characterize the optimal choice for each new firm. If a firm chooses to adopt (i.e., not to innovate), the expected gain is

$$E_t \sum_{s=0}^{\infty} \beta^s \pi_{t+s}(i) = E_t \sum_{s=0}^{\infty} \beta^s h(x_{t+s}) - \frac{k}{1-\beta}. \tag{4.2}$$

If the firm chooses to innovate, on the other hand, the expected gain is

$$E_t \sum_{s=0}^{\infty} \beta^s \pi_{t+s}(i) = E_t \sum_{s=1}^{\infty} \beta^s h(x_{t+s}). \tag{4.3}$$

The firm thus chooses not to innovate iff

$$E_t \sum_{s=0}^{\infty} \beta^s h(x_{t+s}) - \frac{k}{1-\beta} \geq E_t \sum_{s=1}^{\infty} \beta^s h(x_{t+s}), \tag{4.4}$$

which can be written as

$$h(x_t) \geq \frac{k}{1-\beta}. \tag{4.5}$$

Conversely, the firm chooses to innovate iff

$$\frac{k}{1-\beta} > h(x_t). \tag{4.6}$$

The optimal choice for each new firm is thus characterized by a simple form.[5] The production cost $k/(1-\beta)$ signifies the value of innovation. The cost of innovation is the opportunity cost of forgoing a production opportunity in period t. Each new firm chooses to innovate if the value of innovation exceeds the opportunity cost of forgoing production.

4.4 Dynamics

4.4.1 Threshold Nonlinearity

Since $\theta \leq h(x_t) \leq 1$ for any $x_t \in [0,1]$, there are three cases to consider depending upon the value of innovation $k/(1-\beta)$:

Case 1: $\theta(1-\beta) > k$;

Case 2: $k > 1-\beta$;

Case 3: $\theta(1-\beta) \leq k \leq 1-\beta$.

In Case 1, it is never profitable to innovate, and each new firm always chooses not to innovate, that is, $n_t = 0$ for any t. By Eq.(4.1), the sequence $\{x_t\}$ obeys the following dynamic process for any initial conditions $x_0 \in [0,1]$:

$$x_{t+1} = \delta x_t. \tag{4.7}$$

Thus x_t tends to the unique steady state 0 as t goes to infinity, implying that the project of every firm is of low-quality in the end.

[5]This simple representation of the optimal choice stems from the fact that each innovation requires only one gestation period. The model can potentially be much more complicated if each innovation requires a gestation periods for some arbitrary natural number $a > 1$. We do not explore this general case in this paper as it is out of the scope of this paper.

In Case 2, on the contrary, each new firm always chooses to innovate, that is, $n_t = 1 - \delta$ for any t. The resulting dynamic process is then given by

$$x_{t+1} = \delta x_t + \delta(1-\delta) \tag{4.8}$$

with a unique globally stable steady state $\delta \in (0,1)$.

In Case 3, the situation is a little more complicated. Note first that for each k satisfying $\theta(1-\beta) \leq k \leq 1-\beta$, there exists, by the intermediate value theorem and strict monotonicity of h, a unique point $c = c(k) \in [0,1]$ such that

$$h(c) = \frac{k}{(1-\beta)}. \tag{4.9}$$

Note here that the point $c = c(k)$ increases continuously with $k \in [\theta(1-\beta), (1-\beta)]$. Since h is strictly increasing, it follows that $x_t < c$ iff $h(x_t) < k/(1-\beta)$ and also that $c \leq x_t$ iff $k/(1-\beta) \leq h(x_t)$. Therefore, in Case 3, each new firm in period t chooses to innovate if $x_t < c$ (i.e., $n_t = 1-\delta$) and chooses not to innovate if $c \leq x_t$ (i.e., $n_t = 0$). Hereafter, we refer to the point c as the threshold. The law of motion for x_t in Case 3 is thus characterized by the following piecewise linear difference equation with a discontinuity at the threshold c:

$$x_{t+1} = f(x_t) = \begin{cases} \delta x_t + \delta(1-\delta) \equiv f_L(x_t) & \text{if } x_t < c, \\ \delta x_t \equiv f_R(x_t) & \text{if } c \leq x_t. \end{cases} \tag{4.10}$$

We say that the economy is in the *contraction phase* if $x_t < c$ and in the *expansion phase* if $c \leq x_t$. Note that if $c = 0$ (i.e., $k = \theta(1-\beta)$), Eq.(4.10) collapses to Eq.(4.7) with 0 being the globally attracting steady state. Furthermore, it is easy to show that if $\delta \leq c$, then x_t converges to δ as t goes to infinity for any initial condition x_0.[6] See figure 1 for a graphic illustration

[6]For $c < \delta$, the point δ is the unique steady state of Eq.(4.10), that is, $f(\delta) = \delta$. For $c = \delta$, however, $f(\delta) \neq \delta$ due to the discontinuity of f.

of this argument. In the sequel, we will explore the dynamics generated by Eq.(4.10) in more detail.

4.4.2 Periodic Cycles

In the absence of a steady state, the dynamics generated by a piecewise linear difference equation can lead to an asymmetric periodic cycle in which the expansion and the contraction phase alternate each other asymmetrically. This type of dynamics has potentially profound implications as it is able to capture an important feature of business cycles. As already stated, it is often pointed out that many economic indicators, such as GDP, exhibit significant asymmetry as if they repeatedly switch between different regimes. Examining international evidence, for instance, Hess and Iwata (1997) find that: (i) positive shocks are more persistent than negative socks in the United States and France; (ii) negative shocks are more persistent than positive shocks in the United Kingdom and Canada; (iii) there is almost no asymmetry in persistence in Italy, Japan and West Germany.[7] Their findings seem to suggest that there is no consistent asymmetric pattern of business cycles across countries. Those differences are indeed puzzling considering that those countries, roughly at the same stage of development, are supposedly subject to a similar set of stochastic shocks. Our model suggests that those differences in the asymmetric nature of business cycle can be reconciled within a very simple framework of endogenous cycle. It should be noted that the asymmetric nature of business cycle does not need to rest on inherent properties of stochastic shocks.

In a period-n cycle, $x_t = x_{t+n}$ for all t and $x_t \neq x_{t+s}$ for $s = 1, 2, \cdots, n-1$. Let $\{p_1, p_2, \cdots, p_n\}$ be a periodic cycle of prime period n for Eq.(4.10):

[7] Also, see Razzak (2001) who finds significant asymmetry in international GDP fluctuations.

that is, $f^s(p_1) = p_{1+s}$, $f^s(p_1) \neq p_1$ for $s = 1, 2, \cdots, n-1$ and $f^n(p_1) = p_1$.

The complete characterization of the dynamics of the model is clearly out of the scope of the paper as it is extremely tedious. To illustrate the gist of the model, therefore, we focus on two polar examples of some 'simple' periodic cycle. See Nagumo and Sato (1972) for the computation of a wider class of periodic cycles.

We first consider the following type of periodic cycle $\{p_1, p_2, \cdots, p_{n-1}, p_n\}$ such that

$$p_1 < p_2 < \cdots < p_{n-1} < c \leq p_n, \tag{4.11}$$

for some natural number n. This type of cycle corresponds to the case of the United Kingdom and Canada in which negative shocks are more persistent. To compute p_1 (if it exists), we simply need to solve $f^n(p_1) = p_1$. Since $p_i < c$ for $i = 1, 2, \cdots, n-1$, we have $f^n(p_1) = f_R \circ f_L^{n-1}(p_1)$. It follows that

$$f_R \circ f_L^{n-1}(p_1) = \delta^n p_1 + \delta^2(1 - \delta^{n-1}) = p_1.$$

Solving this for p_1, we obtain $p_1 = \delta^2(1 - \delta^{n-1})/(1 - \delta^n)$. For this to be consistent, this p_1 must satisfy the following conditions:

$$f_L^{n-2}(p_1) < c \quad \text{and} \tag{4.12}$$

$$f_R(c) \leq p_1. \tag{4.13}$$

It follows from condition (4.12) that

$$\begin{aligned} c &> f_L^{n-2}(p_1) \\ &= \delta^{n-2}p_1 + \delta - \delta^{n-1} = \frac{\delta^n - \delta^{2n-1}}{1 - \delta^n} + \delta - \delta^{n-1} \\ &= \frac{\delta^n - \delta^{n+1} - \delta^{n-1} + \delta}{1 - \delta^n} \equiv L_n. \end{aligned}$$

Similarly, it follows from condition (4.13) that

$$\begin{aligned} c &\leq \frac{p_1}{\delta} \\ &= \frac{\delta - \delta^n}{1 - \delta^n} \equiv R_n. \end{aligned}$$

Conditions (4.12) and (4.13) are then represented by

$$L_n < c \leq R_n. \tag{4.14}$$

Note that $R_n - L_n = \delta^{n-1}(\delta - 1)^2/(1 - \delta^n) > 0$ so that $(L_n, R_n]$ is well-defined.

Conversely, suppose that condition (4.14) is satisfied. Further, define $\hat{c}$ such that $f_L^{n-2}(\hat{c}) = c$. Solving this for $\hat{c}$, we obtain

$$\hat{c} = \frac{c - \delta + \delta^{n-1}}{\delta^{n-2}}.$$

Then, we can show that the mapping f^n restricted to the half interval $T = [\delta c, \hat{c})$ is well-defined (that is, f^n maps that interval into itself) where $f^n|T$ is linear with a constant slope $\delta^n \in (0, 1)$ and a (unique) fixed point $p_1 \in [\delta c, \hat{c})$. This indicates that condition (4.14) is a necessary and sufficient condition for the trajectory of any initial value to converge to a period-n cycle of the form (4.11). See figure 3 for a graphic illustration of this argument. For instance, a period-2 cycle appears when

$$\frac{\delta^2}{1 + \delta} < c \leq \frac{\delta}{1 + \delta}. \tag{4.15}$$

When c is in this range, a period-2 cycle is globally stable.

Similarly, we also consider another type of periodic cycle such that

$$p_n < c \leq p_{n-1} < \cdots < p_2 < p_1 = \frac{\delta(1 - \delta)}{1 - \delta^n}. \tag{4.16}$$

This type of cycle corresponds to the case of the United States and France in which positive shocks are more persistent. We do not explore this in detail as the argument here basically parallels to the one above. A periodic cycle of this form appears when

$$\sup_{x<c} f_L(x) = \delta c + \delta(1-\delta) > p_1 \quad \text{and} \tag{4.17}$$

$$c \leq f_R^{n-2}(p_1) = \delta^{n-2} p_1. \tag{4.18}$$

Equivalently, we can restate these conditions as

$$\tilde{L}_n < c \leq \tilde{R}_n, \tag{4.19}$$

where

$$\tilde{L}_n \equiv \frac{\delta^n(1-\delta)}{1-\delta^n} \quad \text{and} \quad \tilde{R}_n \equiv \frac{\delta^{n-1}(1-\delta)}{1-\delta^n}. \tag{4.20}$$

4.4.3 The Expansion Rate

The dynamics of the model can also be characterized from a different perspective. To this end, we now convert a sequence generated by f into a 0-1 sequence using the following transformation:

$$s_t = \mathbf{1}[x_t] = \begin{cases} 0 & \text{if } x < c, \\ 1 & \text{if } c \leq x. \end{cases} \tag{4.21}$$

We can then denote a periodic sequence in which 1 appears u consecutive times after 0 appears v consecutive times by $\{0^v 1^u\}$.

According to this definition, a periodic cycle of the form $p_1 < p_2 < \cdots < p_{n-1} < c \leq p_n$ can be expressed as $\{0^{n-1}1\}$. It then directly follows from above that a periodic sequence $\{0^{n-1}1\}$ appears when $L_n < c \leq R_n$. Note also that, even if $c = L_n$, the trajectory of any initial value converges to a periodic sequence $\{0^{n-1}1\}$ after a finite number of iterations. This implies that the condition

$$L_n \leq c \leq R_n, \tag{4.22}$$

can be regarded as the condition which leads to a periodic sequence $\{0^{n-1}1\}$. Similarly, we can show that a periodic sequence $\{01^{n-1}\}$ appears when

$$\tilde{L}_n \leq c \leq \tilde{R}_n. \tag{4.23}$$

Now consider some sequence $\{x_i\}_{i=0}^{\infty}$ generated by f for a given cutoff point c and the corresponding sequence $\{s_i\}_{i=0}^{\infty}$. We now define the *expansion rate*, $\rho(c)$, as

$$\rho(c) \equiv \lim_{m \to \infty} \frac{\sum_{i=0}^{m} s_i}{m}. \tag{4.24}$$

The expansion rate is analogous to the averaging firing rate for a piecewise linear neuron model (see e.g. Nagumo and Sato 1972) or to the rotation number for a homeomorphism on the circle (see e.g. Devaney 1986). In particular, for a periodic sequence of period n, $\{s_1, s_2, \cdots, s_n\}$ $(s_i \in \{0,1\})$, the expansion rate is given by

$$\rho(c) = \frac{\text{number of 1's in } \{s_1, s_2, \cdots, s_n\}}{n}. \tag{4.25}$$

The expansion rate is well-defined as a sequence starting from any initial value x_0 eventually converges to some 0-1 sequence after a finite number of iterations.[8] That is, given the parameters of the model, the expansion rate is uniquely determined independently of the initial condition. This means that $\rho(c) = 1/n$ if condition (4.22) holds and $\rho(c) = (n-1)/n$ if condition (4.23) holds. Furthermore, $\rho(c) = 0$ if $\delta \leq c$. The expansion rate has important

[8]In case of a period-2 cycle, for instance, there exists no period-2 cycle (= periodic point of period 2) when $p_1 = c$. It is, however, easily verified that, even for $p_1 = c$, a sequence starting from any initial condition x_0 becomes $\{01\}$ after a finite number of iterations.

economic implications. First of all, its denominator gives the period of the cycle. Second, it measures how frequently the economy is in the expansion phase over the course of one cycle.

To describe intuitively how the expansion rate is related to the threshold c, we first explicitly compute the range of c for the expansion rates $1/n$ and $(n-1)/n$. Define $\rho^u(c)$ as the expansion rate for which c is the upper bound. With some algebra, we obtain

$$\rho^u(c) = \begin{cases} \dfrac{\log \delta}{\log\left(\frac{\delta - c}{1-c}\right)} & \text{for } \frac{\delta}{1+\delta} \leq c < \delta, \\ \dfrac{\log\left(\delta + \frac{1-\delta}{c}\right)}{\log\left(\delta + \frac{1-\delta}{c}\right) - \log \delta} & \text{for } 0 < c < \frac{\delta}{1+\delta}. \end{cases} \tag{4.26}$$

Similarly, define $\rho^l(c)$ as the expansion rate for which c is the lower bound. With some algebra, we obtain

$$\rho^l(c) = \begin{cases} -\dfrac{\log \delta}{\log\left(1 + \frac{1-\delta}{\delta(\delta - c)}\right)} & \text{for } \frac{\delta^2}{1+\delta} \leq c < \delta, \\ \dfrac{\log\left(1 + \frac{1-\delta}{c}\right) + \log \delta}{\log\left(1 + \frac{1-\delta}{c}\right)} & \text{for } 0 < c < \frac{\delta^2}{1+\delta}. \end{cases} \tag{4.27}$$

The derivation of $\rho^u(c)$ and $\rho^l(c)$ is placed in appendix. Figure 4 depicts the graphs of $\rho^u(c)$ and $\rho^l(c)$ with $\delta = 0.8$.

In general, we can characterize the expansion rate as a function of c by generating a Farey sequence in the following manner.[9] Suppose that we have two incommensurable fractions $\{\frac{p}{q}, \frac{r}{s}\}$ where $\frac{p}{q} > \frac{r}{s}$. We can then generate a new fraction from these such that $\frac{p+r}{q+s}$. Note that, since

$$\frac{p}{q} > \frac{p+r}{q+s} > \frac{r}{s},$$

[9] See Hardy and Wright (1979) for more detail on the properties of the Farey sequence.

we write this as $\{\frac{p}{q}, \frac{p+r}{q+s}, \frac{r}{s}\}$. This process of generating a new fraction is repeatedly applied to any two neighboring fractions. Starting from $F_1 = \{\frac{1}{1}, \frac{0}{1}\}$, applying this process repeatedly yields

$$
\begin{aligned}
F_1 &= \left\{\frac{1}{1}, \frac{0}{1}\right\}, \\
F_2 &= \left\{\frac{1}{1}, \frac{1}{2}, \frac{0}{1}\right\}, \\
F_3 &= \left\{\frac{1}{1}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{0}{1}\right\}, \\
F_4 &= \left\{\frac{1}{1}, \frac{3}{4}, \frac{2}{3}, \frac{3}{5}, \frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{1}{4}, \frac{0}{1}\right\}, \\
F_5 &= \left\{\frac{1}{1}, \frac{4}{5}, \frac{3}{4}, \frac{5}{7}, \frac{2}{3}, \frac{5}{8}, \frac{3}{5}, \frac{4}{7}, \frac{1}{2}, \frac{3}{7}, \frac{2}{5}, \frac{3}{8}, \frac{1}{3}, \frac{2}{7}, \frac{1}{4}, \frac{1}{5}, \frac{0}{1}\right\}, \\
&\vdots
\end{aligned}
$$

For instance, if we observe a period-2 cycle (*i.e.*, $\rho(c_0) = 1/2$) for some threshold c_0 and a period-5 cycle with $\rho(c_1) = 2/5$ for some other $c_1 (> c_0)$, then we can observe a period-7 cycle with $\rho(c_2) = 3/7$ for some $c_2 \in (c_0, c_1)$, provided that other things are held constant. The function $\rho(c)$ extended on $[0, 1]$ turns out to be a Cantor function (also known as a devil's staircase), that is, a monotone, continuous function with $\rho'(c) = 0$ for almost every c (see Nagumo and Sato 1972 for this point). See figure 5 for the devil's staircase as a function of the threshold point c. Figure 6 depicts a bifurcation diagram which shows how periodic patterns change as the threshold point c increases with δ fixed.

4.5 Conclusion

This paper presents a simple model of endogenous business cycles in the presence of knowledge spillovers and the time-to-build restriction. Under the maintained assumptions, we show that a simple model is capable of generating an asymmetric periodic cycle of arbitrary period. As it turns out, the dynamics of the model is characterized by the expansion rate, which gives the period of a cycle and the frequency of the economy being in the expansion phase over the course of one cycle. Despite its simple structure, the model is able to mimic the asymmetric pattern of business cycles which is often modelled as a threshold autoregressive model. The presented model can thus be considered as providing a theoretical ground for a dynamical system with threshold nonlinearities.

As a final note, it should be pointed out that we deliberately keep the model as simple as possible in order to deliver the main message in a clear manner. We therefore regard this model more as a benchmark with several possible extensions. For instance, we can show that the model exhibits much more complicated, and possibly chaotic, dynamics even with some slight modifications (see e.g. the next chapter). It is of some interest to pursue this possibility in future.

4.6 Appendix

The Derivation of $\rho^h(c)$ and $\rho^l(c)$

We obtain the upper bound of c for the expansion rates $1/n$ and $(n-1)/n$ by solving R_n and $\tilde{R}_n$ for n. Let $n(c)$ be the solution as a function of c. We then translate this into the expansion rate to obtain $\rho^h(c)$. Note that $R_n = \tilde{R}_n = \delta/(1+\delta)$ when $n = 2$. It directly follows from the definitions of

R_n and $\tilde{R}_n$ that R_n is valid when $\delta/(1+\delta) \leq c < \delta$ and $\tilde{R}_n$ is valid when $0 < c \leq \delta/(1+\delta)$. This leads to

$$\rho^h(c) = \begin{cases} \frac{1}{n(c)} & \text{for } \frac{\delta}{1+\delta} \leq c < \delta, \\ \frac{n(c)-1}{n(c)} & \text{for } 0 < c < \frac{\delta}{1+\delta}. \end{cases} \tag{4.28}$$

We can basically follow the same procedure to obtain the lower bound of c. Solving L_n and $\tilde{L}_n$ yields $n(c)$ as above. By the same token, we can show from the definitions of L_n and $\tilde{L}_n$ that L_n is valid when $\delta^2/(1+\delta) \leq c < \delta$ and $\tilde{L}_n$ is valid when $0 < c < \delta^2/(1+\delta)$. This leads to

$$\rho^l(c) = \begin{cases} \frac{1}{n(c)} & \text{for } \frac{\delta^2}{1+\delta} \leq c < \delta, \\ \frac{n(c)-1}{n(c)} & \text{for } 0 < c < \frac{\delta^2}{1+\delta}. \end{cases} \tag{4.29}$$

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List of Figures

Figure 1: The dynamics of the model with a steady state