A study on constructions of modular differential equations

保型微分方程式の構成に関する研究

February, 2023

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Acknowledgements

I would like to express my greatest appreciation to Professor Hiro-Aki Narita for his advice and help. I also thank him for reviewing my thesis as the principal referee.

Furthermore, I would like to express my sincere thanks to Professor Masanobu Kaneko, Professor Takeshi Ikeda, and Professor Tsuyoshi Miezaki for their helpful comments as referees of my thesis.

I would like to thank Shuji Horinaga for his instructive comments about my research. I would also thank all members of the number theory seminar at Waseda University, particularly, Hiroshi Sakata, Takuya Aoki, Naoki Kumakawa, and Sosuke Sasaki for their great support and meaningful discussion.

Finally, I deeply appreciate my family for encouraging and supporting me.

Contents

1 Introduction

In this thesis, we study modular differential equations. In particular, we deal with two topics as follows:

- The second order modular differential equation for skew-holomorphic Jacobi forms;
- Constructions of higher order modular differential equations for several modular forms by Rankin-Cohen brackets.

1.1 The Kaneko-Zagier equation

A modular differential equation is a differential equation whose solution space is modular invariant. For a function $f(\tau)$ on the complex upper half-plane $\mathfrak{H}$, we define a slash operator

$$f\Big|_k \begin{bmatrix} a & b \\ c & d \end{bmatrix}(\tau) := (c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)$$

for $k \in \mathbb{Z}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. An elliptic modular form is invariant under the action of the slash operator by $SL_2(\mathbb{Z})$. If a solution space of a differential equation is invariant under the action of the slash operator by $SL_2(\mathbb{Z})$, we then say that the solution space is modular invariant. The study of modular differential equations originates in the study of the polynomials whose roots are the j-invariants of supersingular elliptic curves by Kaneko-Zagier [12]. Let $f' = (2\pi\sqrt{-1})^{-1} df/d\tau$ and E_k be the Eisenstein series of weight k. They introduced the following modular differential equation for elliptic modular forms:

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0. \qquad (\sharp_k)$$

We call this equation $(\sharp_k)$ the Kaneko-Zagier equation. Kaneko-Koike [9, Proposition 2] shows that the equation $(\sharp_k)$ is a unique second order modular differential equation. In general, a solution of the equation $(\sharp_k)$ is not necessarily an elliptic modular form. However, several variants of modular forms appear as the solution of the equation $(\sharp_k)$. In the case when $k=4$, the solutions of the equation $(\sharp_4)$ are the elliptic modular form $E_4(\tau)$ (cf. [9, Theorem 1]) and the mixed mock modular form $E_4(\tau)\displaystyle\int_\tau^{\sqrt{-1}\infty} \frac{\eta(\tau)^{20}}{E_4(\tau)^2}\frac{d\tau}{2\pi\sqrt{-1}}$

(cf. [8, Theorem 1]) , where $\eta(\tau)$ is the Dedekind Eta function. In the case when $k = 5$, the solution of the equation ($\sharp_5$) is the quasi-modular form of weight 6, $E_4'(\tau) = (E_2(\tau)E_4(\tau) - E_6(\tau))/3$ (cf. [9, Theorem 2]). The study of modular differential equations is recognized as an interesting topic in the field of modular forms. In fact, it has applications in other fields as well. For example, it plays an important role in the classification of vertex operator algebras (cf. [10, 11, 13]).

1.2 The modular differential equation for skew-holomorphic Jacobi forms

We focus on construction methods of modular differential equations. Kaneko-Zagier [12] introduced the equation ($\sharp_k$) as a formal eigenvalue problem by the Ramanujan-Serre differential operator. In a similar way, Kiyuna [17] introduced the modular differential equation for holomorphic Jacobi forms by the modified heat operator:

$$\begin{aligned} f^{[4]}(\tau,z) - 8mf^{[2](1)}(\tau,z) &+ \frac{(2k+1)m}{3}E_2(\tau)f^{[2]}(\tau,z) \\ &+ 16m^2 f^{(2)}(\tau,z) - \frac{4(2k+1)m^2}{3}E_2(\tau)f'(\tau,z) \\ &+ \frac{(2k-1)(2k+1)m^2}{3}E_2'(\tau)f(\tau,z) = 0. \end{aligned} \tag{$\flat_{k,m}$}$$

Here $f^{(n)} := (2\pi\sqrt{-1})^{-n}\partial^n f/\partial\tau^n$ and $f^{[n]} := (2\pi\sqrt{-1})^{-n}\partial^n f/\partial z^n$. The study of modular differential equations for holomorphic Jacobi forms has applications to the elliptic genus of Calabi-Yau manifolds (cf. [1]).

The two cases presented above are of holomorphic modular forms. We are interested in the case of non-holomorphic modular forms, which has not been studied. In this thesis, we construct modular differential equations for skew-holomorphic Jacobi forms, which are non-holomorphic modular forms of two variables. We begin with a detailed description for the case of order two given in [15]. In [19], Skoruppa introduced skew-holomorphic Jacobi forms as automorphic forms on the Jacobi group satisfying the heat equation

$$\left(8\pi\sqrt{-1}m\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2}\right)f(\tau,z) = 0.$$

By following the construction method of the previous studies, we introduce a differential operator and second order modular differential equation for skew-holomorphic Jacobi forms:

$$\ddot{f}(\tau,z) - \frac{2k+1}{12}\overline{E_2(\tau)}\dot{f}(\tau,z) + \frac{(k-\frac{1}{2})(k+\frac{1}{2})}{12}\dot{\overline{E_2(\tau)}}f(\tau,z) = 0. \qquad (\natural_k)$$

Here $\dot{f} := (-2\pi\sqrt{-1})^{-1}df/d\overline{\tau}$ and $\overline{\tau}$ denotes the complex conjugate of τ. We show the fundamental properties of the equation $(\natural_k)$. More specifically, the solution space of the equation $(\natural_k)$ is modular invariant and the differential equation is unique.

Theorem 1.1 (Theorem 3.1)**.** *Let k and m be positive integers. If a function $f(\tau,z)$ is a solution of $(\natural_k)$, then $f\Big|^{skew}_{k,m}\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\tau,z)$ and $f|_m[\lambda,\mu](\tau,z)$ are also solutions of $(\natural_k)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $(\lambda,\mu)\in\mathbb{Z}^2$.*

Theorem 1.2 (Theorem 3.2)**.** *We consider a differential equation on $\mathfrak{H}\times\mathbb{C}$:*

$$\ddot{f}(\tau,z) + A(\tau)\dot{f}(\tau,z) + B(\tau)f(\tau,z) = 0. \qquad (1.1)$$

The equation (1.1) coincides with the equation $(\natural_k)$ if it satisfies the following conditions.

- **Holomorphy**
  The functions $A(\tau), B(\tau)$ are holomorphic on $\overline{\tau}$ and bounded when $\Im(\tau)\to\infty$.
- **Modularity**
  If a function $f(\tau,z)$ is a solution of (1.1), then $f\Big|^{skew}_{k,m}\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\tau,z)$ and $f|_m[\lambda,\mu](\tau,z)$ are also solutions of (1.1) for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $(\lambda,\mu)\in\mathbb{Z}^2$.

Moreover, by the theta decomposition (cf. Proposition 2.2), we consider a relation between the solutions of the modular differential equation for skew-holomorphic Jacobi forms $(\natural_k)$ and those for elliptic modular forms $(\sharp_k)$.

Theorem 1.3 (Theorem 3.3)**.** *The function $f(\tau,z) = \sum_{\mu \pmod{2m}} h_\mu(-\overline{\tau})\theta_{m,\mu}(\tau,z)$ is a solution of the equation $(\natural_k)$ if and only if $h_\mu(-\overline{\tau})$ is a solution of the equation $(\sharp_{k-\frac{1}{2}})$ where the variable τ is replaced by $-\overline{\tau}$ for all $\mu \pmod{2m}$.*

1.3 A general construction of modular differential equations by Rankin-Cohen brackets

Rankin-Cohen brackets are typical tools in the construction of modular forms by differential operators. Firstly, Rankin [18] and Cohen [4] introduced the Rankin-Cohen bracket for elliptic modular forms (cf. Proposition 4.1). Moreover, Choie-Eholzer [3] introduced it for holomorphic Jacobi forms and holomorphic Siegel modular forms of degree two (cf. Proposition 5.1, 5.2). By using them, we construct higher order modular differential equations for several modular forms above.

Theorem 1.4 (Theorem 5.1)**.** *For a function f on $\mathfrak{H}_2$ (resp. $\mathfrak{H}$, $\mathfrak{H}\times\mathbb{C}$) and $g\in S_l^2$ (resp. M_l, $J_{l,m}^{hol}$), the equation*

$$\langle f,g\rangle_{k,l,n}=0 \ (\textit{resp. } [f,g]_{k,l,n}=0, \langle f,g\rangle_{k,m_1,l,m_2,n}=0)$$

is the modular differential equation for holomorphic Siegel modular forms of degree two (resp. elliptic modular forms, holomorphic Jacobi forms).

On the other hand, Gouvêa [7] defines the Rankin-Cohen bracket for elliptic modular forms including the Eisenstein series of weight two. Inspired by the previous studies above, we introduce the extended Rankin-Cohen brackets for holomorphic or skew-holomorphic Jacobi forms paired with elliptic modular forms including E_2 in Proposition 4.1. These provide a unified construction method of higher order modular differential equations for elliptic modular forms, holomorphic Jacobi forms, and skew-holomorphic Jacobi forms. In particular, with these extended Rankin-Cohen brackets, we can reproduce the modular differential equations we have seen above.

Theorem 1.5 (Theorem 4.1)**.** *For a function f on $\mathfrak{H}$ (resp. $\mathfrak{H}\times\mathbb{C}$, $\mathfrak{H}\times\mathbb{C}$), the equation*

$$[f,E_2]_{k,2,n}=0 \ (\textit{resp. } [f,E_2]^{hol}_{k,m,2,n}=0,\ [f,E_2]^{skew}_{k,2,n}=0)$$

is the $n+1$(resp. $2(n+1)$, $n+1$)-th order modular differential equation for elliptic modular forms (resp. holomorphic Jacobi forms, skew-holomorphic Jacobi forms).

In particular, when $n=1$, the equation

$$[f,E_2]_{k,2,1}=0 \ (\textit{resp. } [f,E_2]^{hol}_{k,m,2,1}=0,\ [f,E_2]^{skew}_{k,2,1}=0)$$

is the modular differential equation ($\sharp_k$) (resp. ($\flat_{k,m}$), ($\natural_k$)).

Furthermore, we consider the relations among the solutions of these modular differential equations. The following theorems are generalizations of [16, Theorem 5.18] and Theorem 1.3 to the cases of higher order.

Theorem 1.6 (Theorem 4.2). *The function* $f(\tau, z) = \sum_{\mu \pmod{2m}} h_\mu(\tau)\theta_{m,\mu}(\tau, z)$ *is a solution of the equation* $[f, E_2]^{hol}_{k,m,2,n} = 0$ *if and only if* $h_\mu(\tau)$ *is a solution of the equation* $[h_\mu, E_2]_{k-\frac{1}{2},2,n} = 0$ *for all* $\mu \pmod{2m}$.

Theorem 1.7 (Theorem 4.3). *The function* $f(\tau, z) = \sum_{\mu \pmod{2m}} h_\mu(-\overline{\tau})\theta_{m,\mu}(\tau, z)$ *is a solution of the equation* $[f, E_2]^{skew}_{k,2,1} = 0$ *if and only if* $h_\mu(-\overline{\tau})$ *is a solution of the equation* $[h_\mu, E_2]_{k-\frac{1}{2},2,n} = 0$ *for all* $\mu \pmod{2m}$.

These results are discussed based on [14].

We explain the outline of this thesis. In Section 2, we define holomorphic and skew-holomorphic Jacobi forms and review the notion of their theta decomposition. In Section 3, we introduce the differential operator for skew-holomorphic Jacobi forms in Proposition 3.1. By following the method of Kaneko-Zagier [12], we construct the modular differential equation ($\natural_k$) for skew-holomorphic Jacobi forms. In addition, we show the properties of the equation ($\natural_k$). In Section 4, we introduce the extended Rankin-Cohen brackets for elliptic modular forms, holomorphic or skew-holomorphic Jacobi forms paired with elliptic modular forms including E_2. By using these extended Rankin-Cohen brackets, we construct higher order modular differential equations for the modular forms discussed. In Section 5, by using other Rankin-Cohen brackets, we construct modular differential equations for holomorphic Siegel modular forms, elliptic modular forms, and holomorphic Jacobi forms.

2 Preliminaries

2.1 Holomorphic Jacobi forms and skew-holomorphic Jacobi forms

To review the definitions of holomorphic Jacobi forms and skew-holomorphic Jacobi forms, we introduce the slash operators.

Definition 2.1 (Slash operators)**.** *Let k and m be positive integers. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $(\lambda, \mu) \in \mathbb{Z}^2$, we define the slash operators for a function $f(\tau, z)$ on $\mathfrak{H} \times \mathbb{C}$:*

$$f \Bigg|_{k,m}^{hol} \begin{bmatrix} a & b \\ c & d \end{bmatrix} (\tau, z) := (c\tau + d)^{-k} \mathbf{e}\left(\frac{-cmz^2}{c\tau + d}\right) f\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right),$$

$$f \Bigg|_{k,m}^{skew} \begin{bmatrix} a & b \\ c & d \end{bmatrix} (\tau, z) := (c\overline{\tau}+d)^{-k+1}|c\tau+d|^{-1} \mathbf{e}\left(\frac{-cmz^2}{c\tau + d}\right) f\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right)$$

and

$$f|_m [\lambda, \mu] (\tau, z) := \mathbf{e}\left(\lambda^2 m\tau + 2\lambda mz\right) f(\tau, z + \lambda\tau + \mu),$$

where $\mathbf{e}(w) := \exp(2\pi\sqrt{-1}w)$ for a variable w.

Now we define holomorphic Jacobi forms (cf. [6]).

Definition 2.2. *A holomorphic Jacobi form of weight k and index m with respect to $SL_2(\mathbb{Z})$ is a complex-valued holomorphic function f on $\mathfrak{H} \times \mathbb{C}$ satisfying the following conditions:*

(I) $f \Bigg|_{k,m}^{hol} \begin{bmatrix} a & b \\ c & d \end{bmatrix} (\tau, z) = f(\tau, z)$ *for all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

(II) $f|_m [\lambda, \mu] (\tau, z) = f(\tau, z)$ *for all* $(\lambda, \mu) \in \mathbb{Z}^2$,

(III) $f(\tau, z)$ *has a Fourier expansion of the form*

$$f(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ 4mn\geq r^2}} c(n, r) q^n \zeta^r,$$

where $q := \mathbf{e}(\tau)$ and $\zeta := \mathbf{e}(z)$. The space of holomorphic Jacobi forms of weight k and index m is denoted by $J^{hol}_{k,m}$.

Next, we define skew-holomorphic Jacobi forms (cf. [19]).

Definition 2.3. *A skew-holomorphic Jacobi form of weight k and index m with respect to $SL_2(\mathbb{Z})$ is a complex-valued real-analytic function f on $\mathfrak{H} \times \mathbb{C}$ satisfying the following conditions:*

(I) $f \Big|^{skew}_{k,m} \begin{bmatrix} a & b \\ c & d \end{bmatrix} (\tau, z) = f(\tau, z)$ *for all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

(II) $f \,|_m\, [\lambda, \mu]\, (\tau, z) = f(\tau, z)$ *for all* $(\lambda, \mu) \in \mathbb{Z}^2$,

(III) $\dfrac{\partial}{\partial \overline{z}} f(\tau, z) = \left(8\pi\sqrt{-1}m \dfrac{\partial}{\partial \tau} - \dfrac{\partial^2}{\partial z^2} \right) f(\tau, z) = 0$,

(IV) $f(\tau, z)$ *has a Fourier expansion of the form*

$$f(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4mn \leq r^2}} c(n, r) \mathbf{e} \left(n\tau + \frac{r^2 - 4mn}{2m} \sqrt{-1}y + rz \right),$$

where $\tau = x + \sqrt{-1}y$*. The space of skew-holomorphic Jacobi forms of weight* k *and index* m *is denoted by* $J^{skew}_{k,m}$.

2.2 Theta decomposition

We review the notion of the theta decomposition of holomorphic or skew-holomorphic Jacobi forms (cf. [6, Theorem 5.1]).

Proposition 2.1 (Theta decomposition for holomorphic Jacobi forms)**.** *Let* $f(\tau, z) = \sum_{n,r \in \mathbb{Z}} c(n, r) q^n \zeta^r$ *be a holomorphic Jacobi form of weight* k *and index* m*. Then we can decompose* $f(\tau, z)$ *into*

$$f(\tau, z) = \sum_{\mu \pmod{2m}} h_\mu(\tau) \theta_{m,\mu}(\tau, z),$$

where

$$h_\mu(\tau) := \sum_{N \geq 0} c_\mu(N) q^{\frac{N}{4m}},$$

$$c_\mu(N) := \begin{cases} c\left(\frac{N + r^2}{4m}, r\right) & (N \equiv -r^2 \pmod{4m}, r \equiv \mu \pmod{2m}) \\ 0 & (N \not\equiv -r^2 \pmod{4m}) \end{cases}$$

and

$$\theta_{m,\mu}(\tau, z) := \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \mu \pmod{2m}}} q^{\frac{r^2}{4m}} \zeta^r$$

is the Jacobi theta function.

Skew-holomorphic Jacobi forms also have the theta decomposition like holomorphic Jacobi forms.

Proposition 2.2 (Theta decomposition for skew-holomorphic Jacobi forms)**.** *Let* $f(\tau,z)=\sum_{n,r\in\mathbb{Z}}c(n,r)\mathbf{e}\left(n\tau+\frac{r^2-4mn}{2m}\sqrt{-1}y+rz\right)$ *be a skew-holomorphic Jacobi form of weight* k *and index* m*. Then we can decompose* $f(\tau,z)$ *into*

$$f(\tau,z)=\sum_{\mu\pmod{2m}}h_\mu(-\overline{\tau})\theta_{m,\mu}(\tau,z),$$

where

$$h_\mu(-\overline{\tau})=\sum_{N\geq 0}c_\mu(-N)\mathbf{e}\left(\frac{-N}{4m}\overline{\tau}\right),$$

and

$$c_\mu(-N)=\begin{cases}c\left(\frac{N+r^2}{4m},r\right) & \text{if } N\equiv -r^2\pmod{4m}\ \text{and}\ r\equiv\mu\pmod{2m}\\ 0 & \text{if } N\not\equiv -r^2\pmod{4m}.\end{cases}$$

3 The modular differential equation for skew-holomorphic Jacobi forms

We construct the second order modular differential equation for skew-holomorphic Jacobi forms (cf. [15]). First, we discuss the method for deriving the modular differential equation ($\natural_k$). Next, we show the properties of the equation ($\natural_k$). Finally, we consider a relation between the solutions of the modular differential equation for skew-holomorphic Jacobi forms ($\natural_k$) and those for elliptic modular forms ($\sharp_k$).

3.1 Construction method

We introduce the differential operator which raises weights by two for skew-holomorphic Jacobi forms.

Proposition 3.1. *Let* k *and* m *be positive integers. We define the differential operators:*

$$\overline{\partial_k}:=\frac{-1}{2\pi\sqrt{-1}}\frac{\partial}{\partial\overline{\tau}}-\frac{2k-1}{24}\overline{E_2(\tau)}.$$

Here $\overline{E_2(\tau)}$ denotes the complex conjugate of $E_2(\tau)$. Let $f(\tau,z)$ be a skew-holomorphic Jacobi form of weight k and index m. We then have that $\overline{\partial_k}(f)$ is of weight $k+2$ and index m.

Proof. Let $f(\tau,z)$ be a skew-holomorphic Jacobi form of weight k and index m. The function $f(\tau,z)$ has a Fourier expansion of the form

$$f(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}\\ 4mn\le r^2}} c(n,r)\mathbf{e}\left(n\tau + \frac{r^2-4mn}{2m}\sqrt{-1}y + rz\right).$$

We can verify that $\overline{\partial_k}(f(\tau,z))$ is a real-analytic function and satisfies the conditions (III) and (IV) of Definition 2.3. Therefore we have to check that it satisfies the conditions (I) and (II). More precisely, we show that $\overline{\partial_k}(f(\tau,z))$ is invariant under the action of the slash operators.

We apply the differential operator $(-2\pi\sqrt{-1})^{-1}\partial/\partial\overline{\tau}$ to the slash operators, we then get

$$\left(\frac{-1}{2\pi\sqrt{-1}}\frac{\partial}{\partial\overline{\tau}}f\right)\bigg|_m [\lambda,\mu](\tau,z) = \frac{-1}{2\pi\sqrt{-1}}\frac{\partial}{\partial\overline{\tau}}\left(f\,|_m\,[\lambda,\mu]\,(\tau,z)\right) \tag{3.1}$$

and

$$\begin{aligned}
&\left(\frac{-1}{2\pi\sqrt{-1}}\frac{\partial}{\partial\overline{\tau}}f\right)\Bigg|^{\text{skew}}_{k+2,m}\begin{bmatrix} a & b\\ c & d\end{bmatrix}(\tau,z)\\
&= \frac{-1}{2\pi\sqrt{-1}}\frac{\partial}{\partial\overline{\tau}}\left(f\,\Bigg|^{\text{skew}}_{k,m}\begin{bmatrix} a & b\\ c & d\end{bmatrix}(\tau,z)\right)\\
&\quad -\frac{1}{2\pi\sqrt{-1}}\left(k-\frac{1}{2}\right)c(c\overline{\tau}+d)^{-1}f\,\Bigg|^{\text{skew}}_{k,m}\begin{bmatrix} a & b\\ c & d\end{bmatrix}(\tau,z).
\end{aligned} \tag{3.2}$$

Now we have that

$$\begin{aligned}
&(\overline{\partial_k}f)\,|_m\,[\lambda,\mu]\,(\tau,z)\\
&= \left(\frac{-1}{2\pi\sqrt{-1}}\frac{\partial}{\partial\overline{\tau}}f\right)\bigg|_m [\lambda,\mu]\,(\tau,z) - \frac{2k-1}{24}\overline{E_2(\tau)}f\,|_m\,[\lambda,\mu]\,(\tau,z)\\
&= \frac{-1}{2\pi\sqrt{-1}}\frac{\partial}{\partial\overline{\tau}}f(\tau,z) - \frac{2k-1}{24}\overline{E_2(\tau)}f(\tau,z)\\
&= \overline{\partial_k}(f(\tau,z))
\end{aligned}$$

for all $(\lambda, \mu) \in \mathbb{Z}^2$ from (3.1) and that

$$
\begin{aligned}
&\left(\overline{\partial_k} f\right) \Big|^{\text{skew}}_{k+2,m} \begin{bmatrix} a & b \\ c & d \end{bmatrix} (\tau, z) \\
&= \left(\frac{-1}{2\pi\sqrt{-1}} \frac{\partial}{\partial \overline{\tau}} f \right) \Big|^{\text{skew}}_{k+2,m} \begin{bmatrix} a & b \\ c & d \end{bmatrix} (\tau, z) - \frac{2k-1}{24} \left(\overline{E_2(\tau)} f(\tau, z) \right) \Big|^{\text{skew}}_{k+2,m} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \frac{-1}{2\pi\sqrt{-1}} \frac{\partial}{\partial \overline{\tau}} \left(f \Big|^{\text{skew}}_{k,m} \begin{bmatrix} a & b \\ c & d \end{bmatrix} (\tau, z) \right) \\
&\quad - \frac{2k-1}{24} \left((c\overline{\tau} + d)^{-2} \overline{E_2 \left(\frac{a\tau + b}{c\tau + d} \right)} + \frac{6c}{\pi\sqrt{-1}} (c\overline{\tau} + d)^{-1} \right) f(\tau, z) \\
&= \frac{-1}{2\pi\sqrt{-1}} \frac{\partial}{\partial \overline{\tau}} f(\tau, z) - \frac{2k-1}{24} \overline{E_2(\tau)} f(\tau, z) \\
&= \overline{\partial_k}(f(\tau, z))
\end{aligned}
$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ by using (3.2) and the transformation law of $\overline{E_2(\tau)}$ (see Remark 1). Then $\overline{\partial_k}(f(\tau, z))$ is a skew-holomorphic Jacobi form of weight $k + 2$ and index m. □

Remark 1. *We remark that the Eisenstein series of weight two is*

$$E_2(\tau) = 1 - 24 \sum_{n \geq 1} \left(\sum_{d|n} d \right) \mathbf{e}(n\tau)$$

and satisfies the following transformation law

$$E_2 \left(\frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 E_2(\tau) + \frac{6c}{\pi\sqrt{-1}} (c\tau + d)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$*. By taking the complex conjugate of the equations* *above, we obtain*

$$\begin{aligned}\overline{E_2(\tau)} &= 1 - 24\sum_{n\geq 1}\left(\sum_{d|n} d\right)\overline{\mathbf{e}(n\tau)}\\ &= 1 - 24\sum_{n\geq 1}\left(\sum_{d|n} d\right)\mathbf{e}(-n\overline{\tau})\\ &= E_2(-\overline{\tau})\end{aligned}$$

and

$$\overline{E_2\left(\frac{a\tau+b}{c\tau+d}\right)} = (c\overline{\tau}+d)^2\overline{E_2(\tau)} - \frac{6c}{\pi\sqrt{-1}}(c\overline{\tau}+d)$$

for $\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$.

We shall construct the modular differential equation for skew-holomorphic Jacobi forms by using the differential operator $\overline{\partial_k}$. Let $f(\tau, z)$ be a skew-holomorphic Jacobi form of weight k and index m. The constant term of $\frac{1}{\overline{E_4(\tau)}}\overline{\partial_{k+2}} \circ \overline{\partial_k}(f)$ is equal to $\frac{(2k-1)(2k+3)}{24^2}$ multiplied by the constant term of f. We then consider the formal eigenvalue problem:

$$\frac{1}{\overline{E_4(\tau)}}\overline{\partial_{k+2}} \circ \overline{\partial_k}(f(\tau, z)) = \frac{(2k-1)(2k+3)}{24^2} f(\tau, z).$$

By transforming the equation above, we get the modular differential equation

$$\ddot{f}(\tau, z) - \frac{2k+1}{12}\overline{E_2(\tau)}\dot{f}(\tau, z) + \frac{(k-\frac{1}{2})(k+\frac{1}{2})}{12}\dot{\overline{E_2(\tau)}} f(\tau, z) = 0. \qquad (\natural_k)$$

Here we denote $\dot{f} := (-2\pi\sqrt{-1})^{-1} df/d\overline{\tau}$.

3.2 Properties

The modular differential equations for elliptic modular forms and holomorphic Jacobi forms have the same properties that the solution spaces are modular invariant and that the equations are unique (cf. [9, Proposition 2], [17, Proposition 5.3]). We show that the equation ($\natural_k$) for skew-holomorphic Jacobi forms also satisfies the same properties.

Theorem 3.1. *Let k and m be positive integers. If a function $f(\tau,z)$ is a solution of $(\natural_k)$, then $f \Big|_{k,m}^{skew} \begin{bmatrix} a & b \\ c & d \end{bmatrix} (\tau,z)$ and $f\,|_m\,[\lambda,\mu]\,(\tau,z)$ are also solutions of $(\natural_k)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $(\lambda,\mu) \in \mathbb{Z}^2$.*

Theorem 3.2. *We consider a differential equation on $\mathfrak{H} \times \mathbb{C}$:*

$$\ddot{f}(\tau,z) + A(\tau)\dot{f}(\tau,z) + B(\tau)f(\tau,z) = 0. \tag{3.3}$$

The equation (3.3) coincides with the equation $(\natural_k)$ if it satisfies the following conditions.

- **Holomorphy**
  The functions $A(\tau), B(\tau)$ are holomorphic on $\overline{\tau}$ and bounded when $\Im(\tau) \to \infty$.

- **Modularity**
  If a function $f(\tau,z)$ is a solution of (3.3), then $f \Big|_{k,m}^{skew} \begin{bmatrix} a & b \\ c & d \end{bmatrix} (\tau,z)$ and $f\,|_m\,[\lambda,\mu]\,(\tau,z)$ are also solutions of (3.3) for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $(\lambda,\mu) \in \mathbb{Z}^2$.

Theorem 3.1 follows from Theorem 3.2. We show the uniqueness of the modular differential equation $(\natural_k)$.

Proof. We define

$$\psi(\tau,z) := (c\overline{\tau}+d)^{-k+1}|c\tau+d|^{-1}\mathbf{e}\left(\frac{-cmz^2}{c\tau+d}\right) f\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right).$$

Since this is a solution of (3.3) we then get

$$
\begin{aligned}
0 &= (c\overline{\tau}+d)^{-k-3}|c\tau+d|^{-1}\mathbf{e}\left(\frac{cmz^2}{c\tau+d}\right) \\
&\quad \times \left(\ddot{\psi}(\tau,z)+A(\tau)\dot{\psi}(\tau,z)+B(\tau)\psi(\tau,z)\right) \\
&= \ddot{f}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right) \\
&\quad + \left\{(c\overline{\tau}+d)^2A(\tau)+\frac{2k+1}{2\pi\sqrt{-1}}c(c\overline{\tau}+d)\right\}\dot{f}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right) \\
&\quad + \left\{(c\overline{\tau}+d)^4B(\tau)+\frac{k-\frac{1}{2}}{2\pi\sqrt{-1}}c(c\overline{\tau}+d)^3A(\tau)+\frac{\left(k-\frac{1}{2}\right)\left(k+\frac{1}{2}\right)}{(2\pi\sqrt{-1})^2}c^2(c\overline{\tau}+d)^2\right\} \\
&\quad \times f\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right).
\end{aligned}
$$

By comparing the above equation with

$$
\begin{aligned}
\ddot{f}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)+A\left(\frac{a\tau+b}{c\tau+d}\right)\dot{f}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)& \\
+B\left(\frac{a\tau+b}{c\tau+d}\right)f\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)&=0
\end{aligned}
$$

then we have

$$
A\left(\frac{a\tau+b}{c\tau+d}\right)=(c\overline{\tau}+d)^2A(\tau)+\frac{2k+1}{2\pi\sqrt{-1}}c(c\overline{\tau}+d) \tag{3.4}
$$

and

$$
\begin{aligned}
B\left(\frac{a\tau+b}{c\tau+d}\right)&=(c\overline{\tau}+d)^4B(\tau)+\frac{k-\frac{1}{2}}{2\pi\sqrt{-1}}c(c\overline{\tau}+d)^3A(\tau) \\
&\quad +\frac{\left(k-\frac{1}{2}\right)\left(k+\frac{1}{2}\right)}{(2\pi\sqrt{-1})^2}c^2(c\overline{\tau}+d)^2.
\end{aligned} \tag{3.5}
$$

By the equation (3.4) and the transformation law of $\overline{E_2(\tau)}$, we have

$$
A\left(\frac{a\tau+b}{c\tau+d}\right)+\frac{2k+1}{12}\overline{E_2\left(\frac{a\tau+b}{c\tau+d}\right)}=(c\overline{\tau}+d)^2\left(A(\tau)+\frac{2k+1}{12}\overline{E_2(\tau)}\right).
$$

Since we have assumed the anti-holomorphy of $A(\tau)$ and an elliptic modular form of weight two with respect to $SL_2(\mathbb{Z})$ is only 0, we conclude

$$A(\tau) = -\frac{2k+1}{12}\overline{E_2(\tau)}.$$

By combining the derivative of (3.4) and (3.5), we have

$$B\left(\frac{a\tau+b}{c\tau+d}\right) + \frac{k-\frac{1}{2}}{2}\dot{A}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\overline{\tau}+d)^4\left(B(\tau) + \frac{k-\frac{1}{2}}{2}\dot{A}(\tau)\right).$$

Since we have assumed the anti-holomorphy of $B(\tau)$, we conclude

$$B(\tau) = \frac{(k-\frac{1}{2})(k+\frac{1}{2})}{12}\dot{\overline{E_2(\tau)}} + c\overline{E_4(\tau)}$$

with some constant c. A direct calculation shows that if $f(\tau, z)$ is a solution of

$$\ddot{f}(\tau, z) - \frac{2k+1}{12}\overline{E_2(\tau)}\dot{f}(\tau, z) + \left(\frac{(k-\frac{1}{2})(k+\frac{1}{2})}{12}\dot{\overline{E_2(\tau)}} + c\overline{E_4(\tau)}\right) f(\tau, z) = 0,$$

the function $f(\tau, z)\overline{\Delta(\tau)}^{\beta}$ is then a solution of $(\natural_{k+12\beta})$, where β is a solution of

$$\beta^2 + \frac{2k+1}{12}\beta + c = 0.$$

Hence we can take $c = 0$ without loss of generality and we conclude that the equation $(\natural_k)$ is unique. □

Remark 2. *In terms of comparison with the cases of elliptic modular forms and holomorphic Jacobi forms, we have two remarks as follows:*

1. *The equation $(\natural_k)$ is second order ordinary differential equation and coincides with the equation $(\sharp_{k-\frac{1}{2}})$, where the variable τ is replaced by $-\overline{\tau}$.*

2. *The equation $(\natural_k)$ does not depend on index m. It is different from the equation $(\flat_{k,m})$ for holomorphic Jacobi forms.*

3.3 Relation between the equation ($\natural_k$) and ($\sharp_k$)

Now we show a relation between the solutions of the modular differential equation for skew-holomorphic Jacobi forms ($\natural_k$) and those for elliptic modular forms ($\sharp_k$).

Theorem 3.3. *The function $f(\tau,z)=\sum_{\mu \pmod{2m}} h_\mu(-\overline{\tau})\theta_{m,\mu}(\tau,z)$ is a solution of the equation ($\natural_k$) if and only if $h_\mu(-\overline{\tau})$ is a solution of the equation ($\sharp_{k-\frac{1}{2}}$), where the variable τ is replaced by $-\overline{\tau}$ for all $\mu \pmod{2m}$.*

Proof. We remark that $\{\theta_{m,\mu}(\tau,z)$ for $\mu \mod 2m\}$ are holomorphic on the variable τ and linearly independent. We substitute $f(\tau,z)$ by $\sum_{\mu \pmod{2m}} h_\mu(-\overline{\tau})\theta_{m,\mu}(\tau,z)$ in the equation ($\natural_k$), we then get the theorem above. □

4 A general construction of modular differential equations by Rankin-Cohen brackets

Some results of Sections 4 and 5 are summarized in [14].

4.1 Extended Rankin-Cohen brackets

Gouvêa [7, p.199] defines the extended Rankin-Cohen bracket for elliptic modular forms including E_2. Inspired by the previous studies above, we introduce the extended Rankin-Cohen brackets for holomorphic or skew-holomorphic Jacobi forms paired with elliptic modular forms including E_2.

Proposition 4.1. *Let f be a function on $\mathfrak{H}$ or $\mathfrak{H}\times\mathbb{C}$ and g be a function on $\mathfrak{H}$. For $n\geq -1$, we define the extended Rankin-Cohen brackets by*

- $$[f,g]_{k,l,n}(\tau) := \sum_{r+s=n} (-1)^r \binom{n+k-1}{s}\binom{n+l-1}{r} D_\tau^r(f(\tau))D_\tau^s(g(\tau)),$$
- $$[f,g]^{hol}_{k,m,l,n}(\tau,z) := \sum_{r+s=n} \binom{n+(k-\frac{1}{2})-1}{s}\binom{n+l-1}{r} \times(-4m)^s L_m^r(f(\tau,z))D_\tau^s(g(\tau)),$$

- $[f,g]^{skew}_{k,l,n}(\tau,z) := \sum_{r+s=n} (-1)^r \binom{n+(k-\frac{1}{2})-1}{s} \binom{n+l-1}{r}$

$$\times D^r_{-\overline{\tau}}(f(\tau,z)) D^s_{-\overline{\tau}}(\overline{g(\tau)}),$$

where the notations are given as follows:

1. $0 \le r \le n+1$, $-1 \le s \le n$.
2. $\binom{p}{-1} := \frac{12}{p+1}$.
3. $D_w := (2\pi\sqrt{-1})^{-1} d/dw$ *for a variable* w.
4. $L_m = 4mD_\tau - D_z^2$ *is the heat operator.*
5. $D_\tau^{-1}(g(\tau)) = D^{-1}_{-\overline{\tau}}(\overline{g(\tau)}) := \begin{cases} 0 & \text{if } g \text{ is an elliptic modular form} \\ 1 & \text{if } g = E_2. \end{cases}$

We have the following results.

(I) *If* f *is an elliptic modular form of weight* k *and* g *is either an elliptic modular form of weight* l *or* $E_2(l=2)$, $[f,g]_{k,l,n}$ *is an elliptic modular form of weight* $k+l+2n$.

(II) *If* f *is a holomorphic Jacobi form of weight* k *and index* m *and* g *is either an elliptic modular form of weight* l *or* $E_2(l=2)$, $[f,g]^{hol}_{k,m,l,n}$ *is a holomorphic Jacobi form of weight* $k+l+2n$ *and index* m.

(III) *If* f *is a skew-holomorphic Jacobi form of weight* k *and index* m *and* g *is either an elliptic modular form of weight* l *or* $E_2(l=2)$, $[f,g]^{skew}_{k,l,n}$ *is a skew-holomorphic Jacobi form of weight* $k+l+2n$ *and index* m.

Proof. Firstly, we consider the case of holomorphic Jacobi forms. Since $f(\tau,z)$ and $g(\tau)$ are holomorphic, $[f,g]^{\mathrm{hol}}_{k,m,l,n}(\tau,z)$ are also holomorphic. We have to show the following transformation laws.

1. For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

$$[f,g]^{\mathrm{hol}}_{k,m,l,n} \Bigg|^{\mathrm{hol}}_{k+l+2n,m} \begin{bmatrix} a & b \\ c & d \end{bmatrix} (\tau,z) = [f,g]^{\mathrm{hol}}_{k,m,l,n}(\tau,z). \tag{4.1}$$

2. For any $(\lambda, \mu) \in \mathbb{Z}^2$,

$$[f, g]^{\mathrm{hol}}_{k,m,l,n} |_m [\lambda, \mu] (\tau, z) = [f, g]^{\mathrm{hol}}_{k,m,l,n}(\tau, z). \tag{4.2}$$

Since $L^r_m(f |_m [\lambda, \mu]) = L^r_m(f) |_m [\lambda, \mu]$, we get the equation (4.2). For $f(\tau, z) \in J^{\mathrm{hol}}_{k,m}$, we introduce the generating function $\tilde{f}(\tau, z; X)$ as

$$\tilde{f}(\tau, z; X) = \sum_{n=0}^{\infty} \frac{L^n_m(f(\tau, z))}{n!(n + (k - 1/2) - 1)!}((2\pi\sqrt{-1})^2 X)^n.$$

The generating function $\tilde{f}(\tau, z; X)$ satisfies the following transformation law

$$\tilde{f}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}; \frac{X}{(c\tau + d)^2}\right) = (c\tau + d)^k \mathbf{e}\left(\frac{cmz^2}{c\tau + d}\right) e^{8\pi\sqrt{-1}m\frac{cX}{c\tau+d}} \tilde{f}(\tau, z; X) \tag{4.3}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ (for detail, see [2, Lemma 3.2]). For either an elliptic modular form g of weight l or $E_2 (l = 2)$, we introduce the generating function $\tilde{g}(\tau; X)$ as

$$\tilde{g}(\tau; X) = \sum_{n=-1}^{\infty} \frac{D^n_\tau(g(\tau))}{n!(n + l - 1)!}(2\pi\sqrt{-1}X)^n.$$

Here we define $(-1)! = 1/12$. The generating function $\tilde{g}(\tau; X)$ satisfies the following transformation law

$$\tilde{g}\left(\frac{a\tau + b}{c\tau + d}; \frac{X}{(c\tau + d)^2}\right) = (c\tau + d)^l e^{\frac{cX}{c\tau+d}} \tilde{g}(\tau; X) \tag{4.4}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ (for detail, see [7, p.198]). From the equations (4.3) and (4.4), we have that

$$\begin{aligned} &\tilde{f}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}; \frac{X}{(c\tau + d)^2}\right) \tilde{g}\left(\frac{a\tau + b}{c\tau + d}; \frac{-8\pi\sqrt{-1}mX}{(c\tau + d)^2}\right) \\ &\quad = (c\tau + d)^{k+l} \mathbf{e}\left(\frac{cmz^2}{c\tau + d}\right) \tilde{f}(\tau; X) \tilde{g}(\tau; -8\pi\sqrt{-1}mX). \end{aligned} \tag{4.5}$$

Since

$$\tilde{f}(\tau, z; X)\tilde{g}(\tau; -8\pi\sqrt{-1}mX) = \sum_{n=-1}^{\infty} \frac{[f, g]^{\mathrm{hol}}_{k,m,l,n}(\tau, z)}{(n + (k - 1/2) - 1)!(n + l - 1)!}((2\pi\sqrt{-1})^2 X)^n$$

and the equation (4.5), we get the equation (4.1).

Secondly, we prove the case of skew-holomorphic Jacobi forms in a way similar to the case of holomorphic Jacobi forms. Since $f(\tau,z)$ is real-analytic and $\overline{g(\tau)}$ is anti-holomorphic, $[f,g]^{\text{skew}}_{k,l,n}(\tau,z)$ is real-analytic. We have to show the following transformation laws.

1. For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

$$[f,g]^{\text{skew}}_{k,l,n}\Bigg|^{\text{skew}}_{k+l+2n,m}\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\tau,z) = [f,g]^{\text{skew}}_{k,l,n}(\tau,z). \tag{4.6}$$

2. For any $(\lambda,\mu) \in \mathbb{Z}^2$,

$$[f,g]^{\text{skew}}_{k,l,n}\,|_m\,[\lambda,\mu]\,(\tau,z) = [f,g]^{\text{skew}}_{k,l,n}(\tau,z). \tag{4.7}$$

Since $D^n_{-\overline{\tau}}(f\,|_m\,[\lambda,\mu]\,) = D^n_{-\overline{\tau}}(f)\,|_m\,[\lambda,\mu]$, we get the equation (4.7). For $f(\tau,z) \in J^{\text{skew}}_{k,m}$, we introduce the generating function $\tilde{f}(\tau,z;X)$ as

$$\tilde{f}(\tau,z;X) = \sum_{n=0}^{\infty} \frac{D^n_{-\overline{\tau}}(f(\tau,z))}{n!(n+(k-1/2)-1)!}(2\pi\sqrt{-1}X)^n.$$

The generating function $\tilde{f}(\tau,z;X)$ satisfies the following transformation law

$$\tilde{f}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}; \frac{X}{(c\tau+d)^2}\right) = (c\overline{\tau}+d)^{k-1}|c\tau+d|\mathbf{e}\left(\frac{cmz^2}{c\tau+d}\right)e^{\frac{cX}{c\overline{\tau}+d}}\tilde{f}(\tau,z;X) \tag{4.8}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. For either an elliptic modular form g of weight l or $E_2 (l=2)$, the generating function $\overline{\tilde{g}(\tau;X)}$ satisfies the following transformation law

$$\overline{\tilde{g}\left(\frac{a\tau+b}{c\tau+d}; \frac{X}{(c\tau+d)^2}\right)} = (c\overline{\tau}+d)^l e^{\frac{cX}{c\overline{\tau}+d}}\overline{\tilde{g}(\tau;X)} \tag{4.9}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. From the equations (4.8) and (4.9), we have that

$$\begin{aligned} &\tilde{f}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}; \frac{-X}{(c\tau+d)^2}\right)\overline{\tilde{g}\left(\frac{a\tau+b}{c\tau+d}; \frac{X}{(c\tau+d)^2}\right)} \\ &= (c\overline{\tau}+d)^{k+l-1}|c\tau+d|\mathbf{e}\left(\frac{cmz^2}{c\tau+d}\right)\tilde{f}(\tau,z;-X)\overline{\tilde{g}(\tau;X)}. \end{aligned} \tag{4.10}$$

Since

$$\tilde{f}(\tau, z; -X)\overline{\tilde{g}(\tau; X)} = \sum_{n=-1}^{\infty} \frac{[f, g]^{\mathrm{skew}}_{k,l,n}(\tau, z)}{(n + (k-1/2) - 1)!(n + l - 1)!}(2\pi\sqrt{-1}X)^n$$

and the equation (4.10), we get the equation (4.6). □

4.2 Construction of modular differential equations

By using extended Rankin-Cohen brackets in Proposition 4.1, we can review the following modular differential equations.

For a function f on $\mathfrak{H}$,

$$[f, E_2]_{k,2,1}(\tau) = \frac{12}{k+1}\left(f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau)\right).$$

This coincides with the left-hand side of ($\sharp_k$).

For a function f on $\mathfrak{H} \times \mathbb{C}$,

$$\begin{aligned}[f, E_2]^{\mathrm{hol}}_{k,m,2,1}(\tau, z) = -\frac{3}{m((k-1/2)+1)}\Bigg(& f^{[4]}(\tau, z) - 8mf^{[2](1)}(\tau, z) \\ & + \frac{(2k+1)m}{3}E_2(\tau)f^{[2]}(\tau, z) + 16m^2 f^{(2)}(\tau, z) \\ & - \frac{4(2k+1)m^2}{3}E_2(\tau)f'(\tau, z) + \frac{(2k-1)(2k+1)m^2}{3}E_2'(\tau)f(\tau, z)\Bigg).\end{aligned}$$

This coincides with the left-hand side of ($\flat_{k,m}$).

For a function f on $\mathfrak{H} \times \mathbb{C}$,

$$\begin{aligned}[f, E_2]^{\mathrm{skew}}_{k,2,1}(\tau, z) = \frac{12}{(k-1/2)+1}\Bigg(& \ddot{f}(\tau, z) - \frac{2k+1}{12}\overline{E_2(\tau)}\dot{f}(\tau, z) \\ & + \frac{(k-\frac{1}{2})(k+\frac{1}{2})}{12}\dot{\overline{E_2(\tau)}}f(\tau, z)\Bigg).\end{aligned}$$

This coincides with the left-hand side of ($\natural_k$).
By generalizing these, we construct higher order modular differential equations.

Theorem 4.1. *For a function f on $\mathfrak{H}$ (resp. $\mathfrak{H}\times\mathbb{C}$, $\mathfrak{H}\times\mathbb{C}$), the equation*

$$[f,E_2]_{k,2,n}=0\ (\textit{resp. } [f,E_2]^{hol}_{k,m,2,n}=0,\ [f,E_2]^{skew}_{k,2,n}=0)$$

is the $n+1$(resp. $2(n+1)$, $n+1$)-th order modular differential equation for elliptic modular forms (resp. holomorphic Jacobi forms, skew-holomorphic Jacobi forms).

In particular, when $n=1$, the equation

$$[f,E_2]_{k,2,1}=0\ (\textit{resp. } [f,E_2]^{hol}_{k,m,2,1}=0,\ [f,E_2]^{skew}_{k,2,1}=0)$$

is the modular differential equation ($\sharp_k$) (resp. ($\flat_{k,m}$), ($\natural_k$)).

Proof. Firstly, we consider the case of holomorphic Jacobi forms. We have to show that the solution space of the equation $[f,E_2]^{\mathrm{hol}}_{k,m,2,n}=0$ is modular invariant. More precisely, if a function f is a solution of $[f,E_2]^{\mathrm{hol}}_{k,m,2,n}(\tau,z)=0$, then

$$\left[f\,\Big|^{\mathrm{hol}}_{k,m}\begin{bmatrix} a & b\\ c & d\end{bmatrix}(\tau,z), E_2(\tau)\right]^{\mathrm{hol}}_{k,m,2,n}=0 \tag{4.11}$$

and

$$[f\,|_m\,[\lambda,\mu]\,(\tau,z),E_2(\tau)]^{\mathrm{hol}}_{k,m,2,n}=0 \tag{4.12}$$

for all $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$ and $(\lambda,\mu)\in\mathbb{Z}^2$. Since $L^r_m(f\,|_m\,[\lambda,\mu])=L^r_m(f)\,|_m\,[\lambda,\mu]$,

$$[f\,|_m\,[\lambda,\mu]\,(\tau,z),E_2(\tau)]^{\mathrm{hol}}_{k,m,2,n}=\mathbf{e}\left(\lambda^2m\tau+2\lambda mz\right)[f,E_2]^{\mathrm{hol}}_{k,m,2,n}(\tau,z+\lambda\tau+\mu)=0.$$

Then the equation (4.12) holds.

Now we calculate

$$\left[f\,\Big|^{\mathrm{hol}}_{k,m}\begin{bmatrix} a & b\\ c & d\end{bmatrix}(\tau,z), E_2(\tau)\right]^{\mathrm{hol}}_{k,m,2,n} \quad\text{and}\quad [f,E_2]_{k,m,2,n}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right).$$

Since

$$L^r_m\left(f\,\Big|^{\mathrm{hol}}_{k,m}\begin{bmatrix} a & b\\ c & d\end{bmatrix}(\tau,z)\right)=\sum_{l=0}^{r}(-1)^{r-l}\frac{r!}{l!}\binom{(k-1/2)+r-1}{r-l}\frac{(4mc)^{r-l}}{(2\pi\sqrt{-1})^{r-l}}$$
$$\times\frac{1}{(c\tau+d)^{r-l}}L^l_m(f)\,\Big|^{\mathrm{hol}}_{k+2l,m}\begin{bmatrix} a & b\\ c & d\end{bmatrix}(\tau,z),$$

we get

$$
\begin{aligned}
&(c\tau+d)^{k+2n+2}\mathbf{e}\left(\frac{cmz^2}{c\tau+d}\right)\left[f\,\Big|^{\mathrm{hol}}_{k,m}\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\tau,z), E_2(\tau)\right]^{\mathrm{hol}}_{k,m,2,n} \\
&= \sum_{r+s=n}\binom{n+(k-1/2)-1}{s}\binom{n+1}{r}(-4m)^s\sum_{l=0}^{r}(-1)^{r-l}\frac{r!}{l!}\binom{(k-1/2)+r-1}{r-l} \\
&\quad\times\frac{(4mc)^{r-l}}{(2\pi\sqrt{-1})^{r-l}}(c\tau+d)^{2n-r-l+2}L_m^l(f)\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)D_\tau^s(E_2(\tau)). \quad (4.13)
\end{aligned}
$$

The coefficient of $L_m^p(f)\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)$ $(0\le p\le n+1)$ in the equation (4.13) is

$$
\begin{aligned}
&\sum_{r=p}^{n+1}\binom{n+(k-1/2)-1}{n-r}\binom{n+1}{r}\binom{(k-1/2)+r-1}{r-p}\frac{r!}{p!}(-4m)^{n-p} \\
&\qquad\times\left(\frac{c}{2\pi\sqrt{-1}}\right)^{r-p}(c\tau+d)^{2n-r-p+2}D_\tau^{n-r}(E_2(\tau)) \\
&=\sum_{l=-1}^{n-p}\binom{n+(k-1/2)-1}{l}\binom{n+1}{n-l}\binom{(k-1/2)+n-l-1}{n-p-l}\frac{(n-l)!}{p!}(-4m)^{n-p} \\
&\qquad\times\left(\frac{c}{2\pi\sqrt{-1}}\right)^{n-p-l}(c\tau+d)^{n-p+l+2}D_\tau^{l}(E_2(\tau)). \quad (4.14)
\end{aligned}
$$

Here we set $l=n-r$. On the other hand,

$$
\begin{aligned}
&[f,E_2]_{k,m,2,n}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right) \\
&=\sum_{r+s=n}\binom{n+(k-1/2)-1}{s}\binom{n+1}{r}(-4m)^sL_m^r(f)\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right) \\
&\quad\times D_\tau^s(E_2)\left(\frac{a\tau+b}{c\tau+d}\right) \\
&=\sum_{r+s=n}\binom{n+(k-1/2)-1}{s}\binom{n+1}{r}(-4m)^sL_m^r(f)\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right) \\
&\quad\times\left(\sum_{l=-1}^{s}\frac{s!}{l!}\binom{s+1}{s-l}\left(\frac{c}{2\pi\sqrt{-1}}\right)^{s-l}(c\tau+d)^{s+l+2}D_\tau^l(E_2(\tau))\right). \quad (4.15)
\end{aligned}
$$

The coefficient of $L_m^p(f)\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)$ $(0 \le p \le n+1)$ in the equation (4.15) is

$$\sum_{l=-1}^{n-p} \binom{n+(k-1/2)-1}{n-p}\binom{n+1}{p}\binom{n-p+1}{n-p-l}\frac{(n-p)!}{l!}(-4m)^{n-p} \times \left(\frac{c}{2\pi\sqrt{-1}}\right)^{n-p-l}(c\tau+d)^{n-p+l+2}D_\tau^l(E_2(\tau)). \quad (4.16)$$

From the direct calculation, we have

$$\begin{aligned}
&\binom{n+(k-1/2)-1}{l}\binom{n+1}{n-l}\binom{(k-1/2)+n-l-1}{n-p-l}\frac{(n-l)!}{p!}\\
&= \frac{(n+(k-1/2)-1)!}{l!(n+(k-1/2)-l-1)!}\frac{(n+1)!}{(n-l)!(l+1)!}\frac{((k-1/2)+n-l-1)!}{(n-l-p)!((k-1/2)+p-1)!}\\
&\quad\times\frac{(n-l)!}{p!}\frac{(n-p)!}{(n-p)!}\frac{(n-p+1)!}{(n-p+1)!}\\
&= \frac{(n+(k-1/2)-1)!}{(n-p)!((k-1/2)+p-1)!}\frac{(n+1)!}{p!(n-p+1)!}\frac{(n-p+1)!}{(n-p-l)!(l+1)!}\frac{(n-p)!}{l!}\\
&= \binom{n+(k-1/2)-1}{n-p}\binom{n+1}{p}\binom{n-p+1}{n-p-l}\frac{(n-p)!}{l!}.
\end{aligned} \quad (4.17)$$

Since the equation (4.14) and (4.16) are equal by the equation (4.17), we get

$$(c\tau+d)^{k+2n+2}\mathbf{e}\left(\frac{cmz^2}{c\tau+d}\right)\left[f\Big|_{k,m}^{\mathrm{hol}}\begin{bmatrix} a & b\\ c & d\end{bmatrix}(\tau,z), E_2(\tau)\right]_{k,m,2,n}^{\mathrm{hol}} = [f, E_2]_{k,m,2,n}^{\mathrm{hol}}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = 0.$$

Then the equation (4.11) holds.

Secondly, we prove the case of skew-holomorphic Jacobi forms in a way similar to the case of holomorphic Jacobi forms. We have to show that the solution space of the equation $[f, E_2]_{k,2,n}^{\mathrm{skew}} = 0$ is modular invariant. More precisely, if a function f is a solution of $[f, E_2]_{k,2,n}^{\mathrm{skew}}(\tau, z) = 0$, then

$$\left[f\Big|_{k,m}^{\mathrm{skew}}\begin{bmatrix} a & b\\ c & d\end{bmatrix}(\tau,z), E_2(\tau)\right]_{k,2,n}^{\mathrm{skew}} = 0 \quad (4.18)$$

and

$$[f\,|_m\,[\lambda,\mu]\,(\tau,z),E_2(\tau)]^{\text{skew}}_{k,2,n}=0 \tag{4.19}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $(\lambda,\mu)\in\mathbb{Z}^2$. Since $D^r_{-\overline{\tau}}(f\,|_m\,[\lambda,\mu]) = D^r_{-\overline{\tau}}(f)\,|_m\,[\lambda,\mu]$,

$$[f\,|_m\,[\lambda,\mu]\,(\tau,z),E_2(\tau)]^{\text{skew}}_{k,2,n}=\mathbf{e}\left(\lambda^2 m\tau+2\lambda m z\right)[f,E_2]^{\text{skew}}_{k,2,n}(\tau,z+\lambda\tau+\mu)=0.$$

Then the equation (4.19) holds.

Now we calculate

$$\left[f\,\Bigg|^{\text{skew}}_{k,m}\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\tau,z),E_2(\tau)\right]^{\text{skew}}_{k,2,n} \quad\text{and}\quad [f,E_2]_{k,2,n}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right).$$

Since

$$D^r_{-\overline{\tau}}\left(f\,\Bigg|^{\text{skew}}_{k,m}\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\tau,z)\right)=\sum_{l=0}^{r}(-1)^{r-l}\frac{r!}{l!}\binom{(k-1/2)+r-1}{r-l}\frac{c^{r-l}}{(2\pi\sqrt{-1})^{r-l}}$$
$$\times\frac{1}{(c\overline{\tau}+d)^{r-l}}D^l_{-\overline{\tau}}(f)\,\Bigg|^{\text{skew}}_{k+2l,m}\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\tau,z),$$

we get

$$(c\overline{\tau}+d)^{k+2n+1}|c\tau+d|\mathbf{e}\left(\frac{cmz^2}{c\tau+d}\right)\left[f\,\Bigg|^{\text{skew}}_{k,m}\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\tau,z),E_2(\tau)\right]^{\text{skew}}_{k,2,n}$$
$$=\sum_{r+s=n}\binom{n+(k-1/2)-1}{s}\binom{n+1}{r}(-1)^s\sum_{l=0}^{r}(-1)^{r-l}\frac{r!}{l!}\binom{(k-1/2)+r-1}{r-l}$$
$$\times\frac{c^{r-l}}{(2\pi\sqrt{-1})^{r-l}}(c\overline{\tau}+d)^{2n-r-l+2}D^l_{-\overline{\tau}}(f)\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)D^s_{-\overline{\tau}}(\overline{E_2(\tau)}).$$

By comparing the above equations, we then get

$$(c\overline{\tau}+d)^{k+2n+1}|c\tau+d|\mathbf{e}\left(\frac{cmz^2}{c\tau+d}\right)\left[f\,\Bigg|^{\text{skew}}_{k,m}\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\tau,z),E_2(\tau)\right]^{\text{skew}}_{k,2,n}$$
$$=[f,E_2]^{\text{skew}}_{k,2,n}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=0.$$

In the case of elliptic modular forms, we can also show that the solution space is modular invariant by a similar calculation. □

4.3 Relations between higher order modular differential equations

We show relations between solutions of the modular differential equations for holomorphic or skew-holomorphic Jacobi forms and those for elliptic modular forms. The following theorems are generalizations of [16, Theorem 5.18] and Theorem 3.3 to the cases of higher order.

Theorem 4.2. *The function* $f(\tau,z)=\sum_{\mu \pmod{2m}} h_\mu(\tau)\theta_{m,\mu}(\tau,z)$ *is a solution of the equation* $[f,E_2]^{hol}_{k,m,2,n}=0$ *if and only if* $h_\mu(\tau)$ *is a solution of the equation* $[h_\mu,E_2]_{k-\frac{1}{2},2,n}=0$ *for all* $\mu \pmod{2m}$.

Proof. We remark that

$$L_m^r\left(\sum_{\mu \pmod{2m}} h_\mu(\tau)\theta_{m,\mu}(\tau,z)\right)=(4m)^r\sum_{\mu \pmod{2m}} D_\tau^r(h_\mu(\tau))\theta_{m,\mu}(\tau,z)$$

and $\{\theta_{m,\mu}(\tau,z)$ for $\mu \mod 2m\}$ are linearly independent. By substituting $f(\tau,z)$ by $\sum_{\mu \pmod{2m}} h_\mu(\tau)\theta_{m,\mu}(\tau,z)$ in $[f,E_2]^{\mathrm{hol}}_{k,m,2,n}$,

$$\left[\sum_{\mu \pmod{2m}} h_\mu(\tau)\theta_{m,\mu}(\tau,z),E_2(\tau)\right]^{\mathrm{hol}}_{k,m,2,n}=(4m)^n[h_\mu,E_2]_{k-\frac{1}{2},2,n}(\tau)\theta_{m,\mu}(\tau,z).$$

We then get the theorem above. □

Theorem 4.3. *The function* $f(\tau,z)=\sum_{\mu \pmod{2m}} h_\mu(-\overline{\tau})\theta_{m,\mu}(\tau,z)$ *is a solution of the equation* $[f,E_2]^{skew}_{k,2,1}=0$ *if and only if* $h_\mu(-\overline{\tau})$ *is a solution of the equation* $[h_\mu,E_2]_{k-\frac{1}{2},2,n}=0$ *for all* $\mu \pmod{2m}$.

Proof. We remark that

$$D_{-\overline{\tau}}^r\left(\sum_{\mu \pmod{2m}} h_\mu(-\overline{\tau})\theta_{m,\mu}(\tau,z)\right)=\sum_{\mu \pmod{2m}} D_{-\overline{\tau}}^r(h_\mu(-\overline{\tau}))\theta_{m,\mu}(\tau,z).$$

By substituting $f(\tau,z)$ by $\sum_{\mu \pmod{2m}} h_\mu(-\overline{\tau})\theta_{m,\mu}(\tau,z)$ in $[f,E_2]^{\mathrm{skew}}_{k,2,n}$,

$$\left[\sum_{\mu \pmod{2m}} h_\mu(-\overline{\tau})\theta_{m,\mu}(\tau,z),\overline{E_2(\tau)}\right]^{\mathrm{skew}}_{k,2,n}=[h_\mu,E_2]_{k-\frac{1}{2},2,n}(-\overline{\tau})\theta_{m,\mu}(\tau,z).$$

We then get the theorem above. □

5 Remarks on the construction of modular differential equations

We review the definition of holomorphic Siegel modular forms. Let $M_n(K)$ denote the set of all $n \times n$ matrices over a field K. We define the Siegel upper half-plane of degree n as

$$\mathfrak{H}_n := \{Z = X + \sqrt{-1}Y \in M_n(\mathbb{C}) | X, Y \in M_n(\mathbb{R}), Z = {}^t Z, Y > 0\}.$$

Let $Sp(2n, \mathbb{Z})$ be the symplectic group of size $2n$ over $\mathbb{Z}$. For a holomorphic function $f(Z)$ on $\mathfrak{H}_n$ and $M \in Sp(2n, \mathbb{Z})$, we define a slash operator

$$(f\,|_k M)(Z) := \det(CZ + D)^{-k} f\left((AZ + B)(CZ + D)^{-1}\right),$$

where $Z \in \mathfrak{H}_n$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $n \times n$ matrices A, B, C, D.

Definition 5.1 (holomorphic Siegel modular forms)**.** *For $n \geq 2$, a holomorphic function f on $\mathfrak{H}_n$ is a holomorphic Siegel modular form of degree n and weight k if*

$$(f\,|_k M)(Z) = f(Z) \text{ for all } M \in Sp(2n, \mathbb{Z}).$$

The space of holomorphic Siegel modular forms of weight k and degree n is denoted by S_k^n.

Choie-Eholzer [3, Theorem 1.3] introduced the Rankin-Cohen bracket for holomorphic Jacobi forms.

Proposition 5.1. *For $n \geq 0$, we define the Rankin-Cohen bracket $\langle f, g\rangle_{k,m_1,l,m_2,n}$ for holomorphic Jacobi forms as*

$$\langle f, g\rangle_{k,m_1,l,m_2,n}(\tau, z) = \sum_{r+s+p=n} C_{r,s,p}(k,l) L^p_{m_1+m_2}(L^r_{m_1}(f(\tau, z)) L^s_{m_2}(g(\tau, z))).$$

The coefficients $C_{r,s,p}(k,l)$ are given by

$$C_{r,s,p}(k,l) = \frac{(n-r)!(n-s)!(n-p)!}{r!s!p!} \times \binom{k - \frac{3}{2} + n}{n - r} \binom{l - \frac{3}{2} + n}{n - s} \binom{-(k+l) + \frac{3}{2} + n}{n - p}.$$

If f and g are holomorphic Jacobi forms of weight and index k, m_1 and l, m_2 respectively, $\langle f, g\rangle_{k,m_1,l,m_2,n}$ is a holomorphic Jacobi form of weight $k + l + 2n$ and index $m_1 + m_2$.

Choie-Eholzer [3, Theorem 1.4] introduced the Rankin-Cohen bracket for holomorphic Siegel modular forms of degree two by using the Rankin-Cohen bracket for holomorphic Jacobi forms and the Fourier-Jacobi expansion. We remark that Eholzer-Ibukiyama [5] introduced the same Rankin-Cohen bracket for holomorphic Siegel modular forms in a different way.

Proposition 5.2. *For $n \geq 0$, we define the Rankin-Cohen bracket $\langle f, g\rangle_{k,l,n}$ for holomorphic Siegel modular forms of degree two as*

$$\langle f, g\rangle_{k,l,n}(Z) = \sum_{r+s+p=n} C_{r,s,p}(k,l)\mathbb{D}^p(\mathbb{D}^r(f(Z))\mathbb{D}^s(g(Z))).$$

The differential operator $\mathbb{D} = 4\frac{\partial}{\partial \tau_1}\frac{\partial}{\partial \tau_2} - \frac{\partial^2}{\partial z^2}$, where the variable $Z = \begin{pmatrix} \tau_1 & z \\ z & \tau_2 \end{pmatrix} \in \mathfrak{H}_n$. If f and g are holomorphic Siegel modular forms of degree two and weight k and l respectively, $\langle f, g\rangle_{k,l,n}$ is a holomorphic Siegel modular form of degree two and weight $k+l+2n$.

Rankin-Cohen brackets for elliptic modular forms, holomorphic Jacobi forms, and holomorphic Siegel modular forms satisfy the transformation laws stated in the following three lemmas.

Lemma 5.1. *Let f and g be holomorphic functions on $\mathfrak{H}$ and $[f, g]_{k,l,n}$ be the Rankin-Cohen bracket for elliptic modular forms defined in Proposition 4.1. For $\gamma \in SL_2(\mathbb{Z})$,*

$$[f, g]_{k,l,n} \,|_{k+l+2n}\gamma = [f\,|_k\gamma\,, g\,|_l\gamma]_{k,l,n}\,.$$

Lemma 5.2. *Let f and g be holomorphic functions on $\mathfrak{H}\times\mathbb{C}$ and $\langle f, g\rangle_{k,m_1,l,m_2,n}$ be the Rankin-Cohen bracket for holomorphic Jacobi forms defined in Proposition 5.1. For $\gamma \in SL_2(\mathbb{Z})$ and $Y \in \mathbb{Z}^2$,*

$$\langle f, g\rangle_{k,m_1,l,m_2,n} \,|_{k+l+2n,m_1+m_2}\gamma = \langle f\,|_{k,m_1}\gamma\,, g\,|_{l,m_2}\gamma\rangle_{k,m_1,l,m_2,n}.$$

$$\langle f, g\rangle_{k,m_1,l,m_2,n} \,|_{m_1+m_2}Y = \langle f\,|_{m_1}Y\,, g\,|_{m_2}Y\rangle_{k,m_1,l,m_2,n}.$$

Lemma 5.3. *Let f and g be holomorphic functions on $\mathfrak{H}_2$ and $\langle f, g\rangle_{k,l,n}$ be the Rankin-Cohen bracket for holomorphic Siegel modular forms of degree two defined in Proposition 5.2. For $M \in Sp(4,\mathbb{Z})$,*

$$\langle f, g\rangle_{k,l,n} \,|_{k+l+2n}M = \langle f\,|_kM\,, g\,|_lM\rangle_{k,l,n}.$$

In Section 4, we have constructed modular differential equations by using the extended Rankin-Cohen brackets. Now we can construct them by using other Rankin-Cohen brackets from Lemmas 5.1, 5.2, and 5.3.

Theorem 5.1. *For a function f on $\mathfrak{H}_2$ (resp. $\mathfrak{H}$, $\mathfrak{H} \times \mathbb{C}$) and $g \in S_l^2$ (resp. M_l, $J_{l,m}^{hol}$), the equation*

$$\langle f, g \rangle_{k,l,n} = 0 \ (resp.\ [f, g]_{k,l,n} = 0, \langle f, g \rangle_{k,m_1,l,m_2,n} = 0)$$

is the modular differential equation for holomorphic Siegel modular forms of degree two (resp. elliptic modular forms, holomorphic Jacobi forms).

Proof. We consider the case of holomorphic Siegel modular forms. The other cases can be shown in the same way. We have to show that if a function f is a solution of $\langle f, g \rangle_{k,l,n} = 0$, then a function $f|_k M$ is an also solution for $M \in Sp(4, \mathbb{Z})$. From $g \in S_l^2$ and Lemma 5.3,

$$\langle f|_k M, g \rangle_{k,l,n} = \langle f|_k M, g|_l M \rangle_{k,l,n} = \langle f, g \rangle_{k,l,n}|_{k+l+2n} M$$

Since we have assumed a function f is a solution of $\langle f, g \rangle_{k,l,n} = 0$, we then get $\langle f|_k M, g \rangle_{k,l,n} = 0$. □

We give the example of the simplest modular differential equation for holomorphic Siegel modular forms of degree two.

Example. *Let f be a function on $\mathfrak{H}_2$ and g a holomorphic Siegel modular form of weight l and degree two. Then*

$$\begin{aligned} \langle f, g \rangle_{k,l,1} &= \left(k - \frac{1}{2}\right)\left(l - \frac{1}{2}\right)\mathbb{D}(fg) \\ &\quad - \left(k + l - \frac{1}{2}\right)\left(l - \frac{1}{2}\right)\mathbb{D}(f)g - \left(k + l - \frac{1}{2}\right)\left(k - \frac{1}{2}\right)\mathbb{D}(g)f \end{aligned}$$

and $\langle f, g \rangle_{k,l,1} = 0$ is a modular differential equation for holomorphic Siegel modular forms of degree two.

We will have to study the fundamental properties of modular differential equations above as in the previous studies. It is natural to hope that such a study contributes to the development of the research on modular differential equations of a wider class of modular forms.

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Date Submitted(yyyy/mm/dd): 2023/2/1

種類別 (By Type)	題名、 発表・発行掲載誌名、 発表・発行年月、 連名者（申請者含む） (theme, journal name, date & year of publication, name of authors inc. yourself)
Academic papers	(1) The modular differential equation for skew-holomorphic Jacobi forms, Ramanujan Journal, 59, 2022, 1137-1146, Shotaro Kimura.
Lectures	(1) 保型微分方程式とその構成について , 代数的組合せ論セミナー, オンライン, 2021年12月, 木村 昭太郎. (2) Rankin-Cohen bracketを用いた保型微分方程式の構成について , 2021早稲田整数論研究集会, オンライン, 2021年3月 , 木村 昭太郎. (3) 解空間が保型性を満たす微分方程式について, 静岡複素解析幾何セミナー, 静岡大学, 2019年12月 , 木村 昭太郎. (4) Modular differential equations for skew holomorphic Jacobi forms, 第18回広島仙台整数論集会, 広島大学, 2019年7月 , 木村 昭太郎. (5) 歪正則ヤコビ形式に対する保型微分方程式, 早稲田整数論セミナー, 早稲田大学, 2019年4月 , 木村 昭太郎. (6) 歪正則ヤコビ形式に対する保型微分方程式, 第140回日本数学会九州支部例会, 九州大学, 2019年2月 , 木村 昭太郎.
Others	(1) Rankin-Cohen bracketを用いた保型微分方程式の構成について, 2021早稲田整数論研究集会報告集, 2021年3月 , 木村 昭太郎.