Time-dependent Singularities in the Stokes Equations and the Navier-Stokes Equations

Stokes方程式とNavier-Stokes方程式における動的特異点

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Abstract.

In this doctoral thesis, we study removability of time-dependent singularities of the non-stationary Stokes and Navier-Stokes equations in an n-dimensional bounded domain Ω with the smooth boundary for $n \geq 3$. We also show the existence of the solutions with time-dependent singularities to the non-stationary Navier-Stokes equations in $\mathbb{R}^n$ for $n = 2, 3$. In fact, we first prove the following removability of time-dependent singularities: Suppose that $\xi \in C^\alpha([0,T];\Omega)$ for $0 < \alpha \leq 1/2$. If the solution u of the Stokes or the Navier-Stokes equations in $\Omega \times (0,T)$ except for $\xi(t)$ for $t \in (0,T)$ satisfies that

$$|u(x,t)| = o(|x - \xi(t)|^{2-n+(1/\alpha-2)}) \text{ locally uniformly in } t \in (0,T) \text{ as } x \to \xi(t) ,$$

or

$$|u(x,t)| = o(|x - \xi(t)|^{-n+\beta}) \quad \text{locally uniformly in } t \in (0,T) \text{ as } x \to \xi(t) ,$$

for $\beta = \max\{1/\alpha, n-1\}$, respectively, then the curve $\{\xi(t); 0 < t < T\}$ is a family of removable singularities of u in $\Omega \times (0,T)$. Next, in the Navier-Stokes equations we show the existence of the solution with the time-dependent singular point in case $n = 2$ and of the solution having singularities on the time-dependent sphere whose radius or center changes in time in case $n = 3$.

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Contents.

Chapter 1

Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ for $n \geq 3$ with the smooth boundary $\partial\Omega$. Suppose that $\xi : \mathbb{R} \to \mathbb{R}^n$ be a continuous function satisfying that $\xi(t) \in \Omega$ for $t \in (0, T)$. In this doctorical thesis, we first consider the removability of time-dependent singularities of the non-stationary Stokes and Navier-Stokes equations in $Q_T \equiv \bigcup_{0<t<T} (\Omega \backslash \{\xi(t)\}) \times \{t\}$:

$$\begin{cases} \partial_t u - \Delta u + \nabla \pi = 0 & \text{in } \ Q_T, \\ \text{div } u = 0 & \text{in } \ Q_T, \\ u = 0 & \text{on } \ \partial\Omega \times (0, T), \\ u|_{t=0} = u_0 & \text{in } \ \Omega, \end{cases} \tag{St}$$

and

$$\begin{cases} \partial_t u - \Delta u + (u, \nabla)u + \nabla \pi = 0 & \text{in } \ Q_T, \\ \text{div } u = 0 & \text{in } \ Q_T, \\ u = 0 & \text{on } \ \partial\Omega \times (0, T), \\ u|_{t=0} = u_0 & \text{in } \ \Omega, \end{cases} \tag{NS}$$

where $u = u(x, t) = (u_1(x, t), \cdots, u_n(x, t))$ and $\pi = \pi(x, t)$ denote the velocity vector and the pressure of the fluid at the point $(x, t) \in Q_T$, respectively, while $u_0 = u_0(x) = (u_{0,1}(x), \cdots, u_{0,n}(x))$ is the given initial data with the singularity at $\xi(0)$.

We first consider the removability of the time-dependent singularities. For instace, let us recall a fundamental fact on an isolated removable singularity of harmonic functions. Suppose that u is harmonic in $\Omega \setminus \{x_0\}$. If u behaves like $u(x) = o(|x - x_0|^{2-n})$ as $x \to x_0$, then there exists $\bar{u}$ such that $\bar{u} \in C^\infty(\overline{\Omega})$ is harmonic in Ω and that $\bar{u} \equiv u$ in $\Omega \setminus \{x_0\}$. This is a well-known removable singularity theorem for harmonic functions. By Hsu [7] the corresponding theorem to the the solution $u(x, t)$ of the heat equation $\partial_t u = \Delta u$ in $\Omega \times (0, T)$ was proved under the condition that $u(x, t) = o(|x - x_0|^{2-n})$ locally uniformly in $t \in (0, T)$ as $x \to x_0$. The singular order $o(|x - x_0|^{2-n})$ is optimal since the fundamental solution $\Gamma(x - x_0) = |x - x_0|^{2-n}$ of the Laplace equation in $\Omega \setminus \{x_0\}$ is a typical example of the solution having an irremovable singularity at $x = x_0$. Later, the simple proof of this removability was given by Hui [9].

Concerning the stationary Navier-Stokes equations in the open ball B_R with radius R centered at the origin 0, Dyer and Edmunds [3] first proved such a removability of isolated singularity of u at $x = 0$. However, not only the condition on the velocity u but also the condition on the pressure p are needed: If the smooth solution (u, p) in $B_R \setminus \{0\}$ satisfies that $u \in L^{n+\varepsilon}(B_R)$ and $p \in L^{n+\varepsilon}(B_R)$ for some $\varepsilon > 0$, then the singularity of u at $x = 0$ is removable. Later, this assumption was improved to $u \in L^{n+\varepsilon}(B_R)$ for some $\varepsilon > 0$ by Shapiro [24, 25] and to $u \in L^n(B_R)$ or $u(x) = o(|x|^{-1})$ as $x \to 0$ by Kim-Kozono [15]. In case $n = 3$, their result is optimal since the well-known Landau solution U has the irremovable singularity at the origin such as $U(x) = O(|x|^{-1})$ as $x \to 0$.

For the 3D nonstationary Navier-Stokes equations, Kozono [19] proved that there is a constant ε_0 such that if the Leray-Hopf weak solution u satisfies for some $\delta, \rho > 0$ $\sup_{|t-t_0|<\delta} \|u(t)\|_{L^{3,\infty}(B_\rho(x_0))} \leq \varepsilon_0$, then $u \in C^\infty(B_{\rho/2}(x_0) \times (t_0 - \delta/2, t_0 + \delta/2))$, where $L^{3,\infty}$ denotes the weak L^3-space and $B_\rho(x_0)$ is an open ball with the radius ρ centered at x_0. Notice that $L^\infty(0, \infty; L^n(\mathbb{R}^n))$ is the marginal case of Serrin's scaling invariant class in which both uniqueness and regularity of Leray-Hopf weak solutions are obtained. As an application, under the hypothesis $u(x, t) = o(|x - x_0|^{-1})$ locally uniformly near $t = t_0$ as $x \to x_0$, we see that (x_0, t_0) is a removable singularity of u.

In comparison with these problems on removable singularities for the time-independent isolated point, the corresponding question to the time-dependent case becomes more complicated. In this direction, recently Takahashi-Yanagida [29] considered solutions u of the heat equation in Q_T defined by

$$Q_T \equiv \bigcup_{0<t<T} (\Omega \setminus \{\xi(t)\}) \times \{t\}, \tag{1.1}$$

where $\xi : t \in (0, T) \mapsto \Omega$ is a curve in Ω with 1/2-Hölder continuity in $t \in (0, T)$. Under the hypothesis that

$$u(x, t) = o(|x - \xi(t)|^{2-n}) \quad \text{locally uniformly in } t \in (0, T) \text{ as } x \to \xi(t) \tag{1.2}$$

they proved that u is, in fact, extended to the smooth solution in $\Omega \times (0, T)$. The condition (1.2) together with the Hölder exponent 1/2 is optimal in the sense that even for $\xi \in C^\alpha((0, T); \Omega)$ with $\alpha > 1/2$ there exists a solution of the heat equation in Q_T having $(\xi(t), t)$ as irremovable singularities whose singular order is $O(|x - \xi(t)|^{2-n})$. However, the corresponding problem is still open for the Stokes and Navier-Stokes equations since it is difficult to handle the pressure. In this thesis, we discuss the removability of the time-dependent singularities of the Stokes and Navier-Stokes equations. Now we define the removable time-dependent singularities as follows.

Definition 1.1. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. Let $\xi \in C([0, T]; \Omega)$. Suppose that u is a smooth solution of (NS) in Q_T, where

Q_T is the non-cylindrical domain in $\mathbb{R}^n \times (0,T)$ defined by (1.1). We say that the curve $\{\xi(t); 0 < t < T\}$ is a family of removable singularities of u in $\Omega \times (0,T)$ if there exists a smooth solution $\tilde{u}$ of (NS) in $\Omega \times (0,T)$ such that $\tilde{u} \equiv u$ in Q_T.

Our result may be regarded as a generalization of Takahashi-Yanagida [26] to the Stokes and Navier-Stokes equations. By assuming (1.2), they succeeded to show that u is in fact a *weak solution* of the heat equation in the whole space and time $\Omega \times (0,T)$, from which with the aid of the well-known Weyl lemma, $\xi(t)$ is in fact a family of removable singularities of u. Our method is based on the uniqueness and smoothness of very weak solutions of the Stokes equations. The crucial difference of weak solutions between the heat and the Stokes (Navier-Stokes) equations stems from the divergence free condition. In fact, for the proof that u is a very weak solution of the Stokes equations in $\Omega \times (0,T)$, it is necessary to make use of the cut-off procedure around the time-dependent singularities $\xi(t)$. Since test functions of the Stokes equations need to be solenoidal, the correction recovering divergence free property is carried out by means of the Bogovskii lemma. It is not obvious that the remainder arising from the cut-off procedure may be handled as a small perturbation of the integral identity defining the usual Stokes equations.

We next consider the following Cauchy problem of the Navier-Stokes equations in $\mathbb{R}^n (n = 2, 3)$:

$$\begin{cases} \partial_t u - \Delta u + (u, \nabla) u + \nabla \pi = f & \text{in } \mathbb{R}^n \times \mathbb{R}^+, \\ \mathrm{div}\ u = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}^+, \\ u|_{t=0} = a & \text{in } \mathbb{R}^n, \end{cases} \tag{NS2}$$

Concerning the solutions with time-dependent irremovable singularities, the pioneer work is the Takahashi-Yanagida [29]. For the heat equation they showed that there exists a singular solution with the same singular order as that of the fundamental solution of the Laplace equation at the singularities $x = \xi(t)$. The existence of the solutions with time-dependent singularities and the asymptotic behavior of those solutions at the singularities were proved for the semilinear parabolic equations [11],[26],[27], nonlinear deffusion equations [4],[5], and the Navier-Stokes equations [13]. Karch-Zheng [13] constructed a solution to the 3D Navier-Stokes equations with the aid of the space of pseudo measures $\mathcal{PM}^k = \{a \in \mathcal{S}'; \sup_{\xi \in \mathbb{R}^3} |\xi|^k |\hat{a}(\xi)| < \infty\}$. That solution has the same singular order as that of the Landau solution, that is, $O(|x - \xi(t)|^{-1})$ as $x \to \xi(t)$.

Recently, time-dependent high dimensional singular sets have been studied. For the superlinear parabolic equation Htoo-Takahashi-Yanagida [8] and Takahashi-Yamamoto [30] showed the existence of solutions with time-dependent m dimensional submanifold M_t and the asymptotic behavior at M_t. In this thesis, we construct a solution to (NS2) with a time-dependent singular point in $\mathbb{R}^2$ or singular sets on tine-dependent sphere in $\mathbb{R}^3$ similarly to Kozono-Shimizu [18] by using the maximal Lorentz regularity theorem in the Besov spaces. Our method is related to that of Takahashi-Yanagida [29] and Karch-Zheng [13] which constucted solutions with irremovable singularities for the given external force having time-dependent singularities. Their method seems to be indirect since it is necessary to show that the Duhamel term in the integral equation has singularities. On the other hand, we show the existence of singular solution for the given external force having time-dependent singularities. The advantage of our method is that we have only to investigate whether or not the external force belongs to some Lorentz-Besov space. However, we do not know the asymptotic behavior of the solution at the singularities.

This paper is organized as follows. In Chapter 2, we first introduce some function spaces such as the Besov spaces in $\mathbb{R}^n$ or bounded domains. Next we define the very weak solution of the Stokes equations in whole space and time $\Omega \times (0,T)$ and of the perturbed Stokes equations in $Q(r,T) \equiv \bigcup_{0<t<T} B_r(\xi(t)) \times \{t\}$ where $B_r(\xi(t))$ denotes the ball in Ω with radius r centered at $\xi(t)$. In the case of the Navier-Stokes equations, we need to take a small subdomain in such a way that the scale invariant norm $L^\infty(0,\infty; L^{n,\infty}(B_r(\xi(t))))$ of u can be small since we do not obtain the existence of smooth solution in the whole space and time $\Omega \times (0,T)$ is not garranteed. We also state the existence theorem of the strong solution, which yields the existence of the very weak solution with additional regularity. To handle the solution u as a very weak solution we need to show an integral identity. To this end, the precise cut-off procedure such as Takahashi-Yanagida [26] is fully used. Introducing the maximal regularity theorem of the Stokes equations in the Besov space, we show the existence and uniqueness of the strong solutions to the Navier-Stokes equations.

In Chapter 3, we prove the removability of time-dependent singularities in the Stokes and Navier-Stokes equations. Our proof consists of two steps. In the case of the Stokes eqations, at first, assuming $|u(x,t)| = o(|x-\xi(t)|^{2-n+(1/\alpha-2)})$ as $x \to \xi(t)$ locally uniformly in $t \in (0,T)$, we show that u satisfies an integral identity, which implies that u is a very weak solution of the Stokes equations in the whole space and time $\Omega \times (0,T)$. Since we make use of the cut-off procedure, it is necessary to recover the divergence free condition of the test function. Hence we shall establish a variant of Bogovskii's lemma in $\Omega \times (0,T)$. By the precise estimate of the cut-off function as well as the remainder caused by the Bogovskii operator, we see that the solution u is in fact a very weak solution in $\Omega \times (0,T)$. It should be noticed that the class of weak solutions is large enough such as $u \in L^1_{loc}(\Omega \times (0,T))$. On the other hand, it is rather well-known that even for $u_0 \in L^p_\sigma(\Omega)$ with $1 < p < \infty$ there exists a very weak solution $\bar{u}$ with $\bar{u}(\cdot,0) = u_0$ in $\Omega \times (0,T)$, which necessarily becomes a smooth solution in the classical sense. In the next step, we may show that $u \equiv \bar{u}$ in $\Omega \times (0,T)$. To this end, it is necessary to prove the uniqueness in the large class $L^1_{loc}(\Omega \times (0,T))$ with the initial data in $L^p_\sigma(\Omega)$ for some $1 < p < \infty$. We shall establish such a uniqueness result by duality argument like Lions-Masmoudi [21].

In Chapter 4, we consider the corresponding problem on removability of time-dependent singularities of the Navier-Stokes equations. Indeed, we saw that if u behaves near singularities $\{\xi(t)\}_{0<t<T} \subset \Omega$ like

$$|u(x,t)| = o(|x - \xi(t)|^{-n+\beta}) \quad \text{locally uniformly in } t \in (0,T) \text{ as } x \to \xi(t)$$

for $\beta = \max\{1/\alpha, n-1\}$, then u can be extended as the smooth solution in the whole space and time $\Omega \times (0,T)$. We first regard the Navier-Stokes equations as the perturbed Stokes equations for v with the convection term $(u, \nabla)v$ in $\Omega \times (0,T)$. Since u is not regular, it is difficult to show the existence, uniqueness, and regularity of the perturbed Stokes equations. Hence, it is necessary to take a small non-cylindrical space-time region $Q(r,T)$ near singularities so that the norm of u in $L^\infty(0,T; L^{n,\infty}(B_r(\xi(t))))$ is sufficiently small. To this end, the hypothesis plays an important role. Simultaneously, we show that u is a very weak solution in $Q(r,T)$. Next, we construct the strong solution v of the perturbed Stokes equations in $Q(r,T)$. Using the uniqueness of very weak solutions in the class $L^2_{loc}(Q(r,T))$ to the perturbed Stokes equations with small coefficient u in $L^\infty(0,T; L^{n,\infty}(B_r(\xi(t))))$, we may identify u with v in $Q(r,T)$. As a result, it turns out that $\{\xi(t)\}_{0<t<T}$ is a family of removable singularities of u.

In Chapter 5, we construct the solution of the Navier-Stokes equations having the the time-dependent singularities. Indeed, in $\mathbb{R}^2$, our solution behaves like the Dirac measure with supports at $\{\xi(t)\}_{0<t<T}$. In $\mathbb{R}^3$, we solve the Navier-Stokes equations with the external forces $\delta_{S_{\rho(t)}(0)}$ and $\delta_{S_\rho(\xi(t))}$ for $\{\rho(t)\}_{0<t<T}$ denoting the radius and for $\{\xi(t)\}_{0<t<T}$ denoting the center, respectively, where $\delta_{S_\rho(\xi)}$ is the single layer potential supported by the sphere $S_\rho(\xi) \equiv \{y \in \mathbb{R}^3; |y - \xi| = \rho\}$.

In Chapter 6, as Appendix we prove here the expression of the compensation term for recovering divergence-free condition of the test function in the Bogovskii lemma which we have admitted in the main parts without proof.

Chapter 2

Preliminaries

2.1 Function space

Let $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ be the space of tempered distibutions and $\mathcal{P} = \mathcal{P}(\mathbb{R}^n)$ be the space of polynomials. Let us recall $\{\phi_j\}_{j\in\mathbb{Z}}$ the Littlewood-Paley decomposition. We take a function $\phi \in C_0^\infty(\mathbb{R}^n)$ with its support supp $\varphi = \{\xi \in \mathbb{R}^n; 1/2 \le |\xi| \le 2\}$ such that $\sum_{j\in\mathbb{Z}} \phi(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. The functions φ_j is defined by $\mathcal{F}\varphi_j(\xi) = \phi(2^{-j}\xi)$ where $\mathcal{F}$ is the Fourier transform.

Definition 2.1.1. *For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is defined by*

$$\dot{B}^s_{p,q}(\mathbb{R}^n) \equiv \{f \in \mathcal{S}'/\mathcal{P}; \|f\|_{\dot{B}^s_{p,q}} < \infty\}$$

where

$$\|f\|_{\dot{B}^s_{p,q}} \equiv \begin{cases} \left(\sum_{j\in\mathbb{Z}} (2^{sj}\|\varphi_j * f\|_{L^p(\mathbb{R}^n)})^q\right)^{1/q}, & 1 \le q < \infty \\ \sup_{j\in\mathbb{Z}} (2^{sj}\|\varphi_j * f\|_{L^p(\mathbb{R}^n)}), & q = \infty. \end{cases}$$

Let P be the Helmholtz projection. P is bounded from $L^p(\mathbb{R}^n)(1 < p < \infty)$ onto the solenoidal subspace $L^p_\sigma(\mathbb{R}^n) \equiv \{f \in L^p(\mathbb{R}^n); \mathrm{div} f = 0\}$. It is known that P is expressed by

$$P = (P_{i,j})_{1\le i,j\le n}, \quad P_{i,j} = \delta_{i,j} + R_i R_j, \quad i, j = 1, \cdots, n,$$

where $\delta_{i,j}$, $i, j = 1, \cdots, n$, is the Kronecker symbol and $R_i = \frac{\partial}{\partial x_i}(-\Delta)^{-\frac{1}{2}}$, $i = 1, \cdots, n$, is the Riesz transform. Then the solenoidal subspace $\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^n)$ of $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is defined by $\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^n) \equiv P\dot{B}^s_{p,q}(\mathbb{R}^n)$.

Proposition 2.1.2. (i) *Let $1 \le p_0 \le p_1 \le \infty$, $1 \le q \le \infty$, and $s_0, s_1 \in \mathbb{R}$ satisfy $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$. Then, it holds that $\dot{B}^{s_0}_{p_0,q}(\mathbb{R}^n) \subset \dot{B}^{s_1}_{p_1,q}(\mathbb{R}^n)$ with the estimate*

$$\|f\|_{\dot{B}^{s_1}_{p_1,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{B}^{s_0}_{p_0,q}(\mathbb{R}^n)}$$

for all $f \in \dot{B}^{s_0}_{p_0,q}(\mathbb{R}^n)$*, where* $C = C(n, p_0, p_1, s_0, s_1)$*.*

(ii) *Let* $1 \le p \le \infty$*,* $1 \le q \le \infty$*,* $s > 0$*,* $\sigma > 0$*, and* $\mu > 0$*. We take* $1 \le p_1, p_2, r_1, r_2 \le \infty$ *so that* $1/p = 1/p_1 + 1/p_2 = 1/r_1 + 1/r_2$*. For all* $f \in \dot{B}^{s+\sigma}_{p_1,q}(\mathbb{R}^n) \cap \dot{B}^{-\mu}_{r_1,\infty}(\mathbb{R}^n)$ *and* $g \in \dot{B}^{-\sigma}_{p_2,\infty}(\mathbb{R}^n) \cap \dot{B}^{s+\mu}_{r_2,q}(\mathbb{R}^n)$*, it holds that* $f \cdot g \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ *with the estimate*

$$\|f \cdot g\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \le C\left(\|f\|_{\dot{B}^{s+\sigma}_{p_1,q}(\mathbb{R}^n)}\|g\|_{\dot{B}^{-\sigma}_{p_2,\infty}(\mathbb{R}^n)} + \|f\|_{\dot{B}^{-\mu}_{r_1,\infty}(\mathbb{R}^n)}\|g\|_{\dot{B}^{s+\mu}_{r_2,q}(\mathbb{R}^n)}\right)$$

where $C = C(n, p, p_1, p_2, r_1, r_2, s, \sigma, \mu)$*.*

Outline of the proof of Proposition 2.1.2. (i) If $p_0 = p_1$, then the two Besov spaces coincide and the claim is correct. Let $p_0 \neq p_1$. From the Young inequality, we obtain

$$\begin{aligned}
\|f\|_{\dot{B}^{s_1}_{p_1,q}(\mathbb{R}^n)} &= \left(\sum_{j\in\mathbb{Z}} 2^{js_1q}\|(\varphi_{j-1} + \varphi_j + \varphi_{j+1}) * \varphi_j * f\|^q_{L^{p_1}(\mathbb{R}^n)}\right)^{\frac{1}{q}} \\
&\le \left(\sum_{j\in\mathbb{Z}} 2^{js_1q}\|\varphi_{j-1} + \varphi_j + \varphi_{j+1}\|^q_{L^r(\mathbb{R}^n)}\|\varphi_j * f\|^q_{L^{p_0}(\mathbb{R}^n)}\right)^{\frac{1}{q}} \\
&\le \left(\sum_{j\in\mathbb{Z}} 2^{js_1q}(\|\varphi_{j-1}\|_{L^r(\mathbb{R}^n)} + \|\varphi_j\|_{L^r(\mathbb{R}^n)} + \|\varphi_{j+1}\|_{L^r(\mathbb{R}^n)})^q\|\varphi_j * f\|^q_{L^{p_0}(\mathbb{R}^n)}\right)^{\frac{1}{q}} \\
&\le \left(\sum_{j\in\mathbb{Z}} 2^{js_1q}(2^{(j-1)\frac{n}{r'}} + 2^{j\frac{n}{r'}} + 2^{(j+1)\frac{n}{r'}})^q\|\varphi_j * f\|^q_{L^{p_0}(\mathbb{R}^n)}\right)^{\frac{1}{q}} \\
&\le C\left(\sum_{j\in\mathbb{Z}} 2^{js_1q}2^{j\frac{n}{r'}q}\|\varphi_j * f\|^q_{L^{p_0}(\mathbb{R}^n)}\right)^{\frac{1}{q}} \\
&= C\left(\sum_{j\in\mathbb{Z}} 2^{j\{(s_1-\frac{n}{p_1})-(s_0-\frac{n}{p_0})\}q}2^{js_0q}\|\varphi_j * f\|^q_{L^{p_0}(\mathbb{R}^n)}\right)^{\frac{1}{q}} \\
&= C\|f\|_{\dot{B}^{s_0}_{p_0,q}},
\end{aligned}$$

where $1/r' = 1 - 1/r = 1/p_0 - 1/p_1$.

(ii)Since this estimate was shown by Kaneko-Kozono-Shimizu [12], we omit its proof.

□

We next define the Besov space in the bounded domain. To this end, we introduce the real interpolation spaces. Let X_0 and X_1 be a pair of Banach spaces with the norm $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$, respectively, and continuously embedded in the same topological vector space. There are two well-known methods of defining the real interpolation space $(X_0, X_1)_{\theta,q}$ for $0 < \theta < 1$ and $1 \le q \le \infty$.

Definition 2.1.3. *For $t \in \mathbb{R}^+$ and $u \in X_0 + X_1$, let*

$$K(t,u) = \inf\{\|u\|_{X_0} + \|u\|_{X_1}; u = u_0 + u_1, u_0 \in X_0, u_1 \in X_1\}. \tag{2.1}$$

The real interpolation space $(X_0, X_1)_{\theta,q}$ consists of $u \in X_0 + X_1$ such that the norm

$$\|u\|_{\theta,q;K} = \begin{cases} \sup_{t>0} t^{-\theta} K(t,u) & \text{if } q = \infty, \\ \left\{ \int_0^\infty (t^{-\theta} K(t,u))^q \frac{dt}{t} \right\}^{1/q} & \text{if } 1 < q < \infty, \end{cases}$$

is finite.

This method is called K-method. The other method is called J-method. J-method is as foliows.

Definition 2.1.4. *For $t \in \mathbb{R}^+$ and $u \in X_0 \cap X_1$, let*

$$J(t,u) = \max\{\|u\|_{X_0}, \|u\|_{X_1}; u = u_0 + u_1, u_0 \in X_0, u_1 \in X_1\}. \tag{2.2}$$

The real interpolation space $(X_0, X_1)_{\theta,q}$ consists of $u \in X_0 + X_1$ such that u is expressed by

$$u = \int_0^\infty v(t) \frac{dt}{t}$$

with $v(t) \in X_0 \cap X_1$ for all $t > 0$ and that the norm

$$\|u\|_{\theta,q,J} = \inf \begin{cases} \sup_{t>0} t^{-\theta} J(t,v(t)) & \text{if } q = \infty, \\ \left\{ \int_0^\infty (t^{-\theta} J(t,v(t)))^q \frac{dt}{t} \right\}^{1/q} & \text{if } 1 < q < \infty, \end{cases}$$

is finite.

Now the Besov space $B^s_{p,q}(\Omega)$ and the solenoidal Besov space $\mathcal{B}^s_{p,q}(\Omega)$ are defined by

$$\begin{aligned} B^s_{p,q}(\Omega) &= (H^{s_1,p}(\Omega), H^{s_2,p}(\Omega))_{\theta,q}, \\ \mathcal{B}^s_{p,q}(\Omega) &= (H^{s_1,p}_\sigma(\Omega), H^{s_2,p}_\sigma(\Omega))_{\theta,q} \end{aligned} \tag{2.3}$$

with $s = (1-\theta)s_1 + \theta s_2$.

2.2 Cut-off function

We construct and estimate the cut-off function along some continuous curve introduced by Takahashi-Yanagida [29].

Lemma 2.2.1. *Let $n \geq 1$, and $t_1, t_2 \in \mathbb{R}(t_1 < t_2)$. Suppose that $\xi(t)$ is locally α-Hölder continuous in $t \in \mathbb{R}$ for some $\alpha \in (0,1]$. Then there exist $r_0 = r_0(\alpha, n, t_1, t_2)$ and $C = C(\alpha, n, t_1, t_2)$ with the following properties; For any $r \in (0, r_0)$, there exists a family of cut-off functions $\{\eta_r\}_{r>0} \subset C^\infty(\mathbb{R}^n \times \mathbb{R})$ such that $0 \leq \eta_r \leq 1$,*

$$\eta_r(x,t) = \begin{cases} 1 & \text{if } |x - \xi(t)| > r, \\ 0 & \text{if } |x - \xi(t)| < r/2, \end{cases} \tag{2.4}$$

$$|\nabla \eta_r(x,t)| \leq Cr^{-1}, |\Delta \eta_r(x,t)| \leq Cr^{-2}, |\partial_t \eta_r(x,t)| \leq Cr^{-1/\alpha}, \tag{2.5}$$

$$|\Delta \nabla \eta_r(x,t)| \leq Cr^{-3}, |\partial_t \nabla \eta_r(x,t)| \leq Cr^{-1-1/\alpha} \tag{2.6}$$

for all $(x,t) \in \mathbb{R}^n \times [t_1, t_2]$.

Proof. We make use of the argument due to Takahashi-Yanagida [26, Lemma 2.1]. Let us take $\rho \in C_0^\infty([-1,1])$ with $\int_{\mathbb{R}} \rho(t)dt = 1$ such as

$$\rho(t) := \begin{cases} Ae^{-1/(1-t^2)} & \text{if } |t| \leq 1, \\ 0 & \text{if } |t| > 1. \end{cases}$$

We define $\xi^\varepsilon(t) = \xi * \rho^\varepsilon(t)$ where $\rho^\varepsilon(t) = \varepsilon^{-1}\rho(\varepsilon^{-1}t)$ for $\varepsilon \in (0,1)$. Then we have

$$|\xi(t) - \xi^\varepsilon(t)| \leq \int_{-1}^{1} \rho(s)|\xi(t) - \xi(t - \varepsilon s)|ds.$$

Since ξ is loccally α-Hölder continuous in $t \in \mathbb{R}$, there exists a constant $L = L(t_1, t_2)$ such that $|\xi(t) - \xi(t - \varepsilon s)| \leq L\varepsilon^\alpha |s|^\alpha \leq L\varepsilon^\alpha$, which yields

$$|\xi(t) - \xi^\varepsilon(t)| \leq L\varepsilon^\alpha \int_{-1}^{1} \rho(s)ds = L\varepsilon^\alpha \tag{2.7}$$

for $t \in [t_1, t_2]$. Since ρ' is an odd function, we have

$$\begin{aligned} \frac{d\xi^\varepsilon}{dt}(t) &= \frac{1}{\varepsilon^2} \int_{t-\varepsilon}^{t+\varepsilon} \rho'\left(\frac{t-s}{\varepsilon}\right) \xi(s)ds \\ &= \frac{1}{\varepsilon} \int_{-1}^{1} \rho'(\tau)\xi(t - \varepsilon\tau)d\tau \\ &= \frac{1}{\varepsilon} \int_{-1}^{1} \rho'(\tau)\{\xi(t - \varepsilon\tau) - \xi(t)\}d\tau. \end{aligned}$$

Similarly, we obtain from the Hölder continuity of ξ that

$$\begin{aligned} \left|\frac{d\xi^\varepsilon}{dt}(t)\right| &\leq L\varepsilon^{\alpha-1} \int_{-1}^{1} |\rho'(\tau)|d\tau \\ &\leq 4AL\varepsilon^{\alpha-1} \int_0^1 \frac{\tau}{(1-\tau^2)^2} e^{-1/(1-\tau^2)} d\tau \\ &= \frac{2}{e} AL\varepsilon^{\alpha-1} \end{aligned} \tag{2.8}$$

for all $t \in [t_1, t_2]$. Now we define a function $\eta_r \in C^\infty(\mathbb{R}^N \times \mathbb{R})$ for $r > 0$ by

$$\eta_r(x,t) := \begin{cases} \dfrac{e^{-1/\sigma}}{e^{-1/\sigma} + e^{-1/(1-\sigma)}} & \text{if } \dfrac{7}{10}r < |x - \xi^\varepsilon(t)| < \dfrac{4}{5}r, \\ 1 & \text{if } |x - \xi^\varepsilon(t)| \geq \dfrac{4}{5}r, \\ 0 & \text{if } |x - \xi^\varepsilon(t)| \leq \dfrac{7}{10}r, \end{cases} \tag{2.9}$$

where

$$\sigma := \frac{10}{r}(|x - \xi^\varepsilon(t)| - \frac{7}{10}r).$$

Set $r_0 = 10L$ and let $r < r_0$. Taking $\varepsilon = \varepsilon_r = (r/10L)^{1/\alpha}$, it holds by (2.7) that $|\xi(t) - \xi^{\varepsilon_r}(t)| \leq r/10$. From this estimate, we have $|x - \xi^\varepsilon(t)| \leq |x - \xi(t)| + |\xi(t) - \xi^\varepsilon(t)| < 7r/10$ if $|x - \xi(t)| < r/2$ and $|x - \xi^\varepsilon(t)| > |x - \xi(t)| - |\xi(t) - \xi^\varepsilon(t)| > 4r/5$ if $|x - \xi(t)| > r$. Therefore, we see that (2.4).

Finally we estimate the spacial and time derivatives of η_r. It suffices to show (2.5) and (2.6) in the case where $7r/10 < |x - \xi^{\varepsilon_r}(t)| < 4r/5$. By the direct calculation, for $0 < \sigma < 1$, i.e., $7r/10 < |x - \xi^{\varepsilon_r}(t)| < 4r/5$, we have that

$$\begin{aligned} \nabla \eta_r(x,t) &= \frac{10}{r} X(\sigma) \frac{x - \xi^{\varepsilon_r}(t)}{|x - \xi^{\varepsilon_r}(t)|}, \\ \Delta \eta_r(x,t) &= \frac{100}{r^2} Y(\sigma) \end{aligned}$$

with

$$\begin{aligned} X(\sigma) &:= \frac{e^{-1/\sigma} e^{-1/(1-\sigma)}}{(e^{-1/\sigma} + e^{-1/(1-\sigma)})^2} \left(\frac{1}{\sigma^2} + \frac{1}{(1-\sigma)^2} \right), \\ Y(\sigma) &:= \frac{e^{-1/\sigma} e^{-1/(1-\sigma)}}{(e^{-1/\sigma} + e^{-1/(1-\sigma)})^2} \left[\left(\frac{n-1}{\sigma + 7} \right) \left(\frac{1}{\sigma^2} + \frac{1}{(1-\sigma)^2} \right) + \left(\frac{1}{\sigma^4} + \frac{1}{(1-\sigma)^4} \right) (1 - 2\sigma) \right. \\ &\qquad \left. - \frac{2}{e^{-1/\sigma} + e^{-1/(1-\sigma)}} \left(\frac{e^{-1/\sigma}}{\sigma^4} + \frac{e^{-1/\sigma} - e^{-1/(1-\sigma)}}{\sigma^2 (1-\sigma)^2} - \frac{e^{-1/(1-\sigma)}}{(1-\sigma)^4} \right) \right]. \end{aligned}$$

$$\begin{aligned} \partial_t \nabla \eta_r(x,t) &= \frac{100}{r^2} \left\{ \left(Y(\sigma) - \frac{n}{\sigma + 7} X(\sigma) \right) \frac{(x - \xi^{\varepsilon_r}(t)) \cdot \frac{d}{dt} \xi^{\varepsilon_r}(t)}{|x - \xi^{\varepsilon_r}(t)|} \frac{x - \xi^{\varepsilon_r}(t)}{|x - \xi^{\varepsilon_r}(t)|} \right. \qquad (2.10) \\ &\qquad \left. - \frac{1}{\sigma + 7} X(\sigma) \frac{d}{dt} \xi^{\varepsilon_r}(t) \right\}, \\ \Delta \nabla \eta_r(x,t) &= \frac{10^3}{r^3} \left\{ \frac{d^2}{d\sigma^2} X(\sigma) + \frac{n-1}{\sigma + 7} \left(Y(\sigma) - \frac{n-1}{\sigma + 7} X(\sigma) \right) \right\} \frac{x - \xi^{\varepsilon_r}(t)}{|x - \xi^{\varepsilon_r}(t)|} \qquad (2.11) \end{aligned}$$

with

$$\frac{d^2}{d\sigma^2}X(\sigma) = \frac{e^{-1/\sigma}e^{-1/(1-\sigma)}}{(e^{-1/\sigma}+e^{-1/(1-\sigma)})^2}\Bigg[\frac{6}{\sigma^4}+\frac{6}{(1-\sigma)^4}$$
$$-4\left\{\frac{1}{\sigma^2}-\frac{1}{(1-\sigma)^2}-\frac{2}{e^{-1/\sigma}+e^{-1/(1-\sigma)}}\left(\frac{e^{-1/\sigma}}{\sigma^2}-\frac{e^{-1/(1-\sigma)}}{(1-\sigma)^2}\right)\right\}\left(\frac{1}{\sigma^3}-\frac{1}{(1-\sigma)^3}\right)$$
$$+\left\{-\frac{4}{e^{-1/\sigma}+e^{-1/(1-\sigma)}}\left(\frac{1}{\sigma^2}-\frac{1}{(1-\sigma)^2}\right)\left(\frac{e^{-1/\sigma}}{\sigma^2}-\frac{e^{-1/(1-\sigma)}}{(1-\sigma)^2}\right)\right.$$
$$-\frac{2}{\sigma^3}-\frac{2}{(1-\sigma)^3}+\left(\frac{1}{\sigma^2}-\frac{1}{(1-\sigma)^2}\right)^2+\frac{6}{(e^{-1/\sigma}+e^{-1/(1-\sigma)})^2}\left(\frac{e^{-1/\sigma}}{\sigma^2}-\frac{e^{-1/(1-\sigma)}}{(1-\sigma)^2}\right)^2$$
$$\left.-\frac{2(1-2\sigma)}{e^{-1/\sigma}+e^{-1/(1-\sigma)}}\left(\frac{e^{-1/\sigma}}{\sigma^4}-\frac{e^{-1/(1-\sigma)}}{(1-\sigma)^4}\right)\right\}\left(\frac{1}{\sigma^2}+\frac{1}{(1-\sigma)^2}\right)\Bigg].$$

Since $X, Y, \frac{d^2}{d\sigma^2}X \in C^\infty(0,1)$ satisfies

$$\lim_{\sigma\searrow 0} X(\sigma) = \lim_{\sigma\nearrow 1} X(\sigma) = 0,$$
$$\lim_{\sigma\searrow 0} Y(\sigma) = \lim_{\sigma\nearrow 1} Y(\sigma) = 0,$$
$$\lim_{\sigma\searrow 0} \frac{d^2}{d\sigma^2}X(\sigma) = \lim_{\sigma\nearrow 1} \frac{d^2}{d\sigma^2}X(\sigma) = 0,$$

$X(\sigma)$, $Y(\sigma)$, and $\frac{d^2}{d\sigma^2}X(\sigma)$ are bounded for $\sigma \in (0,1)$. Therefore, it follows from (2.4), (2.10), and (2.11) that for $(x,t) \in \mathbb{R}^N \times [t_1,t_2]$,

$$|\nabla\eta_r(x,t)| \le Cr^{-1}, |\Delta\eta_r(x,t)| \le Cr^{-2}, |\partial_t\eta_r(x,t)| \le Cr^{-1/\alpha},$$
$$|\Delta\nabla\eta_r(x,t)| \le Cr^{-3}, |\partial_t\nabla\eta_r(x,t)| \le Cr^{-1-1/\alpha}$$

where $C = C(\alpha, n, t_1, t_2)$. □

2.3 Very weak solutions

2.3.1 Stokes equations

We first introduce some definitions. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary for $n \ge 3$. We define a very weak solution of the Stokes equations in $\Omega \times (0,T)$ as follows.

Definition 2.3.1. *Let $w \in L^p(\Omega \times (0,T))$ for some $p \in (1,\infty]$. Assume that w is smooth in some neighborhood of $\partial\Omega$ with $w(\cdot,t)|_{\partial\Omega} = 0$ for $0 < t < T$. We say that w is a weak solution of the Stokes equations in $\Omega \times (0,T)$ with the initial data $w_0 \in L^p_\sigma(\Omega)$ if w satisfies that*

$$\int_\Omega w_0(x)\cdot\varphi(x,0)dx + \int_0^T\int_\Omega w\cdot(\varphi_t + \Delta\varphi)dxdt = 0 \tag{2.12}$$

for all $\varphi \in C^\infty(\overline{\Omega} \times [0,T])$ *with* $\operatorname{div} \varphi = 0$ *in* $\overline{\Omega} \times [0,T]$, $\varphi = 0$ *on* $\partial\Omega \times [0,T]$, *and* $\varphi(\cdot, T) = 0$ *in* $\overline{\Omega}$, *and that*

$$\int_\Omega w(t) \cdot \nabla \varrho \, dx = 0, \qquad 0 < t < T \tag{2.13}$$

for all $\varrho \in C^\infty(\overline{\Omega})$.

2.3.2 Perturbed Stokes equations

Let us consider the following Stokes equations with the convection term whose coefficient is the solution u of the Navier-Stokes equations.

$$\begin{cases} \partial_t w - \Delta w + (u, \nabla)w + \nabla p = 0 & \text{in } Q(r,T) \equiv \bigcup\limits_{0<t<T} B_r(\xi^{\varepsilon_r}(t)) \times \{t\}, \\ \operatorname{div} w = 0 & \text{in } Q(r,T), \\ w = u & \text{on } \bigcup\limits_{0<t<T} \partial B_r(\xi^{\varepsilon_r}(t)) \times \{t\}, \\ w|_{t=0} = u_0|_{B_r(\xi^{\varepsilon_r}(0))} & \text{in } B_r(\xi^{\varepsilon_r}(0)), \end{cases} \tag{PS}$$

where $B_r(\xi^{\varepsilon_r}(t)) = \{x \in \mathbb{R}^n; |x - \xi^{\varepsilon_r}(t)| < r\}$ for $0 < t < T$. Here $\xi^{\varepsilon_r}(t) = \xi * \rho^{\varepsilon_r}(t)$ is the same in Proposition 2.2.1, and we suppose that u is smooth on $\bigcup_{0<t<T} \partial B_r(\xi^{\varepsilon_r}(t)) \times \{t\}$. We reduce (PS) to the problem in the cylindrical domain by using the method of Inoue-Wakimoto [10] and Miyakawa-Teramoto [22]. Let $\Psi : \overline{Q(r,T)} \to \overline{\widetilde{Q}(r,T)}$ be a diffeomorphism defined by $(y, s) = \Psi(x,t) = (x_1 - \xi_1^{\varepsilon_r}(t), \cdots, x_n - \xi_n^{\varepsilon_r}(t), t)$, where $\widetilde{Q}(r,T) = B_r \times (0,T)$ with $B_r = \{x \in \mathbb{R}^n; |x| < r\}$. It is easy to see that Ψ is a volume-preserving C^∞ diffeomorphism satisfying $(\partial\Psi_i/\partial x_j)_{1 \le i,j \le n} = I_n$, where I_n is the identity matrix on $\mathbb{R}^n$. Defining $\widetilde{w} = \Psi_* w$, $\widetilde{u} = \Psi_* u$, $\widetilde{u_0} = \Psi_* u_0$ and $\widetilde{p}(y,s) = p(\Psi^{-1}(y,s))$, we may transfer (PS) to the following system in $\widetilde{Q}(r,T)$;

$$\begin{cases} \partial_s \widetilde{w} - \Delta_y \widetilde{w} + (\widetilde{u}, \nabla_y)\widetilde{w} + (\frac{d\xi^{\varepsilon_r}}{ds}, \nabla_y)\widetilde{w} + \nabla_y \widetilde{p} = 0 & \text{in } \widetilde{Q}(r,T), \\ \operatorname{div} \widetilde{w} = \displaystyle\sum_{i=1}^N \frac{\partial \widetilde{w}_i}{\partial y_i} = 0 & \text{in } \widetilde{Q}(r,T), \\ \widetilde{w} = \widetilde{u} & \text{on } \partial B_r \times (0,T), \\ \widetilde{w}|_{s=0} = \widetilde{u}_0|_{B_r} & \text{in } B_r. \end{cases} \tag{PS'}$$

The system (PS') has the inhomogeneous boundary condition on ∂B_r, and hence we further transform it with the homogeneous one. Since it holds that

$$\begin{aligned} \int_{\partial B_r} \widetilde{u}(s) \cdot \nu dS &= \int_{\{x - \xi^{\varepsilon_r}(s); x \in \partial\Omega\}} \widetilde{u}(s) \cdot \nu dS - \int_{\{x - \xi^{\varepsilon_r}(s); x \in \Omega \setminus B_r(\xi^{\varepsilon_r}(s))\}} \operatorname{div} \widetilde{u}(s) dy \\ &= 0, \end{aligned}$$

there exists $\widetilde{w}^* \in C^{2,1}(\overline{B_r} \times [0,T])$ such that

$$\begin{cases} \text{div } \widetilde{w}^* = 0 & \text{in } \widetilde{Q}(r,T), \\ \widetilde{w}^* = \widetilde{u} & \text{on } \partial B_r \times (0,T). \end{cases} \tag{2.14}$$

For such $\widetilde{w}^*$ we define $\widetilde{W} := \widetilde{w} - \widetilde{w}^*$. Then, (PS') can be transformed to the following system;

$$\begin{cases} \partial_s \widetilde{W} - \Delta_y \widetilde{W} + (\widetilde{u}, \nabla_y)\widetilde{W} + (\frac{d\xi^{\varepsilon_r}}{ds}, \nabla_y)\widetilde{W} + \nabla_y \widetilde{p} = \widetilde{F} & \text{in } \widetilde{Q}(r,T), \\ \text{div } \widetilde{W} = \displaystyle\sum_{i=1}^N \frac{\partial \widetilde{W}_i}{\partial y_i} = 0 & \text{in } \widetilde{Q}(r,T), \\ \widetilde{W} = 0 & \text{on } \partial B_r \times (0,T), \\ \widetilde{W}|_{s=0} = \widetilde{u}_0|_{B_r} - \widetilde{w}^*|_{s=0} & \text{in } B_r, \end{cases} \tag{PS''}$$

where $\widetilde{F} = -\partial_s \widetilde{w}^* + \Delta_y \widetilde{w}^* - (\widetilde{u}, \nabla_y)\widetilde{w}^* - (\frac{d\xi^{\varepsilon_r}}{ds}, \nabla_y)\widetilde{w}^*$.

Based on (PS"), we introduce the notion of very weak solutions of (PS) in $Q(r,T)$ by the following definition.

Definition 2.3.2. *Let $w \in L^2_{loc}(Q(r,T))$. Assume that $\tilde{w} = \Psi_* w$ is smooth in some neighborhood of ∂B_r. We say w is a very weak solution of (PS) in $Q(r,T)$ with the initial data $u_0 \in L^2_\sigma(B_r(\xi^{\varepsilon_r}(0)))$ if $\widetilde{W} = \widetilde{w} - \widetilde{w}^*$ with $\widetilde{w} = \Psi_* w$ and $\widetilde{w}^*$ in (2.14) satisfies that*

$$\int_{B_r} \widetilde{W}|_{s=0}(y) \cdot \widetilde{\varphi}(y,0)dy + \int_0^T \int_{B_r} \widetilde{W} \cdot \left\{ \partial_s \widetilde{\varphi} + \Delta \widetilde{\varphi} + (\widetilde{u}, \nabla)\widetilde{\varphi} + \left(\frac{d\xi^{\varepsilon_r}}{ds}, \nabla \right) \widetilde{\varphi} \right\} dyds$$
$$+ \int_0^T \int_{B_r} \widetilde{F} \cdot \widetilde{\varphi} dyds = 0 \tag{2.15}$$

for all $\widetilde{\varphi} \in H^{1,2}(0,T; L^2_\sigma(B_r)) \cap L^2(0,T; H^{2,2}(B_r) \cap H^{1,2}_0(B_r) \cap L^\infty(B_r))$ with $\widetilde{\varphi}(\cdot, T) = 0$ in B_r, and if $\widetilde{W}$ satisfies that

$$\int_{B_r} \widetilde{W}(s) \cdot \nabla \widetilde{\varrho} \, dy = 0, \quad 0 < s < T \tag{2.16}$$

for all $\widetilde{\varrho} \in H^{1,2}(B_r)$.

2.4 Strong solutions

We first consider the existence and uniqueness of the Stokes equations in $\Omega \times (0,T)$. In paticular, there exists a smooth very weak solution of the Stokes equations in the sense of Definition 2.3.1. Recall the Stokes operator A in $L^p_\sigma(\Omega)$ defined by $A_p = -P\Delta$, with the domain $D(A_p) = \{u \in H^{2,p}(\Omega) \cap L^p_\sigma(\Omega); u|_{\partial\Omega} = 0\}$, where P denotes the Helmholtz projection from $L^p(\Omega)$ onto $L^p_\sigma(\Omega)$.

Lemma 2.4.1. *Let $1 < p < \infty$. For every $a \in L^p_\sigma(\Omega)$ and $f \in C^\alpha([0,T); L^p(\Omega))$ with $\alpha > 0$, there exists a unique solution u of*

$$\begin{cases} \dfrac{du}{dt} + Au = Pf & \text{in } t \in (0,T), \\ u(0) = a, \end{cases} \tag{St'}$$

in the class $u \in C([0,T); L^p_\sigma(\Omega)) \cap C((0,T); D(A_p)) \cap C^1((0,T); L^p_\sigma(\Omega))$. The solution u is also a weak solution of the Stokes equations in $\Omega \times (0,T)$ in the sense of Definition 2.3.1. Moreover, if $f \in C^\infty(\Omega \times (0,T))$, then such a solution u satisfies in fact $u \in C^\infty(\Omega \times (0,T))$.

Lemma 2.4.2. (Giga-Sohr [6, Theorem 2.8, Lemma 3.2]) *Let $1 < q < \infty$ and $1 < s < \infty$. For every $a \in \mathcal{B}^{2(1-\frac{1}{s})}_{q,s}(\Omega)$ and $f \in L^s(0,T; L^q(\Omega))$, there exists a unique solution u of* (St') *in the class*

$$\begin{gathered} u \in L^s(0,T; D(A_q)),\ \partial_t u \in L^s(0,T; L^q(\Omega)), \\ u \in L^{s_0}(0,T; L^{q_0}(\Omega)) \quad \text{for } \frac{2}{s_0} + \frac{n}{q_0} = \frac{2}{s} + \frac{n}{q} - 2. \end{gathered}$$

Since Lemma 2.4.1 and 2.4.2 are the well-known results, we omit the proof. Finally, we introduce the existence theorem of the Navier-Stokes equations in the Lorentz-Besov spaces.

Proposition 2.4.3. (Kozono-Shimizu [17, Theorem 3]) *Let $1 < p < \infty$, $1 < \alpha < \infty$, $s > -1$ satisfy $2/\alpha + n/p - s = 3$. Let $1 \le q \le \infty$. Assume that $1 \le r \le p$ satisfies*

$$\frac{n}{r} < \frac{2}{\alpha} + \frac{n}{p}. \tag{2.17}$$

Then there exists a constant $\eta = \eta(p,\alpha,s,r,q)$ with the following property: if $a \in \dot{\mathcal{B}}^{-1+n/r}_{r,q}(\mathbb{R}^n)$ and $f \in L^{\alpha,q}(0,\infty; \dot{B}^s_{p,\infty}(\mathbb{R}^n))$ satisfy

$$\|a\|_{\dot{B}^{-1+n/r}_{r,q}} + \|f\|_{L^{\alpha,q}(0,\infty;\dot{B}^s_{p,\infty})} \le \eta, \tag{2.18}$$

there exists a solution u of

$$\begin{cases} \dfrac{du}{dt} + Au + P(u,\nabla)u = Pf & \text{a.e. } t \in \mathbb{R}^+ \text{ in } \dot{\mathcal{B}}^s_{p,\infty}(\mathbb{R}^n), \\ u(0) = a & \text{in } \dot{\mathcal{B}}^{-1+n/r}_{r,q}(\mathbb{R}^n), \end{cases} \tag{NS3}$$

in the class

$$u_t, Au \in L^{\alpha,q}(0,\infty;\dot{\mathcal{B}}^s_{p,\infty}(\mathbb{R}^n)) \tag{2.19}$$

Moreover, u satisfies that

$$u \in L^{\alpha_0,q}(0,\infty;\dot{\mathcal{B}}^{s_0}_{p_0,1}(\mathbb{R}^n)) \ for\ 2/\alpha_0 + n/p_0 - s_0 = 1 \tag{2.20}$$

with $p \le p_0$, $\alpha < \alpha_0$, *and* $\max\{s, n/r-1\} < s_0$.

Outline of the proof of Proposition 2.4.3. We first consider the linear problem:

$$\begin{cases} \dfrac{du}{dt} + Au = Pf \quad \text{a.e. } t \in (0,T) \text{ in } \dot{\mathcal{B}}^s_{p,\beta}(\mathbb{R}^n), \\ u(0) = a \quad \text{in } \dot{\mathcal{B}}^{-1+n/r}_{r,q}(\mathbb{R}^n). \end{cases} \tag{S}$$

Take $0<\theta<1$ and $k_0 < -1+n/r < k_1 < s+2$ so that $-1+n/r = (1-\theta)k_0 + \theta k_1$. By the estimate of the heat semigroup in Besov space proved by Kozono-Ogawa-Taniuchi [16][Lemma 2.2 (ii)] it holds that

$$\|Ae^{-tA}a\|_{\dot{B}^s_{p,1}(\mathbb{R}^n)} = \|e^{-tA}a\|_{\dot{B}^{s+2}_{p,1}(\mathbb{R}^n)} \le Ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})-\frac{1}{2}(s+2-k_i)}\|a\|_{\dot{B}^{k_i}_{r,\infty}(\mathbb{R}^n)}$$

for $i=0,1$. Thus, we see that the mapping

$$a \in \dot{\mathcal{B}}^{k_i}_{r,\infty}(\mathbb{R}^n) \to \|Ae^{-tA}a\|_{\dot{B}^s_{p,1}(\mathbb{R}^n)} \in L^{\alpha_i,\infty}(0,\infty)$$

is a bounded sub-additive operator for

$$\frac{1}{\alpha_i} = \frac{n}{2}\left(\frac{1}{r}-\frac{1}{p}\right) + \frac{1}{2}(s+2-k_i), \quad i=0,1.$$

By the real interpolation thorem it holds that

$$a \in (\dot{\mathcal{B}}^{k_0}_{r,\infty}(\mathbb{R}^n), \dot{\mathcal{B}}^{k_1}_{r,\infty}(\mathbb{R}^n))_{\theta,q} \to \|Ae^{-tA}a\|_{\dot{B}^s_{p,1}(\mathbb{R}^n)} \in (L^{\alpha_0,\infty}(0,\infty), L^{\alpha_1,\infty}(0,\infty))_{\theta,q}.$$

Since $(\dot{\mathcal{B}}^{k_0}_{r,\infty}(\mathbb{R}^n), \dot{\mathcal{B}}^{k_1}_{r,\infty}(\mathbb{R}^n))_{\theta,q} = \dot{\mathcal{B}}^{-1+n/r}_{r,q}(\mathbb{R}^n)$ and $(L^{\alpha_0,\infty}(0,\infty), L^{\alpha_1,\infty}(0,\infty))_{\theta,q} = L^{\alpha,q}(0,\infty)$, we see that the mapping

$$a \in \dot{\mathcal{B}}^{-1+n/r}_{r,q}(\mathbb{R}^n) \to \|Ae^{-tA}a\|_{\dot{B}^s_{p,1}(\mathbb{R}^n)} \in L^{\alpha,q}(0,\infty)$$

is a bounded sub-additive operator.

Let $s_0 < s < s_1 \le 1+n/r$ and $1 \le \beta \le \infty$. The usual maximal regularity in $\dot{H}^s_p(\mathbb{R}^n)$ implies that the mapping

$$S : f \in L^\alpha(0,T;\dot{H}^{s_i}_p(\mathbb{R}^n)) \to (u_t, Au) \in L^\alpha(0,T;\dot{H}^{s_i}_p(\mathbb{R}^n))^2, \quad i=0,1,$$

is a bounded operator with its norm independent of T. Since $(\dot{H}_p^{s_0}(\mathbb{R}^n)), \dot{H}_p^{s_1}(\mathbb{R}^n))_{\theta,\beta} = \dot{B}_{p,\infty}^s(\mathbb{R}^n)$ with $s = (1-\theta)s_0 + \theta s_1$, we see from the real interpolation that

$$S : f \in L^\alpha(0,T;\dot{B}_{p,\beta}^s(\mathbb{R}^n)) \to (u_t, Au) \in L^\alpha(0,T;\dot{B}_{p,\beta}^s(\mathbb{R}^n))^2$$

is a bounded operator with its norm independent of T. Similarly, by using the real interpolation in terms of the time we see that

$$S : f \in L^{\alpha,q}(0,T;\dot{B}_{p,\beta}^s(\mathbb{R}^n)) \to (u_t, Au) \in L^{\alpha,q}(0,T;\dot{B}_{p,\beta}^s(\mathbb{R}^n))^2$$

is a bounded operator with its norm independent of T. For $a \in \mathcal{B}_{r,q}^{-1+n/r}(\mathbb{R}^n)$ and $f \in L^{\alpha,q}(0,T;\dot{B}_{p,\beta}^s(\mathbb{R}^n))$, it holds that

$$u(t) = e^{-tA}a + Sf(t), \quad 0 < t < T$$

solves (S). As a consequence, we see that the followng result holds in the case of the linear problem:

Lemma 2.4.4. *Let $1 < p < \infty$, $1 < \alpha < \infty$, $1 \le \beta \le \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$ satisfy $2/\alpha + n/p - s = 3$. Assume that $1 \le r \le p$ satisfies*

$$\frac{n}{r} < \frac{2}{\alpha} + \frac{n}{p}.$$

For every $a \in \dot{\mathcal{B}}_{r,q}^{-1+n/r}(\mathbb{R}^n)$ and every $f \in L^{\alpha,q}(0,\infty;\dot{B}_{p,\beta}^s(\mathbb{R}^n))$ there exists a unique solution u of

$$\begin{cases} \dfrac{du}{dt} + Au = Pf & \text{a.e. } t \in \mathbb{R}^+ \text{ in } \dot{\mathcal{B}}_{p,\beta}^s(\mathbb{R}^n)\ , \\ u(0) = a & \text{in } \dot{\mathcal{B}}_{r,q}^{-1+n/r}(\mathbb{R}^n), \end{cases} \tag{S}$$

in the class

$$u_t, Au \in L^{\alpha,q}(0,\infty;\dot{\mathcal{B}}_{p,\beta}^s(\mathbb{R}^n)). \tag{2.21}$$

In the next step, we consider the nonlinear problem (NS3). Let $1 < p \le p_0 < \infty$, $1 < \alpha < \alpha_0 < \infty$, and $-\infty < s < s_0 < \infty$ satisfying

$$\frac{2}{\alpha_0} + \frac{n}{p_0} - s_0 = \frac{2}{\alpha} + \frac{n}{p} - s - 2 = 1$$

Let $1 < r \le p$ satisfy

Let $f(t) = u_t + Au$. Then, it holds that $u(t) = e^{-tA}a + Ff(t)$ where $Ff(t) = \displaystyle\int_0^t e^{-(t-\tau)A} f(\tau)d\tau$. Similarly to Proposition 2.4.3, we see that

$$\|e^{-tA}a\|_{L^{\alpha_0,q}(0,\infty;\dot{B}_{p_0,1}^{s_0}(\mathbb{R}^n))} \le C\|a\|_{\dot{\mathcal{B}}_{r,q}^{-1+n/r}(\mathbb{R}^n)} \tag{2.22}$$

where $C = C(n, p, \alpha, s, r, q)$ is independent of u, a, and T. Hence we determine $Ff(t)$. From Kozono-Ogawa-Taniuchi [16][Lemma 2.2] it follows that

$$\begin{aligned}\|Ff(t)\|_{\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n)} &\le \int_0^t \|e^{-(t-\tau)A} f(\tau)\|_{\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n)} d\tau \\ &\le C \int_0^t (t-\tau)^{\{1-\frac{n}{2}(\frac{1}{p}-\frac{1}{p_0})-\frac{1}{2}(s_0-s)\}-1} \|f(\tau)\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}\end{aligned}$$

It is noted that $1-\frac{n}{2}(\frac{1}{p}-\frac{1}{p_0})-\frac{1}{2}(s_0-s) = \frac{1}{\alpha}-\frac{1}{\alpha_0} < \frac{1}{\alpha}$. By the Hardy-Littlewood-Sobolev inequality in the Lorentz space we have that

$$\begin{aligned}\|Ff(t)\|_{L^{\alpha_0,q}(0,T;\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n))} &\le C\|f(\tau)\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p_0,\infty}(\mathbb{R}^n))} \\ &\le C\left(\|u_t\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p_0,\infty}(\mathbb{R}^n))} + \|Au\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p,\infty}(\mathbb{R}^n))}\right). \end{aligned} \tag{2.23}$$

We next estimate the non-linear term in the following lemma:

Lemma 2.4.5. *Let $1 < p < \infty$, $1 < \alpha < \infty$, $s > -1$ satisfy $2/\alpha + n/p - s = 3$. Let $1 < r \le p$, $1 \le q \le \infty$ and $0 < T \le \infty$. For measurable functions u and v in $\mathbb{R}^n \times (0, T)$ satisfying that*

$$\begin{gathered}u_t,\ Au,\ v_t,\ Av \in L^{\alpha,q}(0,T;\dot{\mathcal{B}}^s_{p,\infty}(\mathbb{R}^n)),\\ u(0) = a,\ v(0) = b \in \dot{\mathcal{B}}^{-1+n/r}_{r,q}(\mathbb{R}^n),\end{gathered}$$

it holds that $P(u,\nabla)v \in L^{\alpha,q}(0,T;\dot{B}^s_{p,\infty}(\mathbb{R}^n))$ with the estimate

$$\begin{aligned}&\|P(u,\nabla)v\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p,\infty}(\mathbb{R}^n))}\\ &\le C\left(\|e^{-tA}a\|_{L^{\alpha_0,q}(0,T;\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n))} + \|u_t\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p_0,\infty}(\mathbb{R}^n))} + \|Au\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p,\infty}(\mathbb{R}^n))}\right)\\ &\quad\times\left(\|e^{-tA}b\|_{L^{\alpha_0,q}(0,T;\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n))} + \|v_t\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p_0,\infty}(\mathbb{R}^n))} + \|Av\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p,\infty}(\mathbb{R}^n))}\right)\end{aligned} \tag{2.24}$$

for some $\alpha < \alpha_0 < \infty$, $1 < p \le p_0 < \infty$, and $s < s_0 < \infty$ such that $2/\alpha_0+n/p_0-s_0 = 1$, where $C = C(n, p, \alpha, p_0, \alpha_0, r, q)$ is independent of T.

Outline of the proof of Lemma 2.4.5. We take $\alpha_0 = 2\alpha$, $p_0 \le p$, and $s_0 \in \mathbb{R}$ so that

$$\max\{n/r - 1/\alpha, s + 2 - 1/\alpha\}, \quad \max\{s + 1, n/r - 1\} < s_0,$$

and

$$2/\alpha_0 + n/p_0 - s_0 = 1.$$

Since $0 < s + 1 < s_0$, we can take $\sigma > 0$ so that $\sigma < s_0 - (s + 1)$. Let us define p_1 and p_2 so that

$$n/p_1 = n/p_0 - \{s_0 - (s + 1) - \sigma\}, \quad n/p_2 = n/p_0 - (s_0 + \sigma) \tag{2.25}$$

Hence, we see from (2.25) and Proposition 2.1.2 (i) that

$$\dot{B}^{s_0}_{p_0,\infty}(\mathbb{R}^n) \hookrightarrow \dot{B}^{s+1+\sigma}_{p_1,\infty}(\mathbb{R}^n), \quad \dot{B}^{s_0}_{p_0,\infty}(\mathbb{R}^n) \hookrightarrow \dot{B}^{-\sigma}_{p_2,\infty}(\mathbb{R}^n) \tag{2.26}$$

It is noticed that $1/p_1 + 1/p_2 = 1/p$. By (2.26) and Proposition 2.1.2 (ii) we have that

$$\begin{aligned}
\|P(u,\nabla)v\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} &= \|\nabla\cdot P(u\otimes v)\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} = \|P(u\otimes v)\|_{\dot{B}^{s+1}_{p,\infty}(\mathbb{R}^n)} \\
&\le C\|u\otimes v\|_{\dot{B}^{s+1}_{p,\infty}(\mathbb{R}^n)} \\
&\le C\left(\|u\|_{\dot{B}^{s+1+\sigma}_{p_1,\infty}(\mathbb{R}^n)}\|v\|_{\dot{B}^{-\sigma}_{p_2,\infty}(\mathbb{R}^n)} + \|u\|_{\dot{B}^{-\sigma}_{p_2,\infty}(\mathbb{R}^n)}\|v\|_{\dot{B}^{s+1+\sigma}_{p_1,\infty}(\mathbb{R}^n)}\right) \\
&\le C\|u\|_{\dot{B}^{s_0}_{p_0,\infty}(\mathbb{R}^n)}\|v\|_{\dot{B}^{s_0}_{p_0,\infty}(\mathbb{R}^n)}.
\end{aligned}$$

By the Hölder inequality in the Lorentz space it holds that

$$\begin{aligned}
\|P(u,\nabla)v\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p,\infty}(\mathbb{R}^n))} &\le C\|u\|_{L^{\alpha_0,2q}(0,T;\dot{B}^{s_0}_{p_0,\infty}(\mathbb{R}^n))}\|v\|_{L^{\alpha_0,2q}(0,T;\dot{B}^{s_0}_{p_0,\infty}(\mathbb{R}^n))} \\
&\le C\|u\|_{L^{\alpha_0,q}(0,T;\dot{B}^{s_0}_{p_0,\infty}(\mathbb{R}^n))}\|v\|_{L^{\alpha_0,q}(0,T;\dot{B}^{s_0}_{p_0,\infty}(\mathbb{R}^n))},
\end{aligned} \tag{2.27}$$

where $C = C(n,p,\alpha,p_0,\alpha_0,q)$ is independnet of T. Since $n/p \le n/r < s_0 + 1 = 2/\alpha_0 + n/p_0$, it follows from (2.22) and (2.23) that

$$\begin{aligned}
&\|u\|_{L^{\alpha_0,q}(0,T;\dot{B}^{s_0}_{p_0,\infty}(\mathbb{R}^n))} \\
&\le C\left(\|e^{-tA}a\|_{L^{\alpha_0,q}(0,T;\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n))} + \|u_t\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p,\infty}(\mathbb{R}^n))} + \|Au\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p,\infty}(\mathbb{R}^n))}\right),
\end{aligned} \tag{2.28}$$

$$\begin{aligned}
&\|v\|_{L^{\alpha_0,q}(0,T;\dot{B}^{s_0}_{p_0,\infty}(\mathbb{R}^n))} \\
&\le C\left(\|e^{-tA}b\|_{L^{\alpha_0,q}(0,T;\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n))} + \|v_t\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p,\infty}(\mathbb{R}^n))} + \|Av\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p,\infty}(\mathbb{R}^n))}\right),
\end{aligned} \tag{2.29}$$

where $C = C(n,p,p_0,\alpha,\alpha_0,q)$ is independent of T. The desired estimate (2.24) is obtained from (2.27), (2.28), and (2.29). □

Finally, we construct the solution of (NS3) on $(0,T_*)$ for some $0 < T_* \le T$ by the successive approximation. For simplicity we show only the case $1 \le q < \infty$. Let $a \in \dot{\mathcal{B}}^{-1+n/r}_{r,q}(\mathbb{R}^n)$ and $f \in L^{\alpha,q}(0,T;\dot{B}^s_{p,\infty}(\mathbb{R}^n))$ for $2/\alpha + n/p - s = 3$ with $1 \le r \le p < \infty$, $1 < \alpha < \infty$, and $-1 < s$ satisfying (2.17). We define u_0 by

$$u_0(t) = e^{-tA}a + \int_0^t e^{-(t-\tau)A}Pf(\tau)d\tau,\ 0 < t < T.$$

By setting $u = u_0 + v$, we reduce the solvability of (NS3) to the construction of the solution v in the following equation:

$$\begin{cases} \dfrac{dv}{dt} + Av = -P((u_0,\nabla)v + (v,\nabla)u_0 + (v,\nabla)v + (u_0,\nabla)u_0) \quad \text{a.e. } t \in (0,T_*) \text{ in } \dot{\mathcal{B}}^s_{p,\infty}(\mathbb{R}^n), \\ v(0) = 0, \end{cases} \tag{NS3'}$$

It follows from (2.22) and (2.23) that

$$\left\|\frac{du_0}{dt}\right\|_{L^{\alpha,q}(0,\infty;\dot{B}^s_{p,\infty}(\mathbb{R}^n))}+\|Au_0\|_{L^{\alpha,q}(0,\infty;\dot{B}^s_{p,\infty}(\mathbb{R}^n))}\le C\left(\|a\|_{\dot{\mathcal{B}}^{-1+n/r}_{r,q}(\mathbb{R}^n)}+\|f(\tau)\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p_0,\infty}(\mathbb{R}^n))}\right) \tag{2.30}$$

where $C=C(n,p,\alpha,s,r,q)$ is independent of $0<T\le\infty$. Now we solve (NS2') by the following successive approximation:

$$\begin{cases}\dfrac{dv_0}{dt}+Av_0=-P(u_0,\nabla)u_0 \quad \text{a.e. } t\in(0,T) \text{ in } \dot{\mathcal{B}}^s_{p,\infty}(\mathbb{R}^n),\\ v_0(0)=0,\end{cases} \tag{NS3$'_0$}$$

$$\begin{cases}\dfrac{dv_{j+1}}{dt}+Av_{j+1}=-P((u_0,\nabla)v_j+(v_j,\nabla)u_0+(v_j,\nabla)v_j+(u_0,\nabla)u_0) \quad \text{a.e. } t\in(0,T) \text{ in } \dot{\mathcal{B}}^s_{p,\infty}(\mathbb{R}^n),\\ v_{j+1}(0)=0,\ \ j=0,1,\cdots.\end{cases} \tag{NS3$'_j$}$$

Set

$$X_T=\left\{v:\mathbb{R}^n\times(0,T)\to\mathbb{R}^n; v_t, Av\in L^{\alpha,q}(0,T;\dot{B}^s_{p_0,\infty}(\mathbb{R}^n)), v(0)=0, \|v\|_{X_T}<\infty\right\}$$

where

$$\|v\|_{X_T}=\|v_t\|_{L^{\alpha,q}(0,\infty;\dot{B}^s_{p,\infty}(\mathbb{R}^n))}+\|Av\|_{L^{\alpha,q}(0,\infty;\dot{B}^s_{p,\infty}(\mathbb{R}^n))}.$$

X_T is a Banach space equipped with the norm $\|\cdot\|_{X_T}$. We see from Lemma 2.4.5 and $a\in\dot{\mathcal{B}}^{-1+n/r}_{r,q}(\mathbb{R}^n)$ that

$$P(u_0,\nabla)u_0\in L^{\alpha,q}(0,T;\dot{B}^s_{p,\infty}(\mathbb{R}^n))$$

with the estimate

$$\|P(u_0,\nabla)u_0\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p,\infty}(\mathbb{R}^n))}\le C\left(\|e^{-tA}a\|_{L^{\alpha_0,q}(0,T;\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n))}+\|u_0\|_{X_T}\right)^2,$$

where $C=C(n,p,p_0,\alpha,\alpha_0,r,q)$ is a constant independent of $0<T\le\infty$. Therefore, it follows from Lemma 2.4.4 that there exists a unique solution v_0 of (NS3$'_0$) in the class X_T. Assuming that $v_j\in X_T$, it follows from Lemma 2.4.5 that

$$P((u_0,\nabla)v_j+(v_j,\nabla)u_0+(v_j,\nabla)v_j+(u_0,\nabla)u_0)\in L^{\alpha,q}(0,T;\dot{B}^s_{p,\infty}(\mathbb{R}^n))$$

with the estimate

$$\begin{aligned}&\|P((u_0,\nabla)v_j+(v_j,\nabla)u_0+(v_j,\nabla)v_j+(u_0,\nabla)u_0)\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p,\infty}(\mathbb{R}^n))}\\ \le& C\Bigg(2\left(\|e^{-tA}a\|_{L^{\alpha_0,q}(0,T;\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n))}+\|u_0\|_{X_T}\right)\|v_j\|_{X_T}+\|v_j\|^2_{X_T}\\ &+\left(\|e^{-tA}a\|_{L^{\alpha_0,q}(0,T;\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n))}+\|u_0\|_{X_T}\right)^2\Bigg)\end{aligned}$$

for some $\alpha_0 > \alpha$, $p_0 > p$, and $s_0 > s$ such that $2/\alpha_0 + n/p_0 - s_0 = 1$, where $C = C(n, p, p_0, \alpha, \alpha_0, r, q)$ is a constant independent of $0 < T \leq \infty$. Hence, we see from Lemma 2.4.4 that there exists a unique solution v_{j+1} of (NS3$'_j$) in the class X_T with the estimate

$$\begin{aligned}\|v_{j+1}\|_{X_T} \leq & C\left(2\left(\|e^{-tA}a\|_{L^{\alpha_0,q}(0,T;\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n))} + \|u_0\|_{X_T}\right)\|v_j\|_{X_T} + \|v_j\|^2_{X_T}\right. \\ & \left. + \left(\|e^{-tA}a\|_{L^{\alpha_0,q}(0,T;\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n))} + \|u_0\|_{X_T}\right)^2\right),\end{aligned} \tag{2.31}$$

where $C = C(n, p, p_0, \alpha, \alpha_0, r, q)$ is a constant independent of $0 < T \leq \infty$. By induction, we have that $v_j \in X_T$ for all $j \in \mathbb{N}$. Therefore, if there is $0 < T_* < T$ such that

$$\|e^{-tA}a\|_{L^{\alpha_0,q}(0,T_*;\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n))} + \|u_0\|_{X_{T_*}} < \frac{1}{4C}, \tag{2.32}$$

then we obtain from that

$$\begin{aligned}\|v_j\|_{X_{T_*}} \leq & \frac{1}{2C}\left(1 - 2C\left(\|e^{-tA}a\|_{L^{\alpha_0,q}(0,T_*;\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n))} + \|u_0\|_{X_{T_*}}\right)\right. \\ & \left. - \sqrt{1 - 4C\left(\|e^{-tA}a\|_{L^{\alpha_0,q}(0,T_*;\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n))} + \|u_0\|_{X_{T_*}}\right)}\right), \\ & \equiv K\end{aligned} \tag{2.33}$$

It should be noted that all constants C in (2.31), (2.32), and (2.33) are the same and independent of $0 < T \leq \infty$.

Define $w_j \equiv v_j - v_{j-1} (v_{-1} = 0)$. Then, we obtain from (NS3$'_j$) that

$$\begin{cases} \dfrac{dw_{j+1}}{dt} + Aw_{j+1} = -P((u_0, \nabla)w_j + (w_j, \nabla)u_0 + (v_j, \nabla)w_j + (w_j, \nabla)v_{j-1}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } t \in (0, T) \text{ in } \dot{\mathcal{B}}^s_{p,\infty}(\mathbb{R}^n), \\ w_{j+1}(0) = 0, \ j = 0, 1, \cdots . \end{cases}$$

Similarly to (2.31), we have by (2.33) that

$$\begin{aligned}\|w_{j+1}\|_{X_{T_*}} \leq & C\left(2\left(\|e^{-tA}a\|_{L^{\alpha_0,q}(0,T_*;\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n))} + \|u_0\|_{X_{T_*}}\right)\|w_j\|_{X_{T_*}}\right. \\ & \left. + \|v_j\|_{X_{T_*}}\|w_j\|_{X_{T_*}} + \|v_{j-1}\|_{X_{T_*}}\|w_j\|_{X_{T_*}}\right), \\ \leq & \ 2C\left(\|e^{-tA}a\|_{L^{\alpha_0,q}(0,T_*;\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n))} + \|u_0\|_{X_{T_*}} + K\right)\|w_j\|_{X_{T_*}}, \ j = 0, 1, \cdots .\end{aligned}$$

This yields that

$$\|w_j\|_{X_{T_*}} \leq \left\{2C\left(\|e^{-tA}a\|_{L^{\alpha_0,q}(0,T_*;\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n))} + \|u_0\|_{X_{T_*}} + K\right)\right\}^j \|v_0\|_{X_{T_*}}, \ j = 1, 2, \cdots .$$

By (2.33) it holds that

$$
\begin{aligned}
&2C\left(\|e^{-tA}a\|_{L^{\alpha_0,q}(0,T_*;\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n))}+\|u_0\|_{X_{T_*}}+K\right)\\
=&1-\sqrt{1-4C\left(\|e^{-tA}a\|_{L^{\alpha_0,q}(0,T_*;\dot{B}^{s_0}_{p_0,1}(\mathbb{R}^n))}+\|u_0\|_{X_{T_*}}\right)}<1,
\end{aligned}
$$

from which it follows that

$$\sum_{j=0}^{\infty}\|w_j\|_{x_{T_*}}<\infty.$$

Thus, there exists a limit function $v\in X_{T_*}$ of $\{v_j\}_{j=0}^{\infty}$ in X_{T_*}. Letting $j\to\infty$ in both sides of (NS3$'_j$), we see from Lemma 2.4.5 that v is a solution of (NS3') on $(0,T_*)$ if the hypothesis (2.32) is fulfilled. Since $1\le q<\infty$, we see from (2.22) and (2.30) that there exists $0<T_*\le T$ such that the condition (2.32) holds. Therefore, we have shown the existence of a solution u of (NS3) on $(0,T_*)$.

□

Chapter 3

Removable time-dependent singularities in the Stokes equations

Theorem 3.1. *Let $n \geq 3$ and let Ω be a bounded domain in $\mathbb{R}^n$ with the smooth boundary $\partial\Omega$. Suppose that $\xi \in C^\alpha([0,T];\mathbb{R}^n)$ for $0 < \alpha \leq \frac{1}{2}$ with the property that $\{\xi(t); 0 \leq t \leq T\} \subset \Omega$. Assume that $u_0 \in L^p_\sigma(\Omega)$ for some p satisfying $1 < p < \infty$. If u is a solution of (St) satisfying*

$$|u(x,t)| = o(|x-\xi(t)|^{2-n+(1/\alpha-2)}) \text{ as } x \to \xi(t) \text{ locally uniformly in } t \in (0,T), \qquad (*)$$

then the curve $\{\xi(t); 0 < t < T\}$ is a family of removable singularities of u in $\Omega \times (0,T)$.

Remark 3.2. (1) For $n = 2$, we need to change the condition $(*)$ of the solution u. Indeed, if u satisfies for any $\varepsilon \in (0,1)$, $|u(x,t)| \leq \varepsilon |x-\xi(t)|^{1/\alpha-2} \log \frac{1}{|x-\xi(t)|}$ for (x,t) with $0 < |x-\xi(t)| < \varepsilon$ uniformly in $t \in (0,T)$, then the curve $\{\xi(t); 0 < t < T\}$ is a family of removable singularities of u in $\Omega \times (0,T)$.

(2) For $\alpha > \frac{1}{2}$, Theorem 3.1.1 also holds if we assume that

$$|u(x,t)| = o(|x-\xi(t)|^{2-n}) \text{ as } x \to \xi(t) \text{ locally uniformly in } t \in (0,T).$$

(3) It should be noticed that the behavior $|x-\xi(t)|^{2-n}$ as $x \to \xi(t)$ coincides with that of the fundamental solution of the Laplace equation. Takahashi and Yanagida [29] obtained almost optimal condition on removability of moving singularities in the sense that there exist a curve $\xi \in C^1((0,T);\Omega)$ and a solution $u(x,t)$ of the heat equation in Q_T such that

$$|u(x,t)| = O(|x-\xi(t)|^{2-n}) \text{ as } x \to \xi(t) \text{ locally uniformly in } t \in (0,T).$$

On the other hand, by assuming the stronger condition as $(*)$ for $0 < \alpha \leq \frac{1}{2}$, we may handle more general moving singularities $\xi(t)$ with $\xi \in C^\alpha([0,T];\Omega)$. Hence, our result makes it clear the relation on removable time-dependent singularities between the Hölder exponent α of $\xi(t)$ and the singular order of the solution u around $x = \xi(t)$.

In this section, we give some lemmas for the proof of the main theorem. Our first aim is to show that the solution u of (St) is in fact a weak solution in the sense of Definition 2.3.1. To this end, we need to choose an appropriate family $\{\eta_r\}_{r>0}$ of cut-off functions with the property that $\eta_r(x,t) = 0$ near $x = \xi(t)$.

Proof of Theorem 3.1. We first prove that u is a very weak solution of the Stokes equations in $\Omega \times (0,T)$, in the sense of Definition 2.3.1. To show it we use the following proposition.

Proposition 3.3. *Let $\varphi \in C^\infty(\overline{\Omega} \times [0,T])$ with $\mathrm{div}\ \varphi = 0$ in $\overline{\Omega} \times [0,T]$, $\varphi = 0$ on $\partial\Omega \times [0,T]$, and $\varphi(\cdot,T) = 0$ in $\overline{\Omega}$, and $f_r := \mathrm{div}\ (\eta_r\varphi) = \nabla\eta_r \cdot \varphi$ where η_r is the same cut-off function as in (2.9). Let $D_{r,t} := \{x \in \mathbb{R}^n; 7r/10 < |x - \xi^{\varepsilon_r}(t)| < 4r/5\}$ for $t \in [0,T]$, where ξ^{ε_r} is the same as in the proof of Proposition 2.2.1. Then there exists constant $r_1 = r_1(\alpha, n, T)$ with the following properties: for every $r \in (0, r_1)$, there exists a function $v_r \in C^\infty(\bigcup_{0\le t\le T} D_{r,t} \times \{t\})$ such that $\mathrm{div}\ v_r = f_r$ in $\Omega \times [0,T]$, $v_r = 0$ in $\Omega \setminus D_{r,t}$ for $t \in [0,T]$, and such that*

$$|\Delta v_r(x,t)| \le Cr^{-2}, |\partial_t v_r(x,t)| \le Cr^{-1/\alpha} \text{ for } (x,t) \in \mathbb{R}^n \times [0,T], \tag{3.1}$$

where C is a constant independent of x, t, and r.

Proof. We first show the method of the construction of v_r following from Borchers-Sohr [2, Theorem 2.4]. Let $r_0 = 10L$ with the Hölder constant of ξ in $[0,T]$, $\varepsilon_r = (r/r_0)^{1/\alpha}$, and $r < r_1 := \min\{r_0, d\}$ with $d = \min_{t\in[0,T]} d(\xi(t), \partial\Omega)$. Then, $f_r(\cdot,t) \in C^\infty(D_{r,t})$ with $f_r(\cdot,t) = 0$ in $\Omega \setminus D_{r,t}$ for $t \in [0,T]$ satisfies

$$\begin{aligned}
\int_{D_{r,t}} f_r(\cdot,t)\, dx &= \int_{D_{r,t}} \nabla\eta_r \cdot \varphi\, dx \\
&= \int_{\partial B_{4r/5}(\xi^{\varepsilon_r}(t))} \eta_r\varphi \cdot \nu\, dS - \int_{\partial B_{7r/10}(\xi^{\varepsilon_r}(t))} \eta_r\varphi \cdot \nu\, dS + \int_{D_{r,t}} \eta_r \mathrm{div}\ \varphi\, dx \\
&= \int_{\partial B_{4r/5}(\xi^{\varepsilon_r}(t))} \varphi \cdot \nu\, dS = \int_{\partial\Omega} \varphi \cdot \nu\, dS - \int_{\Omega\setminus B_{4r/5}(\xi^{\varepsilon_r}(t))} \mathrm{div}\ \varphi\, dx = 0.
\end{aligned}$$

Since $D_{r,t}$ is an annulus region in $\mathbb{R}^n$, there exist an integer $N_0 = N_0(n)$ independent of $r \in (0, r_1)$ and $t \in [0,T]$, and a family $\{U_t^i\}_{i=1}^{N_0}$ of N_0 domains in $\mathbb{R}^n$ such that $\bigcup_{i=1}^{N_0} U_t^i \supset D_{r,t}$ and such that $D_{r,t}^i \equiv D_{r,t} \cap U_t^i$ is starshaped with respect to some ball $B_{r,t}^i$ in $\mathbb{R}^n$ for all $i = 1, \cdots, N_0$. It should be noticed that N_0 may be chosen independently of r and t since the time variable t plays a roll only for translation along the curve ξ^{ε_r} and since the radius r has no influence to the number of decomposition so that $D_{r,t}^i$ may be starshaped. Furthermore, there exists a family $\{\phi_t^i\}_{i=1}^{N_0}$ of N_0 smooth functions compactly supported in U_t^i such that $0 \leq \phi_t^i \leq 1$, and such that $\sum_{i=1}^{N_0} \phi_t^i(x) = 1$ for all $x \in D_{r,t}$. Let $J := \{(i,j) \in \{1, \cdots, N_0\} \times \{1, \cdots, N_0\}; D_{r,t}^i \cap D_{r,t}^j \neq \emptyset\}$. Define $\psi_{r,t}^{i,j} \in C_0^\infty(D_{r,t}^i \cap D_{r,t}^j)$ for $(i,j) \in \{1, \cdots, N_0\}^2$ in such a way that

$$\begin{cases} \displaystyle\int_{D_{r,t}} \psi_{r,t}^{i,j}(x)dx = 1, & (i,j) \in J, \\ \psi_{r,t}^{i,j}(x) \equiv 0, & (i,j) \in \{1, \cdots, N_0\}^2 \setminus J, \\ \psi_{r,t}^{i,j}(x) = \psi_{r,t}^{j,i}(x), & i,j = 1, \cdots, N_0. \end{cases} \tag{3.2}$$

Now, let us define a family $\{f_{r,t}^i\}_{i=1}^{N_0}$ by

$$f_{r,t}^i(y) = \phi_t^i(y) f_r(y,t) + \sum_{\substack{1 \leq j \leq N_0 \\ 1 \leq k \leq N_0}} a_{j,k}^i \left(\int_{D(r,t)} \phi_t^j(z) f_r(z,t) dz \right) \psi_{r,t}^{i,k}(y), \tag{3.3}$$

Here $\{a_{j,k}^i\}_{1 \leq i,j,k \leq n_0}$ may be chosen in such a way that

$$\begin{cases} f_r(y,t) = \displaystyle\sum_{i=1}^{N_0} f_{r,t}^i(y) \quad \text{for all} \;\; y \in D_{r,t}, \\ \displaystyle\int_{D_{r,t}^i} f_{r,t}^i(y)\, dy = 0, \quad i = 1, \cdots, N_0. \end{cases} \tag{3.4}$$

For the detail, see the Appendix below. Then we define the function $v_{r,t}$ by

$$v_{r,t}(x) = \sum_{i=1}^{N_0} \int_{D_{r,t}^i} G_{r,t}^i(x,y) f_{r,t}^i(y) dy, \qquad x \in D_{r,t} \tag{3.5}$$

Here $G_{r,t}^i$ has an expression

$$G_{r,t}^i(x,y) = \frac{x-y}{|x-y|^n} \int_{|x-y|}^\infty h_{r,t}^i \left(y + \frac{x-y}{|x-y|} s \right) s^{n-1} ds, \tag{3.6}$$

where $h_{r,t}^i \in C_0^\infty(B_{r,t}^i)$ with $\int_{B_{r,t}^i} h_{r,t}^i(x)dx = 1$. Since for every $i = 1, ..., N_0$ $h_{r,t}^i(y + \frac{x-y}{|x-y|}s) = 0$ without relation to r and $t \in [0,T]$ if $s \in [|x-y|, \infty)$, $x \in \Omega \setminus D_{r,t}$, and $y \in D_{r,t}$, $G_{r,t}^i(x,y)$ also does, which implies that $v_{r,t} = 0$ in $\Omega \setminus D_{r,t}$ for all $r > 0$ and $t \in [0,T]$. We next show the divergence condition. Set $v_{r,t}^i(x) = \int_{D_{r,t}^i} G_{r,t}^i(x,y) f_{r,t}^i(y) dy$ and $v_{r,t,\varepsilon}^i(x) = \int_{\varepsilon \leq |x-y|} G_{r,t}^i(x,y) f_{r,t}^i(y) dy$ for $\varepsilon > 0$. Then, we have that $\lim_{\varepsilon \to +0} v_{r,t,\varepsilon}^i =$

$v^i_{r,t}$. By the direct culculation we have that

$$
\begin{aligned}
\frac{\partial (G^i_{r,t})_k}{\partial x_k}(x,y) &= \left\{ \frac{1}{|x-y|^n} - \frac{n(x_k-y_k)^2}{|x-y|^{n+2}} \right\} \int_{|x-y|}^{\infty} h^i_{r,t}\left(y + \frac{x-y}{|x-y|}s \right) s^{n-1} ds \\
&\quad + \frac{x_k-y_k}{|x-y|^{n+1}} \int_{|x-y|}^{\infty} \frac{\partial h^i_{r,t}}{\partial x_k}\left(y + \frac{x-y}{|x-y|}s \right) s^n ds \\
&\quad - \sum_{l=1}^{n} \frac{(x_l-y_l)(x_k-y_k)^2}{|x-y|^{n+3}} \int_{|x-y|}^{\infty} \frac{\partial h^i_{r,t}}{\partial x_l}\left(y + \frac{x-y}{|x-y|}s \right) s^n ds \\
&\quad - h^i_{r,t}(x) \frac{(x_k-y_k)^2}{|x-y|^2}
\end{aligned}
\tag{3.7}
$$

where $(G^i_{r,t})_k$ is the k-th element of $G^i_{r,t}$ for $k = 1, \cdots, n$. Changing variables $z = (x-y)/\varepsilon$ and $s = \varepsilon + \tau$, we have that

$$
\int_{|x-y|=\varepsilon} \frac{x_k-y_k}{|x-y|} (G^i_{r,t})_k(x,y) f^i_{r,t}(y) dS_y = \int_{|z|=1} z_k^2 \int_0^\infty h^i_{r,t}(x+\tau z)(\varepsilon+\tau)^{n-1} d\tau f^i_{r,t}(x-\varepsilon z) dS_z.
\tag{3.8}
$$

From (3.4), (3.7), and (3.8) it follows that

$$
\begin{aligned}
\operatorname{div} v^i_{r,t,\varepsilon}(x) &= \sum_{k=1}^{n} \left(\int_{\varepsilon \le |x-y|} \frac{\partial (G^i_{r,t})_k}{\partial x_k}(x,y) f^i_{r,t}(y) dy + \int_{|x-y|=\varepsilon} \frac{x_k-y_k}{|x-y|} (G^i_{r,t})_k(x,y) f^i_{r,t}(y) dS_y \right) \\
&= \int_{|z|=1} \int_0^\infty h^i_{r,t}(x+\tau z)(\varepsilon+\tau)^{n-1} d\tau f^i_{r,t}(x-\varepsilon z) dS_z \\
&\xrightarrow[\varepsilon \to +0]{} \int_{|z|=1} \int_0^\infty h^i_{r,t}(x+\tau z) \tau^{n-1} d\tau f^i_{r,t}(x) dS_z \\
&\quad = \int_{\mathbb{R}^n} h^i_{r,t}(y) dy f^i_{r,t}(x) = f^i_{r,t}(x)
\end{aligned}
$$

Therefore, $\operatorname{div} v_{r,t} = \sum_{i=1}^{N_0} \operatorname{div} v^i_{r,t} = \sum_{i=1}^{N_0} f^i_{r,t} = f_{r,t}$.

Finally, we show the estimation (3.1). Define $v_r(x,t) := v_{r,t}(x)$, $G^i_r(x,y,t) := G^i_{r,t}(x,y)$, $f^i_r(x,t) := f^i_{r,t}(x)$, $\phi_i(x,t) := \phi^i_t(x)$, $h^i_r(x,t) := h^i_{r,t}(x)$, $\psi^{i,j}_r(x,t) := \psi^{i,j}_{r,t}(x)$, and $v^i_r(x,t) := v^i_{r,t}(x)$. Since $\sum_{i=1}^{N_0} v^i_r = v_r$ and since N_0 is independent of t and r, it suffices to show that the following estimates hold: for every $1 \le i \le N_0$,

$$
|\Delta v^i_r(x,t)| \le C r^{-2}, |\partial_t v^i_r(x,t)| \le C r^{-2} \quad \text{for } (x,t) \in \mathbb{R}^n \times [0,T].
\tag{3.9}
$$

where C is a constant independent of x, t, r.

Let B_r be the open ball with the radius r centered at the origin 0 for $r > 0$. Let us take $h_i \in C_0^\infty(B_1)$ and $\psi_{i,j} \in C_0^\infty(D^i_{1,t} \cap D^j_{1,t})$ in such a way that $\int_{B_1} h_i(x) dx = 1$ for $i = 1, \cdots, N_0$, $\int_{D^i_{1,t} \cap D^j_{1,t}} \psi_{i,j}(x) dx = 1$ for $(i,j) \in J$. The functions $\{h^i_r\}_{i=1}^{N_0}$ in (3.6) and

$\{\psi_r^{i,j}\}_{(i,j)\in J}$ in (3.3) may be chosen as

$$h_r^i(x,t) = \frac{1}{r^n} h_i \left(\frac{x - x_r^i + \xi^{\varepsilon_r}(0) - \xi^{\varepsilon_r}(t)}{r} \right), \qquad i \in \{1, \cdots, N_0\}, \tag{3.10}$$

$$\psi_r^{i,j}(x,t) = \frac{1}{r^n} \psi_{i,j} \left(\frac{x - \xi^{\varepsilon_r}(t)}{r} \right), \qquad (i,j) \in J, \tag{3.11}$$

for $(x,t) \in \mathbb{R}^n \times [0,T]$ respectively. Here x_r^i is the center of the ball $B_{r,0}^i$, $i = 1, \cdots, N_0$. It is noticed that

$$D_{r,t}^i \subset \{y \in \mathbb{R}^n; |x-y| \le 2r\} \quad \text{for any } x \in D_{r,t}^i. \tag{3.12}$$

We have by (3.3), (3.5), and (3.12) that

$$\begin{aligned}
|\Delta v_r^i(x,t)| &\le \left| \Delta_x \int_{|x-y|\le 2r} G_r^i(x,y,t) f_r^i(y,t) dy \right| \\
&\le \left| \Delta_x \int_{|x'|\le 2r} G_r^i(x,x+x',t) f_r^i(x+x',t) dx' \right| \\
&\le \left| \Delta_x \int_{|x'|\le 2r} G_r^i(x,x+x',t) \phi_i(x+x',t) f_r(x+x',t) dx' \right| \\
&+ \left| \Delta_x \int_{|x'|\le 2r} G_r^i(x,x+x',t) \sum_{\substack{1\le j\le N_0 \\ 1\le k\le N_0}} a_{j,k}^i \left(\int_{D(r,t)} \phi_t^j(z) f_r(z,t) dz \right) \psi_r^{i,k}(x+x',t) dx' \right| \\
&\equiv I_1(r,t) + I_2(r,t),
\end{aligned}$$

$$\begin{aligned}
|\partial_t v_r^i(x,t)| &\le \left| \partial_t \int_{|x-y|\le 2r} G_r^i(x,y,t) f_r^i(y,t) dy \right| \\
&\le \left| \partial_t \int_{|x'|\le 2r} G_r^i(x,x+x',t) f_r^i(x+x',t) dx' \right| \\
&\le \left| \partial_t \int_{|x'|\le 2r} G_r^i(x,x+x',t) \phi_i(x+x',t) f_r(x+x',t) dx' \right| \\
&\quad + \left| \partial_t \int_{|x'|\le 2r} G_r^i(x,x+x',t) \sum_{\substack{1\le j\le N_0 \\ 1\le k\le N_0}} a_{j,k}^i \left(\int_{D(r,t)} \phi_t^j(z) f_r(z,t) dz \right) \psi_r^{i,k}(x+x',t) dx' \right| \\
&\equiv I_3(r,t) + I_4(r,t).
\end{aligned}$$

In the next step, we investigate $I_j(r,t)$ for $j = 1, \cdots, 4$. Since $h_i \in C_0^\infty(B_1)$, we have by (2.8), (3.6), and (3.10) that

$$\begin{aligned}
&|G_r^i(x, x+x', t)| \\
=&\left| \frac{x'}{|x'|^n} \int_{|x'|}^{|x'|+2r} \frac{1}{r^n} h_i \left(\frac{x + x' - \frac{x'}{|x'|}s - x_r^i + \xi^{\varepsilon_r}(0) - \xi^{\varepsilon_r}(t)}{r} \right) s^{n-1} ds \right| \\
\le& C|x'|^{1-n} r^{-n} \int_{|x'|}^{|x'|+2r} s^{n-1} ds,
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
&|\nabla_x G_r^i(x, x+x', t)| \\
=&\left| \nabla_x \frac{x'}{|x'|^n} \int_{|x'|}^{|x'|+2r} \frac{1}{r^n} h_i \left(\frac{x + x' - \frac{x'}{|x'|}s - x_r^i + \xi^{\varepsilon_r}(0) - \xi^{\varepsilon_r}(t)}{r} \right) s^{n-1} ds \right| \\
\le& C|x'|^{1-n} r^{-n-1} \int_{|x'|}^{|x'|+2r} s^{n-1} ds,
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
&|\Delta_x G_r^i(x, x+x', t)| \\
=&\left| \Delta_x \frac{x'}{|x'|^n} \int_{|x'|}^{|x'|+2r} \frac{1}{r^n} h_i \left(\frac{x + x' - \frac{x'}{|x'|}s - x_r^i + \xi^{\varepsilon_r}(0) - \xi^{\varepsilon_r}(t)}{r} \right) s^{n-1} ds \right| \\
\le& C|x'|^{1-n} r^{-n-2} \int_{|x'|}^{|x'|+2r} s^{n-1} ds,
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
&|\partial_t G_r^i(x, x+x', t)| \\
=&\left| \partial_t \frac{x'}{|x'|^n} \int_{|x'|}^{|x'|+2r} \frac{1}{r^n} h_i \left(\frac{x + x' - \frac{x'}{|x'|}s - x_r^i + \xi^{\varepsilon_r}(0) - \xi^{\varepsilon_r}(t)}{r} \right) s^{n-1} ds \right| \\
\le& C|x'|^{1-n} r^{-n-1} \left| \frac{d\xi^{\varepsilon_r}}{dt}(t) \right| \int_{|x'|}^{|x'|+2r} s^{n-1} ds \\
\le& C|x'|^{1-n} r^{-n-1/\alpha} \int_{|x'|}^{|x'|+2r} s^{n-1} ds.
\end{aligned} \tag{3.16}$$

Hence, it follows from (2.8), (3.11), (3.13) - (3.16), and Lemma 2.1 that

$$\begin{aligned}
I_1(r, t) \le& \left| \Delta_x \int_{|x'| \le 2r} G_r^i(x, x+x', t) \phi_i(x+x', t) \nabla \eta_r(x+x', t) \cdot \varphi(x+x', t) dx' \right| \\
\le& C \int_{|x'| \le 2r} |x'|^{1-n} \Bigg\{ (r^{-1} + r^{-2} + r^{-3}) r^{-n} \int_{|x'|}^{|x'|+2r} s^{n-1} ds
\end{aligned} \tag{3.17}$$

$$+(r^{-1}+r^{-2})r^{-n-1}\int_{|x'|}^{|x'|+2r} s^{n-1}ds + r^{-1}r^{-n-2}\int_{|x'|}^{|x'|+2r} s^{n-1}ds\Bigg\} dx'$$
$$\le Cr^{-2},$$

$$\begin{aligned} I_2(r,t) \le &\Bigg|\Delta_x \int_{|x'|\le 2r} G_r^i(x, x+x', t) \\ &\sum_{\substack{1\le j\le N_0\\1\le k\le N_0}} a_{j,k}^i \left(\int_{D_{r,t}} \phi_j(z,t)\nabla\eta_r(z,t)\cdot\varphi(z,t)dz\right)\frac{1}{r^n}\psi_{i,k}\left(\frac{x+x'-\xi^{\varepsilon_r}(t)}{r}\right)dx'\Bigg| \\ &\le Cr^{-2n-3}\int_{D_{r,t}} dz \int_{|x'|\le 2r}|x'|^{1-n}\left\{\int_{|x'|}^{|x'|+2r} s^{n-1}ds\right\}dx' \\ &\le Cr^{-2}, \end{aligned} \tag{3.18}$$

$$\begin{aligned} I_3(r,t) \le &\left|\partial_t\int_{|x'|\le 2r} G_r^i(x,x+x',t)\phi_i(x+x',t)\nabla\eta_r(x+x',t)\cdot\varphi(x+x',t)dx'\right| \\ \le & C\int_{|x'|\le 2r}|x'|^{1-n}\Bigg\{r^{-1}r^{-n-1/\alpha}\int_{|x'|}^{|x'|+2r}s^{n-1}ds \\ &+(r^{-1}+r^{-1-1/\alpha})r^{-n}\int_{|x'|}^{|x'|+2r}s^{n-1}ds\Bigg\}dx' \\ \le & Cr^{-1/\alpha}, \end{aligned} \tag{3.19}$$

$$\begin{aligned} I_4(r,t) \le &\Bigg|\partial_t \int_{|x'|\le 2r} G_r^i(x, x+x', t) \\ &\sum_{\substack{1\le j\le N_0\\1\le k\le N_0}} a_{j,k}^i \left(\int_{D_{r,t}} \phi_j(z,t)\nabla\eta_r(z,t)\cdot\varphi(z,t)dz\right)\frac{1}{r^n}\psi_{i,k}\left(\frac{x+x'-\xi^{\varepsilon_r}(t)}{r}\right)dx'\Bigg| \\ &\le Cr^{-n}\int_{|x'|\le 2r}|x'|^{1-n}\Bigg\{r^{-n-1/\alpha}\int_{|x'|}^{|x'|+2r}s^{n-1}ds\int_{D_{r,t}}r^{-1}dz \\ &+r^{-n}\int_{|x'|}^{|x'|+2r}s^{n-1}ds\int_{D_{r,t}}r^{-1}\left|\frac{d\xi^{\varepsilon_r}}{dt}(t)\right|dz\Bigg\}dx' \\ &\le Cr^{-1/\alpha}. \end{aligned} \tag{3.20}$$

Now, the desired estimate (3.1) is a consequence of (3.17) - (3.20). This proves Proposition 3.3. □

Next, we show that u is in fact a weak solution of the Stokes equations in $\Omega \times (0,T)$.

Proposition 3.4. *Let $n \geq 3$ and let Ω be a bounded domain in $\mathbb{R}^n$ with the smooth boundary $\partial\Omega$. Suppose that $\xi \in C^\alpha([0,T];\Omega)$ for $0 < \alpha \leq \frac{1}{2}$. If u is a solution of (St) satisfying the condition $(*)$, then u is necessarily a very weak solution of the Stokes equations in $\Omega \times (0,T)$ in the sense of Definition 2.3.1.*

Proof. Let $\varrho \in C^\infty(\overline{\Omega})$. Since $|u(x,t)| = o(|x-\xi(t)|^{2-n+(1/\alpha-2)})$ as $x \to \xi(t)$ uniformly in $t \in (0,T)$, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that $|u(x,t)| \leq \varepsilon|x-\xi(t)|^{2-n+(1/\alpha-2)}$ for all $(x,t) \in \bigcup_{0<t<T} B_\delta(\xi(t)) \times \{t\}$ where $B_\delta(\xi(t)) = \{x \in \mathbb{R}^n \ ; \ |x-\xi(t)| < \delta\}$. Then, we easily see that $u \in L^q(\Omega \times (0,T))$ for $1 < q < \infty$ in the case $0 < \alpha \leq \frac{1}{n}$, or for $1 < q < \frac{n}{n-1/\alpha}$ in the case $\frac{1}{n} < \alpha \leq \frac{1}{2}$. By the boundedness of Ω, $u_0 \in L^{p_1}(\Omega)$ and $u \in L^{p_1}(\Omega \times (0,T))$ hold for some $1 < p_1 \leq p < \infty$.

Take $r_1 > 0$ and the function v_r given by Proposition 3.3 and let $r < r_1$. Since $\eta_r\varphi - v_r$ is identically zero near some neighborhood of $\xi(t)$ fot $t \in [0,T]$ with div $(\eta_r\varphi - v_r) = 0$ in $\Omega \times [0,T]$, by multiplying (St) by $\eta_r\varphi - v_r$ and then by integrating by parts, we have that

$$\int_\Omega u_0 \cdot (\eta_r(\cdot,0)\varphi(\cdot,0) - v_r(\cdot,0))dx + \int_0^T \int_\Omega u \cdot (\partial_t(\eta_r\varphi - v_r) + \Delta(\eta_r\varphi - v_r))dxdt = 0. \quad (3.21)$$

Taking $r < \min\{r_1, \varepsilon, \delta\}$, we have by (2.4), (2.5) and (3.1) that

$$\begin{aligned}\left|\int_\Omega u_0 \cdot ((\eta_r(\cdot,0)\varphi(\cdot,0) - v_r(\cdot,0))dx\right| &\leq C\|u_0\|_{L^p(\Omega)} \left(\int_{B_r(\xi(0))} dx\right)^{1-\frac{1}{p}} \\ &\leq Cr^{n(1-\frac{1}{p})} \\ &\leq C\varepsilon^{n(1-\frac{1}{p})},\end{aligned}$$

$$\begin{aligned}\left|\int_0^T \int_\Omega u \cdot \{\partial_t(\eta_r\varphi - v_r) - \partial_t\varphi\}dxdt\right| &\leq C\varepsilon(1+r^{-1/\alpha}) \int_0^T \int_{B_r(\xi(t))} |x-\xi(t)|^{2-n+(1/\alpha-2)}dxdt \\ &\leq C\varepsilon,\end{aligned}$$

$$\begin{aligned}\left|\int_0^T \int_\Omega u \cdot \{\Delta(\eta_r\varphi - v_r) - \Delta\varphi\}dxdt\right| &\leq C\varepsilon(1+r^{-1}+r^{-2}) \int_0^T \int_{B_r(\xi(t))} |x-\xi(t)|^{2-n+(1/\alpha-2)}dxdt \\ &\leq C\varepsilon r^{1/\alpha-2}.\end{aligned}$$

Since ε is arbitrary and since $0 < \alpha \leq 1/2$, from the above estimate, we obtain the identity (2.12).

From (St) we have that

$$\int_\Omega u(t)\cdot\nabla\varrho\;dx = \int_{\Omega\setminus B_r(\xi(t))} \mathrm{div}\;u(t)\;\varrho\;dx + \int_{B_r(\xi(t))} u(t)\cdot\nabla\varrho\;dx \qquad (3.22)$$
$$= \int_{B_r(\xi(t))} u(t)\cdot\nabla\varrho\;dx.$$

For $r<\delta$, we have by (3.22) that

$$\left|\int_{B_r(\xi(t))} u(t)\cdot\nabla\varrho\;dx\right| \le C\int_{B_r(\xi(t))} \varepsilon|x-\xi(t)|^{2-n+(1/\alpha-2)}dx \le C\varepsilon r^{1/\alpha}.$$

Since ε is arbitrary and since $0<\alpha\le 1/2$, this shows (2.13). Consequently, we see that u is a very weak solution of the Stokes equations in $\Omega\times(0,T)$ in the sense of Definition 2.3.1. □

Finally, for the proof of Theorem 3.1 we may show the following proposition.

Proposition 3.5. *Let u and v be very weak solutions of the Stokes equations in $\Omega\times(0,T)$ in the sense of Definition 2.3.1. Then, it holds that $u\equiv v$ in $\Omega\times(0,T)$.*

Proof. Let u and v be two very weak solutions of the Stokes equations in $\Omega\times(0,T)$ in the sense of Definition 2.3.1. It suffices to show that

$$\int_0^T\int_\Omega (u-v)\cdot F\;dxdt = 0$$

for all $F\in C_0^\infty(\Omega\times(0,T))$. By Lemma 2.2.1, for every $F\in C_0^\infty(\Omega\times(0,T))$, there exists a unique solution $\{\varphi,p\}\in C^\infty(\overline{\Omega}\times[0,T])\times C^\infty(\overline{\Omega}\times(0,T))$ such that

$$\begin{cases} \partial_t\varphi+\Delta\varphi+\nabla p = F & \text{in } \Omega\times(0,T),\\ \mathrm{div}\;\varphi = 0 & \text{in } \Omega\times(0,T),\\ \varphi = 0 & \text{on } \partial\Omega,\\ \varphi|_{t=T} = 0 & \text{in } \Omega, \end{cases} \qquad (\text{St''})$$

By (2.13), it holds that

$$\int_0^T\int_\Omega (u-v)\cdot\nabla p\;dxdt = 0, \qquad (3.23)$$

Applying (2.12) with w replaced by $u-v$, together with the fact that $w_0=u(0)-v(0)=0$, we obtain from (3.23) that

$$\begin{aligned}\int_0^T\int_\Omega (u-v)\cdot F\;dxdt &= \int_0^T\int_\Omega (u-v)\cdot(\partial_t\varphi+\Delta\varphi)dxdt\\ &= 0\end{aligned}$$

This proves Proposition 3.5. □

Completion of the proof of Theorem

Since $u_0 \in L^p_\sigma(\Omega)$ for some p satisfying $1 < p < \infty$, by Lemma 2.2.1 there exists a unique solution v of the Stokes equations with $v(0) = u_0$ having the property that $v \in C^\infty(\Omega \times (0,T))$. Hence, it follows from Proposition 3.5 that $u(t) = v(t)$ for all $t \in (0,T)$, which shows that $u \in C^\infty(\Omega \times (0,T))$. This completes the proof of Theorem.

$\square$

Chapter 4

Removable time-dependent singularities in the Navier-Stokes equations

Theorem 4.1. *Let $n \geq 3$ and let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Suppose that $\xi \in C^\alpha([0,T];\Omega)$ for $1/n < \alpha \leq 1$ and that*

$$u_0 \in \begin{cases} L^{n,\infty}_\sigma(\Omega) & \text{for } n = 3, \\ \mathcal{B}^{2(1-\frac{1}{s})}_{q,s}(\Omega) = (L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{s},s} & \text{for } n \geq 4, \end{cases} \tag{4.1}$$

where $\frac{2}{s} + \frac{n}{q} = 3$ with $\max\{\frac{n}{3}, 2\} < q < n$. If u is a smooth solution of (NS) *in Q_T satisfying*

$$|u(x,t)| = o(|x - \xi(t)|^{-n+\beta}) \quad \text{locally uniformly in } t \in (0,T) \text{ as } x \to \xi(t) \tag{4.2}$$

for $\beta = \max\{1/\alpha, n-1\}$, then the curve $\{\xi(t); 0 < t < T\}$ is a family of removable singularities of u in $\Omega \times (0,T)$.

Remark 4.2. (1) In the Stokes equations, we see in the previous chapter that the exponent α of Hölder continuity of singularities for $\xi(t)$ may be handled for all $0 < \alpha \leq 1$. On the other hand, in the case of the Navier-Stokes equations, it seems to be difficult to deal with the exponent for $0 < \alpha \leq 1/n$. Indeed, our method is related to the cut-off procedure, and it is necessary to take some δ-neighborhood around singularities $\{\xi(t)\}_{0<t<T}$ on which the $L^{n,\infty}$-norm of the solution u is sufficiently small. In such a procedure, we need to handle remainder term with its singular order $\delta^{n-\frac{1}{\alpha}}$ arising from the time-derivative of the cut-off function. It should be noticed that such a harmful term dose not appear in the case of the Stokes equations since we do not need any smallness of the solution in the whole region $\Omega \times (0,T)$. This is the reason why we impose the Hölder continuity α on the restriction that $\alpha > \frac{1}{n}$.

(2) Even if the Hölder exponent α of moving singularity $\xi(t)$ satisfies $\alpha > \frac{1}{n-1}$, it is required that $|u(x,t)| = o(|x - \xi(t)|^{-1})$ as $x \to \xi(t)$ locally uniformly in $t \in (0,T)$,

that is, $\beta = n-1$. This seems to be natural since there exists a singular solution with the homogeneous degree -1, so-called the Landau solution to the 3-D stationary Navier-Stokes equations.

(3) Takahashi-Yanagida [29] introduced a family $\{\eta_r\}_{r>0}$ of cut-off functions near singularities $\{\xi(t)\}_{0<t<T}$, and showed that the singular behavior near $r=0$ of $\partial_t\eta_r$ is in proportion to $-1/\alpha$. To cancel such a behavior as $\partial_t\eta_r = O(r^{-\frac{1}{\alpha}})$ as $r\to+0$, we need to impose $\beta\geq 1/\alpha$ on u as in (4.2). Therefore, $\alpha = 1/(n-1)$ is an expected borderline of the Hölder exponent of $\xi(t)$.

We first introduce some lemmta and propositions to show Theorem 5.1. The following lemma is essentially due to Bogovskii [1].

Proposition 4.3. *Let $n\geq 3$ and $1\leq p<n$. Let Ω be a bounded domain in $\mathbb{R}^n$ with the smooth boundary $\partial\Omega$ and let $r<d$ with $d=\min_{t\in[0,T]} d(\xi(t),\partial\Omega)$. Assume that η_δ be the same cut-off function as in Lemma 2.1 with r replaced by δ. Suppose that $\xi\in C^\alpha([0,T];\Omega)$ for $0<\alpha\leq 1$ and that $\widetilde{\varphi}\in H^{1,2}(0,T;L^p_\sigma(B_r))\cap L^2(0,T;H^{2,p}(B_r)\cap H^{1,2}_0(B_r))\cap L^2(0,T;L^\infty(B_r))$ with $\widetilde{\varphi}(\cdot,T)=0$ in B_r. Then there exists $r_1=r_1(\alpha,r,n,T)$ with the following properties: for every $\delta\in(0,r_1)$, there exists a function v_δ such that*

$$\operatorname{supp} v_\delta(\cdot,t)\subset D_{\delta,t} := \{x\in\mathbb{R}^n; 7\delta/10<|x-\xi^{\varepsilon_\delta}(t)|<4\delta/5\}\ \mathit{for}\ t\in[0,T]; \tag{i}$$

$$\operatorname{div} v_\delta = \varphi\cdot\nabla\eta_\delta\ \mathit{in}\ \bigcup_{0\leq t\leq T} D_{\delta,t}\times\{t\}\ \mathit{with}\ \varphi=\Psi_*^{-1}\widetilde{\varphi}; \tag{ii}$$

$$\begin{aligned}
&\|v_\delta(t)\|_{L^p(D_{\delta,t})}\leq C\delta^{\frac{n}{p}}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))}, \qquad\qquad \text{(iii)}\\
&\|\nabla v_\delta(t)\|_{L^p(D_{\delta,t})}\leq C\delta^{\frac{n}{p}-1}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))}+C\|\varphi(t)\|_{\dot H^{1,p}(B_{3\delta}(\xi^{\varepsilon_\delta}(t)))},\\
&\|\Delta v_\delta(t)\|_{L^p(D_{\delta,t})}\leq C\delta^{\frac{n}{p}-2}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))}+C\|\varphi(t)\|_{\dot H^{2,p}(B_{3\delta}(\xi^{\varepsilon_\delta}(t)))},\\
&\|\partial_t v_\delta(t)\|_{L^p(D_{\delta,t})}\leq C\delta^{\frac{n}{p}-\frac{1}{\alpha}}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))}+C\|\partial_t\varphi(t)\|_{L^p(B_{3\delta}(\xi^{\varepsilon_\delta}(t)))}
\end{aligned}$$

for all $t\in(0,T)$, where $C=C(\alpha,n,T)$ is independent of t and δ, $\varepsilon_r=(r/10L)^{1/\alpha}$, and $\varepsilon_\delta=(\delta/10L)^{1/\alpha}$ with the Hölder constant L of ξ in $[0,T]$.

Proof. Similar construction of v_δ for the given $\widetilde{\varphi}\in C^\infty(\overline{\Omega}\times[0,T])$ with the properties (i) and (ii) is carried out by Proposition 3.3. However, we need more precise estimates in L^p such as (iii). Let $r_0=10L$ and $\delta<r_0$. Define $f_\delta=\varphi\cdot\nabla\eta_\delta$. Then, from the proof of Proposition 3.3 v_δ is expressed by

$$v_\delta(x,t)=\sum_{i=1}^{N_0}\int_{D^i_{\delta,t}} G^i_\delta(x,y,t)f^i_\delta(y,t)dy, \qquad (x,t)\in\bigcup_{0\leq t\leq T} D_{\delta,t}\times\{t\} \tag{4.3}$$

where G^i_δ and f^i_δ are similarly defined by (3.2), (3.3), (3.4), and (3.6) with r repleced by δ.

Define $v_\delta^i(x,t) := \int_{D^i_{\delta,t}} G^i_\delta(x,y,t) f^i_\delta(y,t)dy$. Since $\sum_{i=1}^{N_0} v^i_\delta = v_\delta$ and since N_0 is independent of t and δ, it suffices to show that there exists a constant $r_1 > 0$ such that the following estimates hold: for every $1 \le i \le N_0$ and every $\delta \in (0, r_1)$,

$$\|v^i_\delta(t)\|_{L^p(D^i_{\delta,t})} \le C\delta^{\frac{n}{p}} \|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))}, \tag{4.4}$$

$$\|\nabla v^i_\delta(t)\|_{L^p(D^i_{\delta,t})} \le C\delta^{\frac{n}{p}-1} \|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))} + C\|\varphi(t)\|_{\dot{H}^{1,p}(B_{3\delta}(\xi^{\varepsilon_\delta}(t)))}, \tag{4.5}$$

$$\|\Delta v^i_\delta(t)\|_{L^p(D^i_{\delta,t})} \le C\delta^{\frac{n}{p}-2} \|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))} + C\|\varphi(t)\|_{\dot{H}^{2,p}(B_{3\delta}(\xi^{\varepsilon_\delta}(t)))}, \tag{4.6}$$

$$\|\partial_t v^i_\delta(t)\|_{L^p(D^i_{\delta,t})} \le C\delta^{\frac{n}{p}-\frac{1}{\alpha}} \|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))} + C\|\partial_t\varphi(t)\|_{L^p(B_{3\delta}(\xi^{\varepsilon_\delta}(t)))} \tag{4.7}$$

for $1 \le p < n$ and for all $t \in (0,T)$, where C is a constant independent of t and δ. We may show only (4.6) and (4.7) since (4.4) and (4.5) can be handled in the same way.

We take $\psi_{i,j} \in C_0^\infty(D^i_{1,0} \cap D^j_{1,0})$ in such a way that $\int_{D^i_{1,0}\cap D^j_{1,0}} \psi_{i,j}(x)dx = 1$ for $(i,j) \in J$. Let us recall the function $\{\psi^{i,j}_\delta\}_{(i,j)\in J}$ in (3.3) may be chosen as

$$\psi^{i,j}_\delta(x,t) = \frac{1}{\delta^n}\psi_{i,j}\left(\frac{x + \xi^{\varepsilon_\delta}(0) - \xi^{\varepsilon_\delta}(t)}{\delta}\right), \qquad (i,j) \in J \tag{4.8}$$

for $(x,t) \in \mathbb{R}^n \times [0,T]$, respectively. Here x^i_δ is the center of the ball $B^i_{\delta,0}$, $i = 1, \cdots, N_0$. It is noticed that for each $t \in [0,T]$

$$D^i_{\delta,t} \subset \{y \in \mathbb{R}^n; |x-y| \le 2\delta\} \quad \text{for all } x \in D^i_{\delta,t}. \tag{4.9}$$

We have by (3.3), (3.5), and (4.9) that

$$\begin{aligned}
&\|v^i_\delta(t)\|_{L^p(D^i_{\delta,t})} \le \left\| \int_{|x-y|\le 2\delta} G^i_\delta(x,y,t) f^i_\delta(y,t)dy \right\|_{L^p(D^i_{\delta,t})} \\
&= \left\| \int_{|x'|\le 2\delta} G^i_\delta(x,x+x',t)\phi_i(x+x',t) f_\delta(x+x',t)dx' \right\|_{L^p(D^i_{\delta,t})} \\
&\le \left\| \int_{|x'|\le 2\delta} G^i_\delta(x,x+x',t)\phi_i(x+x',t) f_\delta(x+x',t)dx' \right\|_{L^p(D^i_{\delta,t})} \\
&+ \left\| \int_{|x'|\le 2\delta} G^i_\delta(x,x+x',t) \sum_{\substack{1\le j\le N_0\\ 1\le k\le N_0}} a^i_{j,k} \left(\int_{D_{\delta,t}} \phi^j_t(z) f_\delta(z,t)dz \right) \psi^{i,k}_\delta(x+x',t)dx' \right\|_{L^p(D^i_{\delta,t})} \\
&\equiv I_1(\delta,t) + I_2(\delta,t),
\end{aligned}$$

$$\|\nabla_x v_\delta^i(t)\|_{L^p(D_{\delta,t}^i)} \leq \left\| \nabla_x \int_{|x-y|\leq 2\delta} G_\delta^i(x,y,t) f_\delta^i(y,t) dy \right\|_{L^p(D_{\delta,t}^i)}$$
$$= \left\| \nabla_x \int_{|x'|\leq 2\delta} G_\delta^i(x,x+x',t) f_\delta^i(x+x',t) dx' \right\|_{L^p(D_{\delta,t}^i)}$$
$$\leq \left\| \nabla_x \int_{|x'|\leq 2\delta} G_\delta^i(x,x+x',t) \phi_i(x+x',t) f_\delta(x+x',t) dx' \right\|_{L^p(D_{\delta,t}^i)}$$
$$+ \left\| \nabla_x \int_{|x'|\leq 2\delta} G_\delta^i(x,x+x',t) \sum_{\substack{1\leq j\leq N_0 \\ 1\leq k\leq N_0}} a_{j,k}^i \left(\int_{D_{\delta,t}} \phi_t^j(z) f_\delta(z,t) dz \right) \psi_\delta^{i,k}(x+x',t) dx' \right\|_{L^p(D_{\delta,t}^i)}$$
$$\equiv I_3(\delta,t) + I_4(\delta,t),$$

$$\|\Delta_x v_\delta^i(t)\|_{L^p(D_{\delta,t}^i)} \leq \left\| \Delta_x \int_{|x-y|\leq 2\delta} G_\delta^i(x,y,t) f_\delta^i(y,t) dy \right\|_{L^p(D_{\delta,t}^i)}$$
$$= \left\| \Delta_x \int_{|x'|\leq 2\delta} G_\delta^i(x,x+x',t) f_\delta^i(x+x',t) dx' \right\|_{L^p(D_{\delta,t}^i)}$$
$$\leq \left\| \Delta_x \int_{|x'|\leq 2\delta} G_\delta^i(x,x+x',t) \phi_i(x+x',t) f_\delta(x+x',t) dx' \right\|_{L^p(D_{\delta,t}^i)}$$
$$+ \left\| \Delta_x \int_{|x'|\leq 2\delta} G_\delta^i(x,x+x',t) \sum_{\substack{1\leq j\leq N_0 \\ 1\leq k\leq N_0}} a_{j,k}^i \left(\int_{D_{\delta,t}} \phi_t^j(z) f_\delta(z,t) dz \right) \psi_\delta^{i,k}(x+x',t) dx' \right\|_{L^p(D_{\delta,t}^i)}$$
$$\equiv I_5(\delta,t) + I_6(\delta,t),$$

$$\|\partial_t v_\delta^i(t)\|_{L^p(D_{\delta,t}^i)} \leq \left\| \partial_t \int_{|x-y|\leq 2\delta} G_\delta^i(x,y,t) f_\delta^i(y,t) dy \right\|_{L^p(D_{\delta,t}^i)}$$
$$= \left\| \partial_t \int_{|x'|\leq 2\delta} G_\delta^i(x,x+x',t) f_\delta^i(x+x',t) dx' \right\|_{L^p(D_{\delta,t}^i)}$$
$$\leq \left\| \partial_t \int_{|x'|\leq 2\delta} G_\delta^i(x,x+x',t) \phi_i(x+x',t) f_\delta(x+x',t) dx' \right\|_{L^p(D_{\delta,t}^i)}$$
$$+ \left\| \partial_t \int_{|x'|\leq 2\delta} G_\delta^i(x,x+x',t) \sum_{\substack{1\leq j\leq N_0 \\ 1\leq k\leq N_0}} a_{j,k}^i \left(\int_{D_{\delta,t}} \phi_t^j(z) f_\delta(z,t) dz \right) \psi_\delta^{i,k}(x+x',t) dx' \right\|_{L^p(D_{\delta,t}^i)}$$
$$\equiv I_7(\delta,t) + I_8(\delta,t).$$

In the next step, we investigate $I_j(\delta,t)$ for $j = 1, \cdots, 8$. Now we take $\delta < r_1 := \min\{r_0, 9r/31\}$ so that $B_{3\delta}(\xi^{\varepsilon_\delta}(t)) \subset B_r(\xi^{\varepsilon_r}(t))$. Then, it follows from (2.8), (3.13) – (3.16), (4.8), Lemma 2.2.1, and the Hölder, the Minkowski, and the Sobolev inequalities that

$$I_1(\delta,t) = \left\| \int_{|x'|\le 2\delta} G^i_\delta(x, x+x', t)\phi_i(x+x',t)\nabla\eta_\delta(x+x',t)\cdot\varphi(x+x',t)dx' \right\|_{L^p(D^i_{\delta,t})} \tag{4.10}$$

$$\begin{aligned}
&\le C\int_{|x'|\le 2\delta} \|G^i_\delta(\cdot, \cdot + x', t)\nabla\eta_\delta(\cdot+x',t)\|_{L^\infty(D^i_{\delta,t})}\|\varphi(\cdot+x',t)\|_{L^p(D^i_{\delta,t})}dx' \\
&\le C\int_{|x'|\le 2\delta} |x'|^{1-n}\delta^{-n-1}\|\varphi(t)\|_{L^p(B_{3\delta}(\xi^{\varepsilon_\delta}(t)))}\int_{|x'|}^{|x'|+2\delta} s^{n-1}dsdx' \\
&\le C\delta^{\frac{n}{p}}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))},
\end{aligned}$$

$$I_2(\delta,t) = \left\| \int_{|x'|\le 2\delta} G^i_\delta(x,x+x',t) \right. \tag{4.11}$$

$$\begin{aligned}
&\left.\sum_{\substack{1\le j\le N_0\\1\le k\le N_0}} a^i_{j,k}\left(\int_{D_{\delta,t}}\phi_j(z,t)\nabla\eta_\delta(z,t)\cdot\varphi(z,t)dz\right)\frac{1}{\delta^n}\psi_{i,k}\left(\frac{x+x'+\xi^{\varepsilon_\delta}(0)-\xi^{\varepsilon_\delta}(t)}{\delta}\right)dx'\right\|_{L^p(D^i_{\delta,t})} \\
&\le C\delta^{-\frac{n}{p}-1}\sum_{1\le k\le N_0}\int_{|x'|\le 2\delta}\|\varphi(t)\|_{L^p(D_{\delta,t})}\left\|G^i_\delta(\cdot,\cdot+x',t)\psi_{i,k}\left(\frac{\cdot+x'+\xi^{\varepsilon_\delta}(0)-\xi^{\varepsilon_\delta}(t)}{\delta}\right)\right\|_{L^p(D^i_{\delta,t})}dx' \\
&\le C\delta^{-n(1-\frac{1}{p})-1}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))}\int_{|x'|\le 2\delta}|x'|^{1-n}\left\{\int_{|x'|}^{|x'|+2\delta}s^{n-1}ds\right\}dx' \\
&\le C\delta^{\frac{N}{p}}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))},
\end{aligned}$$

$$I_3(\delta,t) = \left\|\nabla_x\int_{|x'|\le 2\delta} G^i_\delta(x,x+x',t)\phi_i(x+x',t)\nabla\eta_\delta(x+x',t)\cdot\varphi(x+x',t)dx'\right\|_{L^p(D^i_{\delta,t})} \tag{4.12}$$

$$\begin{aligned}
&\le C\int_{|x'|\le 2\delta}\sum_{|\alpha_1|+|\alpha_2|=1}\|\partial_{\alpha_1}G^i_\delta(\cdot,\cdot+x',t)\partial_{\alpha_2}\nabla\eta_\delta(\cdot+x',t)\|_{L^\infty(D^i_{\delta,t})}\|\varphi(\cdot+x',t)\|_{L^p(D^i_{\delta,t})}dx' \\
&\quad + C\int_{|x'|\le 2\delta}\|G^i_\delta(\cdot,\cdot+x',t)\nabla\eta_\delta(\cdot+x',t)\|_{L^\infty(D^i_{\delta,t})}\|\nabla\varphi(\cdot+x',t)\|_{L^p(D^i_{\delta,t})}dx' \\
&\le C\int_{|x'|\le 2\delta}|x'|^{1-n}\left(\sum_{|\alpha_1|+|\alpha_2|=1}\delta^{-n-(|\alpha_1|+|\alpha_2|)-1}\|\varphi(t)\|_{L^p(B_{3\delta}(\xi^{\varepsilon_\delta}(t)))}\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \delta^{-n-1}\|\nabla\varphi(t)\|_{L^p(B_{3\delta}(\xi^{\varepsilon\delta}(t)))}\Big)\int_{|x'|}^{|x'|+2\delta} s^{n-1}dsdx' \\
\leq& C\delta^{\frac{n}{p}-1}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon r}(t)))} + C\|\varphi(t)\|_{\dot{H}^{1,p}(B_{3\delta}(\xi^{\varepsilon\delta}(t)))},
\end{aligned}
$$

$$
\begin{aligned}
I_4(\delta,t) =& \Bigg\|\nabla_x \int_{|x'|\leq 2\delta} G^i_\delta(x,x+x',t) \qquad (4.13)\\
&\sum_{\substack{1\leq j\leq N_0\\ 1\leq k\leq N_0}} a^i_{j,k}\left(\int_{D_{\delta,t}}\phi_j(z,t)\nabla\eta_\delta(z,t)\cdot\varphi(z,t)dz\right)\frac{1}{\delta^n}\psi_{i,k}\left(\frac{x+x'+\xi^{\varepsilon\delta}(0)-\xi^{\varepsilon\delta}(t)}{\delta}\right)dx'\Bigg\|_{L^p(D^i_{\delta,t})} \\
\leq& C\delta^{-\frac{n}{p}-1}\sum_{1\leq k\leq N_0}\int_{|x'|\leq 2\delta}\|\varphi(t)\|_{L^p(D_{\delta,t})} \\
&\sum_{|\alpha_1|+|\alpha_2|=1}\left\|\partial_{\alpha_1}G^i_\delta(\cdot,\cdot+x',t)\partial_{\alpha_2}\psi_{i,k}\left(\frac{\cdot+x'+\xi^{\varepsilon\delta}(0)-\xi^{\varepsilon\delta}(t)}{\delta}\right)\right\|_{L^p(D^i_{\delta,t})}dx' \\
\leq& C\delta^{-1}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon r}(t)))}\sum_{|\alpha_1|+|\alpha_2|=1}\delta^{-n+\frac{n}{p}-(|\alpha_1|+|\alpha_2|)}\int_{|x'|\leq 2\delta}|x'|^{1-n}\left\{\int_{|x'|}^{|x'|+2\delta}s^{n-1}ds\right\}dx' \\
\leq& C\delta^{\frac{n}{p}-1}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon r}(t)))},
\end{aligned}
$$

$$
I_5(\delta,t) = \left\|\Delta_x\int_{|x'|\leq 2\delta}G^i_\delta(x,x+x',t)\phi_i(x+x',t)\nabla\eta_\delta(x+x',t)\cdot\varphi(x+x',t)dx'\right\|_{L^p(D^i_{\delta,t})} \qquad (4.14)
$$

$$
\begin{aligned}
\leq& C\int_{|x'|\leq 2\delta}\sum_{|\alpha_1|+|\alpha_2|=2}\|\partial_{\alpha_1}G^i_\delta(\cdot,\cdot+x',t)\partial_{\alpha_2}\nabla\eta_\delta(\cdot+x',t)\|_{L^\infty(D^i_{\delta,t})}\|\varphi(\cdot+x',t)\|_{L^p(D^i_{\delta,t})}dx' \\
&+ C\int_{|x'|\leq 2\delta}\sum_{|\alpha'_1|+|\alpha'_2|=1}\|\partial_{\alpha'_1}G^i_\delta(\cdot,\cdot+x',t)\partial_{\alpha'_2}\nabla\eta_\delta(\cdot+x',t)\|_{L^\infty(D^i_{\delta,t})}\|\nabla\varphi(\cdot+x',t)\|_{L^p(D^i_{\delta,t})}dx' \\
&+ C\int_{|x'|\leq 2\delta}\|G^i_\delta(\cdot,\cdot+x',t)\nabla\eta_\delta(\cdot+x',t)\|_{L^\infty(D^i_{\delta,t})}\|\Delta\varphi(\cdot+x',t)\|_{L^p(D^i_{\delta,t})}dx' \\
\leq& C\int_{|x'|\leq 2\delta}|x'|^{1-n}\Bigg(\sum_{|\alpha_1|+|\alpha_2|=2}\delta^{-n-(|\alpha_1|+|\alpha_2|)-1}\|\varphi(t)\|_{L^p(B_{3\delta}(\xi^{\varepsilon\delta}(t)))} \\
&+\sum_{|\alpha'_1|+|\alpha'_2|=1}\delta^{-n-(|\alpha'_1|+|\alpha'_2|)-1}\|\nabla\varphi(t)\|_{L^p(B_{3\delta}(\xi^{\varepsilon\delta}(t)))}+\delta^{-n-1}\|\Delta\varphi(t)\|_{L^p(B_{3\delta}(\xi^{\varepsilon\delta}(t)))}\Bigg)\int_{|x'|}^{|x'|+2\delta}s^{n-1}dsdx' \\
\leq& C\delta^{\frac{n}{p}-2}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon r}(t)))}+C\delta^{-1}\|1\|_{L^n(B_{3\delta}(\xi^{\varepsilon\delta}(t)))}\|\nabla\varphi(t)\|_{L^{\frac{pn}{n-p}}(B_{3\delta}(\xi^{\varepsilon\delta}(t)))}+C\|\Delta\varphi(t)\|_{L^p(B_{3\delta}(\xi^{\varepsilon\delta}(t)))} \\
\leq& C\delta^{\frac{n}{p}-2}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon r}(t)))}+C\|\varphi(t)\|_{\dot{H}^{2,p}(B_{3\delta}(\xi^{\varepsilon\delta}(t)))},
\end{aligned}
$$

$$
\begin{aligned}
I_6(\delta,t)=&\Bigg\|\Delta_x\int_{|x'|\leq 2\delta}G_\delta^i(x,x+x',t) \qquad (4.15)\\
&\sum_{\substack{1\leq j\leq N_0\\1\leq k\leq N_0}}a_{j,k}^i\left(\int_{D_{\delta,t}}\phi_j(z,t)\nabla\eta_\delta(z,t)\cdot\varphi(z,t)dz\right)\frac{1}{\delta^n}\psi_{i,k}\left(\frac{x+x'+\xi^{\varepsilon_\delta}(0)-\xi^{\varepsilon_\delta}(t)}{\delta}\right)dx'\Bigg\|_{L^p(D^i_{\delta,t})}\\
\leq& C\delta^{-\frac{n}{p}-1}\sum_{1\leq k\leq N_0}\int_{|x'|\leq 2\delta}\|\varphi(t)\|_{L^p(D_{\delta,t})}\\
&\quad\sum_{|\alpha_1|+|\alpha_2|=2}\left\|\partial_{\alpha_1}G_\delta^i(\cdot,\cdot+x',t)\partial_{\alpha_2}\psi_{i,k}\left(\frac{\cdot+x'+\xi^{\varepsilon_\delta}(0)-\xi^{\varepsilon_\delta}(t)}{\delta}\right)\right\|_{L^p(D^i_{\delta,t})}dx'\\
\leq& C\delta^{-1}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))}\sum_{|\alpha_1|+|\alpha_2|=2}\delta^{-n+\frac{n}{p}-(|\alpha_1|+|\alpha_2|)}\int_{|x'|\leq 2\delta}|x'|^{1-n}\left\{\int_{|x'|}^{|x'|+2\delta}s^{n-1}ds\right\}dx'\\
\leq& C\delta^{\frac{n}{p}-2}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))},
\end{aligned}
$$

$$
\begin{aligned}
I_7(\delta,t)=&\left\|\partial_t\int_{|x'|\leq 2\delta}G_\delta^i(x,x+x',t)\phi_i(x+x',t)\nabla\eta_\delta(x+x',t)\cdot\varphi(x+x',t)dx'\right\|_{L^p(D^i_{\delta,t})} \qquad (4.16)\\
\leq& C\int_{|x'|\leq 2\delta}|x'|^{1-n}\Big(\delta^{-n-\frac{1}{\alpha}-1}\|\varphi(\cdot+x',t)\|_{L^p(D^i_{\delta,t})}\\
&+\delta^{-n-1}\|\partial_t\varphi(\cdot+x',t)\|_{L^p(D^i_{\delta,t})}\Big)\int_{|x'|}^{|x'|+2\delta}s^{n-1}dsdx'\\
\leq& C\delta^{\frac{n}{p}-\frac{1}{\alpha}}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))}+C\|\partial_t\varphi(t)\|_{L^p(B_{3\delta}(\xi^{\varepsilon_\delta}(t)))},
\end{aligned}
$$

$$
\begin{aligned}
I_8(\delta,t)=&\Bigg\|\partial_t\int_{|x'|\leq 2\delta}G_\delta^i(x,x+x',t) \qquad (4.17)\\
&\sum_{\substack{1\leq j\leq N_0\\1\leq k\leq N_0}}a_{j,k}^i\left(\int_{D_{\delta,t}}\phi_j(z,t)\nabla\eta_\delta(z,t)\cdot\varphi(z,t)dz\right)\frac{1}{\delta^n}\psi_{i,k}\left(\frac{x+x'+\xi^{\varepsilon_\delta}(0)-\xi^{\varepsilon_\delta}(t)}{\delta}\right)dx'\Bigg\|_{L^p(D^i_{\delta,t})}\\
\leq& C\int_{|x'|\leq 2\delta}|x'|^{1-n}\left(\delta^{-n-\frac{1}{\alpha}-1}\|\varphi(t)\|_{L^p(D_{\delta,t})}+\delta^{-n-1}\|\partial_t\varphi(t)\|_{L^p(D_{\delta,t})}\right)\int_{|x'|}^{|x'|+2\delta}s^{n-1}ds\\
\leq& C\delta^{\frac{n}{p}-\frac{1}{\alpha}}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))}+C\|\partial_t\varphi(t)\|_{L^p(D_{\delta,t})}.
\end{aligned}
$$

Now, the desired estimates (4.4)–(4.7) are consequences of (4.10)–(4.17). This proves Proposition 4.3. □

Lemmata 2.2.1 and 4.3 play an important role in showing that the solution of (NS) in Q_T is also the very weak solution of (PS) in $Q(r,T)$. We next investigate the following Stokes equations with the convection term.

$$\begin{cases} \partial_t w_r - \Delta w_r + (h_1, \nabla) w_r + (h_2, \nabla) w_r + \nabla \pi = f & \text{in } \widetilde{Q}(r,T), \\ \operatorname{div} w_r = 0 & \text{in } \widetilde{Q}(r,T), \\ w_r = 0 & \text{on } \partial B_r, \\ w_r(0) = a & \text{in } B_r, \end{cases} \tag{E}$$

where $\widetilde{Q}(r,T) \equiv B_r \times (0,T)$ with $B_r = \{x \in \mathbb{R}^n \ ; \ |x| < r\}$.

Lemma 4.4. *For $1 < s < \infty$, $1 < q < n$ satisfying $2/s + n/q > 2$, and $r > 0$, there is a constant $\varepsilon_0 = \varepsilon_0(s,q,n,r)$ with the following property. For every $a \in \mathcal{B}^{2(1-\frac{1}{s})}_{q,s}(B_r)$, $h_1 \in L^\infty(0,T)$, $h_2 \in L^\infty(0,T; L^{n,\infty}(B_r))$ with*

$$\sup_{0<t<T} \|h_2(t)\|_{L^{n,\infty}(B_r)} \le \varepsilon_0, \tag{4.18}$$

and every $f \in L^s(0,T;L^q(B_r))$ there exists a solution $\{w_r, \nabla\pi\}$ to (E) *in the class*

$$w_r \in L^s(0,T; H^{2,q}_\sigma(B_r)), \ \ \partial_t w_r \in L^s(0,T; L^q_\sigma(B_r)), \tag{4.19}$$

$$w_r \in L^{s_0}(0,T; L^{q_0}_\sigma(B_r)) \tag{4.20}$$

$$\quad for \ \ \tfrac{2}{s_0} + \tfrac{n}{q_0} = \tfrac{2}{s} + \tfrac{n}{q} - 2 \ with \ \ s < s_0 < \infty, \ \ q < q_0 < \infty,$$

$$\nabla \pi \in L^s(0,T; L^q(B_r)). \tag{4.21}$$

In addition, if $q \ge 2$, it holds that

$$w_r \in C([0,T]; L^2_\sigma(B_r)) \cap L^2(0,T; H^{1,2}_{0,\sigma}(B_r)). \tag{4.22}$$

Proof. We define a Banach space $X^s_{q,T}(B_r)$ by

$$X^s_{q,T}(B_r) \equiv \{w \in L^s(0,T; H^{2,q}(B_r)); \partial_t w \in L^s(0,T; L^q_\sigma(B_r))\} \tag{4.23}$$

with the norm $\|w\|_{X^s_{q,T}(B_r)} \equiv \|\partial_t w\|_{L^s(0,T;L^q_\sigma(B_r))} + \|D^2 w\|_{L^s(0,T;L^q(B_r))}$. To find the solution $\{w_r, \nabla\pi\}$ to (E), we use the successive approximation $\{w_{r,j}, \nabla\pi_j\}_{j=0}^\infty$ as

$$w_{r,0}(t) = e^{-t\Delta} a \tag{4.24}$$

$$\begin{cases} \partial_t w_{r,j+1} + \Delta w_{r,j+1} + \nabla \pi_{j+1} = -\left(h_1, \nabla\right) w_{r,j} - (h_2, \nabla) w_{r,j} + f & \text{in } \widetilde{Q}(r,T), \\ \operatorname{div} w_{r,j+1} = 0 & \text{in } \widetilde{Q}(r,T), \\ w_{r,j+1} = 0 & \text{on } \partial B_r, \\ w_{r,j+1}(0) = a & \text{in } B_r. \end{cases} \tag{4.25}$$

By using the maximal regularity in the usual space $L^s(0,T;L^q_\sigma(B_r))$, we can find unique solutions $\{w_{r,j+1}, \nabla\pi_{j+1}\}$ to (4.25) in $X^s_{q,T}(B_r) \times L^s(0,T;L^q(B_r))$ if the right hand side belongs to $L^s(0,T;L^q(B_r))$. In fact, the following estimates hold.

Lemma 4.5. *Let* $1 < s < \infty$, $1 < q < n$, *and* $\dfrac{2}{s_0} + \dfrac{n}{p_0} = 1$ *with* $s < s_0 < \infty, q < p_0 < \infty$. *Then, it holds that*

$$\|(h_1, \nabla)w\|_{L^s(0,T;L^q_\sigma(B_r))} \tag{4.26}$$
$$\leq C\|h_1\|_{L^{s_0}(0,T;L^{p_0}(B_r))}\left(\|w\|_{X^s_{q,T}(B_r)} + \|w(0)\|_{\mathcal{B}^{2(1-\frac{1}{s})}_{q,s}(B_r)}\right)$$
$$\|(h_2, \nabla)w\|_{L^s(0,T;L^q_\sigma(B_r))} \tag{4.27}$$
$$\leq C\sup_{0<t<T}\|h_2(t)\|_{L^{n,\infty}(B_r)}\left(\|w\|_{X^s_{q,T}(B_r)} + \|w(0)\|_{\mathcal{B}^{2(1-\frac{1}{s})}_{q,s}(B_r)}\right)$$

for $h_1 \in L^{s_0}(0,T;L^{p_0}(B_r))$, $h_2 \in L^\infty(0,T;L^{n,\infty}(B_r))$, *and* $w \in X^s_{q,T}(B_r)$ *where* $C = C(s, q, s_0, p_0, n, r)$ *is independent of* h_1, h_2, *and* w.

Proof of Lemma 4.5. We first show the estimate (4.26). Now we consider the case where $w(0) = 0$. By Hölder inequality, we have that

$$\|(h_1, \nabla)w\|_{L^s(0,T;L^q_\sigma(B_r))} \leq C\|h_1\|_{L^{s_0}(0,T;L^{p_0}(B_r))}\|\nabla w\|_{L^{s_1}(0,T;L^{p_1}(B_r))} \tag{4.28}$$

for $\dfrac{1}{s_0} + \dfrac{1}{s_1} = \dfrac{1}{s}$ and $\dfrac{1}{p_0} + \dfrac{1}{p_1} = \dfrac{1}{q}$. Set $f(t) = \partial_t w(t) + \tilde{A}w(t)$. Then we have that

$$\begin{aligned} \|\nabla w(t)\|_{L^{p_1}(B_r)} &\leq \int_0^t \|\nabla e^{-(t-\tau)\tilde{A}} f(\tau)\|_{L^{p_1}(B_r)}\, d\tau \\ &\leq \int_0^t (t-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p_1})+\frac{1}{2}-1}\|f\|_{L^q(B_r)}\, d\tau. \end{aligned} \tag{4.29}$$

From Hardy-Littlewood-Sobolev inequality, we have by (4.29) that

$$\|\nabla w\|_{L^{s_1}(0,T;L^{p_1}_\sigma(B_r))} \leq C\|f\|_{L^s(0,T;L^q(B_r))} \leq C\|w\|_{X^s_q(T)} \tag{4.30}$$

for $\dfrac{1}{s_1} = \dfrac{1}{s} + \dfrac{n}{2}\left(\dfrac{1}{q} - \dfrac{1}{p_1}\right) - \dfrac{1}{2}$, i.e., $\dfrac{2}{s_0} + \dfrac{n}{p_0} = 1$.

Next we prove the general case. We consider the following function.

$$W(t) = \begin{cases} \theta(t)e^{t\tilde{A}}w(0) & \text{for } t \leq 0 \\ e^{-t\tilde{A}}w(0) - \displaystyle\int_0^t e^{-(t-s)\tilde{A}}f(s)ds & \text{for } 0 \leq t \end{cases}$$

where θ is a $C^\infty(\mathbb{R})$ function satisfying that $\theta(t) = 0$ for $t \leq -1$, $\theta(t) = 1$ for $0 \leq t$, and $0 \leq \theta \leq 1$. From (4.30), we have that

$$\begin{aligned} \|\nabla w\|_{L^{s_1}(0,T;L^{p_1}_\sigma(B_r))} &\leq \|\nabla W\|_{L^{s_1}(-1,T;L^{p_1}_\sigma(B_r))} \\ &\leq C\left(\|\partial_t W\|_{L^s(-1,T;L^q(B_r))} + \|\tilde{A}W\|_{L^s(-1,T;L^q(B_r))}\right). \end{aligned} \tag{4.31}$$

We also have that

$$\|\tilde{A}W\|^s_{L^s(-1,0;L^q(B_r))} \le \int_{-1}^0 \|\tilde{A}e^{t\tilde{A}}w(0)\|^s_{L^q(B_r)}dt \le \int_0^\infty \|\tilde{A}e^{-t\tilde{A}}w(0)\|^s_{L^q(B_r)}dt, \tag{4.32}$$

$$\begin{aligned}\|\partial_t W\|_{L^s(-1,0;L^q(B_r))} &\le \left(\int_{-1}^0 \|\theta'(t)e^{t\tilde{A}}w(0)\|^s_{L^q(B_r)}dt\right)^{\frac{1}{s}} + \|\tilde{A}W\|_{L^s(-1,0;L^q(B_r))} \\ &\le C\left\{\|w(0)\|_{L^q(B_r)} + \left(\int_0^\infty \|\tilde{A}e^{-t\tilde{A}}w(0)\|^s_{L^q(B_r)}dt\right)^{\frac{1}{s}}\right\} \\ &\le C\|w(0)\|_{D^{1-\frac{1}{s},s}_{A_q}}\end{aligned} \tag{4.33}$$

From (4.31), (4.32), and (4.33), (4.26) holds.

Finally we show the estimate (4.27). We take r_0 and r_1 satisfying that $1 < r_0 < q < r_1 < n$. From the Hölder inequality, it follows that

$$\|(h_2,\nabla)w\|_{L^{r_j,\infty}(B_r)} \le \|h_2\|_{L^{n,\infty}(B_r)}\|\nabla w\|_{L^{q_j,\infty}(B_r)} \le \|h_2\|_{L^{n,\infty}(B_r)}\|\nabla w\|_{L^{q_j}(B_r)}$$

where $\frac{1}{q_j} = \frac{1}{r_j} - \frac{1}{n}(j=1,2)$. By the Sobolev inequality, it holds that

$$\|(h_2,\nabla)w\|_{L^{r_j,\infty}(B_r)} \le C\|h_2\|_{L^{n,\infty}(B_r)}\|w\|_{H^{2,r_j}(B_r)}$$

for all $w \in H^{2,r_j}(B_r)$ with $C = C(n, r_0, r_1)$, which implies that the map $S_{h_2} : w \mapsto (h_2,\nabla)w$ is a bounded operator from $H^{2,r_j}(B_r)$ to $L^{r_j,\infty}(B_r)$ for $j = 1, 2$. From the Marcinkiewicz interpolation theorem we see that the map S_{h_2} is also a bounded operator from $H^{2,q}(B_r)$ to $L^q(B_r)$. This proves (4.27). □

Hence, it follows from (4.26), (4.27), and Lemma 2.4.2 that there exists a pair of unique solutions $\{w_{r,j+1}, \nabla\pi_{j+1}\}$ to (4.25) in $X^s_{q,T}(B_r) \times L^s(0,T;L^q(B_r))$ with the estimate

$$\begin{aligned}&\|w_{r,j+1}\|_{X^s_{q,T}(B_r)} + \|\nabla\pi_{j+1}\|_{L^s(0,T;L^q(B_r))} \\ &\quad\le \widetilde{C}\left\{\left(\|h_1\|_{L^{s_0}(0,T;L^{p_0}(B_r))} + \sup_{0<t<T}\|h_2(t)\|_{L^{n,\infty}(B_r)}\right)\|w_{r,j}\|_{X^s_{q,T}(B_r)}\right. \\ &\qquad\left. + \|f\|_{L^s(0,T;L^q(B_r))} + \|a\|_{\mathcal{B}^{2(1-\frac{1}{s})}_{q,s}(B_r)}\right\},\end{aligned} \tag{4.34}$$

where $\widetilde{C} = \widetilde{C}(s,q,n,r)$ is independent of j and T. Now we take ε_0 in (4.18) and T^* so that

$$0 < \varepsilon_0 \le \frac{1}{4\widetilde{C}}, \quad T^* \equiv \left(4\sup_{0<t<T}|h_1(t)|\,\mathrm{vol}(B_r)^{\frac{1}{p_0}}\widetilde{C}\right)^{-s_0}. \tag{4.35}$$

Defining $W_{r,j}$ by $W_{r,j} \equiv w_{r,j} - w_{r,j-1}$ ($w_{r,-1} = 0$), we have by (4.18) and (4.35) that

$$
\begin{aligned}
\|W_{r,j+1}\|_{X^s_{q,T^*}(B_r)} &\le \widetilde{C}\left(\|h_1\|_{L^{s_0}(0,T;L^{p_0}(B_r))} + \sup_{0<t<T}\|h_2(t)\|_{L^{N,\infty}(B_r)}\right)\|W_{r,j}\|_{X^s_{q,T^*}(B_r)} \\
&\le \frac{1}{2}\|W_{r,j}\|_{X^s_{q,T^*}(B_r)} \le \cdots \le \left(\frac{1}{2}\right)^j \|D^2 e^{-t\Delta}a\|_{L^s(0,T;L^q(B_r))} \\
&\le C\left(\frac{1}{2}\right)^j \|a\|_{\mathcal{B}^{2(1-\frac{1}{s})}_{q,s}(B_r)}
\end{aligned}
$$

for $j = 0, 1, \ldots$. Since $w_{r,j} = \sum_{k=0}^{j} W_{r,k}$, there exists a limiting function w_r of $w_{r,j}$ in $X^s_{q,T^*}(B_r)$ as $j \to \infty$. Similarly, we obtain a limiting function $\nabla\pi$ of $\nabla\pi_j$ in $L^s(0,T;L^q(B_r))$. Taking a limit $j \to \infty$ in (4.25), we see that $\{w_r, \nabla\pi\}$ is solutions to (E) for $0 < t < T^*$. Since T^* is chosen by (4.35), we see that T^* is taken independently of the initial data a. Hence, starting from T^* with the initial data $w_r(T^*)$ we may solve (E) on $[T^*, 2T^*]$. Repeating this argument beyond $2T^*$, after finitely many steps, we have a pair of solutions $\{w_r, \nabla\pi\}$ of (E) on $[0,T]$ in the class (4.19). Further, (4.20) is a consequence of Lemma 2.4.2. From (4.19) we obtain that $w_r \in C([0,T]; L^2_\sigma(B_r))$. Therefore, we have (4.22) by the interpolation $\|\nabla w_r\|_{L^2(B_r)} \le \|w_r\|^{\frac{1}{2}}_{L^2(B_r)}\|D^2 w_r\|^{\frac{1}{2}}_{L^2(B_r)}$. This proves Lemma 4.4. □

Remark 4.6. Similarly to (4.34), it follows from the maximal regularity that

$$
\begin{aligned}
&\|w_r^*\|_{X^s_{q,T}(B_1)} + \|\nabla\pi_r\|_{L^s(0,T;L^q(B_1))} \\
&\quad\le C\Bigg\{\left(\|h_r^1\|_{L^{s_0}(0,T;L^{p_0}(B_1))} + \sup_{0<t<T}\|h_r^2(t)\|_{L^{n,\infty}(B_1)}\right)\|w_r^*\|_{X^s_{q,T}(B_1)} \\
&\qquad + \|f_r\|_{L^s(0,T;L^q(B_1))} + \|a_r\|_{\mathcal{B}^{2(1-\frac{1}{s})}_{q,s}(B_1)}\Bigg\},
\end{aligned}
$$

where $w_r^*(x,t) = w_r(rx, r^2t)$, $\pi_r(x,t) = r\pi(rx, r^2t)$, $h_r^1(x,t) = rh_1(rx, r^2t)$, $h_r^2(x,t) = rh_2(rx, r^2t)$, $f_r(x,t) = f(rx, r^2t)$, $a_r(x) = a(rx)$, and where $C = C(s, q, s_0, p_0, n)$ is independent of r. From such change of scaling parameter r of dilation, for the solution $\{w_r, \nabla\pi\}$ of (E), we obtain the following estimate

$$
\begin{aligned}
&\|w_r\|_{X^s_{q,r^2T}(B_r)} + \|\nabla\pi\|_{L^s(0,r^2T;L^q(B_r))} \\
&\quad\le C\Bigg\{\left(\|h_1\|_{L^{s_0}(0,r^2T;L^{p_0}(B_r))} + \sup_{0<t<r^2T}\|h_2(t)\|_{L^{n,\infty}(B_r)}\right)\|w_r\|_{X^s_{q,r^2T}(B_r)} \\
&\qquad + \|f\|_{L^s(0,r^2T;L^q(B_r))} + \|a\|_{\mathcal{B}^{2(1-\frac{1}{s})}_{q,s}(B_r)}\Bigg\}
\end{aligned}
$$

with the same constant C as above independent of r. Notice also that

$$
\|D^2 e^{-t\Delta}a_r\|_{L^s(0,T;L^q(B_1))} \le C\|a_r\|_{\mathcal{B}^{2(1-\frac{1}{s})}_{q,s}(B_1)},
$$

where $C = C(s, q, n)$ is independent of r and T. From such change of scaling parameter r of dilation, we obtain that

$$\|D^2 e^{-t\Delta} a\|_{L^s(0,r^2T;L^q(B_r))} \leq C\|a\|_{\mathcal{B}^{2(1-\frac{1}{s})}_{q,s}(B_r)}$$

with the same constant C independent of r. Since we may choose T arbitrarily, the constant $\widetilde{C}$ in (4.35) may be chosen independently of r. As a result, the consatnt ε_0 in (4.35) is also taken independently of r.

The following lemma gives us the regularity of very weak solutions of (PS).

Proposition 4.7. (Serrin [23]-Takahashi [29]) *Let $a \in L^2_\sigma(\Omega)$. Suppose that w be a weak solution of (NS) on $\Omega \times (0,T)$ in the Leray-Hopf class, which means that w belongs to $L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; H^{1,2}_{0,\sigma}(\Omega))$ and that it w satisfies*

$$\int_\Omega a(x) \cdot \varphi(x,0)\, dx + \int_0^T \int_\Omega (w \cdot \varphi_t + \nabla w \cdot \nabla\varphi + (w,\nabla)w \cdot \varphi)\, dxdt = 0 \qquad (4.36)$$

for all $\varphi \in C^1([0,T); H^{1,2}_{0,\sigma}(\Omega) \cap L^n(\Omega))$ with $\varphi(\cdot,T) = 0$. Assume that $w \in L^s(t_0,t_1; L^q(D))$ for $\frac{2}{s} + \frac{n}{q} = 1$ with $n < q \leq \infty$ and that $\partial_t w \in L^\alpha(t_0,t_1; L^2(D))$ for $\alpha \geq 1$, where $D \times (t_0,t_1) \subset \Omega \times (0,T)$. Then, it holds that

$$\partial_t w,\ \frac{\partial^{\alpha_1+\cdots+\alpha_3} w}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \in C(K) \qquad (4.37)$$

for all multi-indices $(\alpha_1, \cdots, \alpha_n)$ and all compact subsets K in $D \times (t_0,t_1)$.

Serrin [23] first proved this type interior regularity for $\frac{2}{s} + \frac{n}{q} < 1$. Later, Takahashi [29] extended the range to $\frac{2}{s} + \frac{n}{q} = 1$ with $q > n$.

Proof of Theorem 4.1. *Step 1.* We first show that the solution u of (NS) in Q_T is a very weak solution of (PS) in $Q(r,T)$ under the hypothesis (4.2).

Proposition 4.8. *Suppose that $\xi \in C^\alpha([0,T]; \Omega)$ for $1/n < \alpha \leq 1$. Let u be a solution of (NS) in Q_T satisfying (4.2). Then, u is a very weak solution of (PS) in $Q(r,T)$ for all $r > 0$ in the sense of Definition 2.3.2.*

Proof of Proposition 4.8. It is easy to show that $u \in L^2_{loc}(Q(r,T))$. Let $\widetilde{\varphi} \in H^{1,2}(0,T; L^2_\sigma(B_r)) \cap L^2(0,T; H^{2,2}(B_r)) \cap L^2(0,T; L^\infty(B_r))$ with $\widetilde{\varphi}|_{\partial B_r} = 0$ and $\widetilde{\varphi}(\cdot,T) = 0$ in B_r, and define $\varphi := \Psi_*^{-1}\widetilde{\varphi}$. By Lemma 2.2.1, there exists some constant $r_0 = r_0(\alpha,n,T)$ such that under the assumption $\delta < r_0$ we obtain $\eta_\delta \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ satisfying that $0 \leq \eta_\delta \leq 1$,

$$\eta_\delta(x,t) = \begin{cases} 1 & \text{if } |x - \xi(t)| > \delta \\ 0 & \text{if } |x - \xi(t)| < \delta/2, \end{cases} \qquad (4.38)$$

and that

$$|\nabla\eta_\delta| \le C\delta^{-1}, |\Delta\eta_\delta| \le C\delta^{-2}, |\partial_t\eta_\delta| \le C\delta^{-1/\alpha} \tag{4.39}$$

on $\mathbb{R}^n \times [0,T]$ where $C = C(\alpha, n, T)$. By Proposition 4.3 with $p = 2$, there is some constant $r_1 = r_1(\alpha, r, n, T)$ such that if $\delta < r_1$, then there exists a function v_δ with its support in $\bigcup_{0\le t\le T} D_{\delta,t} \times \{t\}$ satisfying that

$$\text{div } v_\delta = \varphi \cdot \nabla\eta_\delta \quad \text{in} \bigcup_{0\le t\le T} D_{\delta,t} \times \{t\}$$

and that

$$\|v_\delta(t)\|_{L^2(D_{\delta,t})} \le C\delta^{\frac{n}{2}}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))}, \tag{4.40}$$

$$\|\nabla v_\delta(t)\|_{L^2(D_{\delta,t})} \le C\delta^{\frac{n}{2}-1}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))} + C\|\varphi(t)\|_{\dot{H}^{1,2}(B_{3\delta}(\xi^{\varepsilon_\delta}(t)))}, \tag{4.41}$$

$$\|\Delta v_\delta(t)\|_{L^2(D_{\delta,t})} \le C\delta^{\frac{n}{2}-2}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))} + C\|\varphi(t)\|_{\dot{H}^{2,2}(B_{3\delta}(\xi^{\varepsilon_\delta}(t)))}, \tag{4.42}$$

$$\|\partial_t v_\delta(t)\|_{L^2(D_{\delta,t})} \le C\delta^{\frac{n}{2}-\frac{1}{\alpha}}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))} + C\|\partial_t\varphi(t)\|_{L^2(B_{3\delta}(\xi^{\varepsilon_\delta}(t)))} \tag{4.43}$$

for all $t \in (0,T)$, where C is independent of t and δ.

From (4.2) we see that $u \in L^\infty(0,T; L^{\frac{4n}{n+2}}(B_r(\xi^{\varepsilon_r}(t)))) \subset L^\infty(0,T; L^2(B_r(\xi^{\varepsilon_r}(t))))$. Since $\varepsilon_\delta = (\delta/10L)^{1/\alpha}$ with $L = L(T) = \inf\{L; |\xi(T) - \xi(0)| \le LT^\alpha\}$, by (2.7) it should be noted that $B_\delta(\xi^{\varepsilon_\delta}(t)) \subset B_{\frac{11\delta}{10}}(\xi(t))$ and $D_{\delta,t} \subset B_\delta(\xi(t))$ for all $0 \le t \le T$ and $\delta > 0$. Define $w^* := \Psi_*^{-1}\widetilde{w}^*$, $W = u - w^*$ and take $\delta_0 = \frac{10}{11}\min\{r_0, r_1\}$. Applying Ψ_* to both sides of (PS") and then multiplying the result equation by $v_\delta - \varphi\eta_\delta$, we have by integration by parts on $Q(r,T)$ that

$$\begin{aligned}
&\int_{B_r(\xi^{\varepsilon_r}(0))} W(x,0)\cdot(\varphi(x,0)\eta_\delta(x,0) - v_\delta(x,0))dx \\
&+\int_0^T\int_{B_r(\xi^{\varepsilon_r}(t))} W\cdot\{\partial_t(\varphi\eta_\delta - v_\delta) + \Delta(\varphi\eta_\delta - v_\delta) + (u,\nabla)(\varphi\eta_\delta - v_\delta)\}dxdt \\
&+\int_0^T\int_{B_r(\xi^{\varepsilon_r}(t))} F\cdot(\varphi\eta_\delta - v_\delta)dxdt = 0,
\end{aligned} \tag{4.44}$$

where $F = -\partial_t w^* + \Delta w^* - (u,\nabla)w^*$. Thus, if we take $\delta < \delta_0$, then we have by (4.38)–(4.43), and the Hölder and the Sobolev inequalities that

$$\begin{aligned}
&\left|\int_{B_r(\xi^{\varepsilon_r}(0))} W(x,0)\cdot(\varphi(x,0)\eta_\delta(x,0) - \varphi(x,0) - v_\delta(x,0))dx\right| \\
\le& C(\|u_0\|_{L^2(B_\delta(\xi^{\varepsilon_\delta}(0)))} + \|w^*(0)\|_{L^2(B_\delta(\xi^{\varepsilon_\delta}(0)))})(\|\varphi(0)\|_{L^2(B_\delta(\xi^{\varepsilon_\delta}(0)))} + \|v_\delta(0)\|_{L^2(D_{\delta,0})}) \\
\le& C(\|u_0\|_{L^{n,\infty}(B_\delta(\xi^{\varepsilon_\delta}(0)))}\|1\|_{L^{\frac{2n}{n-2},1}(B_\delta(\xi^{\varepsilon_\delta}(0)))} + \|w^*(0)\|_{L^\infty(B_\delta(\xi^{\varepsilon_\delta}(0)))}\delta^{\frac{n}{2}})\|\varphi(0)\|_{L^2(B_r(\xi^{\varepsilon_r}(0)))} \\
\le& C(\delta^{\frac{n-2}{2}} + \delta^{\frac{n}{2}}),
\end{aligned} \tag{4.45}$$

$$
\begin{aligned}
&\left|\int_0^T \int_{B_r(\xi^{\varepsilon_r}(t))} W\cdot\{\partial_t(\varphi\eta_\delta)-\partial_t\varphi\}dxdt\right| \qquad (4.46)\\
&\leq \int_0^T \|u(t)-w^*(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))}\|\partial_t(\varphi\eta_\delta)(t)-\partial_t\varphi(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))}dt\\
&\leq C\int_0^T\left\{\varepsilon\left\{\int_{B_{\frac{11\delta}{10}}(\xi(t))}|x-\xi(t)|^{-2n+2\beta}dx\right\}^{\frac{1}{2}}+\|w^*(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))}\right\}\\
&\qquad\qquad\times\left(\delta^{-\frac{1}{\alpha}}\|\varphi(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))}+\|\partial_t\varphi(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))}\right)dt\\
&\leq C(\varepsilon\delta^{-\frac{n}{2}+\beta}+\delta^{\frac{n}{2}})\int_0^T\left(\delta^{\frac{n}{2}-\frac{1}{\alpha}}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))}+\|\partial_t\varphi(t)\|_{L^2(B_r(\xi^{\varepsilon_r}(t)))}\right)dt\\
&\leq C(\varepsilon\delta^{\beta-\frac{1}{\alpha}}+\delta^{n-\frac{1}{\alpha}}+\varepsilon\delta^{-\frac{n}{2}+\beta}+\delta^{\frac{n}{2}}),
\end{aligned}
$$

$$
\begin{aligned}
&\left|\int_0^T \int_{B_r(\xi^{\varepsilon_r}(t))} W\cdot\{\Delta(\varphi\eta_\delta)-\Delta\varphi\}dxdt\right| \qquad (4.47)\\
&\leq C\int_0^T \|u(t)-w^*(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))}\left(\delta^{-2}\|\varphi(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))}\right.\\
&\qquad\qquad\left.+\delta^{-1}\|\nabla\varphi(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))}+\|\Delta\varphi(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))}\right)dt\\
&\leq C\int_0^T\left\{\varepsilon\left\{\int_{B_{\frac{11\delta}{10}}(\xi(t))}|x-\xi(t)|^{-2n+2\beta}dx\right\}^{\frac{1}{2}}+\|w^*(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))}\right\}\\
&\quad\times\left(\delta^{\frac{n}{2}-2}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))}+\delta^{-1}\|1\|_{L^n(B_{\frac{11\delta}{10}}(\xi(t)))}\|\nabla\varphi(t)\|_{L^{\frac{2n}{n-2}}(B_{\frac{11\delta}{10}}(\xi(t)))}\right.\\
&\qquad\qquad\left.+\|\Delta\varphi(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))}\right)dt\\
&\leq C(\varepsilon\delta^{-\frac{n}{2}+\beta}+\delta^{\frac{n}{2}})\int_0^T\left(\delta^{\frac{n}{2}-2}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))}+\|\varphi(t)\|_{\dot{H}^{2,2}(B_r(\xi^{\varepsilon_r}(t)))}\right)dt\\
&\leq C(\varepsilon\delta^{\beta-2}+\delta^{n-2}+\varepsilon\delta^{-\frac{n}{2}+\beta}+\delta^{\frac{n}{2}}),
\end{aligned}
$$

$$
\begin{aligned}
&\left|\int_0^T \int_{B_r(\xi^{\varepsilon_r}(t))} W\cdot\{(u,\nabla)\varphi\eta_\delta-(u,\nabla)\varphi\}dxdt\right| \qquad (4.48)\\
&\leq\int_0^T\left\{\|u(t)\|^2_{L^{\frac{4n}{n+2}}(B_{\frac{11\delta}{10}}(\xi(t)))}\left(\delta^{-1}\|\varphi(t)\|_{L^{\frac{2n}{n-2}}(B_{\frac{11\delta}{10}}(\xi(t)))}+\|\nabla\varphi(t)\|_{L^{\frac{2n}{n-2}}(B_{\frac{11\delta}{10}}(\xi(t)))}\right)\right.\\
&\quad\left.+\|w^*(t)u(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))}\left(\delta^{-1}\|\varphi(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))}+\|\nabla\varphi(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))}\right)\right\}dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq C \int_0^T \left\{ \varepsilon^2 \left\{ \int_{B_{\frac{11\delta}{10}}(\xi(t))} |x-\xi(t)|^{\frac{4n(-n+\beta)}{n+2}} dx \right\}^{\frac{n+2}{2n}} \left(\delta^{\frac{n}{2}-2} \|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))} + \|\varphi(t)\|_{\dot{H}^{2,2}(B_{\frac{11\delta}{10}}(\xi(t)))} \right) \right. \\
&\quad \left. + \varepsilon \left\{ \int_{B_{\frac{11\delta}{10}}(\xi(t))} |x-\xi(t)|^{2(-n+\beta)} dx \right\}^{\frac{1}{2}} \left(\delta^{\frac{n}{2}-1} \|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))} + \|\nabla\varphi(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))} \right) \right\} dt \\
&\leq C\varepsilon^2 \delta^{-\frac{3n}{2}+2\beta+1} \int_0^T \left(\delta^{\frac{n}{2}-2} \|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))} + \|\varphi(t)\|_{\dot{H}^{2,2}((B_r(\xi^{\varepsilon_r}(t)))} \right) dt \\
&\quad + C\varepsilon\delta^{-\frac{n}{2}+\beta} \int_0^T \left(\delta^{\frac{n}{2}-1} \|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))} + \|\nabla\varphi(t)\|_{L^2(B_r(\xi^{\varepsilon_r}(t)))} \right) dt \\
&\leq C \left\{ \varepsilon^2 \left(\delta^{-n+2\beta-1} + \delta^{-\frac{3n}{2}+2\beta+1} \right) + \varepsilon \left(\delta^{\beta-1} + \delta^{-\frac{n}{2}+\beta} \right) \right\},
\end{aligned}
$$

$$
\begin{aligned}
&\left| \int_0^T \int_{B_r(\xi^{\varepsilon_r}(t))} W \cdot (\partial_t v_\delta + \Delta v_\delta + (u,\nabla) v_\delta) dx dt \right| \qquad (4.49) \\
&\leq C \int_0^T \|u(t) - w^*(t)\|_{L^2(D_{\delta,t})} \left\{ (\delta^{-2} + \delta^{-\frac{1}{\alpha}}) \delta^{\frac{n}{2}} \|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))} \right. \\
&\qquad \left. + \|\varphi(t)\|_{\dot{H}^{2,2}(B_{3\delta}(\xi^{\varepsilon\delta}(t)))} + \|\partial_t\varphi(t)\|_{L^2(B_{3\delta}(\xi^{\varepsilon\delta}(t)))} \right\} dt \\
&\quad + \int_0^T \left(\|u(t)\|^2_{L^{\frac{4n}{n+2}}(D(\delta,t))} \|\nabla v_\delta(t)\|_{L^{\frac{2n}{n-2}}(D(\delta,t))} + \|w^*(t) u(t)\|_{L^2(D_{\delta,t})} \|\nabla v_\delta(t)\|_{L^2(D_{\delta,t})} \right) dt \\
&\leq C \int_0^T \left\{ \varepsilon \left\{ \int_{B_{\frac{11\delta}{10}}(\xi(t))} |x-\xi(t)|^{-2n+2\beta} dx \right\}^{\frac{1}{2}} + \|w^*(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))} \right\} \\
&\quad \times \left\{ (\delta^{-2} + \delta^{-\frac{1}{\alpha}}) \delta^{\frac{n}{2}} \|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))} + \|\varphi(t)\|_{\dot{H}^{2,2}(B_r(\xi^{\varepsilon_r}(t)))} + \|\partial_t\varphi(t)\|_{L^2(B_r(\xi^{\varepsilon_r}(t)))} \right\} dt \\
&\quad + C \int_0^T \varepsilon^2 \left\{ \int_{B_{\frac{11\delta}{10}}(\xi(t))} |x-\xi(t)|^{\frac{4n(-n+\beta)}{n+2}} dx \right\}^{\frac{n+2}{2n}} \|v_\delta(t)\|_{\dot{H}^{2,2}(D_{\delta,t})} dt \\
&\quad + C \int_0^T \varepsilon \left\{ \int_{B_{\frac{11\delta}{10}}(\xi(t))} |x-\xi(t)|^{-2n+2\beta} dx \right\}^{\frac{1}{2}} \left\{ \delta^{\frac{n}{2}-1} \|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))} + \|\varphi(t)\|_{\dot{H}^{1,2}(B_{3\delta}(\xi^{\varepsilon\delta}(t)))} \right\} dt \\
&\leq C \left\{ \varepsilon (\delta^{\beta-2} + \delta^{\beta-\frac{1}{\alpha}} + \delta^{-\frac{n}{2}+\beta}) + \delta^{n-\frac{1}{\alpha}} + \delta^{n-2} + \delta^{\frac{n}{2}} \right\} \\
&\quad + C\varepsilon^2 \delta^{-\frac{3n}{2}+2\beta+1} \int_0^T \left(\delta^{\frac{n}{2}-2} \|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))} + \|\varphi(t)\|_{\dot{H}^{2,2}(B_{\frac{11\delta}{10}}(\xi(t)))} \right) dt \\
&\quad + C\varepsilon\delta^{-\frac{n}{2}+\beta} \int_0^T \left(\delta^{\frac{n}{2}-1} \|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))} + \|\nabla\varphi(t)\|_{L^2(B_r(\xi^{\varepsilon_r}(t)))} \right) dt
\end{aligned}
$$

$$\leq C\left\{\varepsilon^2(\delta^{-n+2\beta-1}+\delta^{-\frac{3n}{2}+2\beta+1})+\varepsilon(\delta^{\beta-2}+\delta^{\beta-\frac{1}{\alpha}}+\delta^{-\frac{n}{2}+\beta}+\delta^{\beta-1})+\delta^{n-\frac{1}{\alpha}}+\delta^{n-2}+\delta^{\frac{n}{2}}\right\},$$

$$\left|\int_0^T\int_{B_r(\xi^{\varepsilon_r}(t))}F\cdot(\varphi\eta_\delta-\varphi-v_\delta)dxdt\right| \tag{4.50}$$
$$\leq C\int_0^T\Big(\|-\partial_t w^*(t)+\Delta w^*(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))}+\|(u(t),\nabla)w^*(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))}\Big)$$
$$\times\Big(\|\varphi(t)\|_{L^2(B_{\frac{11\delta}{10}}(\xi(t)))}+\|v_\delta(t)\|_{L^2(D_{\delta,t})}\Big)dt$$
$$\leq C\int_0^T\left\{\delta^{\frac{n}{2}}+\varepsilon\left\{\int_{B_\delta(\xi(t))}|x-\xi(t)|^{2(-n+\beta)}dx\right\}^{\frac{1}{2}}\right\}\delta^{\frac{n}{2}}\|\varphi(t)\|_{L^\infty(B_r(\xi^{\varepsilon_r}(t)))}dt$$
$$\leq C(\delta^n+\varepsilon\delta^\beta)$$

Since $0<\delta<\delta_0$ is arbitrarily taken and since $\beta=\max\{1/\alpha,n-1\}$, by letting $\delta\to+0$ in (4.44), we obtain from (4.45)–(4.50) that

$$\int_{B_r(\xi^{\varepsilon_r}(0))}W(x,0)\cdot\varphi(x,0)dx+\int_0^T\int_{B_r(\xi^{\varepsilon_r}(t))}W\cdot\{\varphi_t+\Delta\varphi+(u,\nabla)\varphi\}dxdt$$
$$+\int_0^T\int_{B_r(\xi^{\varepsilon_r}(t))}F\cdot\varphi dxdt=0.$$

Since $\widetilde{W}=\Psi_*W$, by changing variable $y=\Psi x$ in the above, we obtain the first desired identity (2.15) in Definition 2.3.2.

We shall next show the second identity (2.16). For $\varrho\in H^{1,2}(B_r(\xi^{\varepsilon_r}(t)))$, we have that

$$\left|\int_{B_r(\xi^{\varepsilon_r}(t))}W(t)\cdot\nabla\varrho\,dx\right| \tag{4.51}$$
$$=\left|\int_{B_r(\xi^{\varepsilon_r}(t))\setminus B_\delta(\xi(t))}\mathrm{div}\,u(t)\,\varrho\,dx+\int_{B_\delta(\xi(t))}u(t)\cdot\nabla\varrho\,dx-\int_{B_r(\xi^{\varepsilon_r}(t))}\mathrm{div}\,w^*(t)\,\varrho\,dx\right|$$
$$=\left|\int_{B_\delta(\xi(t))}u(t)\cdot\nabla\varrho\,dx\right|$$
$$\leq C\varepsilon\left\{\int_{B_\delta(\xi(t))}|x-\xi(t)|^{2(-n+\beta)}dx\right\}^{\frac{1}{2}}\|\rho\|_{H^{1,2}(B_r(\xi^{\varepsilon_r}(t)))}$$
$$\leq C\delta^{-\frac{n}{2}+\beta}.$$

Since $0<\delta<\delta_0$ is arbitrary and since $-\frac{n}{2}+\beta>0$, implied by $\beta=\max\{1/\alpha,n-1\}$, by letting $\delta\to+0$ in (4.51) we have that

$$\int_{B_r(\xi^{\varepsilon_r}(t))}W(t)\cdot\nabla\varrho\,dx=0,\quad 0<t<T.$$

Since $\widetilde{W} = \Psi_* W$, again by changing variable $x \to y = \Psi^{-1}(x)$ in the above identity, we obtain (2.16). As a result, we see that u is a very weak solution of (PS) in $Q(r,T)$ in the sense of Definition 2.3.2. This proves Proposition 4.8. $\square$

Step 2. We next show the uniqueness of very weak solutions of (PS). Indeed, we have the following proposition.

Proposition 4.9. *Let u be a solution of (NS) in Q_T satisfying the hypothesis (4.2). Suppose that v is a very weak solution of (PS) in $Q(r,T)$ with the initial data v_0 satisfying $v_0 = u_0|_{B_r(\xi^{\varepsilon_r}(0))}$. There exists $r_2 > 0$ such that if $r < r_2$, then it holds that $u \equiv v$ in $Q(r,T)$.*

Proof. Let u be a solution of (NS) in Q_T satisfying (4.2) and let v be a very weak solution of (PS) in $Q(r,T)$. It suffices to show that

$$\int_0^T \int_{B_r(\xi^{\varepsilon_r}(t))} (u-v)\cdot H dxdt = \int_0^T \int_{B_r} (\widetilde{u}-\widetilde{v})\cdot \widetilde{H} dyds = 0$$

for all $H \in C_0^\infty(Q(r,T))$, where $\widetilde{u} = \Psi_* u$, $\widetilde{v} = \Psi_* v$, and $\widetilde{H} = \Psi_* H$. To show this integral identity, we make use of the duality argument due to Lions-Masmoudi [21]. For every given $\widetilde{H} \in C_0^\infty(\widetilde{Q}(r,T))$ we consider existence of the solution to the following perturbed Stokes equations;

$$\begin{cases} \partial_s \widetilde{\Phi}_r + \Delta_y \widetilde{\Phi}_r + (\frac{d\xi^{\varepsilon_r}}{ds}, \nabla_y)\widetilde{\Phi}_r + (\widetilde{u}, \nabla_y)\widetilde{\Phi}_r - \nabla\widetilde{\pi} = -\widetilde{H} & \text{in } \widetilde{Q}(r,T), \\ \text{div } \widetilde{\Phi}_r = 0 & \text{in } \widetilde{Q}(r,T), \\ \widetilde{\Phi}_r = 0 & \text{on } \partial B_r \times [0,T], \\ \widetilde{\Phi}_r|_{s=T} = 0 & \text{in } B_r. \end{cases}$$

Set $\overline{\Phi}_r(y,\tau) = \widetilde{\Phi}_r(y, T-\tau)$, $\overline{u}$, $\overline{\pi}$, and $\overline{H}$ similarly, and $\overline{\xi}^{\varepsilon_r}(\tau) = \xi^{\varepsilon_r}(T-\tau)$. Changing variables $s = T-\tau$, we obtain that

$$\begin{cases} \partial_\tau \overline{\Phi}_r - \Delta_y \overline{\Phi}_r + (\frac{d\overline{\xi}^{\varepsilon_r}}{d\tau}, \nabla_y)\overline{\Phi}_r - (\overline{u}, \nabla_y)\overline{\Phi}_r + \nabla\overline{\pi} = \overline{H} & \text{in } \widetilde{Q}(r,T), \\ \text{div } \overline{\Phi}_r = 0 & \text{in } \widetilde{Q}(r,T), \\ \overline{\Phi}_r = 0 & \text{on } \partial B_r \times [0,T], \\ \overline{\Phi}_r|_{\tau=0} = 0 & \text{in } B_r. \end{cases} \tag{E'}$$

Since u satisfies the hypothesis (4.2), there exists some constant $r_2 > 0$ such that if $|x - \xi(t)| < \frac{11r_2}{10}$ $(t \in [0,T])$, it holds that $|u(x,t)| \leq \varepsilon |x-\xi(t)|^{-1}$ for all small $\varepsilon > 0$. From this inequality, we have that $|\overline{u}(y, T-\tau)| \leq \varepsilon |y - (\xi(\tau) - \xi^{\varepsilon_r}(\tau))|^{-1}$ for $y \in B_{\frac{11r_2}{10}}(\xi(\tau) - \xi^{\varepsilon_r}(\tau))$ and $\tau \in [0,T]$. Note that $B_r \subset B_{\frac{11r_2}{10}}(\xi(\tau) - \xi^{\varepsilon_r}(\tau))$ for all

$\tau \in [0, T]$ provided $r < r_2$. Hence, taking $r < r_2$, we have that

$$\begin{aligned} \sup_{0<\tau<T} \|\overline{u}(\tau)\|_{L^{n,\infty}(B_r)} &= \sup_{0<\tau<T} \|\overline{u}(T-\tau)\|_{L^{n,\infty}(B_r)} \\ &\leq \sup_{0<\tau<T} \|\overline{u}(T-\tau)\|_{L^{n,\infty}(B_{\frac{11r_2}{10}}(\xi(\tau)-\xi^{\varepsilon_r}(\tau)))} \\ &\leq C\varepsilon \end{aligned} \tag{4.52}$$

Hence, if we take $\varepsilon \leq \varepsilon_0/C$, wee see by (4.52) that $\overline{u}$ satisfies (4.18). Then it follows from Lemma 4.4 that for every $r < r_2$ there exists a pair of solutions $\{\overline{\Phi}_r, \overline{\pi}\}$ to (E') in the class (4.19). Since $\overline{H} \in L^2(0, T; L^2(B_r) \cap L^{\frac{2n}{3}}(B_r))$, from (4.19) we see that $\overline{\Phi} \in H^{1,2}(0, T; L^2_\sigma(B_r)) \cap L^2(0, T; H^{2,2}(B_r)) \cap L^2(0, T; L^\infty(B_r))$, and so does $\widetilde{\Phi}$. From Proposition 4.8 it follows that u is also a very weak solution of (PS) in $Q(r, T)$. Since u and v are very weak solutions of (PS) in $Q(r, T)$, we have by (2.15) and (2.16) that

$$\begin{aligned} &\int_0^T \int_{B_r(\xi^{\varepsilon_r}(t))} (u-v)\cdot H dxdt = \int_0^T \int_{B_r} (\widetilde{u}-\widetilde{v})\cdot \widetilde{H} dyds \\ &= \int_0^T \int_{B_r} (-\widetilde{u}+\widetilde{v})\cdot \left\{\partial_s \widetilde{\Phi}_r + \Delta_y \widetilde{\Phi}_r + \left(\frac{d\xi^{\varepsilon_r}}{ds}, \nabla_y\right)\widetilde{\Phi}_r + (\widetilde{u}, \nabla_y)\widetilde{\Phi}_r - \nabla\widetilde{\pi}\right\} dyds = 0. \end{aligned} \tag{4.53}$$

This proves Proposition 4.9. □

Step 3. We next show that the very weak solution in Proposition 4.9 may be chosen as the Leray-Hopf weak solution in the Serrin class given by Lemma 4.7. Let us consider the problem (PS") again. We first deal with the case where $n = 3$. It is easy to show that $\widetilde{F} \in L^2(B_r \times (0, T))$. Since $\widetilde{W}(0) \in L^{n,\infty}_\sigma(B_r) \subset L^2_\sigma(B_r)$ and $\widetilde{F} \in L^2(B_r \times (0, T))$, by the standard procedure such as the Galerkin method, we can construct a weak solution $\widetilde{W}$ of (PS") in the Leray-Hopf class $L^\infty(0, T; L^2_\sigma(B_r)) \cap L^2(0, T; H^{1,2}_{0,\sigma}(B_r))$. Therefore, we may assume that $\widetilde{W}(\varepsilon) \in H^{1,2}_{0,\sigma}(B_r)$ for any $\varepsilon > 0$. It is noticed that $H^{1,2}_{0,\sigma}(B_r) \subset \mathcal{B}^{\frac{1}{2}}_{2,\frac{4}{3}}(B_r)$. In fact, we see that

$$\begin{aligned} \int_0^\infty \|t^{1-\frac{1}{4}}\Delta e^{t\Delta} f\|^{\frac{4}{3}}_{L^2(B_r)} \frac{dt}{t} &= \int_0^1 \|\mathrm{div}\, e^{t\Delta}\nabla f\|^{\frac{4}{3}}_{L^2(B_r)} dt + \int_1^\infty \|\Delta e^{t\Delta} f\|^{\frac{4}{3}}_{L^2(B_r)} dt \\ &\leq \int_0^1 \left(t^{-\frac{1}{2}}\|\nabla f\|_{L^2(B_r)}\right)^{\frac{4}{3}} dt + \int_1^\infty \left(t^{-1}e^{-t}\|f\|_{L^2(B_r)}\right)^{\frac{4}{3}} dt \\ &\leq \|\nabla f\|^{\frac{4}{3}}_{L^2(B_r)} \int_0^1 t^{-\frac{2}{3}} dt + \|f\|^{\frac{4}{3}}_{L^2(B_r)} \int_1^\infty t^{-\frac{4}{3}} dt \\ &\leq C\|f\|^{\frac{4}{3}}_{H^{1,2}(B_r)} \end{aligned}$$

for all $f \in \mathcal{B}^{\frac{1}{2}}_{2,\frac{4}{3}}(B_r)$. Let $r < r_2$, where r_2 is the same constant as in the proof of Proposition 4.9. Then, similarly to (4.52), it follows from the assumption (4.2) that $\sup_{0<\tau<T}\|\widetilde{u}(\tau)\|_{L^{n,\infty}(B_r)} \le \varepsilon_0$. Now we consider (PS") in $B_r \times (\varepsilon, T)$ for $r < r_2$ with the initial data $\widetilde{W}(\varepsilon)$. Since $\widetilde{w}^* \in C^{2,1}(\overline{B}_r \times [0,T])$, we can easily show that $\widetilde{F} \in L^{\frac{4}{3}}(\varepsilon, T; L^2(B_r))$. Applying Lemma 4.4 to (PS") in $B_r \times (\varepsilon, T)$ for $s = \frac{4}{3}$, $q = 2$, $s_0 = 4$, and $q_0 = 6$, we obtain a strong solution $\widetilde{W}^*$ of (PS") in the class

$$\widetilde{W}^* \in L^4(\varepsilon, T; L^6(B_r)). \tag{4.54}$$

Therefore, if $r < r_2$, we may choose a very weak solution v of (PS) in $Q(r,T)$ in Proposition 4.9 as

$$v = \begin{cases} \Psi_*^{-1}\widetilde{W} + \Psi_*^{-1}\widetilde{w}^* & \text{in } Q(r,\varepsilon), \\ \\ \Psi_*^{-1}\widetilde{W}^* + \Psi_*^{-1}\widetilde{w}^* & \text{in } \displaystyle\bigcup_{\varepsilon \le t < T} B_r(\xi^{\varepsilon_r}(t)) \times \{t\} \end{cases} \tag{4.55}$$

with the initial data $v|_{t=0} = u_0|_{B_r(\xi^{\varepsilon_r}(0))}$.

Since u is a smooth solution of (NS) in Q_T satisfying (4.2), it follows from (4.54), (4.55) and Proposition 4.9 that u is a weak solution of (NS) in $\Omega \times (\varepsilon, T)$ and that

$$u \in L^4(\varepsilon, T; L^6(\Omega)), \quad \partial_t u \in L^{\frac{4}{3}}(\varepsilon, T; L^2(\Omega)). \tag{4.56}$$

Hence it follows from Lemma 4.7 that

$$\partial_t u, \ \frac{\partial^{\alpha_1+\cdots+\alpha_n} u}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \in C(K)$$

for all multi-indices $(\alpha_1, \cdots, \alpha_n)$ and all compact subsets K of $\Omega \times (\varepsilon, T)$, which implies that the curve $\{\xi(t); 0 < t < T\}$ is a family of removable singularities of u in the sense of Definition 1.1.

In the case when $n \ge 4$, since we assume that the initial data u_0 belongs to $\mathcal{B}_{q,s}^{2(1-\frac{1}{s})}(\Omega)$ for $\frac{2}{s} + \frac{n}{q} = 3$ with $\max\{\frac{n}{3}, 2\} < q < n$, it holds that $\Psi_* u_0|_{B_r(\xi^{\varepsilon_r}(0))} \in \mathcal{B}_{q,s}^{2(1-\frac{1}{s})}(B_r)$. From Lemma 4.4 for $a = \Psi_* u_0|_{B_r(\xi^{\varepsilon_r}(0))}$ we obtain a strong solution $\widetilde{W}^*$ of (PS") in $B_r \times (0,T)$ in the class

$$\widetilde{W}^* \in L^{s_0}(0, T; L^{q_0}(B_r)) \tag{4.57}$$

for $\frac{2}{s_0} + \frac{n}{q_0} = 1$ with $s < s_0 < \infty$ and $q < q_0 < \infty$. Choosing a very weak solution v of (PS) in $Q(r,T)$ as $v = \Psi_*^{-1}\widetilde{W} + \Psi_*^{-1}\widetilde{w}^*$, we see from Proposition 4.9 that $u \equiv v$ in $Q(r,T)$, which implies that u is a weak solution of (NS) in $\Omega \times (0,T)$ with the properties that

$$u \in L^4(0, T; L^{2n}(\Omega)), \quad \partial_t u \in L^{\frac{4}{3}}(0, T; L^2(\Omega)).$$

Hence, the desired result is also a consequence of Lemma 4.7. This completes the proof of Theorem 4.1. $\square$

Chapter 5

Solutions with time-dependent singularities

We first define the time-dependent Dirac measure and single layer potential on $\mathbb{R}^n$. Let $\gamma : \mathbb{R}^n \to \mathbb{R}$ and $R : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $\rho(t) > 0$ for all $t \in \mathbb{R}$. Let $S_R(\gamma(t)) = \{x \in \mathbb{R}^n; |x - \gamma(t)| = R\}$ and $S_{\rho(t)} = \{x \in \mathbb{R}^n; |x| = \rho(t)\}$ for each $t \in \mathbb{R}$. Then we define the time-dependent Dirac measure $\delta_{\gamma(t)}$ and single layer potentials $\delta_{S_R(\gamma(t))}$ and $\delta_{S_{\rho(t)}}$ as distributions such as

$$(\delta_{\gamma(t)}, \Phi) = \Phi(\gamma(t)), \tag{5.1}$$

$$(\delta_{S_R(\gamma(t))}, \Phi) = \int_{S_R(\gamma(t))} \Phi(x) d\sigma(x), \tag{5.2}$$

$$(\delta_{S_{\rho(t)}}, \Phi) = \int_{S_{\rho(t)}} \Phi(x) d\sigma(x) \tag{5.3}$$

for $\Phi \in C_0^\infty(\mathbb{R}^n)$. Now our theorems read as follows.

Theorem 5.1. *Let $n = 2$, $1 < p < 2$, $1 \le r \le p$, and $\gamma : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Suppose that a and f are expressed by*

$$a(x) = -\varepsilon(-\Delta)^{-1} \mathrm{rot}\ \delta_{\gamma(0)}(x), \tag{5.4}$$

$$f(x,t) = \varepsilon t^{-\frac{1}{p}} \delta_{\gamma(t)}(x), \tag{5.5}$$

with a small parameter $\varepsilon > 0$. Then there exists a solution on $(0, \infty)$ of

$$\begin{cases} \dfrac{du}{dt} + Au + P(u, \nabla)u = Pf & a.e.\ t \in \mathbb{R}^+ \ in\ \dot{\mathcal{B}}^{-2+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2)\ , \\ u(0) = a & in\ \dot{\mathcal{B}}^{-1+\frac{2}{r}}_{r,\infty}(\mathbb{R}^2), \end{cases} \tag{NS4}$$

in the class

$$u_t, Au \in L^{2,\infty}(0, \infty; \dot{\mathcal{B}}^{-2+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2)) \tag{5.6}$$

Moreover, u satisfies that

$$u \in L^{\alpha_0,\infty}(0, \infty; \dot{\mathcal{B}}^{s_0}_{p_0,1}(\mathbb{R}^2))\ for\ 2/\alpha_0 + 2/p_0 - s_0 = 1 \tag{5.7}$$

with $p \le p_0$, $2 < \alpha_0$, and $-1 + \frac{2}{r} < s_0$.

Theorem 5.2. *Let $n = 3$, $2 \le p < \frac{5}{2}$, $R > 0$, $\gamma : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function, and $\rho : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying that $\rho_0 \le \rho(t) \le \rho_1$ for some $\rho_0, \rho_1 > 0$. Suppose that a and f are expressed by*

$$a(x) = \varepsilon(-\Delta)^{-1}\mathrm{rot}\ P(\delta_{S_{\rho(0)}}(x)e_i) \ \textit{or}\ \varepsilon(-\Delta)^{-1}\mathrm{rot}\ P(\delta_{S_R(\gamma(0))}(x)e_i), \tag{5.8}$$

$$f(x,t) = \varepsilon t^{-\frac{1}{p}}\delta_{S_{\rho(t)}}(x) \ \textit{or}\ \varepsilon t^{-\frac{1}{p}}\delta_{S_R(\gamma(t))}(x), \tag{5.9}$$

with a small parameter $\varepsilon > 0$, where $e_i (i = 1, 2, 3)$ is the canonical basis. Then there exists a solution on $(0,\infty)$ of

$$\begin{cases} \dfrac{du}{dt} + Au + P(u,\nabla)u = Pf & \textit{a.e. } t \in \mathbb{R}^+ \textit{ in } \dot{\mathcal{B}}^{-3+\frac{5}{p}}_{p,\infty}(\mathbb{R}^3)\ , \\ u(0) = a & \textit{in } \dot{\mathcal{B}}^{-1+3/2}_{2,\infty}(\mathbb{R}^3), \end{cases} \tag{NS5}$$

in the class

$$u_t, Au \in L^{p,\infty}(0,\infty; \dot{\mathcal{B}}^{-3+\frac{5}{p}}_{p,\infty}(\mathbb{R}^3)) \tag{5.10}$$

Moreover, u satisfies that

$$u \in L^{\alpha_0,\infty}(0,\infty; \dot{\mathcal{B}}^{s_0}_{p_0,1}(\mathbb{R}^3)) \ for\ 2/\alpha_0 + 3/p_0 - s_0 = 1 \tag{5.11}$$

with $p \le p_0$, $p < \alpha_0$, and $\frac{1}{2} < s_0$.

Remark 5.3. (1)We construct a solution with time-dependent singular point or sets to the Navier-Stokes equations easier than [13]. Moreover, in our main theorem it is enough for us to suppose that γ is continuous. However, we don't know how the solution behaves near the singularities.
(2)We cannot replace $n = 3$ in Theorem 5.1 in our method since we cannot find $1 < \alpha < \infty$ satisfying the condition $2/\alpha + 3/p - s = 3$ in Proposition 2.4.3.

Proof of Theorem 5.1. Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be the Littlewood-Paley decomposition. Since $\delta_{\gamma(t)}$ is the translation operator by $\gamma(t)$ of convolution for each $t \in \mathbb{R}^+$, we have

$$\varphi_j * \delta_{\gamma(t)}(x) = \varphi_j(x - \gamma(t)) = \mathcal{F}^{-1}[\phi(2^{-j}\cdot)](x-\gamma(t)) = 2^{nj}[\mathcal{F}^{-1}\phi](2^j(x-\gamma(t))).$$

Hence, it holds that

$$\|\varphi_j * \delta_{\gamma(t)}\|_{L^p(\mathbb{R}^n)} = 2^{nj}\|[\mathcal{F}^{-1}\phi](n^j(\cdot - \gamma(t)))\|_{L^p(\mathbb{R}^n)} = 2^{nj(1-\frac{1}{p})}\|[\mathcal{F}^{-1}\phi]\|_{L^p(\mathbb{R}^n)} \tag{5.12}$$

From (5.12) we obtain that

$$\delta_{\gamma(t)}(x) \in \dot{B}^{-n+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n), \quad 1 \le p \le \infty,\ n \in \mathbb{N} \tag{5.13}$$

for all $t \in \mathbb{R}^+$. If we take $\varepsilon > 0$ so small, it holds by (5.4), (5.5), and (5.13) that

$$a(x) = -\varepsilon(-\Delta)^{-1}\mathrm{rot}\ \delta_{\gamma(0)}(x) \in \dot{\mathcal{B}}^{-1+\frac{2}{r}}_{r,\infty}(\mathbb{R}^2),\ 1 \le r \le p,$$
$$f(x,t) = \varepsilon t^{-\frac{1}{p}}\delta_{\gamma(t)}(x) \in L^{2,\infty}(0,\infty; \dot{B}^{-2+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2))$$

for $1 < p < \infty$ with

$$\|a\|_{\dot{B}^{-1+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2)} + \|f\|_{L^{p,\infty}(0,\infty;\dot{B}^{-2+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2))} \le \eta.$$

Therefore, from Proposition 2.4.3 with $s = -2 + \frac{2}{p}$, $\alpha = 2$, and $q = \infty$ we obtain the global solution

$$u \in L^{\alpha_0,\infty}(0,\infty; \dot{\mathcal{B}}^{s_0}_{p_0,1}(\mathbb{R}^2)) \text{ for } 2/\alpha_0 + 2/p_0 - s_0 = 1$$

of (NS4) with $p \le p_0$, $2 < \alpha_0$, and $-1 + \frac{2}{p} < s_0$. □

Proof of Theorem 5.2. Let $\delta_{S_R(\gamma(t))}$ and $\delta_{S_{\rho(t)}}$ be the time-dependent single layer potentials defined by (5.2) and (5.3). By the Minkovski inequality we have that

$$\begin{aligned}\|\varphi_j * \delta_{S_R(\gamma(t))}\|_{L^p(\mathbb{R}^3)} &= \left\{\int_{\mathbb{R}^3}\left|\int_{S_R(\gamma(t))}\varphi_j(x-y)d\sigma(y)\right|^p dx\right\}^{\frac{1}{p}} \\ &\le \int_{S_R(\gamma(t))}\left\{\int_{\mathbb{R}^3}|\varphi_j(x-y)|^p dx\right\}^{\frac{1}{p}} d\sigma(y) \\ &= |S_R|\|\varphi_j\|_{L^p(\mathbb{R}^3)}\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}\|\varphi_j * \delta_{S_{\rho(t)}}\|_{L^p(\mathbb{R}^3)} &= \left\{\int_{\mathbb{R}^3}\left|\int_{S_{\rho(t)}}\varphi_j(x-y)d\sigma(y)\right|^p dx\right\}^{\frac{1}{p}} \\ &\le \int_{S_{\rho(t)}}\left\{\int_{\mathbb{R}^3}|\varphi_j(x-y)|^p dx\right\}^{\frac{1}{p}} d\sigma(y) \\ &= |S_{\rho(t)}|\|\varphi_j\|_{L^p(\mathbb{R}^3)} \\ &\le |S_{\rho_1}|\|\varphi_j\|_{L^p(\mathbb{R}^3)}\end{aligned} \tag{5.15}$$

for all $t \in \mathbb{R}^+$ and $1 \le p \le \infty$ where $|S_R| = \int_{S_R} d\sigma$. Taking $p = \infty$, it holds that

$$\|\varphi_j * \delta_{S_R(\gamma(t))}\|_{L^\infty(\mathbb{R}^3)} \le |S_R|\|\varphi_j\|_{L^\infty(\mathbb{R}^3)} = |S_R|\|\mathcal{F}^{-1}[\phi(2^{-j}\cdot)]\|_{L^\infty(\mathbb{R}^3)} = 2^{3j}C_\phi|S_R|,$$
$$\|\varphi_j * \delta_{S_{\rho(t)}}\|_{L^\infty(\mathbb{R}^3)} \le |S_{\rho_1}|\|\varphi_j\|_{L^\infty(\mathbb{R}^3)} = |S_{\rho_1}|\|\mathcal{F}^{-1}[\phi(2^{-j}\cdot)]\|_{L^\infty(\mathbb{R}^3)} = 2^{3j}C_\phi|S_{\rho_1}|,$$

where $C_\phi = \|[\mathcal{F}^{-1}\phi](2^j\cdot)\|_{L^\infty(\mathbb{R}^3)}$. Therefore, we obtain that

$$\sup_{j\in\mathbb{Z}} 2^{-3j}\|\varphi_j * \delta_{S_R(\gamma(t))}\|_{L^\infty(\mathbb{R}^3)}, \sup_{j\in\mathbb{Z}} 2^{-3j}\|\varphi_j * \delta_{S_{\rho(t)}}\|_{L^p(\mathbb{R}^3)} < \infty$$

for all $t \in \mathbb{R}^+$, that is,

$$\delta_{S_R(\gamma(t))}, \delta_{S_{\rho(t)}} \in \dot{B}^{-3}_{\infty,\infty}(\mathbb{R}^3) \tag{5.16}$$

In order to obtain more precise estimates, we use the Fourier transform. We first deal with the case of $\delta_{S_R(\gamma(t))}$. By the symmetry of the sphere we may set $S_R(\gamma(t)) = \{x = R\omega + \gamma(t)\}$, $\omega = (\sin\theta\cos\varphi, \sin\theta\cos\varphi, \cos\theta)$, and $\xi = \xi' + \gamma(t)$ with the angles $0 \leq \theta \leq \pi$ and $0 \leq \varphi < 2\pi$ and $\xi' = (0, 0, -|\xi'|)$. Hence, we have

$$\begin{aligned}
\mathcal{F}[\delta_{S_R(\gamma(t))}](\xi - \gamma(t)) &= \int_{S_R(\gamma(t))} e^{-ix\cdot(\xi-\gamma(t))} dx \\
&= \int_0^\pi d\theta \int_0^{2\pi} e^{i(R|\xi'|\cos\theta - \gamma_3(t)|\xi'|)} R^2 \sin\theta \, d\theta d\varphi \\
&= 4\pi R e^{-\gamma_3(t)|\xi'|} \frac{\sin R|\xi'|}{|\xi'|},
\end{aligned}$$

which yields

$$\mathcal{F}[\delta_{S_R(\gamma(t))}](\xi) = 4\pi R e^{-\gamma_3(t)|\xi|} \frac{\sin R|\xi|}{|\xi|}.$$

By the Plancherel theorem we have

$$\begin{aligned}
\|\varphi_j * \delta_{S_R(\gamma(t))}\|^2_{L^2(\mathbb{R}^3)} = \|\mathcal{F}[\varphi_j]\mathcal{F}[\delta_{S_R(\gamma(t))}]\|^2_{L^2(\mathbb{R}^3)} &\leq CR^2 \int_{2^{j-1} \leq |\xi| \leq 2^{j+1}} \frac{|\phi(2^{-j}\xi)|^2}{|\xi|^2} d\xi \\
&= CR^2 2^j \int_{\frac{1}{2} \leq |\xi| \leq 2} \frac{|\phi(\xi)|^2}{|\xi|^2} d\xi \leq C 2^j
\end{aligned} \tag{5.17}$$

It follows from (5.17) that

$$2^{-\frac{j}{2}} \|\varphi_j * \delta_{S_R(\gamma(t))}\|_{L^2(\mathbb{R}^3)} \leq C$$

for all $t \in \mathbb{R}^+$ and $j \in \mathbb{Z}$, which shows that

$$\delta_{S_R(\gamma(t))} \in \dot{B}^{-\frac{1}{2}}_{2,\infty}(\mathbb{R}^3). \tag{5.18}$$

From the interpolation of (5.16) and (5.18) we obtain that

$$\delta_{S_R(\gamma(t))} \in \dot{B}^{-3+\frac{5}{p}}_{p,\infty}(\mathbb{R}^3), \; 2 \leq p \leq \infty. \tag{5.19}$$

Next we deal with the case of $\delta_{S_{\rho(t)}}$. Similarly, we may set $S_{\rho((t)} = \{x = \rho(t)\omega\}$, $\omega = (\sin\theta\cos\varphi, \sin\theta\cos\varphi, \cos\theta)$, and $\xi = (0, 0, -|\xi|)$. Hence, we have

$$\begin{aligned}
\mathcal{F}[\delta_{S_{\rho(t)}}](\xi) &= \int_{S_{\rho(t)}} e^{-ix\cdot\xi} dx \\
&= \int_0^\pi \int_0^{2\pi} e^{i\rho(t)|\xi|\cos\theta} \rho(t)^2 \sin\theta \, d\theta d\varphi \\
&= 4\pi\rho(t) \frac{\sin\rho(t)|\xi|}{|\xi|}
\end{aligned}$$

By the Plancherel theorem we have

$$\|\varphi_j * \delta_{S_{\rho(t)}}\|^2_{L^2(\mathbb{R}^3)} = \|\mathcal{F}[\varphi_j]\mathcal{F}[\delta_{S_{\rho(t)}}]\|^2_{L^2(\mathbb{R}^3)} \leq C\rho(t)^2 \int_{2^{j-1}\leq|\xi|\leq 2^{j+1}} \frac{|\phi(2^{-j}\xi)|^2}{|\xi|^2} d\xi \tag{5.20}$$
$$\leq C\rho_1^2 2^j \int_{\frac{1}{2}\leq|\xi|\leq 2} \frac{|\phi(\xi)|^2}{|\xi|^2} d\xi \leq C2^j$$

It follows from (5.20) that

$$2^{-\frac{j}{2}}\|\varphi_j * \delta_{S_{\rho(t)}}\|_{L^2(\mathbb{R}^3)} \leq C$$

for all $t \in \mathbb{R}^+$ and $j \in \mathbb{Z}$, which shows that

$$\delta_{S_{\rho(t)}} \in \dot{B}^{-\frac{1}{2}}_{2,\infty}(\mathbb{R}^3). \tag{5.21}$$

From the interpolation of (5.16) and (5.21) we obtain that

$$\delta_{S_{\rho(t)}} \in \dot{B}^{-3+\frac{5}{p}}_{p,\infty}(\mathbb{R}^3),\ 2 \leq p \leq \infty. \tag{5.22}$$

If we take $\varepsilon > 0$ so small, it holds by (5.8), (5.9), and (5.22) that

$$a(x) = \varepsilon(-\Delta)^{-1}\text{rot } P(\delta_{S_{\rho(0)}}(x)e_i) \ or \ \varepsilon(-\Delta)^{-1}\text{rot } P(\delta_{S_{R(\gamma(0))}}(x)e_i) \in \dot{\mathcal{B}}^{-1+\frac{3}{2}}_{2,\infty}(\mathbb{R}^3),$$
$$f(x,t) = \varepsilon t^{-\frac{1}{p}}\delta_{S_{\rho(t)}}(x) \ or \ \varepsilon t^{-\frac{1}{p}}\delta_{S_{R(\gamma(t))}}(x) \in L^{p,\infty}(0,\infty;\dot{B}^{-3+\frac{5}{p}}_{p,\infty}(\mathbb{R}^3))$$

for $2 \leq p \leq \infty$ with

$$\|a\|_{\dot{B}^{-1+3/2}_{2,\infty}(\mathbb{R}^3)} + \|f\|_{L^{p,\infty}(0,\infty;\dot{B}^{-3+\frac{5}{p}}_{p,\infty}(\mathbb{R}^3))} \leq \eta.$$

Therefore, from Proposition 2.4.3 with $s = -3 + \frac{5}{p}$, $\alpha = p$, $q = \infty$, and $r = 2$ we obtain the global solution

$$u \in L^{\alpha_0,\infty}(0,\infty;\dot{\mathcal{B}}^{s_0}_{p_0,1}(\mathbb{R}^3)) \text{ for } 2/\alpha_0 + 3/p_0 - s_0 = 1$$

of (NS5) with $p \leq p_0$, $p < \alpha_0$, and $1/2 < s_0$. This proves Theorem 5.2. $\square$

Chapter 6

Appendix

In this Appendix, we prove the existence of functions $f^i_{r,t}$, $i = 1, \cdots, N_0$ satisfying in the proof of Proposition 3.3. More precisely, we have the following proposition.

Proposition. *Let* $\psi^{i,j}_{r,t}$, $(i,j) \in \{1, \cdots, N_0\}^2$ *be as in* (3.2). *There exist* $\{a^i_{j,k}\}_{1\le i,j,k\le N_0}$ *such that the family* $\{f^i_{r,t}\}_{i=1}^{N_0}$ *defined by* (3.3) *satisfies the property* (3.4).

Proof. Since $D_{r,t}$ is the annulus region with the radius between $7r/10$ and $4r/5$ for all $t \in [0,T]$, it suffices to prove (3.4) for $t = 0$. Let $L^i_r := \int_{D_{r,0}} \phi^i(x,0) f_r(x,0)dx$. Since $\{f^i_{r,0}\}_{i=1}^{N_0}$ satisfies $\int_{D_{r,0}} f^i_{r,0}\, dx = 0$, $i = 1, \cdots, N_0$, $\{a^i_{j,k}\}_{1\le i,j,k\le N_0}$ needs to fulfill that

$$L^i_r + \sum_{\substack{1\le j\le N_0 \\ 1\le k\le N_0}} a^i_{j,k} L^j_r \langle i,k\rangle = 0, \tag{6.1}$$

where

$$\langle i,j\rangle = \begin{cases} 0 & i = j \ \text{ or } \ D^i_{r,0} \cap D^j_{r,0} = \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, since $f_r(x,0) = \sum_{i=1}^{N_0} f^i_{r,0}(x)$, we see that $\{a^i_{j,k}\}_{1\le i,j,k\le N_0}$ needs to fulfill that

$$\begin{aligned} &\sum_{1\le j\le N_0} a^1_{j,1} L^j_r \psi^{1,1}_{r,0}(x) \ + \cdots + \sum_{1\le j\le N_0} a^1_{j,N_0} L^j_r \psi^{1,N_0}_{r,0}(x) + \cdots \\ &\cdots + \sum_{1\le j\le N_0} a^{N_0}_{j,1} L^j_r \psi^{N_0,1}_{r,0}(x) \ + \cdots + \sum_{1\le j\le N_0} a^{N_0}_{j,N_0} L^j_r \psi^{N_0,N_0}_{r,0}(x) \\ &= 0. \end{aligned} \tag{6.2}$$

Since $\psi^{i,j}_{r,0}(x) = \psi^{j,i}_{r,0}(x)$ for $1 \leq i, j \leq N_0$, it follows from (6.2) that

$$\begin{cases} \displaystyle\sum_{1\leq j\leq N_0} a^i_{j,i} L^j_r = 0 & \text{for } 1 \leq i \leq N_0, \\ \displaystyle\sum_{1\leq j\leq N_0} (a^i_{j,k} + a^k_{j,i}) L^j_r = 0 & \text{for } 1 \leq i, k \leq N_0,\ i \neq k. \end{cases} \tag{6.3}$$

Hence, for the validity of (3.4), it suffices to show that there exist $\{a^i_{j,k}\}_{1\leq i,j,k\leq N_0}$ such that (6.1) and (6.3) fulfilled for any $r > 0$. By using $\sum_{i=1}^{N_0} L^i_r = \int_{D_{r,0}} f_r(x, 0)\, dx = 0$, we obtain the following equations from (6.1) and (6.3):

$$\begin{cases} \displaystyle -\sum_{1\leq k\leq n_0} a^i_{i,k}\langle i, k\rangle + \sum_{1\leq k\leq N_0} a^i_{j,k}\langle i, k\rangle = 1 & \text{for } 1 \leq i, j \leq N_0,\ i \neq j, \\ a^k_{1,k} = \cdots = a^k_{N_0,k} & \text{for } 1 \leq k \leq N_0, \\ a^i_{1,j} + a^j_{1,i} = \cdots = a^i_{N_0,j} + a^j_{N_0,i} & \text{for } 1 \leq i, j \leq N_0,\ i \neq j, \end{cases} \tag{6.4}$$

It is easy to see that (6.4) is equivalent to the following linear system:

$$A\boldsymbol{x} = b \tag{6.5}$$

with

$$A = \begin{bmatrix} F_1 & \cdots & F_{N_0^2} \\ G_1 & \cdots & G_{N_0^2} \end{bmatrix},\ b = {}^t\big[\underbrace{0\ \cdots\ \cdots\ ,\ \cdots\ ,\ 0}_{\frac{1}{2}N_0(N_0-1)(N_0+1)}\ \underbrace{1\ \cdots\ 1}_{N_0(N_0-1)}\big],$$

$$\boldsymbol{x} = {}^t\underbrace{\big[a^1_{1,1}\ \cdots\ a^1_{N_0,1}\ a^1_{1,2}\ \cdots\ a^1_{N_0,N_0}\ a^2_{1,1}\ \cdots\ a^{N_0}_{N_0,N_0}\big]}_{N_0^3}.$$

Here, by a direct calculation, we see that the $\frac{1}{2}N_0(N_0-1)(N_0+1) \times N_0$ matrix F_i $(1 \leq i \leq N_0^2)$ and the $N_0(N_0-1) \times N_0$ matrix G_i $(1 \leq i \leq N_0^2)$ are expressed by

$$F_{(k-1)N_0+k} = \underbrace{\left[\begin{array}{cccc} & O & & \\ \hline 1 & -1 & & O \\ \vdots & & \ddots & \\ \vdots & & & \ddots \\ 1 & O & & -1 \\ \hline & O & & \end{array}\right]}_{N_0} \begin{array}{l} \left.\vphantom{O}\right\}(k-1)(N_0-1) \\ \left.\vphantom{\begin{array}{c}1\\ \vdots\\ \vdots\\ 1\end{array}}\right\}N_0-1 \\ \left.\vphantom{O}\right\}\frac{1}{2}(N_0-1)(N_0^2+N_0-2k) \end{array}$$

for $1 \le k \le N_0$,

$$F_{(k-1)N_0+l} = F_{(l-1)N_0+k} = \begin{array}{c} \left[\begin{array}{cccc} & O & & \\ \hline & O & & \\ \hline 1 & -1 & & O \\ \vdots & & \ddots & \\ \vdots & & & \ddots \\ 1 & O & & -1 \\ \hline & O & & \end{array}\right] \\ \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{N_0} \end{array} \begin{array}{l} \}\sum_{i=1}^{k}(N_0-i+1)(N_0-1) \\ \}(l-k-1)(N_0-1) \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\}N_0-1 \\ \}\frac{1}{2}(N_0-1)\{N_0^2+(1-2k)N_0+k^2+k-2l\} \\ \end{array}$$

for $1 \le k < l \le N_0$, and

$$G_{(k-1)N_0+l} = \begin{array}{c} \left[\begin{array}{ccc|c|ccc} & & & O & & & \\ \hline \langle k,l\rangle & & O & -\langle k,l\rangle & & O & \\ & \ddots & & \vdots & & & \\ O & & \langle k,l\rangle & \vdots & & & \\ \hline & & & \vdots & \langle k,l\rangle & & O \\ & O & & \vdots & & \ddots & \\ & & & -\langle k,l\rangle & O & & \langle k,l\rangle \\ \hline & & & O & & & \end{array}\right] \\ \underbrace{\qquad\qquad\qquad\qquad}_{k-1}\qquad\qquad\underbrace{\qquad\qquad\qquad\qquad}_{N_0-k} \end{array} \begin{array}{l} \}(k-1)(N_0-1) \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\\1\end{matrix}}\right\}N_0-1 \\ \}(N_0-k)(N_0-1) \\ \end{array}$$

for $1 \le k, l \le N_0$. To solve the linear system (6.5), we need to investigate the rank of the matrix A. For this purpose, we find an elementary matrix A' equivalent to A by elementary row and column operations.

1. Add the $(k-1)N_0+1$-st column of A to the $(k-1)N_0+2$-nd column and then subtract the $(k-1)N_0+2$-nd column from $(k-1)N_0+1$-st column $1 \le k \le N_0^2$. By adding the $(k-1)N_0+l$-th column of A to the $(k-1)N_0+l+1$-th column and then subtract the $(k-1)N_0+l+1$-th column from $(k-1)N_0+l$-th column in order from

$l = 2$ to $l = N_0 - 1$, we have that

$$F_{(k-1)N_0+k} \longrightarrow \underbrace{\left[\begin{array}{cc} \multicolumn{2}{c}{O} \\ \hline E_{N_0-1} & 0 \\ \hline \multicolumn{2}{c}{O} \end{array}\right]}_{N_0} \begin{array}{l} \left.\right\} (k-1)(N_0-1) \\ \left.\right\} N_0 - 1 \\ \left.\right\} \frac{1}{2}(N_0-1)(N_0^2+N_0-2k) \end{array}$$

for $1 \leq k \leq N_0$ with the identity matrix E_{N_0-1} of size $N_0 - 1$,

$$F_{(k-1)N_0+l} = F_{(l-1)N_0+k} \longrightarrow \underbrace{\left[\begin{array}{cc} \multicolumn{2}{c}{O} \\ \hline \multicolumn{2}{c}{O} \\ \hline E_{N_0-1} & 0 \\ \hline \multicolumn{2}{c}{O} \end{array}\right]}_{N_0} \begin{array}{l} \left.\right\} \sum_{i=1}^{k}(N_0-i+1)(N_0-1) \\ \left.\right\} (l-1)(N_0-1) \\ \left.\right\} N_0 - 1 \\ \left.\right\} \frac{1}{2}(N_0-1)(N_0^2+N_0-2k) \end{array}$$

for $1 \leq k < l \leq N_0$,

$$G_k \longrightarrow \underbrace{\left[\begin{array}{cccc|c} -\langle 1,k\rangle & & & O & 0 \\ & \ddots & & & \vdots \\ & & \ddots & & \vdots \\ O & & & -\langle 1,k\rangle & 0 \\ \hline \multicolumn{5}{c}{O} \end{array}\right]}_{N_0} \begin{array}{l} \left.\vphantom{\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}}\right\} N_0 - 1 \\ \left.\right\} (N_0-1)^2 \end{array}$$

for $1 \leq k \leq N_0$, and

$$
G_{(k-1)N_0+l} \longrightarrow
\begin{array}{c}
\left[\begin{array}{ccc|c|cccc}
 & & & O & & & & \\
\hline
0 & \cdots & 0 & \langle k,l\rangle & & & & \\
-\langle k,l\rangle & & O & \vdots & & O & & \\
 & \ddots & & \vdots & & & & \\
O & & -\langle k,l\rangle & \vdots & & & & \\
\cline{1-3}\cline{5-8}
 & & & \vdots & -\langle k,l\rangle & & O & 0 \\
 & O & & \vdots & & \ddots & & \vdots \\
 & & & \langle k,l\rangle & O & & -\langle k,l\rangle & 0 \\
\hline
 & & & O & & & & \\
\end{array}\right] \\
\underbrace{\hphantom{-\langle k,l\rangle \quad \ddots \quad -\langle k,l\rangle}}_{k-2} \qquad\quad \underbrace{\hphantom{-\langle k,l\rangle \quad \ddots \quad -\langle k,l\rangle \quad 0}}_{N_0-k+1}
\end{array}
\begin{array}{l}
\left.\vphantom{\begin{array}{c}O\\O\end{array}}\right\}(k-1)(N_0-1) \\
\left.\vphantom{\begin{array}{c}0\\0\\0\\0\\0\\0\\0\end{array}}\right\}N_0-1 \\
\left.\vphantom{\begin{array}{c}O\\O\end{array}}\right\}(N_0-k)(N_0-1)
\end{array}
$$

for $2 \leq k \leq N_0$ and $1 \leq l \leq N_0$.

2. We can transform

$$
\begin{aligned}
F_{(k-1)N_0+1} &\longrightarrow F_{(k-1)N_0+1} - F_k = O, \\
G_{(k-1)N_0+1} &\longrightarrow G_{(k-1)N_0+1} - G_k \equiv \begin{bmatrix} G^{+}_{(k-1)N_0+1} \\ G^{-}_{(k-1)N_0+1} \end{bmatrix}
\end{aligned}
$$

with

$$
G^{+}_{(k-1)N_0+1} =
\begin{array}{c}
\underbrace{\left[\begin{array}{cccc|c}
\langle 1,k\rangle & & & O & 0 \\
 & \ddots & & & \vdots \\
 & & \ddots & & \vdots \\
O & & & \langle 1,k\rangle & 0 \\
\hline
 & & O & & \\
\end{array}\right]}_{N_0}
\end{array}
\begin{array}{l}
\left.\vphantom{\begin{array}{c}0\\0\\0\\0\end{array}}\right\}N_0-1 \\
\left.\vphantom{\begin{array}{c}O\\O\end{array}}\right\}(k-2)(N_0-1)
\end{array}
$$

and

$$G^{-}_{(k-1)N_0+1} = \begin{array}{ll}
\left[\begin{array}{ccc|c|cccc}
0 & \cdots & 0 & \langle k,1\rangle & & & & \\
-\langle k,1\rangle & & O & \vdots & & & & \\
& \ddots & & \vdots & & & O & \\
O & & -\langle k,1\rangle & \vdots & & & & \\
\cline{1-3}\cline{5-8}
& & & \vdots & -\langle k,1\rangle & & O & 0 \\
& O & & \vdots & & \ddots & & \vdots \\
& & & \langle k,1\rangle & O & & -\langle k,1\rangle & 0 \\
\hline
& & & & & & & \\
& & & O & & & & \\
& & & & & & &
\end{array}\right]
&
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\end{array}}\right\} N_0-1 \\
\left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} (N_0-k)(N_0-1)
\end{array}
\\
\quad \underbrace{\hphantom{-\langle k,1\rangle \quad \cdots \quad -\langle k,1\rangle}}_{k-2} \hphantom{\langle k,1\rangle} \underbrace{\hphantom{-\langle k,1\rangle \quad \ddots \quad -\langle k,1\rangle \quad 0}}_{N_0-k+1} &
\end{array}$$

for $2 \leq k \leq N_0$. Organizing G_k by using F_k $1 \leq k \leq N_0$, we have that for $1 \leq k \leq N_0$

$$G_k \longrightarrow O.$$

3. As in the operation **2**, we can transform

$$\begin{aligned}
F_{(l-1)N_0+k} &\longrightarrow F_{(l-1)N_0+k} - F_{(k-1)N_0+l} = O, \\
G_{(l-1)N_0+k} &\longrightarrow G_{(l-1)N_0+k} - G_{(k-1)N_0+l} \equiv \begin{bmatrix} G^{+}_{(l-1)N_0+k} \\ G^{-}_{(l-1)N_0+k} \end{bmatrix}
\end{aligned}$$

with

$$G^{+}_{(l-1)N_0+k} = \begin{array}{c} \left[\begin{array}{ccc|c|cccc}
\multicolumn{8}{c}{O} \\ \hline
0 & \cdots & 0 & -\langle k,l\rangle & & & & \\
\langle k,l\rangle & & O & \vdots & & & & \\
 & \ddots & & \vdots & & O & & \\
O & & \langle k,l\rangle & \vdots & & & & \\ \cline{1-3}\cline{5-8}
 & & & \vdots & \langle k,l\rangle & & O & 0 \\
 & O & & \vdots & & \ddots & & \vdots \\
 & & & -\langle k,l\rangle & O & & \langle k,l\rangle & 0 \\ \hline
\multicolumn{8}{c}{O}
\end{array}\right] \\
\underbrace{\hphantom{0\cdots0\cdots0}}_{k-2} \hphantom{-\langle k,l\rangle} \underbrace{\hphantom{\langle k,l\rangle\cdots O\cdots 0}}_{N_0-k+1}
\end{array}
\begin{array}{l}
\}\,(k-1)(N_0-1) \\
\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} N_0-1 \\
\}\,(l-k-1)(N_0-1)
\end{array}$$

and

$$G^{-}_{(l-1)N_0+k} = \begin{array}{c} \left[\begin{array}{ccc|c|cccc}
0 & \cdots & 0 & \langle l,k\rangle & & & & \\
-\langle l,k\rangle & & O & \vdots & & & & \\
 & \ddots & & \vdots & & O & & \\
O & & -\langle l,k\rangle & \vdots & & & & \\ \cline{1-3}\cline{5-8}
 & & & \vdots & -\langle l,k\rangle & & O & 0 \\
 & O & & \vdots & & \ddots & & \vdots \\
 & & & \langle l,k\rangle & O & & -\langle l,k\rangle & 0 \\ \hline
\multicolumn{8}{c}{O}
\end{array}\right] \\
\underbrace{\hphantom{0\cdots0\cdots0}}_{l-2} \hphantom{\langle l,k\rangle} \underbrace{\hphantom{\langle l,k\rangle\cdots O\cdots 0}}_{N_0-l+1}
\end{array}
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}\right\} N_0-1 \\
\}\,(N_0-l)(N_0-1)
\end{array}$$

for $2 \le k < l \le N_0$. Organizing $G_{(k-1)N_0+l}$ by using $F_{(k-1)N_0+l}$, we have that for $2 \le k < l \le N_0$

$$G_{(k-1)N_0+l} \longrightarrow O.$$

From the operations **1**, **2**, and **3** we see that by excluding 0 vectors and then by organizing A we can transform

$$A \longrightarrow A' \equiv \left[\begin{array}{c|c} E_{\frac{1}{2}N_0(N_0-1)(N_0+1)} & O \\ \hline O & B \end{array}\right],$$

with the identity matrix $E_{\frac{1}{2}N_0(N_0-1)(N_0+1)}$ of size $\frac{1}{2}N_0(N_0-1)(N_0+1)$ where the $N_0(N_0-1) \times \frac{1}{2}N_0(N_0-1)^2$ matrix B is the arrangement of $G_{(k-1)N_0+l}$ for $1 \leq l < k \leq N_0$, i.e., B is expressed by

$$B = \left[\begin{array}{ccccc} G_{N_0+1} & G_{2N_0+1} & G_{2N_0+2} & \cdots & G_{N_0^2-1} \end{array}\right].$$

Finally, to show the existence of $\{a^i_{j,k}\}_{1\leq i,j,k\leq N_0}$ satisfying (6.4) we show that A' has the same rank as the augmented matrix $[A'|b]$, that is, rank A' = rank $[A'|b]$. It is noted that b is unchanged by the operation **1-3** and that $N_0(N_0-1) < \frac{1}{2}N_0(N_0-1)^2$ since in general it holds that $N_0 > 3$.

If necessary by changing the annulus $D^2_{r,0}$ to another, we may suppose that $\langle 1, 2\rangle = 1$. By elementary row and column operations we can transform

$$G_{(k-1)n_0+1} \longrightarrow G_{(k-1)N_0+1} - \langle 1, k\rangle G_{N_0+1}$$

for $3 \leq k \leq N_0$. Since the 1st, $\cdots$, (N_0-1)-th row of $G_{(k-1)N_0+l}$ is 0 vector for $1 \leq l < k \leq N_0$ with $k \geq 3$, organizing G_{N_0+1} we have that

$$G_{N_0+1} \longrightarrow \underbrace{\left[\begin{array}{c} E_{N_0-1} \quad 0 \\ \hline O \end{array}\right]}_{N_0} \begin{array}{l} \}N_0-1 \\ \}(N_0-1)^2 \end{array}$$

Now we define H_l by $H_l := \langle 1, l\rangle + \cdots \langle l-1, l\rangle$ and let $k \geq 3$. Generally, since $\bigcup_{i=1}^{l-1} D^i_{r,0}$ and $\bigcup_{i=l}^{N_0} D^i_{r,0}$ have an intersection, if necessary we can rearrange $D^l_{r,0}$ so that

$\bigcup_{i=1}^{l-1} D_{r,0}^i$ and $D_{r,0}^l$ have an intersection. Then, it holds that $H_l \geq 1$ for $2 \leq l \leq N_0$. By elementary row and column operations we have that

$$G_{(k-1)N_0+1} \longrightarrow \sum_{i=1}^{k-1} G_{(k-1)N_0+i} \tag{6.6}$$

and then we obtain that

$$G_{(k-1)N_0+m} \longrightarrow G_{(k-1)N_0+m} - \frac{\langle k, m\rangle}{H_k} G_{(k-1)N_0+1} = O \tag{6.7}$$

for $2 \leq m \leq k-1$ and

$$G_{(l-1)N_0+m} \longrightarrow G_{(l-1)N_0+m} - \frac{\langle l, m\rangle}{H_k} G_{(k-1)N_0+1} \tag{6.8}$$

for $1 \leq m \leq k-1$ and $k+1 \leq l \leq N_0$. Since the 1st, $\cdots$, $(k-1)(N_0-1)$-th row of $G_{(l-1)N_0+m}$ is 0 vector for $1 \leq m < l \leq N_0$ with $l \geq k+1$, multiplying $G_{(k-1)N_0+1}$ by $1/H_k$ and then organizing $G_{(k-1)N_0+1}$, we have that

$$G_{(k-1)N_0+1} \longrightarrow \underbrace{\left[\begin{array}{c} O \\ \hline E_{N_0-1} \quad 0 \\ \hline O \end{array}\right]}_{N_0} \begin{array}{l} \}(k-2)(N_0-1) \\ \}N_0-1 \\ \}(N_0-k+1)(N_0-1) \end{array}$$

We see from repeating elementary row and colume operations (6.6)$-$(6.9) from $k=3$ to $k=N_0$ that rank $B=$ rank $\left[\begin{array}{ccccc} G_{N_0+1} & G_{2N_0+1} & G_{2N_0+2} & \cdots & G_{N_0^2-1}\end{array}\right] = (N_0-1)^2$. Since by these operations b is transformed into

$$b \longrightarrow {}^t[\underbrace{0 \ \cdots \ \cdots, \ \cdots, \ 0}_{\frac{1}{2}N_0(N_0-1)(N_0+1)} \ 0 \ \underbrace{1 \ \cdots \ 1}_{N_0-2} \ 0 \ 0 \ \underbrace{1 \ \cdots \ 1}_{N_0-3} \ \cdots \ \underbrace{0 \ \cdots \ 0}_{N_0-1}],$$

rank $A' =$ rank $[A'|b] = \frac{1}{2}(N_0-1)(N_0^2+3N_0-2)$ holds. This shows the conclusion. $\square$

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List of original papers

The papers composing this doctoral thesis are as follows.

1. Kozono, H., Ushikoshi, E., Wakabayashi, F., Removable time-dependent singularities of solutions to the Stokes equations, Journal of Differential Equations, published online January 2023, Volume 342, 472-489.