Direct decompositions of groups of piecewise linear homeomorphisms of the unit interval

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Introduction

The groups F, T, and V were introduced by Richard Thompson in 1965, and were first used in [18] to construct other finitely presented groups with unsolvable word problems. Since then, these groups have appeared naturally in different branches in mathematics, for example, in homotopy theory [12], dynamical systems [13], and diagram groups over semi-group presentations [16]. The groups have a collection of various exotic properties which made them counterexamples to well-known conjectures in the study of infinite group theory. For example, T and V are the first known examples of infinite, finitely presented, simple groups [9], and F is the first known example of a torsion free group of type FP_∞ and not of type FP [7]. Some generalizations of these groups are also known. For example, Higman generalized V to an infinite family of finitely presented simple groups [17], and Brown extended this to the infinite families $F_n \subset T_n \subset V_n$, where n is an integer greater than or equal to 2 ($F_2 = F$, $T_2 = T$, and $V_2 = V$), and proved that each of the groups is finitely presented [8].

The group F has the following presentation:

$$\langle A, B \mid [AB^{-1}, A^{-1}BA],\ [AB^{-1}, A^{-2}BA^2]\rangle,$$

where $[x, y] = xyx^{-1}y^{-1}$. In fact, it is known that each of the groups F_n has the presentation with n generators and $n(n-1)$ relators. Although these presentations help us understand the structures and properties of the groups, they are somewhat complicated. Hence many different geometric descriptions of the groups have been used. For example, tree pair diagrams [9], diagrams of Guba and Sapir [16], and strand diagrams of Belk and Matucci [2] have been used to describe elements for Thompson's groups F_n. One of the most useful representations is the realization of them as particular subgroups of $\mathrm{PLo}(I)$, where $\mathrm{PLo}(I)$ is the group of piecewise linear orientation-preserving homeomorphisms of the unit interval $I = [0, 1]$ that are differentiable everywhere except at finitely many real numbers, under the operation of composition. In fact, F can be defined as the subgroup of $\mathrm{PLo}(I)$ that are differentiable everywhere except at finitely many dyadic rational numbers (i.e., numbers from $\mathbb{Z}[1/2]$) and such that on the intervals of differentiability the derivatives are integer powers of 2. Many interesting properties of F have been clarified using the interpretation as homeomorphism groups. For example, F is a finitely presented torsion free group, does not contain a nonabelian free subgroup [5], and the commutator subgroup of F is an infinite simple group. A comprehensive introduction is given in [9]. The group F is also one of the most mysterious objects in group theory. For example, one of the main questions about F is whether it is amenable. Incorrect proofs of amenability and non-amenability of F appear quite often, and the reason why all these proofs are wrong is that F is very counter-intuitive. Other problems about F are also challenging, and attract many researchers.

Recently, subgroups of F have been extensively studied. For example, finite index subgroups of F are described in [4], and the solvable subgroups of F are classified in [3] by Bleak. The maximal subgroups and stabilizer subgroups of F are investigated in [14, 15] by Golan and Sapir. However, the structures of subgroups of F are far from well-understood, and there are many open questions.

This dissertation contributes to the study of subgroups of Thompson's group F (and $\mathrm{PLo}(I)$). The focus of our attention will be subgroups of F which are direct products of finitely many indecomposable groups. In fact, we provide a criterion for any two subgroups of $\mathrm{PLo}(I)$ which are direct products of finitely many indecomposable non-commutative groups to be non-isomorphic. As its application we give a necessary and sufficient condition for any two subgroups of the R. Thompson group F that are stabilizers of finite sets of numbers in the interval $(0, 1)$ to be isomorphic, thus solving a problem by Golan and Sapir. We also show that if two stabilizers are isomorphic, then they are conjugate inside a certain group $\mathcal{G}$ (see Introduction 1.1 in Chapter 1, for details).

In Chapter 2, we consider other finitely presented groups in geometric group theory. We explicitly construct Schreier coset graphs of solvable Baumslag-Solitar groups for stabilizers of all points in the real line under a natural action of them on the real line. As its consequence, we classify the Schreier coset graphs up to isomorphism, and obtain a relevance to presentations for the stabilizers (see Introduction 2.1 in Chapter 2, for details).

Chapter 1

Groups of piecewise linear homeomorphisms

1.1 Introduction

Let PLo(I) be the group of piecewise linear orientation-preserving homeomorphisms of the unit interval $I = [0, 1]$ that are differentiable everywhere except at finitely many real numbers, under the operation of composition. The focus of our attention will be subgroups of PLo(I) which are direct products of finitely many indecomposable groups.

Subgroups of PLo(I) have been extensively studied (see [3, 5, 6, 23], for examples). Among those, Thompson's group F is one of the most interesting objects. Recall that it is defined as the group of piecewise linear homeomorphisms from the closed unit interval I onto itself that are differentiable everywhere except at finitely many dyadic rational numbers (i.e., numbers from $\mathbb{Z}[1/2]$) and such that on the intervals of differentiability the derivatives are integer powers of 2.

One of the most interesting open problems about F is whether it is amenable. In [21, 22], Savchuk constructed the Schreier graph of F with respect to the stabilizer H_U of any finite set of real numbers $U \subset (0, 1)$. He proved that all the Schreier graphs are amenable and also showed that if U consists of a single number, then H_U is an infinite index maximal subgroup of F. In [14], Golan and Sapir constructed other maximal subgroups of infinite index which do not fix any real number in $(0, 1)$.

Golan and Sapir [15] continued the study of the subgroups H_U for arbitrary finite sets U. Let $U = \{\alpha_1, \ldots, \alpha_n\} \subset I$, where $\alpha_j < \alpha_{j+1}$. The type $\tau(U)$ was defined as the word of length n in the alphabet $\{1, 2, 3\}$ as follows: for every i, the ith letter in $\tau(U)$ is 1 if α_i is a dyadic rational, 2 if α_i is rational but not a dyadic rational, and 3 if α_i is irrational. They showed that H_U is isomorphic to a certain semidirect product, and also proved that H_U is finitely generated if and only if U does not contain irrational numbers. Moreover, it was proved that if $\tau(U) \equiv \tau(V)$ for finite sets U, $V \subset (0, 1)$, then H_U and H_V are isomorphic ($p \equiv q$ denotes letter-by-letter equality of words p, q). The proof was completed by realizing the subgroups as iterated ascending HNN-extensions. They also noticed that H_U is a direct product where the factors correspond to subwords of $\tau(U)$. They asked [15, Subsection 4.1] what a necessary and sufficient condition for H_U and H_V to be isomorphic is.

In Chapter 1, we will answer the above question by focusing on the fact that the stabilizers are direct products of finitely many subgroups (see Theorem 1.1.3 below). In fact, we establish the following result about a unique expression of a direct product of finitely many non-commutative indecomposable subgroups of $\mathrm{PLo}(I)$. The symmetric group of degree n and the center of a group G are denoted by S_n and $Z(G)$, respectively.

Theorem 1.1.1. *Let $H_1, \dots, H_n$, $K_1, \dots, K_m$ be non-commutative, indecomposable subgroups of $\mathrm{PLo}(I)$. Suppose that $Z(H_i) = \{1\}$ for each i and $Z(K_j) = \{1\}$ for each j. Then, $\prod_{i=1}^{n} H_i$ and $\prod_{j=1}^{m} K_j$ are isomorphic if and only if $n = m$, and there exists a permutation $\sigma \in S_n$ such that H_i and $K_{\sigma(i)}$ are isomorphic for each $i \in \{1, \dots, n\}$.*

Since for any subgroup G of $\mathrm{PLo}(I)$ the center of the quotient group $G/Z(G)$ is trivial (see Proposition 1.3.4), we obtain the following:

Corollary 1.1.2. *Let $H_1, \dots, H_n$, $K_1, \dots, K_m$ be non-commutative, indecomposable subgroups of $\mathrm{PLo}(I)$. If $\prod_{i=1}^{n} H_i$ and $\prod_{j=1}^{m} K_j$ are isomorphic, then $n = m$, and there exists $\sigma \in S_n$ such that $H_i/Z(H_i)$ and $K_{\sigma(i)}/Z(K_{\sigma(i)})$ are isomorphic for each $i \in \{1, \dots, n\}$.*

We will now recall some definitions about a direct decomposition in group theory. Let G be a group. A subgroup H of G is called a *direct factor* of G if there exists a subgroup K of G such that $G = H \times K$. If there are no proper non-trivial direct factors of G, then G is said to be *indecomposable*. A *Remak decomposition* of a group G is a decomposition of G as a direct product of finitely many non-trivial indecomposable subgroups. It is said that a group G satisfies the *maximal* (respectively, *minimal*) *condition on normal subgroups* if each non-empty family of normal subgroups contains at least one maximal (respectively, minimal) element for the inclusion. A classical result, called the Krull-Remak-Schmidt theorem, states that if a group G satisfies both the maximal and minimal conditions on normal subgroups, then its Remak decomposition is unique up to isomorphism of the direct factors and a permutation of the direct factors. We refer to [20, Sec.3.3] for details. Obviously, finite groups satisfy both maximal and minimal conditions on normal subgroups. On the other hand, subgroups of $\mathrm{PLo}(I)$ do not satisfy these conditions in general. For example, the Thompson group F satisfies the maximal condition on normal subgroups, but does not satisfy the minimal condition on normal subgroups since the lattice of normal subgroups of F is isomorphic to the lattice of subgroups of $\mathbb{Z}^2$ (see, for example, [4]). Hence we cannot apply the Krull-Remak-Schmidt theorem directly to the subgroups of $\mathrm{PLo}(I)$, and so establish the above result (see Theorem 1.1.1 and Corollary 1.3.8), which is analogous to the Krull-Remak-Schmidt theorem.

Now we return to the question about stabilizers of F. Every finite subset U of $(0,1)$ is subdivided into three subsets $U = U_1 \sqcup U_2 \sqcup U_3$, where U_1 consists of numbers from $\mathbb{Z}[1/2]$, U_2 consists of rational numbers not in $\mathbb{Z}[1/2]$, and U_3 consists of irrational numbers. Write

$$U_1 \cup U_3 = \{r_1, \dots, r_n\}, \quad r_j < r_{j+1}$$

and let $r_0 = 0$, $r_{n+1} = 1$, and

$$U_{2,k} = \{q \in U_2 \mid r_k < q < r_{k+1}\}.$$

For any word $w_1w_2 \in \{11, 13, 33\}$ and $j \in \{0, \dots, |U_2|\}$, let $\Lambda_{U,w_1w_2,j} =$

$$\{i \in \{0, \dots, n\} \mid \tau(\{r_i, r_{i+1}\} \cup U_{2,i}) \equiv w_1 2^j w_2 \text{ or } \tau(\{r_i, r_{i+1}\} \cup U_{2,i}) \equiv w_2 2^j w_1\}.$$

By applying the result mentioned above to the case of the stabilizers of F, we obtain the following:

Theorem 1.1.3. *Let U and V be finite sets of numbers in $(0,1)$. Then the following statements are equivalent.*

(1) *H_U and H_V are isomorphic.*

(2) *$|U_2| = |V_2|$, and $|\Lambda_{U,w_1w_2,j}| = |\Lambda_{V,w_1w_2,j}|$ for each $w_1w_2 \in \{11, 13, 33\}$ and each $j \in \{0, \dots, |U_2|\}$.*

We note that if $\tau(U) \equiv \tau(V)$, then the statement (2) holds. The implication from (2) to (1) is shown essentially in [15](see Lemma 1.4.3 below). The converse of this implication is the essential part of our theorem.

We also establish a result about conjugacy of the stabilizers. Golan and Sapir [15] defined $\mathcal{F}$ as a group which consists of possibly infinite tree-diagrams with finitely many infinite branches, and described that the standard embedding $F \to \mathrm{Homeo}(I)$ extends to an embedding $\mathcal{F} \to \mathrm{Homeo}(I)$. In addition, they proved that if $\tau(U) \equiv \tau(V)$, then H_U and H_V are conjugate in the group $\mathcal{F}$. We will introduce a group $\mathcal{G}$ into which $\mathcal{F}$ can be naturally embedded, and prove that H_U and H_V (indeed, their images in $\mathcal{G}$) are conjugate in the group $\mathcal{G}$ whenever they are isomorphic (see Theorem 1.5.8).

1.2 Notation, Terminology, and commutativity in $\mathrm{PLo}(I)$

Recall that $\mathrm{PLo}(I)$ is the group of piecewise linear orientation-preserving homeomorphisms of the unit interval I that are differentiable everywhere except at finitely many real numbers, under the operation of composition. We basically follow the notation and terminology used in [5, 6]. Composition and evaluation of functions in

PLo(I) will be in word order. Namely, for any two elements f, g in PLo(I) and any $t \in I$, $tf = f(t)$ and $fg = g \circ f$. Here, we will describe a geometric condition under which two elements of PLo(I) commute, and a geometric condition under which subgroups of PLo(I) are commutative.

The *support* of an element f in PLo(I) is the subset $\mathrm{supp}(f) = \{x \in I \mid xf \neq x\}$. We can easily see that $\mathrm{supp}(f)$ is a finite union of disjoint open intervals. Each of these open intervals will be called an *orbital* of f. If A is an orbital for f, then either $xf > x$ for all points x in A or $xf < x$ for all points x in A. We note that if $f \in$ PLo(I) and A is an orbital of f such that $xf < x$ for some (and therefore all) $x \in A$, then f^{-1} has $xf^{-1} > x$ for all $x \in A$.

Let $[f, g] = fgf^{-1}g^{-1}$. The following is a known fact [24, Lemma 2.1], but we give the proof for the convenience of the reader.

Lemma 1.2.1. *Let f, $g \in$ PLo(I) where $[f, g] = 1$. If A is an orbital of f, then either $A \cap \mathrm{supp}(g) = \emptyset$ or A is also an orbital of g.*

Proof. Suppose that $A \cap \mathrm{supp}(g) \neq \emptyset$. Then there exists an orbital B of g such that $A \cap B \neq \emptyset$. Let $A = (a, c)$ and $B = (b, d)$. Without loss of generality, we can assume that $b \leq a < d \leq c$. Since $[f, g] = 1$, it follows that $[f, g^n] = 1$ for each $n \in \mathbb{Z}$. Since $af = a$, it follows that $ag^n = afg^n = ag^n f$ for each $n \in \mathbb{Z}$. Suppose that $xg > x$ for all points x in B. Since the sequence (ag^n) converges to d as n increases, by continuity of f, we see $df = d$. Thus $d = c$. Suppose that $xg < x$ for all points x in B. Then we see that $xg^{-1} > x$ for all $x \in B$. Replacing g in the argument described above with g^{-1}, we see that $d = c$.

We take an element e in the open interval $(\max\{a, ag^{-1}\}, c(= d))$. Suppose that $yf < y$ for all points y in A. Since $[f^n, g] = 1$ for each $n \in \mathbb{Z}$, it follows that $(ef^n)g = egf^n$ for each $n \in \mathbb{Z}$. Since the sequence (ef^n) converges to a as n increases, and by $c > eg > a$ the sequence (egf^n) also converges to a, it follows that $ag = a$. Thus $b = a$. Suppose that $yf > y$ for all points y in A. Then we see that $yf^{-1} < y$ for all $y \in A$. Replacing f in the argument described above with f^{-1}, we see $b = a$. Therefore we conclude $A = B$. □

Let $f \in$ PLo(I) and A be an open interval of $(0, 1)$ whose two endpoints are fixed by f. Then define f_A as follows:

$$xf_A = \begin{cases} xf & if\ x \in A, \\ x & if\ x \notin A. \end{cases}$$

Clearly, $f_A \in$ PLo(I). If A is an orbital of f, then f_A is called the *bump* of f with *supporting interval* A. Then, A is also an orbital of f_A.

Let f_A be the bump of an element f with supporting interval A and

$$\widetilde{C}(f_A) = \{g_A \in \mathrm{PLo}(I) \mid [f_A, g] = 1\}.$$

Brin and Squier proved in [6, Theorem 4.18] that $\widetilde{C}(f_A)$ is the infinite cyclic subgroup of PLo(I) generated by all roots of f_A.

Lemma 1.2.2. *Let $f, g \in \mathrm{PLo}(I)$. Then $[f, g] = 1$ if and only if*

(1) $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$, *or*

(2) *There exists an integer $k \geq 1$ such that*

$$\mathrm{supp}(f) \cap \mathrm{supp}(g) = \bigsqcup_{i=1}^{k} C_i,$$

where C_i are common orbitals of both f and g, and for any i there exists $h_i \in \mathrm{PLo}(I)$ with exactly one orbital C_i such that $f_{C_i}, g_{C_i} \in \langle h_i \rangle$.

Proof. Suppose that $[f, g] = 1$ and $\mathrm{supp}(f) \cap \mathrm{supp}(g) \neq \emptyset$. It suffices to prove (2). By Lemma 1.2.1, we see that

$$\mathrm{supp}(f) \cap \mathrm{supp}(g) = \bigsqcup_{i=1}^{k} C_i,$$

where C_i are common orbitals of both f and g. Since $[f_{C_i}, g_{C_i}] = 1$, by [6, Theorem 4.18], there exists $h_i \in \mathrm{PLo}(I)$ with exactly one orbital C_i such that $f_{C_i}, g_{C_i} \in \langle h_i \rangle$, which proves (2). The converse implication is immediate, so the details are left to the reader. □

Let G be a subgroup of $\mathrm{PLo}(I)$. The *support* of G is the set

$$\mathrm{supp}(G) = \bigcup_{g \in G} \mathrm{supp}(g).$$

Since $\mathrm{supp}(G)$ is an open subset of $(0, 1)$, it can be written as a disjoint union of a countable (possibly finite) collection of open intervals in $(0, 1)$. Each of these open intervals will be called an *orbital of* G. We note that the complement set of $\mathrm{supp}(G)$ in I is the set of points fixed by all elements in G.

Definition 1.2.3. Let $G < \mathrm{PLo}(I)$. An orbital N of G is called a *commutative orbital* of G if for any $g \in G$, the element g_N has the open interval N as an orbital or $g_N = 1$ in $\mathrm{PLo}(I)$. An orbital of G which is not a commutative orbital is called a *non-commutative orbital* of G.

For any set $S \subset \mathbb{R}$ we denote by ∂S the boundary of S with respect to the standard topology of $\mathbb{R}$.

Lemma 1.2.4. *A subgroup G of $\mathrm{PLo}(I)$ is a non-commutative group if and only if G has a non-commutative orbital.*

Proof. Suppose that G has a non-commutative orbital N. Then, by definition, there exists $g \in G$ such that g_N does not have N as an orbital and $g_N \neq 1$. Thus, there exists an orbital A of g_N such that $A \subsetneq N$. Take a point $x \in \partial A \cap N$. Since N is an orbital of G, there exist $h \in G$ and an orbital B of h such that $x \in B \subset N$. Then, $A \cap B \neq \emptyset$ and $A \neq B$. By Lemma 1.2.2, we see that $[g, h] \neq 1$. Thus, G is a non-commutative group.

Suppose that G is a non-commutative group. Since there exist g and h in G such that $[g, h] \neq 1$, $\mathrm{supp}(g) \cap \mathrm{supp}(h) \neq \emptyset$. By Lemma 1.2.2, (i) there exists a pair (A, B) of orbitals, where A is an orbital of g and B is an orbital of h, such that $A \cap B \neq \emptyset$ and $A \neq B$, or (ii) there exists a common orbital C of both g and h such that for any $r \in \mathrm{PLo}(I)$ with exactly one orbital C, $g_C \notin \langle r \rangle$ or $h_C \notin \langle r \rangle$. If the statement (i) holds, then there exists an orbital N of G such that $A \cup B \subset N$. Hence, the orbital N is a non-commutative orbital of G.

If the statement (ii) holds, then the non-trivial commutator $[g_C, h_C] (\in \mathrm{PLo}(I))$ has slope 1 near each end of C. Thus, there exists an orbital D of $[g_C, h_C]$ such that $D \subsetneq C$. Since $[g, h] \in G$ and $[g, h]_C = [g_C, h_C]$, $[g, h]$ has D as an orbital. Since there exists an orbital N of G such that $D \subsetneq C \subset N$, the orbital N is a non-commutative orbital of G. □

The following is a known fact [3]. Nevertheless, we give the proof for the convenience of the reader.

Proposition 1.2.5. *Let G be a commutative subgroup of* $\mathrm{PLo}(I)$*. Then G is a free abelian group of finite rank or a free abelian group of countably infinite rank.*

Proof. Suppose that G is a commutative subgroup of $\mathrm{PLo}(I)$. By Lemma 1.2.4, all orbitals of G are commutative. Let $\mathrm{supp}(G)$ be the disjoint union of the collection $\{N_i\}$ of the orbitals of G. Since N_i is commutative for each i, there exists $f_i \in G$ such that f_i has N_i as an orbital. Then, since any element g in G commutes with f_i for each i, it follows that

$$\{g_{N_i} \,|\, g \in G\} < \widetilde{C}(f_{iN_i}) (\cong \mathbb{Z})$$

for each i. Let r_i be a generator of $\widetilde{C}(f_{iN_i})$ for each i. If the collection $\{N_i\}$ is finite, then for any $g \in G$ we can write $g = g_{N_1} \cdots g_{N_n}$. Thus, we see $g \in \langle r_1, \ldots, r_n \rangle$. Hence, G is a subgroup of the free abelian group $\langle r_1, \ldots, r_n \rangle$ of rank n, so G is a free abelian group of a rank less than or equal to n.

If the collection $\{N_i\}$ is countable, then for any element $g \in G$ there exist elements $s_1, \ldots, s_k$ in $\{r_i\}$ such that $g \in \langle s_1, \ldots, s_k \rangle$ since g has finitely many break points. Thus, G is a subgroup of the free abelian group $\langle \{r_i\} \rangle$ of countably infinite rank. Since G has infinitely many orbitals, G is also a free abelian group of countably infinite rank. □

By Proposition 1.2.5, any commutative subgroup of $\mathrm{PLo}(I)$ is isomorphic to a subgroup of the direct sum $\bigoplus_{i \in \mathbb{Z}} \mathbb{Z}$ of copies of integers. where $\bigoplus_{i \in \mathbb{Z}} \mathbb{Z}$ is the direct sum of copies of integers.

1.3 Decompositions of subgroups of $\mathrm{PLo}(I)$

In this section, we will prove that any group which is a direct product of finitely many non-commutative, indecomposable subgroups of $\mathrm{PLo}(I)$ has a unique decomposition (Theorem 1.3.7).

Let G and H be subgroups of $\mathrm{PLo}(I)$. We first consider a necessary and sufficient condition for any $g \in G$ and $h \in H$ to commute by extending the argument of orbitals of elements described in the previous section to that of orbitals of groups.

For a subgroup G of $\mathrm{PLo}(I)$ the complement set of $\mathrm{supp}(G)$ is denoted by $\mathrm{Fix}(G)$. Note that

$$\mathrm{Fix}(G) = \bigcap_{g \in G} \mathrm{Fix}(g),$$

where the set of points fixed by $g \in G$ is denoted by $\mathrm{Fix}(g)$.

Lemma 1.3.1. *Let G and H be subgroups of $\mathrm{PLo}(I)$ such that $[g, h] = 1$ for each $g \in G$ and each $h \in H$. Then the following statements hold.*

(1) *If N is an orbital of G, then $\partial N \subset \mathrm{Fix}(\langle G, H \rangle)$.*

(2) *If N is a non-commutative orbital of G, then $h_N = 1$ for all $h \in H$. If N is a commutative orbital of G, then for any $h \in H$ the element h_N has N as an orbital or $h_N = 1$.*

Proof. Suppose that N is an orbital of G. It suffices to prove that $\partial N \subset \mathrm{Fix}(H)$. Let $x \in \partial N$ and $h \in H$. Without loss of generality, we may assume that x is the larger one of the two elements in ∂N. Then, there exists a sequence (y_n) in N which converges to x. Since N is an orbital of G, for each n there exist $g_n \in G$ and an orbital $A_n \subset N$ of g_n such that $y_n \in A_n (\subset \mathrm{supp}(g_n))$. Let z_n be an element of $\partial A_n \cap (y_n, x]$. Since $z_n g_n = z_n$ and $[h, g_n] = 1$, by Lemma 1.2.1, we see $z_n h = z_n$. Since the sequence (z_n) converges to x, it follows that $xh = x$, which proves (1).

Suppose that N is a non-commutative orbital of G. Let $x \in N$. Since N is an orbital of G, there exist $g \in G$ and an orbital $A \subset N$ of g such that $x \in A$. Now, we claim that there exist $g' \in G$ and an orbital B of g' such that $A \cap B \neq \emptyset$, $A \neq B$, and $A \cup B \subset N$. Indeed, if $A \neq N$, then let $y \in \partial A \cap N$. Since N is an orbital of G, there exist $g' \in G$ and an orbital B of g' such that $y \in B \subset N$. If $A = N$, then $\partial A \cap N = \emptyset$. Since N is non-commutative, by Lemmas 1.2.2 and 1.2.4, there exists $g'' \in G$ such that $[g_N, g''_N] \neq 1$. From the latter part in the proof of Lemma 1.2.4, there exists an orbital B of an element g' such that $A \cap B \neq \emptyset$, $A \neq B$ and $A \cup B \subset N$ (recall that we can take g' as either g'' or $[g, g'']$).

Let $h \in H$. Since $[g, h] = 1$ and $[g', h] = 1$, by Lemma 1.2.1, we see that $\partial A \cup \partial B \subset \mathrm{Fix}(h)$. Hence $ah = a$ for all $a \in A \cup B$. In particular, $xh = x$, thus $h_N = 1$.

Suppose that N is commutative. Let $g \in G$ such that g_N has N as an orbital, and $h \in H$. Since $[h, g] = 1$, it follows that $[h_N, g_N] = 1$. By Lemma 1.2.1, h_N has N as an orbital or $h_N = 1$, which proves (2). $\square$

The next lemma will be used in Subsection 1.4.2.

Lemma 1.3.2. *Let G and H be subgroups of* $\mathrm{PLo}(I)$. *Then* $[g,h]=1$ *for each* $g \in G$ *and each* $h \in H$ *if and only if*

(1) $\mathrm{supp}(G) \cap \mathrm{supp}(H) = \emptyset$, *or*

(2) *There exists a collection* $\{C_i\}$ *of common commutative orbitals of both G and H such that*

$$\mathrm{supp}(G) \cap \mathrm{supp}(H) = \bigsqcup_i C_i,$$

and for any i there exists $r_i \in \mathrm{PLo}(I)$ *with exactly one orbital C_i such that* $g_{C_i}, h_{C_i} \in \langle r_i \rangle$ *for each $g \in G$ and each $h \in H$.*

Proof. Suppose that $[g,h]=1$ for each $g \in G$ and each $h \in H$, and $\mathrm{supp}(G) \cap \mathrm{supp}(H) \neq \emptyset$. It suffices to prove (2). Suppose that C is an orbital of G and there exists $x \in C$ such that $xh \neq x$ for some $h \in H$. By Lemma 1.3.1, C is a commutative orbital of G and h_C has C as an orbital. Then, since for any $h \in H$ h_C has C as an orbital or $h_C = 1$, C is also a commutative orbital of H. Thus, there exists a collection $\{C_i\}$ of common commutative orbitals of both G and H, and we can write

$$\mathrm{supp}(G) \cap \mathrm{supp}(H) = \bigsqcup_i C_i.$$

Fix i and let f be an element of G which has C_i as an orbital. Since $[f,h]=1$ for each $h \in H$, it follows that

$$\{h_{C_i} \mid h \in H\} < \widetilde{C}(f_{C_i}) \cong \mathbb{Z}.$$

Hence, the group $\{h_{C_i} \mid h \in H\}$ is infinite cyclic. In a similar way, the group $\{g_{C_i} \mid g \in G\}$ is also infinite cyclic. Thus, we see that

$$\langle \{g_{C_i} \mid g \in G\} \cup \{h_{C_i} \mid h \in H\} \rangle < \widetilde{C}(f_{C_i}),$$

so

$$\langle \{g_{C_i} \mid g \in G\} \cup \{h_{C_i} \mid h \in H\} \rangle$$

is also infinite cyclic. Let r_i be a generator of the group, then we complete the statement (2). The converse implication is easy to prove, so the details are left to the reader. □

Next, we set up the terminology needed to prove Proposition 1.3.4.

Let $\{A,B\}$ be a set of two non-empty open subintervals of $\mathbb{R}$. We call $\{A,B\}$ a *chain of intervals* if $A \cap B$ is a proper subinterval of A and of B. We call $\{A,B\}$ a *nest of intervals* if A is a proper subinterval of B, or if B is a proper subinterval of A. In particular, we call $\{A,B\}$ a *proper nest of intervals* if $\{A,B\}$ is a nest of intervals and $\partial A \cap \partial B = \emptyset$.

We call $\{A,B\}$ a *chain of orbitals* if $\{A,B\}$ is a chain of intervals such that A is an orbital of an element of $\mathrm{PLo}(I)$ and B is an orbital of another element. In an entirely analogous way *a nest of orbitals* and *a proper nest of orbitals* are defined.

Lemma 1.3.3. *Let g, $h \in \mathrm{PLo}(I)$ and $[g,h] \neq 1$. If for any orbital A of g and any orbital B of h such that $A \cap B \neq \emptyset$ and $[g_A, h_B] \neq 1$, the set $\{A, B\}$ is a proper nest of orbitals of g and h, then $[g,[g,h]] \neq 1$ or $[h,[g,h]] \neq 1$. Otherwise, $[g,[g,h]] \neq 1$ and $[h,[g,h]] \neq 1$.*

Proof. Let g and h be elements in $\mathrm{PLo}(I)$ and $[g,h] \neq 1$. Let $\mathcal{A}$ be the set of all orbitals of g and $\mathcal{B}$ be the set of all orbitals of h. Suppose that for any orbital A of g and any orbital B of h such that $A \cap B \neq \emptyset$ and $[g_A, h_B] \neq 1$, $\{A, B\}$ is a proper nest of orbitals of g and h. Now since $[g,h] \neq 1$, there exist $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $A \cap B \neq \emptyset$ and $[g_A, h_B] \neq 1$, and by assumption, $\{A, B\}$ is a proper nest of orbitals of g and h. Without loss of generality, we may assume that $A = (a, c)$, $\{B_1, \dots, B_n\} \subset \mathcal{B}$ and $B_i \subsetneq A$ for all i. Then, there exists $\epsilon > 0$ such that $[g,h]$ has slope 1 on $[a, a+\epsilon] \cup [c-\epsilon, c]$. By Lemma 1.2.2, $[g,h]_A \neq 1$, and then, by Lemma 1.2.1, we see $[g,[g,h]] \neq 1$.

Suppose that there exist $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $A \cap B \neq \emptyset$ and $[g_A, h_B] \neq 1$, and $\{A, B\}$ is not a proper nest of orbitals. Then there are three cases: (i) $A = B$, (ii) $\{A, B\}$ is a nest of orbitals with exactly one common endpoint, and (iii) $\{A, B\}$ is a chain of orbitals.

(i) Let $A = B = (a, c)$. Since $[g,h]_A \neq 1$ and there exists $\epsilon > 0$ such that $[g,h]$ has slope 1 on $[a, a+\epsilon] \cup [c-\epsilon, c]$, by Lemma 1.2.1, it follows that $[g,[g,h]] \neq 1$ and $[h,[g,h]] \neq 1$.

(ii) Without loss of generality we may assume that $A = (a, c)$, $B = (a, d)$, and $a < d < c$. Then there exists $\epsilon > 0$ such that $[g,h]$ has slope 1 on $[a, a+\epsilon]$. Suppose that $xg > x$ for each $x \in A$. Since $dh = d$ and $dg^{-1} \in B$, it follows that

$$dg^{-1}[g,h] = dg^{-1}ghg^{-1}h^{-1} \neq dg^{-1}.$$

Hence, $[g,h] \neq 1$ on B. Suppose that $xg < x$ for each $x \in A$. Since $dg \in B$, $dgh \neq dg$. Thus, $d[g,h] \neq d$. Hence, $[g,h] \neq 1$ on B. In both cases, $[g,h]$ has neither A nor B as orbitals. By Lemma 1.2.1, we see that $[g,[g,h]] \neq 1$ and $[h,[g,h]] \neq 1$.

(iii) We may assume that A is the leftmost orbital of g such that there exists an orbital C of h such that $\{A, C\}$ is a chain of orbitals. In addition, we may assume that B is the leftmost orbital of h such that $\{A, B\}$ is a chain of orbitals. Without loss of generality, we may assume that $A = (a, c)$, $B = (b, d)$, and $b < a < d < c$. If $yh > y$ for all $y \in B$, then $a < ah < dh = d$, thus $ahg^{-1} \neq ah$. Hence

$$a[g,h] = aghg^{-1}h^{-1} = ahg^{-1}h^{-1} \neq a$$

and, by Lemma 1.2.1, $[g,[g,h]] \neq 1$. Now, if there exists an orbital A' of g on the left of A such that $A' \cap B \neq \emptyset$, then by the above assumption, $\{A', B\}$ is a nest of orbitals. Thus, $bg = b$. Since there exists $\epsilon > 0$ such that $[g,h]$ has slope 1 on $[b, b+\epsilon]$, and $a[g,h] \neq a$, by Lemma 1.2.1, it follows that $[h,[g,h]] \neq 1$.

If $yh < y$ for all $y \in B$, then $yh^{-1} > y$ for all $y \in B$. Thus $a[g, h^{-1}] \neq a$. Since $ah^{-1}g^{-1} \neq ah^{-1}$, we see $ah^{-1}[g,h] \neq ah^{-1}$. Since $a < ah^{-1} < dh^{-1} = d$, it follows that $[g, h] \neq 1$ on B. By an argument similar to the above, we see $[h, [g, h]] \neq 1$. Now, assume by contradiction that $[g, [g, h]] = 1$. Since $ah^{-1}[g, h] \neq ah^{-1}$, it follows that $[g, h] \neq 1$ on A. Since g has A as an orbital, by Lemma 1.2.1, the commutator $[g, h]$ also has A as an orbital. Then, $hg^{-1}h^{-1}$ fixes both a and c. Since g_A and $[g, h]_A$ commute, by Lemma 1.2.2, there exist $u \in \mathrm{PLo}(I)$ and $m, n \in \mathbb{Z}$ such that $g_A = u^m$, $[g, h]_A = u^n$. Since $[g, h]_A = g_A(hg^{-1}h^{-1})_A$, it follows that

$$u^{n-m} = (hg^{-1}h^{-1})_A.$$

Since $hg^{-1}h^{-1}$ fixes ah^{-1} and $a < ah^{-1} < d$, we must have $n = m$. Then, $(hg^{-1}h^{-1})_A = 1$. On the other hand, since $h^{-1}(A) = (ah^{-1}, ch^{-1})$ is an orbital of $hg^{-1}h^{-1}$ and $h^{-1}(A) \cap A \neq \emptyset$, we see $(hg^{-1}h^{-1})_A \neq 1$, a contradiction. □

The *center* of a group G is the subset

$$Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$

We note that $Z(G)$ is a normal subgroup of G.

Proposition 1.3.4. *Let G be a subgroup of* $\mathrm{PLo}(I)$. *Then $Z(G/Z(G)) = \{1\}$.*

Proof. We may assume that G is non-commutative. Let $gZ(G) \in Z(G/Z(G))$ and h be any element of G. Since $gZ(G)hZ(G) = hZ(G)gZ(G)$, it follows that $[g, h] \in Z(G)$. Thus, $[x, [g, h]] = 1$ for all $x \in G$. By Lemma 1.3.3, we have $[g, h] = 1$. Hence, $g \in Z(G)$, which implies $Z(G/Z(G)) = \{1\}$. □

Lemma 1.3.5. *Let G, $H < \mathrm{PLo}(I)$. Suppose that H is non-commutative and $[g, h] = 1$ for each $g \in G$ and each $h \in H$. Then* $\mathrm{supp}(G) \subsetneq \mathrm{supp}(\langle G, H\rangle)$.

Proof. Since H is non-commutative, by Lemma 1.2.4, H has a non-commutative orbital N. Since $[g, h] = 1$ for each $g \in G$ and each $h \in H$, by Lemma 1.3.1, it follows that $N \cap \mathrm{supp}(G) = \emptyset$ and $N \subset \mathrm{supp}(\langle G, H\rangle)$. □

Recall that a subgroup H of a group G is called a *direct factor* of G if there exists a subgroup K of G such that $G = H \times K$. If there are no proper non-trivial direct factors of G, then G is said to be *indecomposable*.

Theorem 1.3.6. *Let $H_1, \dots, H_n$, $K_1, \dots, K_m$ be non-commutative, indecomposable subgroups of* $\mathrm{PLo}(I)$. *Suppose that $Z(H_i) = \{1\}$ for each i and $Z(K_j) = \{1\}$ for each j. Then, $\prod_{i=1}^n H_i$ and $\prod_{j=1}^m K_j$ are isomorphic if and only if $n = m$, and there exists a permutation $\sigma \in S_n$ such that H_i and $K_{\sigma(i)}$ are isomorphic for each $i \in \{1, \dots, n\}$.*

Proof. Let $H < \mathrm{PLo}(I)$. Then we know $\mathrm{supp}(H) \subset (0, 1)$. First we claim that for any $a, b \in I$ there exists a group $\widetilde{H}$ such that $\mathrm{supp}(\widetilde{H}) \subset (a, b)$ and $\widetilde{H} \cong H$. Let $f : (0, 1) \to (a, b)$ be an orientation-preserving piecewise linear homeomorphism that is differentiable everywhere except at finitely many real numbers. Then for $h \in H$ we define $\widetilde{h}$ by

$$x\widetilde{h} = \begin{cases} xf^{-1}hf & if\ x \in (a,b), \\ x & if\ \ x \notin (a,b), \end{cases}$$

and set $\widetilde{H} = \{\widetilde{h} \,|\, h \in H\}$. It is easy to check that $\mathrm{supp}(\widetilde{H}) \subset (a,b)$ and $\widetilde{H} \cong H$. Suppose that $\prod_{i=1}^n H_i$ and $\prod_{j=1}^m K_j$ are isomorphic. By the above, we can assume that the direct factors of $\prod_{i=1}^n H_i$ have disjoint supports, that is, if $i \neq j$ then $\mathrm{supp}(H_i) \cap \mathrm{supp}(H_j) = \emptyset$. Similarly, we can assume this for the direct factors of $\prod_{j=1}^m K_j$. Let

$$\phi : \prod_{i=1}^n H_i \to \prod_{j=1}^m K_j$$

be an isomorphism. Since

$$\langle \phi(H_1), \dots, \phi(H_n) \rangle = \langle K_1, \dots, K_m \rangle,$$

it follows that

$$\bigcup_{i=1}^n \mathrm{supp}(\phi(H_i)) = \bigcup_{j=1}^m \mathrm{supp}(K_j).$$

First we prove that for any $i \in \{1, \dots, n\}$, $\mathrm{supp}(\phi(H_i))$ can be written as a disjoint union of finitely many elements of the set

$$\{\mathrm{supp}(K_j) \,|\, j \in \{1, \dots, m\}\}.$$

Fix $k \in \{1, \dots, n\}$ and let J_k be the set of all $j \in \{1, \dots, m\}$ such that $\phi(H_k)$ has an orbital of K_j as an orbital. Then it suffices to prove that

$$\mathrm{supp}(\phi(H_k)) = \bigcup_{j \in J_k} \mathrm{supp}(K_j).$$

Let N be an orbital of $\phi(H_k)$. Since

$$\bigcup_{i=1}^n \mathrm{supp}(\phi(H_i)) = \bigcup_{j=1}^m \mathrm{supp}(K_j),$$

there exists $l \in \{1, \dots, m\}$ such that N is contained in some orbital M of K_l. Since by Lemma 1.3.1 (1), the endpoints of N are fixed by

$$\langle \phi(H_1), \dots, \phi(H_n) \rangle = \langle K_1, \dots, K_m \rangle,$$

it follows that $N = M$. That is, $\phi(H_k)$ has the orbital M of K_l as an orbital. Thus $l \in J_k$ and $N \subset \mathrm{supp}(K_l)$. Hence we have

$$\mathrm{supp}(\phi(H_k)) \subset \bigcup_{j \in J_k} \mathrm{supp}(K_j).$$

Now we prove that

$$\mathrm{supp}(\phi(H_k)) \supset \bigcup_{j \in J_k} \mathrm{supp}(K_j).$$

Assume by contradiction that there exist $l \in J_k$ and an orbital L of K_l such that $L \not\subseteq \mathrm{supp}(\phi(H_k))$. Then since by Lemma 1.3.1 (1), the endpoints of any orbital of $\phi(H_k)$ are fixed by

$$\langle \phi(H_1), \ldots, \phi(H_n) \rangle = \langle K_1, \ldots, K_m \rangle,$$

we see that $L \cap \mathrm{supp}(\phi(H_k)) = \emptyset$. Since $L \subset \mathrm{supp}(K_l)$, there exists $p \in K_l$ such that $\mathrm{supp}(p) \cap L \neq \emptyset$. Then there exist $s \in H_k$ and

$$t \in \langle H_1, \ldots, H_{k-1}, H_{k+1}, \ldots, H_n \rangle$$

such that $p = \phi(s)\phi(t)$, $\mathrm{supp}(\phi(t)) \cap L \neq \emptyset$. Let

$$G_1 = \{\pi_l(\phi(h)) \mid h \in H_k\}$$

and

$$G_2 = \{\pi_l(\phi(h)) \mid h \in \langle H_1, \ldots, H_{k-1}, H_{k+1}, \ldots, H_n \rangle\},$$

where $\pi_l : \prod_{j=1}^m K_j \to K_l$ is the projection. Then $G_i < K_l$ for $i \in \{1, 2\}$. In addition, the following statements hold. (i) $G_i \neq \{1\}$ for each $i \in \{1, 2\}$. (ii) $K_l = \langle G_1, G_2 \rangle$.

(i) Since $l \in J_k$, $\phi(H_k)$ has an orbital O of K_l as an orbital. Thus, there exists $u \in H_k$ such that $\mathrm{supp}(\phi(u)) \cap O \neq \emptyset$. Let $\phi(u) = k_1 \cdots k_m$, where $k_j \in K_j$. Since $\mathrm{supp}(K_l) \cap \mathrm{supp}(K_j) = \emptyset$ for each $j \neq l$, it follows that $\pi_l(\phi(u)) = k_l \neq 1$. Thus $G_1 \neq \{1\}$. On the other hand, we can write $t = t_1 \cdots t_n$, where $t_i \in H_i$, $i \neq k$. Since $\mathrm{supp}(\phi(t)) \cap L \neq \emptyset$, there exists $i \neq k$ such that $\pi_l(\phi(t_i)) \neq 1$. Thus $G_2 \neq \{1\}$.

(ii) Let $q \in K_l$. There exist $h_i \in H_i$, $1 \leq i \leq n$ such that $q = \phi(h_1) \cdots \phi(h_n)$. Then for each i there exist $q_i \in K_l$ and

$$r_i \in \langle K_1, \ldots, K_{l-1}, K_{l+1}, \ldots, K_m \rangle$$

such that $\phi(h_i) = q_i r_i$. Then

$$q = \prod_{i=1}^n q_i r_i = (\prod_{i=1}^n q_i)(\prod_{i=1}^n r_i).$$

Since $K_l \cap \langle K_1, \ldots, K_{l-1}, K_{l+1}, \ldots, K_m \rangle = \{1\}$, it follows that $q = \prod_{i=1}^n q_i$ and $\prod_{i=1}^n r_i = 1$. Then,

$$q = \prod_{i=1}^n q_i = \prod_{i=1}^n \pi_l(\phi(h_i)) \in \langle G_1, G_2 \rangle.$$

Moreover, it is easy to check that $[g_1, g_2] = 1$ for each $g_1 \in G_1$ and each $g_2 \in G_2$, and $G_1 \cap G_2 \subset Z(K_l) = \{1\}$. Hence $K_l = G_1 \times G_2$, a contradiction.

Fix $k \in \{1, \ldots, n\}$, again. Let δ be a bijection from $\{1, \ldots, n\}$ to itself such that $\delta(1) = k$. For each $j \in \{1, \ldots, n\}$, we have

$$\begin{aligned}
\operatorname{supp}\langle \phi(H_{\delta(1)}), \ldots, \phi(H_{\delta(j)}) \rangle &= \bigcup_{i=1}^{j} \operatorname{supp}(\phi(H_{\delta(i)})) \\
&= \bigcup_{i=1}^{j} \bigcup_{l \in J_{\delta(i)}} \operatorname{supp}(K_l) \\
&= \bigcup_{l \in \bigcup_{i=1}^{j} J_{\delta(i)}} \operatorname{supp}(K_l).
\end{aligned}$$

Let $a_j = |\bigcup_{i=1}^{j} J_{\delta(i)}|$. Since for any $i \in \{1, \ldots, n\}$, $\phi(H_{\delta(i)})$ is non-commutative, by Lemma 1.3.5, it follows that $a_{j+1} \geq a_j + 1$ for each $j \in \{1, \ldots, n\}$. Then, $a_j - a_1 \geq j - 1$ for each $j \in \{1, \ldots, n\}$. Since $a_1 \geq 1$, it follows that $a_j \geq j$ for each $j \in \{1, \ldots, n\}$. Hence $m \geq n$. By considering the inverse ϕ^{-1}, we obtain $n \geq m$. Thus $n = m$. Then $a_j = j$. Since $a_1 = 1$, there exists $l \in \{1, \ldots, m = n\}$ such that $\operatorname{supp}(\phi(H_k)) = \operatorname{supp}(K_l)$. Let $h \in H_k$. Then there exist $q \in K_l$ and

$$r \in \langle K_1, \ldots, K_{l-1}, K_{l+1}, \ldots, K_m \rangle$$

such that $\phi(h) = qr$. Since $\operatorname{supp}(\phi(h)) \subset \operatorname{supp}(\phi(H_k)) = \operatorname{supp}(K_l)$, it follows that $r = 1$. Thus $\phi(h) = q \in K_l$. Hence $\phi(H_k) \subset K_l$. By considering ϕ^{-1}, we see that there exists $j \in \{1, \ldots, n\}$ such that $\phi^{-1}(K_l) \subset H_j$. Since $H_k \subset \phi^{-1}(K_l) \subset H_j$, $j = k$. Therefore $\phi(H_k) = K_l$. Since ϕ is bijective, we complete the proof of Theorem 1.3.6. □

Let $G = \prod_{i=1}^{n} H_i$ be the direct product of groups $H_1, \ldots, H_n$. Then we note that $Z(G) = \prod_{i=1}^{n} Z(H_i)$. It is still not known whether there exists a non-commutative, indecomposable subgroup of $\mathrm{PLo}(I)$ whose center is non-trivial. However, by Proposition 1.3.4 and Theorem 1.3.6, we obtain the following:

Corollary 1.3.7. *Let $H_1, \ldots, H_n$, $K_1, \ldots, K_m$ be non-commutative, indecomposable subgroups of* $\mathrm{PLo}(I)$. *If $\prod_{i=1}^{n} H_i$ and $\prod_{j=1}^{m} K_j$ are isomorphic, then $n = m$, and there exists $\sigma \in S_n$ such that $H_i/Z(H_i)$ and $K_{\sigma(i)}/Z(K_{\sigma(i)})$ are isomorphic for each $i \in \{1, \ldots, n\}$.*

Recall that a *Remak decomposition* of a group G is a decomposition of G as a direct product of finitely many non-trivial indecomposable subgroups. We note that any commutative subgroup of $\mathrm{PLo}(I)$ is isomorphic to a subgroup of the direct sum $\bigoplus_{i \in \mathbb{Z}} \mathbb{Z}$ of copies of integers. The infinite cyclic group is indecomposable, and every free abelian group of countably infinite rank is decomposable. If a subgroup G of $\mathrm{PLo}(I)$ has a Remak decomposition, then G may have the form

$$G = H_1 \times \cdots \times H_n \times \mathbb{Z}^p$$

up to order of the direct factors, where H_i are non-commutative and indecomposable, and $\mathbb{Z}^p$ is the free abelian group of rank p as a direct product of the indecomposable group $\mathbb{Z}$.

The following corollary helps us understand whether certain subgroups of $\mathrm{PLo}(I)$ described in the following section are non-isomorphic.

Corollary 1.3.8. *Let $G < \mathrm{PLo}(I)$. Suppose that G has the following Remak decompositions*

$$G = (\prod_{i=1}^{n} H_i) \times \mathbb{Z}^p = (\prod_{j=1}^{m} K_j) \times \mathbb{Z}^q,$$

where H_i and K_j are non-commutative, $Z(H_i) = \{1\}$ for each i, and $Z(K_j) = \{1\}$ for each j. Then, $p = q$, $n = m$, and there exists $\sigma \in S_n$ such that H_i and $K_{\sigma(i)}$ are isomorphic for each $i \in \{1, \ldots, n\}$.

1.4 Subgroups of F which have direct decompositions

1.4.1 Some facts about F

Recall that the group F is a subgroup of $\mathrm{PLo}(I)$ and it is defined as the group of piecewise linear homeomorphisms from the closed unit interval I onto itself that are differentiable everywhere except at finitely many dyadic rational numbers (i.e., numbers from $\mathbb{Z}[1/2]$) and such that on the intervals of differentiability the derivatives are integer powers of 2. As examples of subgroups of F which have direct decompositions, other than the stabilizers of F to be described in the next subsection, the centralizers of F are known. The centralizer of every element in F is a direct product of finitely many cyclic groups and groups isomorphic to F [16, p. 97]. We note that any subgroup of F which has a direct product decomposition of two non-trivial subgroups is an infinite index subgroup of F.

It is known [9] that the commutator subgroup $[F, F]$ is simple, and that $[F, F]$ is the subgroup of all functions with slope 1 both at 0^+ and 1^-.

1.4.2 Isomorphism between stabilizers of finite sets

We say that $f \in F$ *has closure of support in an interval* J if the closure in I of $\mathrm{supp}(f)$ is contained in J. Let F_J be the set of all functions from F with closure of support in J. Then F_J is a subgroup of F. We note that $F_{(0,1)}$ is exactly the group $[F, F]$. It is known [15, Lemma 3.1] that for any $a, b \in I$ with $a < b$, $F_{(a,b)}$ is isomorphic to $F_{(0,1)}$.

For any finite subset X of $(0, 1)$, let H_X be the *stabilizer* of X in F. That is,

$$H_X = \{f \in F \mid xf = x \text{ for each } x \in X\}.$$

Any finite subset Y of I is subdivided into three subsets:

$$Y_1 = Y \cap \mathbb{Z}[1/2],\ Y_2 = Y \cap (\mathbb{Q} \setminus \mathbb{Z}[1/2]),\ \text{and } Y_3 = Y \cap (\mathbb{R} \setminus \mathbb{Q}).$$

Let $Y = \{r_1, \ldots, r_n\} \subset I$, where $r_j < r_{j+1}$ and $r_1, r_n \notin Y_2$. Let

$$B_Y = F_{[r_1,r_n]} \cap H_{Y\setminus\{r_1,r_n\}}.$$

The following proposition generalizes [15, Theorem 3.2] slightly in that r_1 and r_n are any numbers in $Y_1 \cup Y_3$. Since it can be proved by an argument similar to the proof of that theorem, the details are left to the reader.

Proposition 1.4.1. *The group B_Y is isomorphic to a semidirect product*

$$B_Y \cong [F, F]^{n-1} \rtimes \mathbb{Z}^{2|(Y\setminus\{r_1,r_n\})_1|+|Y_2|+|\{r_1,r_n\}_1|}.$$

Corollary 1.4.2. *Let $U = \{\alpha_1, \ldots, \alpha_n\} \subset I$, where $\alpha_j < \alpha_{j+1}$ and $\alpha_1, \alpha_n \notin U_2$, and $V = \{\beta_1, \ldots, \beta_m\} \subset I$, where $\beta_j < \beta_{j+1}$ and $\beta_1, \beta_m \notin V_2$. If the subgroups B_U and B_V are isomorphic, then $n = m$ and*

$$2|(U \setminus \{\alpha_1, \alpha_n\})_1| + |U_2| + |\{\alpha_1, \alpha_n\}_1| = 2|(V \setminus \{\beta_1, \beta_m\})_1| + |V_2| + |\{\beta_1, \beta_m\}_1|.$$

Proof. The commutator subgroup of B_U is isomorphic to the direct product of $n-1$ copies of the simple group $[F, F]$. Thus, it has 2^{n-1} normal subgroups. Since it cannot be isomorphic to a direct power of a different number of simple groups, we have $n = m$. Since

$$B_U/[F, F]^{n-1} \cong B_V/[F, F]^{m-1},$$

we see that

$$\mathbb{Z}^{2|(U\setminus\{\alpha_1,\alpha_n\})_1|+|U_2|+|\{\alpha_1,\alpha_n\}_1|} \cong \mathbb{Z}^{2|(V\setminus\{\beta_1,\beta_m\})_1|+|V_2|+|\{\beta_1,\beta_m\}_1|},$$

which implies

$$2|(U \setminus \{\alpha_1, \alpha_n\})_1| + |U_2| + |\{\alpha_1, \alpha_n\}_1| = 2|(V \setminus \{\beta_1, \beta_m\})_1| + |V_2| + |\{\beta_1, \beta_m\}_1|.$$

□

Let $U = \{\alpha_1, \ldots, \alpha_n\} \subset I$ with $\alpha_j < \alpha_{j+1}$. The type $\tau(U)$ is the word of length n in the alphabet $\{1, 2, 3\}$ as follows: for every i, the ith letter in $\tau(U)$ is 1 if α_i is a dyadic rational, 2 if α_i is rational but not a dyadic rational, and 3 if α_i is irrational. We will use the following lemma.

Lemma 1.4.3 ([15]). *Let $U = \{\alpha_1, \ldots, \alpha_n\} \subset I$, where $\alpha_j < \alpha_{j+1}$ and $\alpha_1, \alpha_n \notin U_2$, and $V = \{\beta_1, \ldots, \beta_m\} \subset I$, where $\beta_j < \beta_{j+1}$ and $\beta_1, \beta_m \notin V_2$. Suppose that $\tau(U) \equiv \tau(V)$ or the word $\tau(U)$ is equal to $\tau(V)$ read backwards. Then the groups B_U and B_V are isomorphic.*

By Lemma 1.4.3 we will use the notation B_w for the group B_U if $w \equiv \tau(U)$.

Let U be a finite subset of $(0,1)$. Write $U_1 \cup U_3 = \{r_1, \ldots, r_n\}$, where $r_j < r_{j+1}$, $|U_1 \cup U_3| = n$. Let $r_0 = 0$, $r_{n+1} = 1$, and

$$U_{2,k} = \{q \in U_2 \,|\, r_k < q < r_{k+1}\} \text{ for } \textit{each } k \in \{0, \ldots, n\}.$$

Then

$$U_2 = \bigsqcup_{k=0}^{n} U_{2,k}.$$

Recall (see [15, subsection 4.2] for details), that

$$H_U = B_{\{r_0,r_1\}\cup U_{2,0}} \times \cdots \times B_{\{r_n, r_{n+1}\}\cup U_{2,n}}.$$

For any word $w_1w_2 \in \{11, 13, 33\}$ and $j \in \{0, \ldots, |U_2|\}$, let $\Lambda_{U,w_1w_2,j} =$

$$\{i \in \{0, \ldots, n\} \,|\, \tau(\{r_i, r_{i+1}\} \cup U_{2,i}) \equiv w_1 2^j w_2 \text{ or } \tau(\{r_i, r_{i+1}\} \cup U_{2,i}) \equiv w_2 2^j w_1\}.$$

Then we have $|\Lambda_{U,w_1w_2,j}| \geq 0$, and

$$\bigsqcup_{j=0}^{|U_2|} \bigsqcup_{w_1w_2 \in \{11,13,33\}} \Lambda_{U,w_1w_2,j} = \{0, \ldots, n\}. \tag{E}$$

By Lemma 1.4.3 and the above notation, we can see that

$$H_U \cong \prod_{0 \leq j \leq |U_2|, w_1w_2 \in \{11,13,33\}} \underbrace{(B_{w_1 2^j w_2} \times \cdots \times B_{w_1 2^j w_2})}_{|\Lambda_{U,w_1w_2,j}|}.$$

Remark 1.4.4. *Let U and V be finite sets of numbers in I such that $\tau(U) \equiv u_1 2^j u_2$ and $\tau(V) \equiv v_1 2^k v_2$, where $u_1u_2, v_1v_2 \in \{11, 13, 33\}$. If the groups $B_{u_1 2^j u_2}$ and $B_{v_1 2^k v_2}$ are isomorphic, then by Corollary* 1.4.2, *$j = k$ and $u_1u_2 \equiv v_1v_2$.*

In order to prove that the factors of H_U are indecomposable, we need a new definition and a lemma. We give them for subgroups of $\mathrm{PLo}(I)$ and then apply them to the factors of H_U.

Definition 1.4.5. Let G be a subgroup of $\mathrm{PLo}(I)$, and M_1 and M_2 be orbitals of G. Let $e_1 \in \partial M_1$ and $e_2 \in \partial M_2$. We say that $\{e_1, e_2\}$ is *a set of linked endpoints of M_1 and M_2* if for any $g \in G$, the following statement holds: g has some orbital contained in M_1 sharing the end e_1 if and only if g also has some orbital contained in M_2 sharing the end e_2. In particular, we say that e is a *linked endpoint* of M_1 and M_2 if $e_1 = e_2 = e$.

Lemma 1.4.6. *Let G and H be subgroups of* $\mathrm{PLo}(I)$. *Suppose that $[g, h] = 1$ for each $g \in G$ and each $h \in H$. Let M_1 and M_2 be disjoint orbitals of $\langle G, H\rangle$, M_1 and M_2 have a set $\{e_1, e_2\}$ of linked endpoints for $\langle G, H\rangle$, and* $\mathrm{supp}(G) \supset M_1$. *Suppose also that there exists $r \in \langle G, H\rangle$ such that r has an orbital which shares the end e_1 and is properly contained in M_1. Then* $\mathrm{supp}(G) \supset M_1 \sqcup M_2$.

Proof. Since $M_1 \subset \mathrm{supp}(G)$, there exists an orbital N of G such that $M_1 \subset N$. Since $\partial M_1 \subset \mathrm{Fix}(\langle G, H\rangle)$, it follows that $M_1 = N$. Thus M_1 is an orbital of G. Then $M_1 \cap \mathrm{supp}(H) = \emptyset$. Indeed, assume by contradiction that $M_1 \cap \mathrm{supp}(H) \neq \emptyset$. Then by Lemma 1.3.1, M_1 is an orbital of H. Thus M_1 is an orbital of both G and H. Since $r \in \langle G, H\rangle$, there exist $s \in G$ and $t \in H$ such that $r = st$. By Lemma 1.3.2, M_1 is a common commutative orbital of both G and H, a contradiction. Hence $M_1 \cap \mathrm{supp}(H) = \emptyset$. Since $r = st$, where $s \in G$, $t \in H$, and $\mathrm{supp}(t) \cap M_1 = \emptyset$, s has an orbital which shares the end e_1 and is properly contained in M_1. Then since $\{e_1, e_2\}$ is a set of linked endpoints, s has an orbital contained in M_2 sharing the end e_2. Hence $\mathrm{supp}(G) \cap M_2 \neq \emptyset$. Thus, M_2 is an orbital of G, which completes the proof. □

Proposition 1.4.7. *Let $Y = \{r_1, \ldots, r_n\} \subset I$, where $r_1, r_n \notin \mathbb{Q} \setminus \mathbb{Z}[1/2]$, $r_j < r_{j+1}$ and $r_2, \ldots, r_{n-1} \in \mathbb{Q} \setminus \mathbb{Z}[1/2]$. Then B_Y is indecomposable, and $Z(B_Y) = \{1\}$.*

Proof. Assume by contradiction that there exist non-trivial subgroups K and L of B_Y such that $B_Y = K \times L$. Since $K \neq \{1\}$, there exist $k \in K$ and $i \in \{1, \ldots, n-1\}$ such that $\mathrm{supp}(k) \cap (r_i, r_{i+1}) \neq \emptyset$. Since

$$\mathrm{supp}(K) \cup \mathrm{supp}(L) = \mathrm{supp}(B_Y) = \bigcup_{j=1}^{n-1} (r_j, r_{j+1})$$

and $[f, g] = 1$ for each $f \in K$ and each $g \in L$, by Lemma 1.3.1, K must have the interval (r_i, r_{i+1}) as an orbital. We notice that since $r_2, \ldots, r_{n-1} \in \mathbb{Q} \setminus \mathbb{Z}[1/2]$, all elements of B_Y are differentiable at those numbers. Thus for each $j \in \{1, \ldots, n-1\}$, r_j is a linked endpoint of (r_{j-1}, r_j) and (r_j, r_{j+1}). By applying Lemma 1.4.6 repeatedly for K and L, we can see that

$$\mathrm{supp}(K) = \bigcup_{j=1}^{n-1} (r_j, r_{j+1})$$

(we can clearly take an element r as in Lemma 1.4.6. Use both [9, Lemma 4.2] and the argument in the proof of [22, Proposition 4.1(b)], for example). Similarly, we see that

$$\mathrm{supp}(L) = \bigcup_{j=1}^{n-1} (r_j, r_{j+1}).$$

By Lemma 1.3.2, for any j the interval (r_j, r_{j+1}) is a common commutative orbital of both K and L. Hence (r_j, r_{j+1}) is a commutative orbital of $\langle K, L\rangle = B_Y$ for each $j \in \{1, \ldots, n-1\}$. By Lemma 1.2.4, B_Y is commutative, a contradiction.

Let $g \in Z(B_Y)(< B_Y)$. Since for each $j \in \{1, \ldots, n-1\}$, (r_j, r_{j+1}) is a non-commutative orbital of B_Y, by Lemma 1.3.1 (2),

$$g_{(r_j, r_{j+1})} = 1 \text{ for each } j \in \{1, \ldots, n-1\}.$$

Then $g = 1$. Thus we see that $Z(B_Y) = \{1\}$. □

Theorem 1.4.8. *Let U and V be finite sets of numbers in $(0,1)$. Then the following statements are equivalent.*

(1) *H_U and H_V are isomorphic.*

(2) *$|U_2| = |V_2|$, and $|\Lambda_{U,w_1w_2,j}| = |\Lambda_{V,w_1w_2,j}|$ for each $w_1w_2 \in \{11, 13, 33\}$ and each $j \in \{0, \ldots, |U_2|\}$.*

Proof. Suppose that the statement (2) holds. Then

$$H_U \cong \prod_{0 \le j \le |U_2| = |V_2|, w_1w_2 \in \{11,13,33\}} \underbrace{(B_{w_1 2^j w_2} \times \cdots \times B_{w_1 2^j w_2})}_{|\Lambda_{U,w_1w_2,j}| = |\Lambda_{V,w_1w_2,j}|} \cong H_V.$$

Suppose that the statement (1) holds. Let

$$U_1 \cup U_3 = \{\alpha_1, \ldots, \alpha_n\},\ \alpha_j < \alpha_{j+1},\ V_1 \cup V_3 = \{\beta_1, \ldots, \beta_m\},\ \beta_j < \beta_{j+1},$$

$\alpha_0 = \beta_0 = 0$, and $\alpha_{n+1} = \beta_{m+1} = 1$. Set

$$C_i = B_{\{\alpha_i, \alpha_{i+1}\} \cup U_{2,i}} \text{ and } D_j = B_{\{\beta_j, \beta_{j+1}\} \cup V_{2,j}},$$

where

$$U_{2,i} = \{q \in U_2 \mid \alpha_i < q < \alpha_{i+1}\} \text{ and } V_{2,j} = \{q \in V_2 \mid \beta_j < q < \beta_{j+1}\}.$$

Then we see that

$$H_U = \prod_{i=0}^{n} C_i \text{ and } H_V = \prod_{j=0}^{m} D_j.$$

Let $\phi : H_U \to H_V$ be an isomorphism. By Proposition 1.4.7, all direct factors of H_U and H_V are indecomposable, and the center of each factor is trivial. By Theorem 1.3.6,

$$n = |U_1 \cup U_3| = |V_1 \cup V_3| = m,$$

and there exists $\sigma \in S_{n+1}$ such that $\phi(C_i) = D_{\sigma(i)}$ for each $i \in \{0, \ldots, n\}$. Then by Remark 1.4.4, $|U_2| = |V_2|$, and

$$\tau(\{\alpha_i, \alpha_{i+1}\} \cup U_{2,i}) \equiv \tau(\{\beta_{\sigma(i)}, \beta_{\sigma(i)+1}\} \cup V_{2,\sigma(i)})$$

or the word $\tau(\{\alpha_i, \alpha_{i+1}\} \cup U_{2,i})$ is equal to $\tau(\{\beta_{\sigma(i)}, \beta_{\sigma(i)+1}\} \cup V_{2,\sigma(i)})$ read backwards. Thus, for any word $w_1w_2 \in \{11, 13, 33\}$ and any $j \in \{0, \ldots, |U_2| = |V_2|\}$,

$$i \in \Lambda_{U,w_1w_2,j} \text{ if and only if } \sigma(i) \in \Lambda_{V,w_1w_2,j}.$$

Therefore the permutation σ satisfies $\sigma(\Lambda_{U,w_1w_2,j}) = \Lambda_{V,w_1w_2,j}$ for each $w_1w_2 \in \{11, 13, 33\}$ and each $j \in \{0, \ldots, |U_2|\}$, which completes the proof. □

It is known [6] that the centralizer in PLo(I) of every element of PLo(I) is a direct product of finitely many cyclic groups and groups isomorphic to PLo(I). Let $\mathcal{C}$ be the family of all centralizers in F of elements of F, all centralizers in PLo(I) of elements of PLo(I), and all stabilizers of finite sets of numbers in F. Note that PLo(I) is uncountable, indecomposable and centerless. By proposition 1.4.7, Remark 1.4.4, Theorem 1.4.8, and Corollary 1.3.8, we can easily see whether any two subgroups in $\mathcal{C}$ are non-isomorphic.

1.5 Conjugacy of stabilizers of finite sets

Let $\mathrm{Aut}(I)$ be the group of all bijections from $I = [0, 1]$ onto itself.

Definition 1.5.1. Let $\mathcal{S}$ be the subset of $\mathrm{Aut}(I)$ defined as follows. A bijection f lies in $\mathcal{S}$ if there exists a finite set $\{r_1, \ldots, r_n\}$ of real numbers such that for any neighborhood U_i of r_i for each i,

(1) f is a piecewise linear homeomorphism on $I \setminus (\bigcup_{i=1}^{n} U_i)$,

(2) f is differentiable on $I \setminus (\bigcup_{i=1}^{n} U_i)$ everywhere except at finitely many dyadic rational numbers (and boundaries of U_i for each i), and

(3) on the intervals of differentiability, absolute values of the derivatives are integer powers of 2.

Lemma 1.5.2. *The set $\mathcal{S}$ is a group under the operation of composition.*

Proof. Let f_1 and f_2 be elements of $\mathcal{S}$. Let X and Y be finite sets of real numbers in I for f_1 and f_2, respectively, and $X \cup f_1^{-1}(Y) = \{z_1, \ldots, z_n\}$. Let N_i be a neighborhood of z_i for each i. Set

$$L = \bigcup_{j \in \{k | z_k \in X\}} N_j.$$

Then f_1 is a piecewise linear homeomorphism from $I \setminus L$ onto $I \setminus f_1(L)$ that has properties (2) and (3). Set

$$M = \bigcup_{j=1}^{n} f_1(N_j).$$

Then f_2 is a piecewise linear homeomorphism from $I \setminus M$ onto $I \setminus f_2(M)$ that has properties (2) and (3). That properties (1) and (2) are preserved under composition and inversion is standard, and property (3) is preserved under composition and inversion from the chain rule. Hence the bijection $f_1 f_2$ is a piecewise linear homeomorphism from $I \setminus \bigcup_{j=1}^{n} N_j$ onto $I \setminus f_2(M)$ that has properties (2) and (3). Hence $f_1 f_2 \in \mathcal{S}$. We can also check that $f_1^{-1} \in \mathcal{S}$. The identity map from I onto itself is also the identity element in $\mathcal{S}$. □

Let $\mathrm{Homeo}(I)$ be the group of all homeomorphisms from the closed unit interval I onto itself, under the operation of composition. Let $\mathcal{F}$ be the subset of $\mathrm{Homeo}(I)$ defined as follows: an element f of $\mathrm{Homeo}(I)$ lies in $\mathcal{F}$ if there exists a finite set $\{r_1, \ldots, r_n\}$ of real numbers that satisfies the condition that $r_i \in \mathbb{Z}[1/2]$ if and only if $f(r_i) \in \mathbb{Z}[1/2]$ and such that for any neighborhood U_i of r_i for each i, there exists $g \in F$ such that

$$xf = xg \text{ for all } x \in I \setminus (\bigcup_{i=1}^{n} U_i).$$

Then we can check that $\mathcal{F}$ is a group.

Remark 1.5.3. *The group $\mathcal{F}$ is the image in* Homeo(I) *of some group, which was defined in* [*15*] *and consists of possibly infinite tree-diagrams with finitely many infinite branches, under an embedding.*

Let N be the set of all elements in $\mathcal{S}$ that are equal to the identity mapping everywhere except at finitely many real numbers. Then it is easy to check that N is a normal subgroup of $\mathcal{S}$.

Definition 1.5.4. Define a group $\mathcal{G}$ to be the quotient group $\mathcal{S}/N$.

Let p be the quotient map from $\mathcal{S}$ to $\mathcal{G}$. Since $F \subset \mathcal{F} \subset \mathcal{S}$ and all elements in $\mathcal{F}$ are continuous, p induces a natural embedding of $\mathcal{F}$ into $\mathcal{G}$.

To prove the later theorem about conjugacy, we need to choose functions which map certain intervals or numbers in I. In doing so, we apply the following Lemmas 1.5.5 and 1.5.6. Lemma 1.5.5 follows directly from the proof of [9, Lemma 4.2].

Lemma 1.5.5. *Let $\alpha_1, \alpha_2, \beta_1$, and β_2 be dyadic rational numbers in I, where $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$. Then there exists a piecewise linear homeomorphism*

$$f : [\alpha_1, \alpha_2] \to [\beta_1, \beta_2]$$

with $\alpha_i f = \beta_i$ that is differentiable everywhere except at finitely many dyadic rational numbers and such that on the intervals of differentiability the derivatives are integer powers of 2*. In addition, there exists a piecewise linear homeomorphism*

$$g : [\alpha_1, \alpha_2] \to [\beta_1, \beta_2]$$

with $\alpha_1 g = \beta_2$, $\alpha_2 g = \beta_1$ that is differentiable everywhere except at finitely many dyadic rational numbers and such that on the intervals of differentiability, signs of values of the derivatives are all negative and absolute values of the derivatives are integer powers of 2.

The next lemma follows immediately from the proof of [15, Lemma 3.1].

Lemma 1.5.6. *Let $\alpha_1, \alpha_2, \beta_1$, and β_2 be real numbers in I, where $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$. Let $(\alpha_1, \alpha_2) \neq (0, 1)$, and $(\beta_1, \beta_2) \neq (0, 1)$. Then there exists a homeomorphism*

$$f : [\alpha_1, \alpha_2] \to [\beta_1, \beta_2]$$

with $\alpha_i f = \beta_i$ such that for any neighborhoods U_1, U_2 of α_1 and α_2, respectively, f has properties (1) *and* (2) *in definition* 1.5.1*, and on the intervals of differentiability the derivatives are integer powers of* 2*. In addition, there exists a homeomorphism*

$$f : [\alpha_1, \alpha_2] \to [\beta_1, \beta_2]$$

with $\alpha_1 f = \beta_2$, $\alpha_2 f = \beta_1$ such that for any neighborhoods U_1, U_2 of α_1 and α_2, respectively, f has properties (1)–(3) *in definition* 1.5.1*, and on the intervals of differentiability signs of values of the derivatives are all negative.*

Let α_1 and α_2 be real numbers in I, where $\alpha_1 < \alpha_2$. We define the type $\widetilde{\tau}\{\alpha_1, \alpha_2\}$ as the word of length 2 in the alphabet $\{1, \overline{1}\}$ as follows: for every i, the ith letter in $\widetilde{\tau}\{\alpha_1, \alpha_2\}$ is 1 if α_i is a dyadic rational, $\overline{1}$ if α_i is not a dyadic rational.

The next lemma also follows essentially from the proof of [15, Lemma 3.1].

Lemma 1.5.7. *Let $\alpha_1, \alpha_2, \beta_1$, and β_2 be real numbers in I, where $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$. Let $(\alpha_1, \alpha_2) \neq (0, 1)$, and $(\beta_1, \beta_2) \neq (0, 1)$. Suppose that $\widetilde{\tau}\{\alpha_1, \alpha_2\} \equiv \widetilde{\tau}\{\beta_1, \beta_2\}$ or the word $\widetilde{\tau}\{\alpha_1, \alpha_2\}$ is equal to $\widetilde{\tau}\{\beta_1, \beta_2\}$ read backwards. Then there exists an element f of $\mathcal{S}$ such that f maps (α_1, α_2) onto (β_1, β_2) homeomorphically, and the groups $p(F_{[\alpha_1,\alpha_2]})$ and $p(F_{[\beta_1,\beta_2]})$ are conjugate by $p(f)$ in $\mathcal{G}$.*

The following shows that the isomorphism between H_U and H_V (provided that the statement (2) in Theorem 1.4.8 holds) is induced by conjugacy in $\mathcal{G}$. Note that we complete the proof without using Lemma 1.4.3.

Theorem 1.5.8. *Let U and V be two finite sets of numbers in $(0, 1)$. Suppose that the statement (2) in Theorem 1.4.8 holds. Then $p(H_U)$ and $p(H_V)$ are conjugate in $\mathcal{G}$.*

Proof. Suppose that the statement (2) in Theorem 1.4.8 holds, where U and V are two finite sets of numbers in $(0, 1)$. Then by the two equations of type (E) (in Subsection 1.4.2) for U and V, respectively, we obtain $|U_1 \cup U_3| = |V_1 \cup V_3|$, and there exists $\sigma \in S_{n+1}$ such that

$$i \in \Lambda_{U,w_1w_2,j} \text{ if and only if } \sigma(i) \in \Lambda_{V,w_1w_2,j}$$

for each $w_1w_2 \in \{11, 13, 33\}$ and for each $j \in \{0, \ldots, |U_2| = |V_2|\}$, where $n = |U_1 \cup U_3| = |V_1 \cup V_3|$. Let

$$U_1 \cup U_3 = \{\alpha_1, \ldots, \alpha_n\},\ \alpha_j < \alpha_{j+1}, V_1 \cup V_3 = \{\beta_1, \ldots, \beta_n\}, \beta_j < \beta_{j+1},$$

$\alpha_0 = \beta_0 = 0$, and $\alpha_{n+1} = \beta_{n+1} = 1$. Set

$$C_i = B_{\{\alpha_i,\alpha_{i+1}\}\cup U_{2,i}} \text{ and } D_j = B_{\{\beta_j,\beta_{j+1}\}\cup V_{2,j}},$$

where

$$U_{2,i} = \{q \in U_2 \mid \alpha_i < q < \alpha_{i+1}\} \text{ and } V_{2,j} = \{q \in V_2 \mid \beta_j < q < \beta_{j+1}\}.$$

Then we have $H_U = \prod_{i=0}^{n} C_i$ and $H_V = \prod_{j=0}^{n} D_j$. Note that for each i,

$$\tau(\{\alpha_i, \alpha_{i+1}\} \cup U_{2,i}) \equiv \tau(\{\beta_{\sigma(i)}, \beta_{\sigma(i)+1}\} \cup V_{2,\sigma(i)})$$

or the word $\tau(\{\alpha_i, \alpha_{i+1}\} \cup U_{2,i})$ is equal to $\tau(\{\beta_{\sigma(i)}, \beta_{\sigma(i)+1}\} \cup V_{2,\sigma(i)})$ read backwards. Set $I_i = (\alpha_i, \alpha_{i+1})$ and $J_i = (\beta_i, \beta_{i+1})$ for each $i \in \{0, \ldots, n\}$. By Lemma 1.5.7, for each i there exists f_i of $\mathcal{S}$ such that f_i maps I_i onto $J_{\sigma(i)}$ homeomorphically. Then define $f : I \to I$ as follows. The function f coincides with f_i on I_i for each i, and $\alpha_i f = \beta_i$ for each i. Then f lies in $\mathcal{S}$. By Lemma 1.5.7,

$$\begin{aligned} p(f)^{-1}p(H_U)p(f) = p(f)^{-1}p(\prod_{i=0}^{n} C_i)p(f) &= \prod_{i=0}^{n} p(f_i)^{-1}p(C_i)p(f_i) \\ &= \prod_{i=0}^{n} p(B_{\{\beta_{\sigma(i)},\beta_{\sigma(i)+1}\}\cup f_i(U_{2,i})}) \\ &= p(H_{\{\beta_i\}_{1\leq i\leq n}\cup f(U_2)}) \end{aligned}$$

in $\mathcal{G}$. Then we see that $\tau(V) \equiv \tau(\{\beta_i\}_{1\leq i\leq n} \cup f(U_2))$. Since $\mathcal{F}$ is embedded into $\mathcal{G}$, by [15, Theorem 7.7], $p(H_V)$ and $p(H_{\{\beta_i\}_{1\leq i\leq n}\cup f(U_2)})$ are conjugate in $\mathcal{G}$. Hence, $p(H_U)$ and $p(H_V)$ are conjugate in $\mathcal{G}$. □

Chapter 2

Geometric description of Schreier graphs of Baumslag-Solitar groups

2.1 Introduction

Let m and n be non-zero integers. The group which has the presentation

$$\langle A, B \mid AB^m = B^n A \rangle$$

is called the *Baumslag-Solitar group* and denoted by $BS(m,n)$. In 1962, Baumslag and Solitar [1] introduced these groups and showed that $BS(3,2)$ is a non-Hopfian group with one defining relation. It is the first example having such property. Since then these groups have served as a proving ground for many new ideas in combinatorial and geometric group theory (see [10, 11] for examples).

Schreier coset graphs are generalizations of the Cayley graph of a group G, which are constructed for each choice of a subgroup of G and a generating set of G. The detail is given in Section 2.2. In general, given a group G and its subgroup H, it is difficult to construct the Cayley graph of G or the Schreier coset graph of all left cosets of H in G. However once we have the appropriate Cayley or Schreier graphs, we can use them as discrete models and may learn, from combinatorial and geometric viewpoints, some properties of the original group or its subgroups. Recently, in [22, 21], Savchuk constructed Schreier graphs of Thompson's group F from a motivation to study the amenability of the group.

In Chapter 2 we focus on the solvable group $BS(1,n)$ for $n \geq 2$. It is known that $BS(1,n)$ is isomorphic to some subgroup G_n with the generator S_n of the affine group $\mathrm{Aff}(\mathbb{R})$ of the real line $\mathbb{R}$, thus it has the natural action on $\mathbb{R}$ (see Section 2.2 for details). For any $x \in \mathbb{R}$, we explicitly construct the Schreier coset graph $(BS(1,n)/\mathrm{Stab}_{BS(1,n)}(x), \{A,B\}^{\pm})$ for the stabilizer $\mathrm{Stab}_{BS(1,n)}(x)$ of x under the action. First, we show that for any $x \in \mathbb{R}$, the Schreier graphs $(\mathrm{Orb}_{G_n}(x), S_n, x)$ and $(BS(1,n)/\mathrm{Stab}_{BS(1,n)}(x), \{A,B\}^{\pm}, \mathrm{Stab}_{BS(1,n)}(x))$ are isomorphic as marked labelled directed graphs, where $\mathrm{Orb}_{G_n}(x)$ is the orbit of x under the natural action on $\mathbb{R}$ (see Proposition 2.2.3 below). Hence, in most of this paper we consider the Schreier graph $(\mathrm{Orb}_{G_n}(x), S_n)$. Let $\mathbb{Z}_n^\omega$ be the set of all infinite words over the finite group $\mathbb{Z}_n$. The following theorem allows us to understand the structure of the Schreier graphs.

Theorem 2.1.1. *Let $n \geq 2$ and x be a real number represented by $w \in \mathbb{Z}_n^\omega$. Then, there exists a homomorphism $h = (f, \psi, \gamma) : (\mathrm{Orb}_{G_n}(x), S_n) \to \Gamma_w$ such that for every $v \in V_w$, the subgraph $h^{-1}(v) = (D_v, D_v \times \{b\}^\pm, S_n, \alpha|, \beta|, l|)$ is isomorphic to $\Gamma_{\mathbb{Z}}$, where $h^{-1}(v) = (f^{-1}(v), \psi^{-1}(v), S_n, \alpha|, \beta|, l|)$.*

See Definition 2.3.10 below for Γ_w and $\Gamma_{\mathbb{Z}}$. As its consequence, we classify the Schreier graphs up to isomorphism.

Theorem 2.1.2. *Let $m, n \geq 2$ with $m \neq n$.*

(1) *For any $x, y \in \mathbb{R}$, the Schreier graph $(\mathrm{Orb}_{G_m}(x), S_m)$ is not isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(y), S_n)$ as labelled directed graphs.*

(2) *For any $\alpha_1, \alpha_2 \in \mathbb{R}\backslash\mathbb{Q}$, the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha_1), S_n, \alpha_1)$ is S_n-isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha_2), S_n, \alpha_2)$ as marked labelled directed graphs.*

(3) *For any $q \in \mathbb{Q}$ and any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the Schreier graph $(\mathrm{Orb}_{G_n}(q), S_n)$ is not isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha), S_n)$ as labelled directed graphs.*

(4) *Let q_1, $q_2 \in \mathbb{Q}$. Then, the following statements are equivalent.*

   (*a*) *The Schreier graph $(\mathrm{Orb}_{G_n}(q_1), S_n)$ is isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(q_2), S_n)$ as labelled directed graphs.*

   (*b*) $\mathrm{Orb}_{G_n}(q_1) = \mathrm{Orb}_{G_n}(q_2)$ *or* $\mathrm{Orb}_{G_n}(-q_1) = \mathrm{Orb}_{G_n}(q_2)$.

This result leads to a relevance to presentations for the stabilizers which turn out to be infinite index subgroups in $BS(1, n)$(Theorem 2.5.5). Thus we expect that this idea may give a way to investigate infinite index subgroups in a suitable group.

In Section 2.2, we set up notation and terminology concerning Schreier graphs and Baumslag-Solitar groups. In Section 2.3, we start to construct Schreier graphs and give a complete description of Schreier graphs of $BS(1, n)$ with respect to any real numbers. In Section 2.4, we classify them up to isomorphism. In Section 2.5, by using the Schreier graphs we determine the group structure of the stabilizers and obtain a relevance to presentations for the stabilizers of rational numbers.

2.2 Schreier graphs and Baumslag-Solitar groups

A *labelled directed graph* denoted by $(V, E, L, \alpha, \beta, l)$ consists of a nonempty set V of vertices, a set E of edges, a set L of labels and three mappings $\alpha : E \to V$, $\beta : E \to V$, and $l : E \to L$. The vertices $\alpha(e)$ and $\beta(e)$ are called the *initial* and the *terminal vertices* of the edge e, respectively.

A *marked labelled directed graph* denoted by $(V, E, L, \alpha, \beta, l, v_0)$ is a labelled directed graph with a distinguished vertex v_0 called the *marked vertex*.

For $i \in \{1, 2\}$ let $\Gamma_i = (V_i, E_i, L_i, \alpha_i, \beta_i, l_i)$ be a labelled directed graph. Let $f : V_1 \to V_2$, $\psi : E_1 \to E_2 \sqcup V_2$ and $\gamma : L_1 \to L_2$ be maps satisfying the following statements:

(1) If $\psi(e) \in E_2$, then $\alpha_2(\psi(e)) = f(\alpha_1(e))$, $\beta_2(\psi(e)) = f(\beta_1(e))$, and $l_2(\psi(e)) = \gamma(l_1(e)) \in L_2$.

(2) If $\psi(e) \in V_2$, then $\psi(e) = f(\alpha_1(e)) = f(\beta_1(e))$.

The triple (f, ψ, γ) of maps is called the *homomorphism from* Γ_1 *to* Γ_2. Labelled directed graphs Γ_1 and Γ_2 are *isomorphic* if there exists a homomorphism $(f, \psi, \gamma) : \Gamma_1 \to \Gamma_2$, called *an isomorphism*, such that both f and γ are bijections and ψ is a injection with $\psi(E_1) = E_2$. In particular, if $L_1 = L_2 = L$ and $\gamma = 1_L$, Γ_1 is said to be *L-isomorphic* to Γ_2.

For $i \in \{1, 2\}$ let Γ_i be a marked labelled directed graph. Γ_1 is said to be *isomorphic* to Γ_2 if Γ_1 is isomorphic to Γ_2 as labelled directed graphs and the mapping between vertices preserves the marked vertices.

Let S be a generating set of a group G. The generating set S is *symmetric* if $S = S^{-1}$.

Let G be a group with a symmetric finite generating set S, M be a set and $\varphi : G \to \mathrm{Aut}(M)$ be a homomorphism, where $\mathrm{Aut}(M)$ is the set of all bijections of M onto itself. The *orbit* of an element m of M is the set

$$\mathrm{Orb}_G(m) = \{\varphi(g)(m) \,|\, g \in G\}.$$

The *stabilizer* of an element m of M is the subgroup

$$\mathrm{Stab}_G(m) = \{g \in G \,|\, \varphi(g)(m) = m\}.$$

Definition 2.2.1. Let G be a group with a symmetric finite generating set S, M be a set and $\varphi : G \to \mathrm{Aut}(M)$ be a homomorphism. The *Schreier graph* denoted by (M, S, φ) is a labelled directed graph $(M, M \times S,\ S,\ \alpha,\ \beta,\ l)$ such that $\alpha(m, s) = m$, $l(m, s) = s$, and $\beta(m, s) = \varphi(s)(m)$. The *Schreier graph with a marked vertex* denoted by (M, S, φ, m_0) is a Schreier graph with a marked vertex $m_0 \in M$.

Let G be a group with a symmetric finite generating set S, H be a subgroup of G and G/H be the set of all left cosets of H in G. The *Schreier coset graph* denoted by $(G/H, S)$ is a Schreier graph $(G/H, S, \varphi_H)$ where $\varphi_H : G \to \mathrm{Aut}(G/H)$ is the usual left action on G/H.

The composition of maps is from right to left.

Remark 2.2.2. *For $i \in \{1, 2\}$ let G_i be a group with a symmetric finite generateing set S_i. The Schreier graph (M_1, S_1, φ_1) is isomorphic to (M_2, S_2, φ_2) as labelled directed graphs if and only if there exist bijections $f : M_1 \to M_2$ and $\gamma : S_1 \to S_2$*

such that $\varphi_1(s) = f^{-1}\varphi_2(\gamma(s))f$ for all $s \in S_1$. In particular, if $S_1 = S_2 = S$, (M_1, S, φ_1) is S-isomorphic to (M_2, S, φ_2) as labelled directed graphs if and only if there exists a bijection $f : M_1 \to M_2$ such that $\varphi_1(s) = f^{-1}\varphi_2(s)f$ for all $s \in S$.

The next proposition will help us to describe Schreier graphs explicitly in the later sections.

Proposition 2.2.3. *Let G be a group with a symmetric finite generating set S, M be a set, $x_0 \in M$, and $\varphi : G \to \mathrm{Aut}(M)$ be a homomorphism. Then the Schreier graph $(\mathrm{Orb}_G(x_0), S, \varphi, x_0)$ with the marked vertex x_0 is S-isomorphic to the Schreier coset graph $(G/H, S, H)$ with the marked vertex $H = \mathrm{Stab}_G(x_0)$ as marked labelled directed graphs.*

Proof. Define $f : G/H \to \mathrm{Orb}_G(x_0)$ by $f(gH) = \varphi(g)(x_0)$. Since $g^{-1}g' \in H = \mathrm{Stab}_G(x_0)$ implies $\varphi(g)(x_0) = \varphi(g')(x_0)$, its map is well-defined. Clearly f is a bijection. Since

$$f(\varphi_H(s)(gH)) = f(sgH) = \varphi(sg)(x_0) = \varphi(s)\varphi(g)(x_0) = \varphi(s)(f(gH)),$$

we have $\varphi_H(s) = f^{-1}\varphi(s)f$ for all $s \in S$, which is the desired conclusion by Remark 2.2.2. □

Let m and n be nonzero integers. The group with the presentation

$$\langle A, B \mid AB^m = B^n A\rangle$$

is called the *Baumslag-Solitar group* and it is denoted by $BS(m, n)$. For any $n \geq 2$, $BS(1, n)$ has a geometric representation. That is, we define two affine maps a and b of the real line $\mathbb{R}$ by $a(x) = nx$ and $b(x) = x+1$ respectively. Let $n \geq 2$, $S_n = \{a, b\}^{\pm}$ and $G_n = \langle S_n\rangle$ be the subgroup of the affine group $\mathrm{Aff}(\mathbb{R})$. Then there exists the isomorphism

$$h_n : BS(1, n) \to G_n$$

with $h_n(A) = a$ and $h_n(B) = b$ (see [4, p.100]). Thus, $BS(1, n)$ has the natural left action

$$\varphi_n : BS(1, n) \to G_n \hookrightarrow \mathrm{Aff}(\mathbb{R}) \hookrightarrow \mathrm{Aut}(\mathbb{R}).$$

By [4, p.102], we note that

$$(*)_n \qquad G_n = \{g : \mathbb{R} \to \mathbb{R} \mid g(x) = n^i x + j/n^k,\ i, j, k \in \mathbb{Z}\}.$$

2.3 Schreier graphs of all real numbers

Let $x \in \mathbb{R}$ and

$$\phi_x : G_n \to \mathrm{Aut}(\mathrm{Orb}_{G_n}(x))$$

be the usual left action. By the isomorphism h_n and Proposition 2.2.3, the Schreier graph $(\mathrm{Orb}_{G_n}(x), S_n, \phi_x, x)$ and the Schreier coset graph

$$(BS(1,n)/\mathrm{Stab}_{BS(1,n)}(x), \{A,B\}^{\pm}, \mathrm{Stab}_{BS(1,n)}(x))$$

with the marked vertexes are isomorphic, so we will consider the Schreier graph $(\mathrm{Orb}_{G_n}(x), S_n, \phi_x)$ for each $x \in \mathbb{R}$. For simplicity of notation, we write g and $(\mathrm{Orb}_{G_n}(x), S_n)$ instead of $\phi_x(g)$ and the Schreier graph $(\mathrm{Orb}_{G_n}(x), S_n, \phi_x)$, respectively.

Remark 2.3.1. *For any $x \in \mathbb{R}$ and any $f \in \mathrm{Stab}_{G_n}(x)$ with $f \neq 1_{\mathbb{R}}$, $bfb^{-1} \notin \mathrm{Stab}_{G_n}(x)$. Thus $\mathrm{Stab}_{G_n}(x)$ is not a normal subgroup of G_n.*

We notice that the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha), S_n)$ for $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ is S_n-isomorphic to the Cayley graph of $BS(1,n)$ relative to the generators $\{A,B\}^{\pm}$ by the above since the stabilizer $\mathrm{Stab}_{BS(1,n)}(\alpha)$ is trivial. However in this section we construct the Schreier graphs $(\mathrm{Orb}_{G_n}(q), S_n)$ for rational numbers q and will compare those descriptions in the later section (see Theorem 2.4.4). Therefore we employ the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha), S_n)$. We construct the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha), S_n)$ by an arrangement of elements in the orbit $\mathrm{Orb}_{G_n}(\alpha)$. The construction of the Cayley graph of $BS(1,n) \cong G_n$ given in [19] depends on the fact that the word problem for $BS(1,n)$ is solvable.

Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ be the finite group with the additive group structure. The set of all finite words over $\mathbb{Z}_n$ and the set of all infinite words over $\mathbb{Z}_n$ are denoted by $\mathbb{Z}_n^*$ and $\mathbb{Z}_n^\omega$ respectively. Let

$$\widetilde{\mathbb{Z}_n} = \mathbb{Z}_n^* \setminus \{\epsilon\},$$

where ϵ denotes the *empty word*. For every word $w = w_1 w_2 \ldots w_k$ in $\mathbb{Z}_n^*$, the *length* of w, denoted by $|w|$, is the number k. Note that $|\epsilon|$ is zero.

Definition 2.3.2. An element w of $\mathbb{Z}_n^\omega$ is called a *rational element* in $\mathbb{Z}_n^\omega$ if there exist $u \in \mathbb{Z}_n^*$ and $v \in \widetilde{\mathbb{Z}_n}$ such that

(1) $w = uv^\infty$,

(2) $v \neq t^k$ whenever $k \geq 2$ and $t \in \widetilde{\mathbb{Z}_n}$, and

(3) $u_{|u|} \neq v_{|v|}$ whenever $u \neq \epsilon$.

Then, we say that the pair (u,v) of words *satisfies* (A). An element w of $\mathbb{Z}_n^\omega$ which is not rational is called an *irrational element* in $\mathbb{Z}_n^\omega$. Let $x \in \mathbb{R}$. Then, there exists $w \in \mathbb{Z}_n^\omega$ such that

$$x - \lfloor x \rfloor = \sum_{i \geq 1} w_i / n^i,$$

where $\lfloor x \rfloor = \max\{k \in \mathbb{Z} \mid k \leq x\}$. We say that x is *represented by* $w \in \mathbb{Z}_n^\omega$. It is easy to see that x is a rational number if and only if it is represented by a rational element in $\mathbb{Z}_n^\omega$.

Lemma 2.3.3. *Let $x, x' \in \mathbb{Z}_n^*$ and y be an irrational element of $\mathbb{Z}_n^\omega$ with $xy = x'y$. Then $x = x'$.*

Proof. Without loss of generality, we can assume that $|x| \leq |x'|$. By assumption,

$$y_{|x'|-|x|+j} = y_j$$

for each $j \geq 1$. Since y is an irrational element in $\mathbb{Z}_n^\omega$, $|x'| = |x|$. Therefore, $x = x'$. □

Lemma 2.3.4. *Suppose that pairs (x, y) and (x', y') of words satisfy (A). Then $xy^\infty = x'y'^\infty$ if and only if $x = x'$ and $y = y'$.*

Proof. Suppose that $xy^\infty = x'y'^\infty$. It suffices to show that $x = x'$ and $y = y'$. First we show that $|x| = |x'|$. On the contrary, suppose that $|x| < |x'|$. For any $k \geq 1$, there exists a unique $\underline{k} \in \{1, \ldots, |y|\}$ such that $k \equiv \underline{k} \mod |y|$. Then

$$x'_{|x'|} = (x'y'^\infty)_{|x'|} = (xy^\infty)_{|x'|} = (y^\infty)_{|x'|-|x|} = y_{\underline{|x'|-|x|}}.$$

On the other hand,

$$y'_{|y'|} = (x'y'^\infty)_{|x'|+|y'|(|y|/g)} = (xy^\infty)_{|x'|+|y|(|y'|/g)} = (y^\infty)_{|x'|-|x|+|y|(|y'|/g)} = y_{\underline{|x'|-|x|}},$$

where $g = \gcd(|y'|, |y|)$. Since $x' \neq \epsilon$, by the assumption of x', we see $x'_{|x'|} \neq y'_{|y'|}$, a contradiction. Thus $|x| = |x'|$. Hence we have that $x = x'$ and $y^\infty = y'^\infty$.

Next we show that $|y| = |y'|$. On the contrary, suppose that $|y| < |y'|$. There exist $\alpha \in \mathbb{Z}$ and $\beta \geq 0$ such that $|y'|\alpha + |y|\beta = g$. For any $i \geq 1$

$$(y'^\infty)_{i+g} = (y'^\infty)_{i+|y'|\alpha+|y|\beta} = (y'^\infty)_{i+|y|\beta} = (y^\infty)_{i+|y|\beta} = (y^\infty)_i = (y'^\infty)_i.$$

Since y'^∞ has the period g, y' has the period $g \leq |y| < |y'|$. This contradicts the assumption of y'. Since $|y| = |y'|$, we conclude $y = y'$. □

Lemma 2.3.5. *Let $x, y \in \widetilde{\mathbb{Z}_n}$. Suppose that $x_{|x|} = y_{|y|}$ and the word y satisfies the condition (2) in Definition 2.3.2. Then $xy^\infty = y^\infty$ if and only if $|x| \equiv 0 \bmod |y|$ and $x = y^{|x|/|y|}$.*

Proof. Suppose that $xy^\infty = y^\infty$. It suffices to show that $|x| \equiv 0 \bmod |y|$ and $x = y^{|x|/|y|}$. Let $m \geq 0$ and $1 \leq r \leq |y|$ such that $|x| = |y|m + r$. Then for any $i \geq 1$

$$\begin{aligned}(y^\infty)_{i+r} = (xy^\infty)_{|x|+i+r} = (xy^\infty)_{|x|+i+r+|y|m} &= (xy^\infty)_{|x|+i+|x|} \\ &= (y^\infty)_{i+|x|} \\ &= (xy^\infty)_{i+|x|} \\ &= (y^\infty)_i.\end{aligned}$$

Thus y^∞ has the period r and $(y_1 \ldots y_{|y|})^\infty = y^\infty = (y_1 \ldots y_r)^\infty$. Since (ϵ, y) and $(\epsilon, y_1 \ldots y_r)$ satisfy (A), by Lemma 2.3.4, we have $|y| = r$. Therefore $|x| \equiv 0 \mod |y|$. Moreover, since $(xy^\infty)_i = (y^\infty)_i$ for all $1 \leq i \leq |x|$, we have $x = y^{|x|/|y|}$. □

Let $\sigma : \mathbb{Z}_n^\omega \to \mathbb{Z}_n^\omega$ be the sift map defined by $\sigma(w_1w_2w_3\ldots) = w_2w_3w_4\ldots$ Write $\sigma^{k-1} = \underbrace{\sigma\sigma\cdots\sigma}_{k-1}$ for each $k \geq 1$, where σ^0 is the identity map. We note that $\sigma^{k-1}(w)_i = w_{k-1+i}$ for any $k, i \geq 1$ and each $w \in \mathbb{Z}_n^\omega$.

Lemma 2.3.6. *Let (x, y) be a pair of words satisfying (A). Then for $|x| \leq j < j'$, $\sigma^j(xy^\infty) = \sigma^{j'}(xy^\infty)$ if and only if $j' - j \equiv 0 \mod |y|$.*

Proof. For any $k \geq 1$, there exists a unique $\underline{k} \in \{1, \ldots, |y|\}$ such that $k \equiv \underline{k}$ mod $|y|$. Then

$$\sigma^j(xy^\infty) = \sigma^{j-|x|}(y^\infty) = (y_{\underline{j-|x|+1}} \cdots y_{\underline{j'-|x|}})\, \sigma^{j'-|x|}(y^\infty), \text{ and}$$

$$\sigma^{j'}(xy^\infty) = \sigma^{j'-|x|}(y^\infty).$$

Thus $\sigma^j(xy^\infty) = \sigma^{j'}(xy^\infty)$ if and only if $(y_{\underline{j-|x|+1}} \cdots y_{\underline{j'-|x|}})\, \sigma^{j'-|x|}(y^\infty) = \sigma^{j'-|x|}(y^\infty)$. By Lemma 2.3.5,

$$(y_{\underline{j-|x|+1}} \cdots y_{\underline{j'-|x|}})\, \sigma^{j'-|x|}(y^\infty) = \sigma^{j'-|x|}(y^\infty)$$

if and only if $j' - j \equiv 0 \mod |y|$. □

For any $v \in \mathbb{Z}_n^\omega$ and any $t \in \mathbb{Z}_n$, set

$$D_v = \mathbb{Z} + \sum_{i \geq 1} v_i/n^i \subset \mathbb{R}, \text{ and } D_v^t = n\mathbb{Z} + t + \sum_{i\geq 1} v_i/n^i \subset \mathbb{R}.$$

Note that

$$0 \leq \sum_{i\geq 1} v_i/n^i \leq 1 \text{ and } D_v = \bigsqcup_{t \in X} D_v^t.$$

Lemma 2.3.7. *Let y and y' be irrational elements in $\mathbb{Z}_n^\omega$. Then, the following statements are equivalent.*

(1) $D_y \cap D_{y'} \neq \emptyset$.

(2) $\sum_{i\geq 1} y_i/n^i = \sum_{i \geq 1} y'_i/n^i$.

(3) $y = y'$.

Proof. It suffices to show that (2) implies (3). On the contrary, suppose that there exists $i \geq 1$ such that $y_i \neq y'_i$. Let $i_0 = \min\{i \mid y_i \neq y'_i\}$. Then,

$$y_{i_0}/n^{i_0} + \sum_{i \geq i_0+1} y_i/n^i = y'_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} y'_i/n^i.$$

Without loss of generality, we can assume that $y_{i_0} < y'_{i_0}$. Since y and y' are irrational elements,

$$1/n^{i_0} < y'_{i_0}/n^{i_0} - y_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} y'_i/n^i = \sum_{i \geq i_0+1} y_i/n^i < 1/n^{i_0},$$

a contradiction. □

Lemma 2.3.8. *Let (x, y) and (x', y') be pairs of words satisfying (A) such that $\min\{|y|, |y'|\} \geq 2$ whenever $y \neq y'$. Then, the following statements are equivalent.*

(1) $D_{xy^\infty} \cap D_{x'y'^\infty} \neq \emptyset$.

(2) $\sum_{i\geq 1} (xy^\infty)_i/n^i = \sum_{i\geq 1} (x'y'^\infty)_i/n^i$.

(3) $xy^\infty = x'y'^\infty$.

Proof. Suppose that $\sum_{i\geq 1} (xy^\infty)_i/n^i = \sum_{i\geq 1} (x'y'^\infty)_i/n^i$. It suffices to prove that $xy^\infty = x'y'^\infty$. On the contrary, suppose that there exists $i \geq 1$ such that $(xy^\infty)_i \neq (x'y'^\infty)_i$. Let $i_0 = \min\{i \mid (xy^\infty)_i \neq (x'y'^\infty)_i\}$. Then

$$(xy^\infty)_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} (xy^\infty)_i/n^i = (x'y'^\infty)_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} (x'y'^\infty)_i/n^i.$$

Without loss of generality, we can assume that $(xy^\infty)_{i_0} < (x'y'^\infty)_{i_0}$.

If $\min\{|y|, |y'|\} \geq 2$, or if $y = y' \in \{1, \dots, n-2\}$, then we have

$$1/n^{i_0} < (x'y'^\infty)_{i_0}/n^{i_0} - (xy^\infty)_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} (x'y'^\infty)_i/n^i = \sum_{i\geq i_0+1} (xy^\infty)_i/n^i < 1/n^{i_0},$$

a contradiction.

If $y = y' = 0$, then $i_0 \leq |x'|$. Then

$$1/n^{i_0} \leq (x'y'^\infty)_{i_0}/n^{i_0} - (xy^\infty)_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} (x'y'^\infty)_i/n^i = \sum_{i\geq i_0+1} (xy^\infty)_i/n^i < 1/n^{i_0},$$

a contradiction.

If $y = y' = n - 1$, then $i_0 \leq |x|$. Then

$$1/n^{i_0} < (x'y'^\infty)_{i_0}/n^{i_0} - (xy^\infty)_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} (x'y'^\infty)_i/n^i = \sum_{i\geq i_0+1} (xy^\infty)_i/n^i \leq 1/n^{i_0},$$

a contradiction. Therefore $xy^\infty = x'y'^\infty$. □

The proof of the following lemma is immediate, so the details are left to the reader.

Lemma 2.3.9. *Let $v \in \mathbb{Z}_n^\omega$ and $t \in \mathbb{Z}_n$. Then,*

(a) $a(D_v) = D^{v_1}_{\sigma(v)}$, $a^{-1}(D^t_v) = D_{tv}$, $a^{-1}(D_v) = \bigsqcup_{t\in\mathbb{Z}_n} D_{tv}$,

(b) $b^{\pm 1}(D^t_v) = D^{t\pm 1}_v$, *and* $b^{\pm 1}(D_v) = D_v$.

Definition 2.3.10. Let $w \in \mathbb{Z}_n^\omega$. Set

$$V_w = \{u\sigma^j(w) \,|\, j \geq 0, u \in \mathbb{Z}_n^*\},\ E_w = V_w \times (\{a\} \sqcup \mathbb{Z}_n),$$

and $L_w = \{a\}^{\pm}$. Define $\alpha_w : E_w \to V_w$, $\beta_w : E_w \to V_w$ and $l_w : E_w \to L_w$ by

$$\alpha_w(v, a) = \alpha_w(v, k) = v,\ \beta_w(v, a) = \sigma(v),\ \beta_w(v, k) = kv,$$

$$l_w(v, a) = a \text{ and } l_w(v, k) = a^{-1}$$

for each $v \in V_w$ and each $k \in \mathbb{Z}_n$. The labelled directed graph $(V_w, E_w, L_w, \alpha_w, \beta_w, l_w)$ and the Schreier graph $(\mathbb{Z}, \{\pm 1\}, \phi)$ will be denoted by Γ_w and $\Gamma_{\mathbb{Z}}$ respectively, where $\phi : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z})$ is the usual action.

Lemma 2.3.11. (1) *If w is an irrational element in $\mathbb{Z}_n^\omega$, then*

$$V_w = \bigsqcup_{j \geq 1} \{\sigma^j(w)\} \sqcup \bigsqcup_{u \in \mathbb{Z}_n^*} \{uw\} \sqcup \bigsqcup_{j \geq 1, s \in \mathbb{Z}_n^*, t \in \mathbb{Z}_n, t \neq w_j} \{st\sigma^j(w)\}.$$

(2) *If $w = uv^\infty$ is a rational element in $\mathbb{Z}_n^\omega$ as in Definition* 2.3.2, *then*

$$V_w = \bigsqcup_{|u| \leq j < |u|+|v|} \{\sigma^j(w)\} \sqcup \bigsqcup_{|u| < j \leq |u|+|v|, s \in \mathbb{Z}_n^*, t \in \mathbb{Z}_n, t \neq w_j} \{st\sigma^j(w)\}.$$

Proof. By Lemmas 2.3.4 and 2.3.6, we can easily show (2). Thus we prove (1). Let $j, j' \geq 1$, $u, u' \in \mathbb{Z}_n^*$, and $t, t' \in \mathbb{Z}_n$ with $t \neq w_j$ and $t' \neq w_{j'}$. It suffices to show the following statements:

(a) $j = j'$ whenever $\sigma^j(w) = \sigma^{j'}(w)$.
(b) $u = u'$ whenever $uw = u'w$.
(c) $u = u'$, $t = t'$, and $j = j'$ whenever $ut\sigma^j(w) = u't'\sigma^{j'}(w)$.
(d) $\sigma^j(w) \neq uw$.
(e) $\sigma^j(w) \neq ut'\sigma^{j'}(w)$.
(f) $uw \neq u't\sigma^j(w)$.

The statements (b) and (d) directly follow from Lemma 2.3.3.

Suppose that $ut\sigma^j(w) = u't'\sigma^{j'}(w)$ and $j \leq j'$. Since $\sigma^j(w) = w_{j+1} \dots w_{j'}\sigma^{j'}(w)$, by Lemma 2.3.3, we have

$$utw_{j+1} \dots w_{j'} = u't'.$$

Since $t' \neq w_{j'}$, we see $j = j'$, thus $u = u'$ and $t = t'$, which proves (c). Similarly, we can show (a).

If $j \geq j'$, by Lemma 2.3.3,

$$ut'\sigma^{j'}(w) = ut'w_{j'+1} \dots w_j\sigma^j(w) \neq \sigma^j(w).$$

Suppose that $j \leq j'$ and $\sigma^j(w) = ut'\sigma^{j'}(w)$. Since $\sigma^j(w) = w_{j+1} \dots w_{j'}\sigma^{j'}(w)$,

$$w_{j+1} \dots w_{j'}\sigma^{j'}(w) = ut'\sigma^{j'}(w).$$

Hence by Lemma 2.3.3 $w_{j+1}\dots w_{j'} = ut'$. Thus $w_{j'} = t'$, a contradiction, and (e) is proved.

Since $w_j \neq t$, $uw_1 \dots w_j \neq u't$. By Lemma 2.3.3,

$$uw = uw_1 \dots w_j \sigma^j(w) \neq u't\sigma^j(w),$$

which proves (f). □

Lemma 2.3.12. *Let $n \geq 2$ and $x \in \mathbb{R}$ represented by $w \in \mathbb{Z}_n^\omega$. Then,*

$$\mathrm{Orb}_{G_n}(x) = \bigsqcup_{v \in V_w} D_v.$$

Proof. By Lemmas 2.3.7, 2.3.8 and 2.3.11,

$$\bigcup_{v \in V_w} D_v = \bigsqcup_{v \in V_w} D_v.$$

Thus it suffices to show that

$$\mathrm{Orb}_{G_n}(x) = \bigcup_{v \in V_w} D_v.$$

Since

$$x \in D_w \subset \bigcup_{v \in V_w} D_v,$$

by Lemma 2.3.9,

$$\mathrm{Orb}_{G_n}(x) \subset \bigcup_{g \in G_n} \bigcup_{v \in V_w} g(D_v) = \bigcup_{v \in V_w} D_v.$$

Let $j \geq 0$ and $u \in \mathbb{Z}_n^*$. It suffices to show that

$$D_{u\sigma^j(w)} \subset \mathrm{Orb}_{G_n}(x).$$

We have

$$\begin{aligned}
D_{u\sigma^j(w)} &= \mathbb{Z} + \sum_{i \geq 1} (u\sigma^j(w))_i / n^i \\
&= \mathbb{Z} + \sum_{i=1}^{|u|} u_i/n^i + \sum_{l \geq j+1} w_l / n^{l-j+|u|} \\
&= \mathbb{Z} + \sum_{i=1}^{|u|} u_i/n^i + n^{j-|u|}\Big(\sum_{l \geq 1} w_l/n^l - \sum_{l=1}^{j} w_l/n^l\Big) \\
&= \mathbb{Z} + n^{-|u|}\Big(\sum_{i=1}^{|u|} n^{|u|-i}u_i - \sum_{i=1}^{j} n^{j-i} w_i + n^j(x - \lfloor x \rfloor)\Big) \\
&= \{b^k a^{-|u|} b^{(\sum_{i=1}^{|u|} n^{|u|-i}u_i - \sum_{i=1}^{j} n^{j-i}w_i)} a^j b^{-\lfloor x \rfloor}(x) \mid k \in \mathbb{Z}\} \subset \mathrm{Orb}_{G_n}(x).
\end{aligned}$$

□

Theorem 2.3.13. *Let $n \geq 2$ and x be a real number represented by $w \in \mathbb{Z}_n^\omega$. Then, there exists a homomorphism*

$$h = (f, \psi, \gamma) : (\mathrm{Orb}_{G_n}(x), S_n) \to \Gamma_w$$

such that for every $v \in V_w$, the subgraph $h^{-1}(v) = (D_v, D_v \times \{b\}^{\pm}, S_n, \alpha|, \beta|, l|)$ is isomorphic to $\Gamma_{\mathbb{Z}}$, where $h^{-1}(v) = (f^{-1}(v), \psi^{-1}(v), S_n, \alpha|, \beta|, l|)$.

Proof. It suffices to find a homomorphism $h = (f, \psi, \gamma) : (\mathrm{Orb}_{G_n}(x), S_n) \to \Gamma_w$ such that for every $v \in V_w$, the subgraph $h^{-1}(v)$ is isomorphic to $\Gamma_{\mathbb{Z}}$. By Lemmas 2.3.11 and 2.3.12, for any $y \in \mathrm{Orb}_{G_n}(x)$, there exists a unique $v_y \in V_w$ and $k \in \mathbb{Z}_n$ such that $y \in D^k_{v_y} \subset D_{v_y}$. Thus, we can define

$$f : \mathrm{Orb}_{G_n}(x) \to V_w,\ \psi : \mathrm{Orb}_{G_n}(x) \times S_n \to E_w \sqcup V_w \text{ and } \gamma : S_n \to L_w$$

by $f(y) = v_y$, $\psi(y, a) = (f(y), a)$, $\psi(y, a^{-1}) = (f(y), k)$, $\psi(y, b) = f(y)$, $\psi(y, b^{-1}) = f(y)$, $\gamma(a) = a$, $\gamma(a^{-1}) = a^{-1}$, $\gamma(b) = a$, and $\gamma(b^{-1}) = a^{-1}$. □

2.4 Classification of Schreier graphs

In this section we classify Schreier graphs described in the previous section.

Lemma 2.4.1. *Let $v \in \widetilde{\mathbb{Z}_n}$. For $i \geq 1$ set*

$$W_i = b^{-(v^\infty)_i} a \text{ and } Z_i = b^{(v^\infty)_i} a.$$

Then, for every $k \geq 1$, $W_k \cdots W_1$ and $Z_k \cdots Z_1$ are nontrivial affine maps with the slopes n^k such that

$$(W_k \cdots W_1)(\sum_{j\geq 1} (v^\infty)_j / n^j) = \sum_{j \geq 1} (v^\infty)_{k+j} / n^j \text{ and}$$

$$(Z_k \cdots Z_1)(-\sum_{j\geq 1} (v^\infty)_j / n^j) = -\sum_{j \geq 1} (v^\infty)_{k+j} / n^j.$$

Proof. The proof is by induction on k. The affine map W_1 has the slope n such that

$$W_1(\sum_{j\geq 1} (v^\infty)_j / n^j) = b^{-(v^\infty)_1}\, a\,(\sum_{j\geq 1} (v^\infty)_j / n^j) = b^{-(v^\infty)_1}\,((v^\infty)_1 + \sum_{j\geq 2} (v^\infty)_j / n^{j-1})$$
$$= \sum_{j\geq 1} (v^\infty)_{1+j} / n^j.$$

Assume the formula holds for $k-1$, we have

$$
\begin{aligned}
(W_kW_{k-1}\cdots W_1)\,(\sum_{j\geq 1}(v^\infty)_j/n^j) &= W_k(\sum_{j\geq 1}(v^\infty)_{k-1+j}/n^j) \\
&= b^{-(v^\infty)_k}\,a\,(\sum_{j\geq 1}(v^\infty)_{k-1+j}/n^j) \\
&= b^{-(v^\infty)_k}\,((v^\infty)_k+\sum_{j\geq 2}(v^\infty)_{k-1+j}/n^{j-1}) \\
&= \sum_{j\geq 1}(v^\infty)_{k+j}/n^j
\end{aligned}
$$

and the affine map $W_k\cdots W_1$ has the slope n^k. Similarly, we can prove it for $Z_k\cdots Z_1$. □

Remark 2.4.2. *Let $x,y\in\mathbb{R}$. By Remark 2.2.2, Schreier graphs $(\mathrm{Orb}_{G_n}(x),S_n)$ and $(\mathrm{Orb}_{G_n}(y),S_n)$ are isomorphic if and only if there exist two bijections*

$$f:\mathrm{Orb}_{G_n}(x)\to\mathrm{Orb}_{G_n}(y)\ \textit{and}\ \gamma:S_n\to S_n$$

such that $\gamma(s)(f(z))=f(s(z))$ for each $z\in\mathrm{Orb}_{G_n}(x)$ and each $s\in S_n$.

Lemma 2.4.3. *Let $x,y\in\mathbb{R}$. Suppose that the Schreier graph $(\mathrm{Orb}_{G_n}(x),S_n)$ is isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(y),S_n)$ by a bijection $\gamma:S_n\to S_n$. Then*

$$\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^n=1_{\mathbb{R}}\ \textit{in}\ G_n$$

if and only if

$$\gamma=1_{S_n}\quad\textit{or}\quad\gamma(a)=a,\gamma(a^{-1})=a^{-1},\gamma(b)=b^{-1},\ \textit{and}\ \gamma(b^{-1})=b.$$

Proof. Let

$$f:\mathrm{Orb}_{G_n}(x)\to\mathrm{Orb}_{G_n}(y)$$

be a bijection as in Remark 2.4.2. For any $s\in S$ and any $x_0\in\mathrm{Orb}_{G_n}(x)$,

$$\gamma(s)\gamma(s^{-1})(f(x_0))=f(ss^{-1}(x_0))=f(x_0)$$

by Remark 2.4.2. Since f is a bijection,

$$\gamma(s)\gamma(s^{-1})=1_{\mathrm{Orb}_{G_n}(y)}.$$

Since $\gamma(s)\gamma(s^{-1})$ is an affine map, $\gamma(s)\gamma(s^{-1})=1_{\mathbb{R}}$, thus $\gamma(s)^{-1}=\gamma(s^{-1})\in\mathrm{Aff}(\mathbb{R})$.

Suppose that

$$\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^n=1_{\mathbb{R}}\text{ and }\gamma\neq 1_{S_n}.$$

Since $a(x)=nx$ and $\gamma(b^{-1})$ has the n-th power, $\gamma(b^{-1})\in\{b\}^{\pm}$.

Suppose that $\gamma(b^{-1}) = b^{-1}$. Then $\gamma(b) = b$. Since $\gamma \neq 1_{S_n}$, we have $\gamma(a) = a^{-1}$. Then

$$\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^n = a^{-1}bab^{-n} \neq 1_{\mathbb{R}},$$

a contradiction. Thus $\gamma(b^{-1}) = b$ and $\gamma(b) = b^{-1}$.

If $\gamma(a) = a^{-1}$, then $\gamma(a^{-1}) = a$ and

$$\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^n = a^{-1}b^{-1}ab^n \neq 1_{\mathbb{R}},$$

a contradiction. Hence $\gamma(a) = a$ and $\gamma(a^{-1}) = a^{-1}$. □

Theorem 2.4.4. *Let $m, n \geq 2$ with $m \neq n$.*

(1) *For any $x, y \in \mathbb{R}$, the Schreier graph $(\mathrm{Orb}_{G_m}(x), S_m)$ is not isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(y), S_n)$ as labelled directed graphs.*

(2) *For any $\alpha_1, \alpha_2 \in \mathbb{R}\backslash\mathbb{Q}$, the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha_1), S_n, \alpha_1)$ is S_n-isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha_2), S_n, \alpha_2)$ as marked labelled directed graphs.*

(3) *For any $q \in \mathbb{Q}$ and any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the Schreier graph $(\mathrm{Orb}_{G_n}(q), S_n)$ is not isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha), S_n)$ as labelled directed graphs.*

(4) *Let q_1, $q_2 \in \mathbb{Q}$. Then, the following statements are equivalent.*

   (*a*) *The Schreier graph $(\mathrm{Orb}_{G_n}(q_1), S_n)$ is isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(q_2), S_n)$ as labelled directed graphs.*

   (*b*) $\mathrm{Orb}_{G_n}(q_1) = \mathrm{Orb}_{G_n}(q_2)$ *or* $\mathrm{Orb}_{G_n}(-q_1) = \mathrm{Orb}_{G_n}(q_2)$.

Proof. On the contrary, suppose that the Schreier graphs

$$(\mathrm{Orb}_{G_m}(x), S_m) \text{ and } (\mathrm{Orb}_{G_n}(y), S_n)$$

are isomorphic by bijections

$$f : \mathrm{Orb}_{G_m}(x) \to \mathrm{Orb}_{G_n}(y) \text{ and } \gamma : S_m \to S_n$$

as in Remark 2.2.2. We check at once that

$$\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^m \neq 1_{\mathbb{R}} \in G_n.$$

By Remark 2.2.2,

$$\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^m(f(z)) = f(aba^{-1}b^{-m}(z)) = f(z)$$

for each $z \in \mathrm{Orb}_{G_m}(x)$, contradiction, which proves (1). Since $\mathrm{Stab}_{G_n}(\alpha) = 1$ for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, by Proposition 2.2.3, the statement (2) is proved.

Let q be a rational number represented by uv^∞ and $x \in \mathbb{R}$ such that the Schreier graph $(\mathrm{Orb}_{G_n}(q), S_n)$ is isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(x), S_n)$ as labelled directed graphs by bijections $f : \mathrm{Orb}_{G_n}(q) \to \mathrm{Orb}_{G_n}(x)$ and $\gamma : S_n \to S_n$ as in Remark 2.4.2. Let

$$q_0 = \sum_{j\geq 1} (v^\infty)_j / n^j \in \mathrm{Orb}_{G_n}(q).$$

Since $aba^{-1}b^{-n}(q') = q'$ for each $q' \in \mathrm{Orb}_{G_n}(q)$, by Remark 2.4.2, we have

$$\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^n(f(q')) = f(aba^{-1}b^{-n}(q')) = f(q').$$

Hence, $\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^n = 1_{\mathbb{R}}$. By Lemma 2.4.3,

$$\gamma = 1_{S_n} \quad \text{or} \quad \gamma(a) = a, \gamma(a^{-1}) = a^{-1}, \gamma(b) = b^{-1}, \text{and}\, \gamma(b^{-1}) = b. \tag{E}$$

On the other hand, by Lemma 2.4.1, there exists a nontrivial affine map $W_{|v|} \cdots W_1 = c_k \cdots c_1$ such that $c_k \cdots c_1(q_0) = q_0$, where $c_i \in \{a, b^{-1}\}$. By Remark 2.4.2, we have

$$\gamma(c_k) \cdots \gamma(c_1)(f(q_0)) = f(c_k \cdots c_1(q_0)) = f(q_0).$$

(*i*) If $\gamma = 1_{S_n}$, then the nontrivial affine map $c_k \cdots c_1$ fixes both q_0 and $f(q_0)$. Hence, $f(q_0) = q_0$.

(*ii*) If $\gamma(a) = a, \gamma(a^{-1}) = a^{-1}, \gamma(b) = b^{-1}$, and $\gamma(b^{-1}) = b$, then by Lemma 2.4.1,

$$\gamma(c_k) \cdots \gamma(c_1)(-q_0) = Z_{|v|} \cdots Z_1(-q_0) = -q_0.$$

Since the nontrivial affine map $\gamma(c_k) \cdots \gamma(c_1)$ fixes both $-q_0$ and $f(q_0)$, we have $-q_0 = f(q_0)$.

We start to prove (3). On the contrary, if $x = \alpha \in \mathbb{R} \setminus \mathbb{Q}$, by the above, we see $f(q_0) \in \mathbb{Q}$, a contradiction, which proves (3).

Next we prove (4). Suppose that the statement (*a*) holds, i.e.,$q = q_1$, $x = q_2 \in \mathbb{Q}$ above. If $\gamma = 1_{S_n}$, by (*i*) above,

$$\mathrm{Orb}_{G_n}(q_1) = \mathrm{Orb}_{G_n}(q_0) = \mathrm{Orb}_{G_n}(q_2).$$

If $\gamma \neq 1_{S_n}$, by (*ii*) above,

$$\mathrm{Orb}_{G_n}(-q_1) = \mathrm{Orb}_{G_n}(-q_0) = \mathrm{Orb}_{G_n}(q_2),$$

which proves (*b*).

Suppose that the statement (*b*) holds. We show that

$$(\mathrm{Orb}_{G_n}(q_1), S_n) \text{ and } (\mathrm{Orb}_{G_n}(q_2), S_n)$$

are isomorphic. Without loss of generality, we can assume that

$$\mathrm{Orb}_{G_n}(-q_1) = \mathrm{Orb}_{G_n}(q_2).$$

Define $\gamma : S_n \to S_n$ by

$$\gamma(a) = a,\ \gamma(a^{-1}) = a^{-1},\ \gamma(b) = b^{-1},\ \text{and } \gamma(b^{-1}) = b.$$

In addition define $f : \mathrm{Orb}_{G_n}(q_1) \to \mathrm{Orb}_{G_n}(q_2)$ by

$$f(c_k \cdots c_1(q_1)) = \gamma(c_k) \cdots \gamma(c_1)(-q_1),$$

where $c_i \in S_n$. By induction on k, we can show that

$$(c_k \cdots c_1)(q_1) + (\gamma(c_k) \cdots \gamma(c_1))(-q_1) = 0$$

for each $k \geq 1$ and each $c_i \in S_n$. Hence, f is well-defined and an injection. By definition, f is a surjection satisfying that $f(s(z)) = \gamma(s)(f(z))$ for each $z \in \mathrm{Orb}_{G_n}(q_1)$ and each $s \in S_n$. By Remark 2.4.2, the Schreier graphs $(\mathrm{Orb}_{G_n}(q_1), S_n)$ and $(\mathrm{Orb}_{G_n}(q_2), S_n)$ are isomorphic by f and γ. □

Corollary 2.4.5. *Let q_1, q_2 be rational numbers. Then, the following statements are equivalent.*

(*a*) *The Schreier graph $(\mathrm{Orb}_{G_n}(q_1), S_n, q_1)$ is isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(q_2), S_n, q_2)$ as marked labelled directed graphs.*

(*b*) $|q_1| = |q_2|$.

Proof. From the latter part of the proof of Theorem 2.4.4, we can show that (*b*) implies (*a*). Suppose that $(\mathrm{Orb}_{G_n}(q_1), S_n, q_1)$ is isomorphic to $(\mathrm{Orb}_{G_n}(q_2), S_n, q_2)$ by bijections $f : \mathrm{Orb}_{G_n}(q_1) \to \mathrm{Orb}_{G_n}(q_2)$ with $f(q_1) = q_2$ and $\gamma : S_n \to S_n$ as in Remark 2.4.2. It suffices to show that $|q_1| = |q_2|$. Let us represent by $uv^\infty \in \mathbb{Z}_n^\omega$ $q_1 \in \mathbb{Q}$. Set

$$q_0 = \sum_{j \geq 1} (v^\infty)_j / n^j \in \mathrm{Orb}_{G_n}(q_1).$$

Then, there exist $d_1, \ldots, d_j \in S_n$ such that $(d_j \cdots d_1)(q_1) = q_0$. From the proof of Theorem 2.4.4, the map γ satisfies (E) in the proof of Theorem 2.4.4, and the map f satisfies

$$f(q_0) = \begin{cases} q_0 & \text{if } \gamma = 1_{S_n} \\ -q_0 & \text{if } \gamma \neq 1_{S_n}. \end{cases}$$

Moreover, there exist $c_1, \ldots, c_k \in S_n$ such that

$$(c_k \cdots c_1)(q_0) = q_0 \text{ and } \gamma(c_k) \cdots \gamma(c_1)(f(q_0)) = f(q_0).$$

Then

$$(d_j \cdots d_1)^{-1}(c_k \cdots c_1)(d_j \cdots d_1)(q_1) = q_1.$$

By Remark 2.4.2

$$\gamma(d_1)^{-1} \cdots \gamma(d_j)^{-1} \gamma(c_k) \cdots \gamma(c_1) \gamma(d_j) \cdots \gamma(d_1)(q_2) = q_2.$$

Thus

$$\gamma(c_k) \cdots \gamma(c_1)(\gamma(d_j) \cdots \gamma(d_1)(q_2)) = \gamma(d_j) \cdots \gamma(d_1)(q_2).$$

Suppose that $\gamma = 1_{S_n}$. Then,

$$(c_k \cdots c_1)((d_j \cdots d_1)(q_2)) = (d_j \cdots d_1)(q_2).$$

Since the nontrivial affine map $c_k \cdots c_1$ fixes both

$$q_0 = (d_j \cdots d_1)(q_1) \text{ and } (d_j \cdots d_1)(q_2),$$

it follows that $(d_j \cdots d_1)(q_1) = (d_j \cdots d_1)(q_2)$. We conclude that $q_1 = q_2$.

Suppose that $\gamma \neq 1_{S_n}$. By Remark 2.4.2,

$$\begin{aligned}
\gamma(d_j) \cdots \gamma(d_1)(q_2) &= (\gamma(d_j) \cdots \gamma(d_1))(f(q_1)) \\
&= f((d_j \cdots d_1)(q_1)) \\
&= f(q_0) \\
&= -q_0 \\
&= -(d_j \cdots d_1)(q_1).
\end{aligned}$$

Since the map γ satisfies (E) in the proof of Theorem 2.4.4, by induction on j, we can show $q_1 = -q_2$. □

2.5 Applications

First we determine the group structure of stabilizers for all rational numbers by using the Schreier graphs described in the previous section. The proof of next proposition allows us to understand a word stood for a generator as well as the group structure. We note that the the stabilizer $\mathrm{Stab}_{G_n}(q)$ is an infinite index subgroup of G_n since the orbit $\mathrm{Orb}_{G_n}(q)$ is an infinite set.

Proposition 2.5.1. *Let $n \geq 2$ and q be a rational number represented by $uv^\infty \in \mathbb{Z}_n^\omega$. Then, there exists $f \in \mathrm{Aff}(\mathbb{R})$ such that $f(x) = n^{|v|}(x - q) + q$ for each $x \in \mathbb{R}$, and $\mathrm{Stab}_{G_n}(q) = \langle f \rangle \cong \mathbb{Z}$.*

Proof. For $i \geq 1$ set $\widetilde{W_i} = b^{-(uv^\infty)_i} a$. By Lemma 2.4.1 we have

$$\begin{aligned}
\widetilde{W}_{|u|+|v|} \cdots \widetilde{W}_{|u|+1} \widetilde{W}_{|u|} \cdots \widetilde{W}_1 (b^{-\lfloor q \rfloor}(q)) &= \widetilde{W}_{|u|+|v|} \cdots \widetilde{W}_{|u|+1} (\sum_{i \geq 1} (v^\infty)_i / n^i) \\
&= W_{|v|} \cdots W_1 (\sum_{i \geq 1} (v^\infty)_i / n^i) \\
&= \sum_{i \geq 1} (v^\infty)_i / n^i \\
&= \widetilde{W}_{|u|} \cdots \widetilde{W}_1 \, (b^{-\lfloor q \rfloor}(q)).
\end{aligned}$$

Set

$$f = b^{\lfloor q \rfloor} \widetilde{W}_1^{-1} \cdots \widetilde{W}_{|u|}^{-1} \widetilde{W}_{|u|+|v|} \cdots \widetilde{W}_{|u|+1} \widetilde{W}_{|u|} \cdots \widetilde{W}_1 b^{-\lfloor q \rfloor}.$$

Then, f is an affine map with the slope $n^{|v|}$ such that $f(q) = q$. Hence $\langle f \rangle < \mathrm{Stab}_{G_n}(q)$.

Let $g \in \mathrm{Stab}_{G_n}(q)$. By $(*)_n$, there exists $i \in \mathbb{Z}$ such that $g(x) = n^i(x-q)+q$ for any $x \in \mathbb{R}$. If $|v| = 1$, f has the slope n, thus $g = f^i$. Hence, we may assume that $|v| \geq 2$. On the contrary, suppose that there exist $h \in \mathrm{Stab}_{G_n}(q) \setminus \langle f \rangle$, $0 < r < |v|$, $j \in \mathbb{Z}$, and $k \geq 0$ such that $h(x) = n^r x + j/n^k$ and $h(q) = q$. Then, we have

$$q = \frac{-j}{n^k(n^r - 1)}.$$

There exist $m \geq 0$ and $z = z_1 z_2 \dots z_r \in \widetilde{\mathbb{Z}_n}$ with $z \neq (n-1)^r$ such that

$$|j| = \left(\sum_{i=0}^{r-1}(n-1)n^i\right) m + \sum_{i=0}^{r-1} z_{r-i} n^i = n^r \left(m \sum_{i=1}^{r} \frac{n-1}{n^i} + \sum_{i=1}^{r} \frac{z_i}{n^i} \right).$$

Since

$$\frac{n^r}{n^r - 1} = \sum_{j \geq 0} \left(\frac{1}{n^r}\right)^j,$$

we have

$$qn^k = m + \sum_{i\geq 1} \frac{(z^\infty)_i}{n^i} \quad \text{or} \quad qn^k = -(m+1) + \sum_{i \geq 1} \frac{(\overline{z}^\infty)_i}{n^i},$$

where $\overline{z} = (n-1-z_1)\dots(n-1-z_r) \in \widetilde{\mathbb{Z}_n}$. Thus, qn^k has a repeating part whose length is the period of z^∞. However,

$$qn^k = \left(\lfloor q \rfloor + \sum_{i \geq 1} \frac{(uv^\infty)_i}{n^i}\right) n^k = \left(\lfloor q \rfloor n^k + \sum_{i=1}^{k} (uv^\infty)_i n^{k-i}\right) + \sum_{i\geq 1} \frac{(uv^\infty)_{i+k}}{n^i},$$

which contradicts (2) in Definition 2.3.2. □

Next we introduce the definition of being isomorphic in presentations for subgroups in order to translate the graphical expression of the Schreier graphs into the algebraic expression of subgroups. Consequently, we get a relevance to presentations for the stabilizers from the previous result about the classification of the Schreier graphs (see Theorem 2.5.5).

For $i \in \{1, 2\}$, let G_i be a group with a generating set T_i. Let $T_i^{-1} = \{t^{-1} \,|\, t \in T_i\}$ and $T_i^{\pm} = T_i \cup T_i^{-1}$. We assume that

$$(*) \qquad t \in T_i \cap T_i^{-1} \ \text{ if and only if } \ t \in T_i, \quad t^2 = 1.$$

For $i \in \{1, 2\}$ let $X_i = \{\, x_t \,|\, t \in T_i \,\}$. Set $X_i^{-1} = \{x_t^{-1} \,|\, t \in T_i\}$, where x_t^{-1} denotes a new symbol corresponding to the element x_t. We assume that $X_i \cap X_i^{-1} = \emptyset$ and that the expression $(x_t^{-1})^{-1}$ denotes the element x_t. For $i \in \{1, 2\}$ the free group with the basis X_i is denoted by $F(X_i)$, and for a subset R_i of $F(X_i)$ the normal closure of the set R_i in $F(X_i)$ is denoted by $\langle\langle R_i\rangle\rangle$. Let G_i be the group with the presentation $\langle X_i \,|\, R_i\rangle$ with respect to the epimorphism $\psi_i : F(X_i) \to G_i$ given by $\psi_i(x_t) = t$.

Definition 2.5.2. For $i \in \{1, 2\}$, let H_i be a subgroup of G_i. The subgroups H_1 and H_2 are *isomorphic in presentations* $\langle X_1|R_1\rangle$ *and* $\langle X_2|R_2\rangle$ *respectively* if there exists a bijection $\gamma : X_1^{\pm} \to X_2^{\pm}$ with $\gamma(x_t^{-1}) = \gamma(x_t)^{-1}$ such that

$$\widetilde{\gamma}\,(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2) \text{ and } \widetilde{\gamma}(\langle\langle R_1\rangle\rangle) = \langle\langle R_2\rangle\rangle,$$

where $\widetilde{\gamma} : F(X_1) \to F(X_2)$ is defined by

$$\widetilde{\gamma}(x_{t_1}^{\epsilon_1} \cdots x_{t_k}^{\epsilon_k}) = \gamma(x_{t_1})^{\epsilon_1} \cdots \gamma(x_{t_k})^{\epsilon_k}$$

for $\epsilon_i \in \{\pm 1\}$. Then, $\widetilde{\gamma}$ is an isomorphism and $H_1 \cong H_2$. Conversely, if there exists an isomorphism $\widetilde{\gamma} : F(X_1) \to F(X_2)$ such that $\widetilde{\gamma}(K_1) = K_2$ for each $K_i \in \{\psi_i^{-1}(H_i), \mathrm{Ker}\psi_i, X_i^{\pm}\}$, then $\gamma = \widetilde{\gamma}|_{X_1^{\pm}}$ satisfies the above condition.

Proposition 2.5.3. *Let*

$$\Gamma_i = (G_i/H_i, T_i^{\pm}, H_i) \textit{ and } \Gamma_i' = (F(X_i)/\psi_i^{-1}(H_i), X_i^{\pm}, \psi_i^{-1}(H_i))$$

be Schreier coset graphs for $i \in \{1, 2\}$*. Then, the following statements are equivalent.*

(*a*) Γ_1 *is isomorphic to* Γ_2 *as marked labelled directed graphs by a bijection* $\gamma : T_1^{\pm} \to T_2^{\pm}$ *such that* $\gamma(t^{-1}) = \gamma(t)^{-1}$ *for every* $t \in T_1$.

(*b*) Γ_1' *is isomorphic to* Γ_2' *as marked labelled directed graphs by a bijection* $\gamma' : X_1^{\pm} \to X_2^{\pm}$ *with* $\gamma'(x_t^{-1}) = \gamma'(x_t)^{-1}$ *for every* $x_t \in X_1$ *satisfying the condition*

$$(B) \qquad \psi_1(x_t)^2 = 1_{G_1} \;\; \textit{if and only if} \;\; \psi_2(\gamma'(x_t))^2 = 1_{G_2}.$$

Proof. Let $\varphi_i : G_i \to \mathrm{Aut}(G_i/H_i)$ and

$$\varphi_i' : F(X_i) \to \mathrm{Aut}(F(X_i)/\psi_i^{-1}(H_i))$$

be the usual left actions for $i \in \{1, 2\}$. We define $\Psi_i : F(X_i)/\psi_i^{-1}(H_i) \to G_i/H_i$ by $\Psi_i(y\,\psi_i^{-1}(H_i)) = \psi_i(y)H_i$. Since $y^{-1}y' \in \psi_i^{-1}(H_i)$ is equivalent to $\psi_i(y)^{-1}\psi_i(y') \in H_i$, Ψ_i is well-defined and an injection. Since ψ_i is a surjection, Ψ_i is also a surjection.

Suppose that the statement (a) holds. Let $f : G_1/H_1 \to G_2/H_2$ be a bijection between vertices such that $f(H_1) = H_2$ and $f\varphi_1(t) = \varphi_2(\gamma(t))f$ for every $t \in T_1$. Set

$$f' = \Psi_2^{-1} f \Psi_1 : F(X_1)/\psi_1^{-1}(H_1) \to F(X_2)/\psi_2^{-1}(H_2).$$

Clearly f' is bijective with $f'(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2)$.

Define $\gamma' : X_1^{\pm} \to X_2^{\pm}$ by

$$\gamma'(x_t^{\epsilon}) = \begin{cases} x_{\gamma(t)}^{\epsilon} & \text{if } \gamma(t) \in T_2 \text{ and } \epsilon \in \{\pm 1\}, \\ x_{\gamma(t)^{-1}}^{-\epsilon} & \text{if } \gamma(t) \notin T_2 \text{ and } \epsilon \in \{\pm 1\}. \end{cases}$$

Then we have $\gamma'(x_t^{-1}) = \gamma'(x_t)^{-1}$. To show that γ' is bijective, we define $\sigma : X_2^{\pm} \to X_1^{\pm}$ by

$$\sigma(x_t^{\epsilon}) = \begin{cases} x_{\gamma^{-1}(t)}^{\epsilon} & \text{if } \gamma^{-1}(t) \in T_1 \text{ and } \epsilon \in \{\pm 1\}, \\ x_{\gamma^{-1}(t)^{-1}}^{-\epsilon} & \text{if } \gamma^{-1}(t) \notin T_1 \text{ and } \epsilon \in \{\pm 1\}. \end{cases}$$

Then

$$\sigma\,\gamma'(x_t^{\epsilon}) = \begin{cases} \sigma(x_{\gamma(t)}^{\epsilon}) & \text{if } \gamma(t) \in T_2 \text{ and } \epsilon \in \{\pm 1\}, \\ \sigma(x_{\gamma(t)^{-1}}^{-\epsilon}) & \text{if } \gamma(t) \notin T_2 \text{ and } \epsilon \in \{\pm 1\}. \end{cases}$$

If $\gamma(t) \in T_2$, $\gamma^{-1}(\gamma(t)) = t \in T_1$. If $\gamma(t) \notin T_2$, $\gamma^{-1}(\gamma(t)^{-1}) = \gamma^{-1}(\gamma(t^{-1})) = t^{-1} \notin T_1$ by $(*)$. Since $\gamma(t^{-1}) = \gamma(t)^{-1}$, we have $\gamma^{-1}(s^{-1}) = \gamma^{-1}(s)^{-1}$. Hence we have

$$\sigma\,\gamma'(x_t^{\epsilon}) = \begin{cases} x_t^{\epsilon} & \text{if } \gamma(t) \in T_2 \text{ and } \epsilon \in \{\pm 1\}, \\ x_t^{\epsilon} & \text{if } \gamma(t) \notin T_2 \text{ and } \epsilon \in \{\pm 1\}, \end{cases}$$

thus $\sigma\,\gamma' = 1_{X_1^{\pm}}$. The similar argument gives $\gamma'\sigma = 1_{X_2^{\pm}}$. Thus γ' is a bijection.

Since $\psi_2(\gamma'(x_t)) = \gamma(t)$ and $t^2 = 1_{G_1}$ if and only if $\gamma(t)^2 = 1_{G_2}$, we have $\psi_1(x_t)^2 = 1_{G_1}$ if and only if $\psi_2(\gamma'(x_t))^2 = 1_{G_2}$, which establishes (B).

Since $\Psi_1\varphi_1'(x_t) = \varphi_1(t)\Psi_1$ and $\Psi_2\varphi_2'(\gamma'(x_t)) = \varphi_2(\gamma(t))\Psi_2$, we have

$$\begin{aligned} \varphi_2'(\gamma'(x_t))f'\varphi_1'(x_t)^{-1} &= \varphi_2'(\gamma'(x_t))\Psi_2^{-1}f\Psi_1\varphi_1'(x_t)^{-1} \\ &= \Psi_2^{-1}\varphi_2(\gamma(t))f\varphi_1(t)^{-1}\Psi_1 \\ &= \Psi_2^{-1}f\Psi_1 \\ &= f'. \end{aligned}$$

By Remark 2.2.2 we obtain (b).

Suppose that the statement (b) holds. Let

$$f' : F(X_1)/\psi_1^{-1}(H_1) \to F(X_2)/\psi_2^{-1}(H_2)$$

be a bijection between vertices such that $f'(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2)$ and $f'\varphi_1'(x_t) = \varphi_2'(\gamma'(x_t))f'$ for every $x_t \in X_1$. Set

$$f = \Psi_2 f' \Psi_1^{-1} : G_1/H_1 \to G_2/H_2.$$

Clearly f is bijective with $f(H_1) = H_2$.

Define $\gamma : T_1^{\pm} \to T_2^{\pm}$ by $\gamma(t^{\epsilon}) = \psi_2(\gamma'(x_t^{\epsilon}))$ for each $t \in T_1$ and $\epsilon \in \{\pm 1\}$. First we show that γ is well-defined. Suppose that $t_1^{\epsilon_1} = t_2^{\epsilon_2}$. If $\epsilon_1 = \epsilon_2$ and $t_1 = t_2$, then $\psi_2(\gamma'(x_{t_1}^{\epsilon_1})) = \psi_2(\gamma'(x_{t_2}^{\epsilon_2}))$. If $\epsilon_1 \neq \epsilon_2$, then $t_1 = t_2$. Since $\psi_2(\gamma'(x_{t_j}))^2 = 1_{G_2}$ by (B),

$$\psi_2(\gamma'(x_{t_1}^{\epsilon_1})) = \psi_2(\gamma'(x_{t_1}^{-\epsilon_1})) = \psi_2(\gamma'(x_{t_2}^{\epsilon_2})).$$

Thus γ is well-defined. Then we have $\gamma(t^{-1}) = \gamma(t)^{-1}$. Next we show that γ is bijective. We define $\rho : T_2^{\pm} \to T_1^{\pm}$ by $\rho(t^{\epsilon}) = \psi_1(\gamma'^{-1}(x_t^{\epsilon}))$ for each $t \in T_2$ and $\epsilon \in \{\pm 1\}$. Since γ' satisfies the condition (B),

$$\psi_2(x_t)^2 = 1_{G_2} \text{ if and only if } \psi_1(\gamma'^{-1}(x_t))^2 = 1_{G_1}.$$

Hence ρ is well-defined. We can easily see that $\gamma\rho = 1_{T_2^{\pm}}$ and $\rho\gamma = 1_{T_1^{\pm}}$. Hence γ is a bijection.

Since $\Psi_1\varphi_1'(x_t) = \varphi_1(t)\Psi_1$ and $\Psi_2\varphi_2'(\gamma'(x_t)) = \varphi_2(\gamma(t))\Psi_2$,

$$\begin{aligned}\varphi_2(\gamma(t))f\varphi_1(t)^{-1} &= \varphi_2(\gamma(t))\Psi_2 f'\Psi_1^{-1}\varphi_1(t)^{-1}\\ &= \Psi_2\varphi_2'(\gamma'(x_t))f'\varphi_1'(x_t)^{-1}\Psi_1^{-1}\\ &= \Psi_2 f'\Psi_1^{-1}\\ &= f.\end{aligned}$$

By Remark 2.2.2 we obtain (a). □

Lemma 2.5.4. *Let $\Gamma_i = (G_i/H_i, T_i^{\pm}, H_i)$ be Schreier coset graphs for $i \in \{1,2\}$. Then the following statements are equivalent.*

- (a) *Γ_1 is isomorphic to Γ_2 as marked labelled directed graphs by a bijection $\gamma : T_1^{\pm} \to T_2^{\pm}$ satisfying the following condition: for any $t_1, \ldots, t_k \in T_1$ and any $\epsilon_1, \ldots, \epsilon_k \in \{\pm 1\}$,*

$$(C) \quad t_1^{\epsilon_1}\cdots t_k^{\epsilon_k} = 1_{G_1} \text{ if and only if } \gamma(t_1^{\epsilon_1})\cdots\gamma(t_k^{\epsilon_k}) = 1_{G_2}.$$

- (b) *H_1 and H_2 are isomorphic in presentations $\langle X_1|R_1\rangle$ and $\langle X_2|R_2\rangle$ respectively.*

Proof. By Proposition 2.5.3, (a) is equivalent to the following statement.

- (a') Γ_1' is isomorphic to Γ_2' as marked labelled directed graphs by a bijection $\gamma' : X_1^{\pm} \to X_2^{\pm}$ such that $\gamma'(x_t^{-1}) = \gamma'(x_t)^{-1}$ for every $x_t \in X_1$ and

$$(C') \quad \psi_1(x_{t_1}^{\epsilon_1})\cdots\psi_1(x_{t_k}^{\epsilon_k}) = 1_{G_1} \text{ if and only if } \psi_2(\gamma'(x_{t_1}^{\epsilon_1}))\cdots\psi_2(\gamma'(x_{t_k}^{\epsilon_k})) = 1_{G_2}.$$

In addition we note that the following statements are equivalent.

1. There exists a bijection $\gamma' : X_1^{\pm} \to X_2^{\pm}$ with $\gamma'(x_t^{-1}) = \gamma'(x_t)^{-1}$ satisfying the condition (C').
2. There exists a group isomorphism $\delta : F(X_1) \to F(X_2)$ such that $\delta(X_1^{\pm}) = X_2^{\pm}$ and $\delta(\langle\langle R_1\rangle\rangle) = \langle\langle R_2\rangle\rangle$.

Suppose that the statement (a) holds. By the above, we may suppose that the statement (a') holds, and can take $\widetilde{\gamma'}$ as δ in (2), where

$$\widetilde{\gamma'} : F(X_1) \to F(X_2)$$

given by

$$\widetilde{\gamma'}(x_{t_1}^{\epsilon_1}\cdots x_{t_k}^{\epsilon_k}) = \gamma'(x_{t_1})^{\epsilon_1}\cdots\gamma'(x_{t_k})^{\epsilon_k}.$$

It suffices to prove that $\widetilde{\gamma'}\,(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2)$. Let

$$f' : F(X_1)/\psi_1^{-1}(H_1) \to F(X_2)/\psi_2^{-1}(H_2)$$

be a bijection between vertices which preserves marked vertices. Now, we note that for $i \in \{1,2\}$,

$$\psi_i^{-1}(H_i) = \{\, l(P) \mid P \text{ is an edge path in } \Gamma_i' \text{ from } \psi_i^{-1}(H_i) \text{ to itself}\},$$

where $l(P) = l(e_n)\dots l(e_1)$ whenever $P = e_1\dots e_n$.

Let $l(P) \in \psi_1^{-1}(H_1)$, where $e_j = (x_{t_{j-1}}^{\epsilon_{j-1}}\cdots x_{t_1}^{\epsilon_1}\psi_1^{-1}(H_1),\, x_{t_j}^{\epsilon_j})$ and $P = e_1\dots e_n$. Since $x_{t_n}^{\epsilon_n}\cdots x_{t_1}^{\epsilon_1}\psi_1^{-1}(H_1) = \beta(e_n) = \psi_1^{-1}(H_1)$, by Remark 2.2.2,

$$\begin{aligned}\widetilde{\gamma'}(l(P))\psi_2^{-1}(H_2) &= \gamma'(x_{t_n}^{\epsilon_n})\cdots\gamma'(x_{t_1}^{\epsilon_1})f'(\psi_1^{-1}(H_1))\\ &= f'(x_{t_n}^{\epsilon_n}\cdots x_{t_1}^{\epsilon_1}\psi_1^{-1}(H_1))\\ &= f'(\psi_1^{-1}(H_1))\\ &= \psi_2^{-1}(H_2).\end{aligned}$$

Thus we have $\widetilde{\gamma'}(\psi_1^{-1}(H_1)) \subset \psi_2^{-1}(H_2)$. Similarly $\widetilde{\gamma'}^{-1}(\psi_2^{-1}(H_2)) \subset \psi_1^{-1}(H_1)$, which proves $\widetilde{\gamma'}\,(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2)$.

Suppose that the statement (b) holds. There exists a bijection $\gamma' : X_1^{\pm} \to X_2^{\pm}$ with $\gamma'(x_t^{-1}) = \gamma'(x_t)^{-1}$ such that

$$\widetilde{\gamma'}\,(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2) \text{ and } \widetilde{\gamma'}(\langle\langle R_1\rangle\rangle) = \langle\langle R_2\rangle\rangle,$$

which establishes (2). Define

$$f' : F(X_1)/\psi_1^{-1}(H_1) \to F(X_2)/\psi_2^{-1}(H_2)$$

by

$$f'(g\psi_1^{-1}(H_1)) = \widetilde{\gamma'}(g)\psi_2^{-1}(H_2).$$

Since $g_2^{-1}g_1 \in \psi_1^{-1}(H_1)$ is equivalent to

$$\widetilde{\gamma'}(g_2^{-1}g_1) \in \widetilde{\gamma'}(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2),$$

f' is well-defined and an injection. Since $\widetilde{\gamma'}$ is a surjection, f' is also a surjection. Since

$$\begin{aligned}f'\varphi_1'(x_t)(g\psi_1^{-1}(H_1)) &= f'(x_t g\psi_1^{-1}(H_1))\\ &= \widetilde{\gamma'}(x_t g)\psi_2^{-1}(H_2)\\ &= \widetilde{\gamma'}(x_t)\widetilde{\gamma'}(g)\psi_2^{-1}(H_2)\\ &= \varphi_2'(\gamma'(x_t))f'(g\psi_1^{-1}(H_1)),\end{aligned}$$

we have $f'\varphi_1'(x_t) = \varphi_2'(\gamma'(x_t))f'$ for every $x_t \in X_1$. Thus Γ_1' is isomorphic to Γ_2' as marked labelled directed graphs by a bijection $\gamma' : X_1^{\pm} \to X_2^{\pm}$, which establishes (a'), i.e., (a). □

By Lemmas 2.4.3 and 2.5.4, Corollary 2.4.5, (1) in Theorem 2.4.4 and the isomorphism h_n, we obtain the following theorem.

Theorem 2.5.5. *Let $m, n \geq 2$ and q_1, $q_2 \in \mathbb{Q}$. Then the following statements are equivalent.*

(*a*) $\mathrm{Stab}_{BS(1,m)}(q_1)$ *and* $\mathrm{Stab}_{BS(1,n)}(q_2)$ *are isomorphic in presentations* $BS(1, m)$ *and* $BS(1, n)$ *respectively.*

(*b*) $m = n$ *and* $|q_1| = |q_2|$.

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